any direction.

July 15, 1977.

W. Hahn: Über Orthogonalpolynome, die q-Differenzen gleichungen genügen, Math. Nach. 2 [redacted] (1949) 4-34.

Suppose L is [redacted] a linear operator on polynomials decreasing degree by 1 such that one has the commutation formula with x as follows:

$$L(xf) = f + (qx+\omega)Lf$$

Then

$$L(x^n) = x^{n-1} + (qx+\omega)\{x^{n-2} + (qx+\omega)Lx^{n-2}\}$$

$$= x^{n-1} + (qx+\omega)x^{n-2} + \cdots + (qx+\omega)^{n-1}$$

$$= [redacted] \frac{(qx+\omega)^n - x^n}{(qx+\omega) - x}$$

So in general

$$(Lf)(x) = \frac{f(qx+\omega) - f(x)}{(qx+\omega) - x}$$

(and when $qx+\omega = x$ this is to be interpreted as $\frac{d}{dx}$).

If we change variables $x = az+b$, then

$$a(\tilde{z})+b \text{ [redacted]} = qx+\omega = q(az+b)+\omega$$

[redacted]

or

$$\tilde{z} = qz + \frac{\omega + qb - b}{a}$$

Hence if $q \neq 1$ we can pick $a = 1$, $\omega + (q-1)b = 0$, and so arrange that $\omega = 0$.

Therefore we see that there are essentially three cases of L to study:

i) $L = \frac{d}{dx}$ or $qx + \omega = x$ whence you have differential equations

ii) $L = \Delta$ or $qx + \omega = x + 1$, whence you have difference equations.

iii) $Lf = \partial f = \frac{f(qx) - f(x)}{(q-1)x}$, whence you have what Hahn calls q-difference equations.

Example: Put

$$y_a(x) = \sum_{n \geq 0} \frac{a(a+1)\cdots(a+n-1)}{n!} x^n$$

Then

$$y_{a+1}(x) - y_a(x) = x\, y_{a+1}(x) \quad \text{or}$$

$$(1-x)\, y_{a+1}(x) = y_a(x)$$

Also

$$\frac{d}{dx} y_a(x) = a\, y_{a+1}(x) \quad , \text{so}$$

$$(1-x) \frac{d}{dx} y_a(x) = a\, y_a(x)$$

$$\frac{y_a'}{y_a} = \frac{a}{1-x} = -\frac{a}{x-1}$$

$$\log y_a = -a \log(1-x) + \cancel{const}$$

$$y_a(x) = (1-x)^{-a}$$

The q-analogue goes as follows. Put $[a] = \frac{q^a - 1}{q-1}$.

$$y_a(x) = \sum_{n \geq 0} \frac{[a] \cdots [a+n-1]}{[1] \cdots [n]} x^n .$$

Then $[a+n] - [a] = \frac{q^{a+n} - q^a}{q-1} = q^a [n]$, so

$$y_{a+1}(x) - y_a(x) = q^a x \, y_{a+1}(x)$$

or

$$\boxed{(1 - q^a x) \, y_{a+1}(x) = y_a(x)}$$

$$(\Theta y_a)(x) = \sum_{n \geq 0} \frac{[a] \cdots [a+n-1]}{[1] \quad [n]} \frac{q^n - 1}{q-1} x^{n-1}$$

$\Theta x^n = [n] x^{n-1}$

so

$$\boxed{(\Theta y_a)(x) = \frac{y_a(qx) - y_a(x)}{(q-1)x} = [a] \, y_{a+1}(x)}$$

$$(1 - q^a x) \frac{y_a(qx) - y_a(x)}{\cancel{(q-1)} x} = \frac{q^a - 1}{\cancel{q-1}} y_a$$

$$\frac{y_a(qx)}{y_a(x)} = 1 + \frac{(q^a - 1)x}{1 - q^a x} = \frac{1 - x}{1 - q^a x}$$

So

$$y_a(x) = \frac{1 - q^a x}{1 - x} y_a(qx) = \frac{(1 - q^a x)(1 - q^{a+1} x)}{(1-x)(1-qx)} y_a(q^2 x)$$

Assuming $|q| < 1$ (think of q as $e^{2\pi i \tau}$ with $\operatorname{Im} \tau > 0$), one has $y_a(q^n x) \to 1$ as $n \to \infty$, so

$$y_a(x) = \prod_{j \geq 0} \left(\frac{1 - q^{a+j} x}{1 - q^j x} \right) .$$

Amazingly enough this approaches $(1-x)^{-a}$ as $q \to 1$.

What is the Δ-analogue of the above calculation?

Let us regard as basic the difference equations with constant coefficients constructed from the operator L. In other words L is the basic gadget; its eigenfunctions are the analogues of exponential functions, perhaps first order L-equations with rational coeffs. will explain what are the analogues of Γ-functions.

Again, let $$(Lf)(x) = \frac{f(qx+\omega) - f(x)}{(qx+\omega) - x}.$$ Put

$$p_n(x) = x(x-[1]\omega)\cdots(x-[n-1]\omega)$$

and note that

$$\begin{aligned}(qx+\omega-[i]\omega) &= (qx+\omega-(1+\cdots+q^{i-1})\omega) \\ &= q(x-[i-1]\omega)\end{aligned}$$

hence

$$\begin{aligned} Lp_n(x) &= \frac{\{(q(x+\omega)q^{n-1} - x) + [n-1]\omega\}}{(q-1)x+\omega}\, x\cdots(x+[n-2]\omega) \\ &= \frac{((q^n-1)x + [n]\omega)}{(q-1)x+\omega}\, x\cdots(x+[n-2]\omega)\end{aligned}$$

$$\boxed{(Lp_n)(x) = [n]\,p_{n-1}(x)}$$

It follows that

$$e(\lambda, x) = \sum_{n\geq 0} \frac{x(x-[1]\omega)\cdots(x-[n-1]\omega)}{[1][2]\cdots\cdots[n]}\lambda^n$$

satisfies

$$\boxed{L e(\lambda, x) = \lambda e(\lambda, x)}$$

Example: $q = 1$, whence $[i] = i$. Then

$$e(\lambda, x) = \sum \frac{x(x-\omega)\cdots(x-n\omega+\omega)}{n!}\lambda^n$$

$$= \sum \frac{\left(\frac{x}{\omega}\right)\left(\frac{x}{\omega}-1\right)\cdots\left(\frac{x}{\omega}-n+1\right)}{n!}(\omega\lambda)^n$$

$$\boxed{e(\lambda, x) = (1+\omega\lambda)^{\frac{x}{\omega}}} \longrightarrow e^{\lambda x} \quad \text{as } \omega \to 0$$

Example: $\omega = 0$. Put $f(x) = e(\lambda, x)$. Then

$$\frac{f(qx) - f(x)}{(q-1)x} = \lambda f(x)$$

or

$$\frac{f(qx)}{f(x)} - 1 = \lambda(q-1)x$$

or

$$f(x) = \frac{1}{1+\lambda(q-1)x} f(qx)$$

$$f(x) = \frac{1}{1-\lambda(1-q)x} \frac{1}{1-\lambda(1-q)qx} \cdots \frac{1}{1-\lambda(1-q)q^n x} \cdots$$

Hence

$$e(\lambda, x) = \left(\prod_{j \geq 0}\left(1 - \lambda(1-q)q^j x\right)\right)^{-1}$$

& for this to converge one needs $|q| < 1$.

In the general case one has

$$\frac{f(qx+\omega)-f(x)}{(q-1)x+\omega} = \lambda f(x)$$

$$\frac{f(qx+\omega)}{f(x)} = 1+\lambda(q-1)x+\lambda\omega$$

sequence $\quad x,\ qx+\omega,\ q^2x+[2]\omega,\ \dots,\ q^nx+[n]\omega \longrightarrow \frac{\omega}{1-q}$

$$\frac{f\left(\frac{\omega}{1-q}\right)}{f(x)} = \prod_{j\geq 0}\Big(1+\lambda\big((q-1)(q^jx+[j]\omega)+\omega\big)\Big)$$

so what is $\quad f\left(\frac{\omega}{1-q}\right)$?

$$\frac{\omega}{1-q}-[i]\omega = \omega\frac{q^i}{1-q}$$

$$f\left(\frac{\omega}{1-q}\right) = \sum_{n\geq 0}\frac{q^{n(n-1)/2}}{[1][2]\cdots[n]}\left(\frac{\omega\lambda}{1-q}\right)^n$$

not useful. Instead put $x=0$ above. Note

$$(q-1)(q^jx+[j]\omega)+\omega = q^j(q-1)x+(q^j-1)\omega+\omega$$
$$= q^j\big((q-1)x+\omega\big).$$

Hence

$$\cancel{\frac{f\left(\frac{\omega}{1-q}\right)}{f(0)} = \prod 1+q^j[(q-1)x+\omega]}$$

$$f\left(\frac{\omega}{1-q}\right) = \prod_{j\geq 0}\left(1+\lambda q^j\omega\right)$$

so

$$e(\lambda, x) = \frac{\prod_{j\geq 0} \left(1+\lambda q^j \omega\right)}{\prod_{j\geq 0} \left(1+\lambda q^j\left((q-1)x+\omega\right)\right)}$$

$$(\Delta u)(x) = \frac{u(x+\omega)-u(x)}{\omega} = \lambda u(x)$$

If $u(x) = e^{\alpha x}$, then

$$\frac{e^{\alpha\omega}-1}{\omega} = \lambda \qquad \text{or} \qquad \alpha = \frac{1}{\omega}\log(1+\lambda\omega)$$

Thus α is determined up to $\frac{2\pi i n}{\omega}$, $n \in \mathbb{Z}$, hence we can alter $e^{\alpha x}$ by any of the ω-periodic functions

$$e^{2\pi i \frac{nx}{\omega}} \qquad n \in \mathbb{Z}.$$

This is important to note perhaps, because the series solution

$$(1+\omega\lambda)^{\frac{x}{\omega}} = \sum_{n\geq 0}{}' \frac{x(x-\omega)\cdots(x-n\omega+\omega)}{n!}\lambda^n$$

converges only for $|\lambda| < 1$, hence as we analytically continue the solution for different λ we get different solutions of the original difference equation.

Consider the equation

$$(\Theta f)(x) = \lambda x f(x)$$

Try a series $\sum' a_n x^n = f(x)$. Then

$$(\Theta f)(x) = \sum' a_n [n] x^{n-1} = \sum' \lambda a_n x^{n+1}$$

leads to the recursion formula

$$a_n [n] = \lambda a_{n-2} \qquad \text{for } n \geq 2$$
$$a_1 = 0$$
$$a_0 \text{ arbitrary.}$$

So

$$f(x) = \sum_{n\geq 0}' \frac{\lambda^n x^{2n}}{[2][4]\cdots[2n]} .$$

As $q \to 1$ this approaches $\displaystyle\sum_{n\geq 0} \frac{\lambda^n x^{2n}}{2^n n!} = e^{\lambda \frac{x^2}{2}}$
which is the solution of the differential equation $\frac{d}{dx} f = \lambda x f$. On the other hand the original difference equation can be written

$$\frac{f(qx) - f(x)}{(q-1)x} = \lambda x f(x)$$

or

$$\frac{f(qx)}{f(x)} - 1 = \lambda (q-1) x^2$$

or

$$f(x) = \frac{1}{1 + \lambda(q-1)x^2} f(qx)$$

so

$$f(x) = \left(\prod_{j\geq 0} \left(1 + \lambda(q-1) q^{2j} x^2\right)\right)^{-1}$$

So if we put $\lambda = \frac{1}{1-q}$ we get the identity

$$\frac{1}{\prod_{j\geq 0} (1 - q^{2j} x^2)} = \sum_{n\geq 0}' \frac{x^{2n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}$$

which is essentially the formula for the q-exponential fn

$$\sum \frac{x^n}{[1]\cdots\cdots[n]} = \prod_{j\geq 0}\left(1+\lambda(q-1)q^j x\right)^{-1}$$

obtained previously.

Gauss identity. Put

$$f_m(x) = (1+x)(1+qx)\cdots\cdots(1+q^{m-1}x)$$

Then

$$(1+x)f_m(qx) = f_{m+1}(x) = f_m(x)(1+q^m x)$$

so if we put $f_m(x) = \sum a_n x^n$, then

$$(1+x)\sum a_n q^n x^n = \sum a_n x^n (1+q^m x)$$

$$a_n q^n + a_{n-1} q^{n-1} = a_n + a_{n-1} q^m$$

$$a_n(-1+q^n) = a_{n-1}(q^m - q^{n-1})$$

so $$a_n = \frac{q^m - q^{n-1}}{-1+q^n}\, a_{n-1} = \frac{(q^m-q^{n-1})\cdots(q^m-1)}{(-1+q^n)\cdots\cdots(-1+q)}$$

$$0+1+\cdots+(n-1) = \frac{n(n-1)}{2}$$ so

$$a_n = q^{\frac{n(n-1)}{2}}\,\frac{(1-q^{m-n+1})\cdots(1-q^m)}{(1-q)\cdots(1-q^n)}$$

$$\boxed{(1+x)(1+qx)\cdots(1+q^{m-1}x) = \sum_{n=0}^{m} q^{\frac{n(n-1)}{2}}\,\frac{[m]\cdots[m-n+1]}{[1]\cdots[n]}\,x^n}$$

$$\boxed{\prod_{j\geq 0}(1+q^j x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\,\frac{1}{(1-q)\cdots(1-q^n)}\,x^n}$$

July 16, 1977

First order difference equations with rational coefficients:

$$\theta f(x) = R(x) f(x)$$

can be written $$\frac{f(qx)}{f(x)} = 1 + (q-1)x R(x).$$

To solve this one can factor the rational function on the right into linear factors. A factor $(1+\lambda x)$ leads to $\prod_{j\geq 0}(1+\lambda q^j x)^{-1}$ as a factor of f.

A singular case occurs possibly if x occurs in the denominator of R.

I should consider first the case

$$\frac{d}{dx} f(x) = R(x) f(x)$$

which has the solution

$$f(x) = e^{\int R(x)dx}$$

This gives a good power series solution at $x=0$ provided R is regular at $x=0$. If $R(x)$ has a simple pole at $x=0$, say $R(x) = \frac{\alpha}{x} + \text{reg.}$, then $x=0$ is a regular singular pt, and the solution is of the form

$$f(x) = x^\alpha(1 + a_1 x + a_2 x^2 + \cdots)$$

So one expects the analogous thing with θ.

The good case is when R is regular at 0 and we get nice power series solutions. Ther next good case is when R has a simple pole. Important special case is $R = \frac{\alpha}{x}$ which leads to the equation

$$\frac{f(qx)}{f(x)} = 1 + (q-1)\alpha = \lambda$$

Try a solution $f(x) = x^{\mu}$. Then

$$\frac{q^{\mu} x^{\mu}}{x^{\mu}} = \lambda \qquad\qquad q^{\mu} = \lambda$$

This determines μ up to an additive constant $\frac{2\pi i}{\log q} n$, $n \in \mathbb{Z}$, but these differences in the choices for μ change f by a q-periodic function.

In the above we supposed that $\lambda \neq 0$. If the equation

$$\frac{f(qx)}{f(x)} = x^{m}$$

has a Laurent series solution $f(x) = \sum a_n x^n$, then

$$\sum_n a_n q^n x^n = \sum_n a_n x^{n+m} = \sum_n a_{n-m} x^n$$

so
$$a_n q^n = a_{n-m}. \qquad \text{Say } m = -1. \text{ Then}$$

$$a_n = q^{n-1} a_{n-1} = q^{(n-1)+\cdots+1+0} a_0 = q^{\frac{n(n-1)}{2}} a_0$$

so
$$f(x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n$$

which is essentially a Θ-function. If $m = +1$, then we find that

$$a_n q^n = a_{n-1}$$

or

$$a_n = q^{-n} a_{n-1} = q^{-n} q^{-n+1} a_{n-2} = q^{-n} q^{-n+1} q^{-n+2} a_{n-3}$$

$$= q^{-n} q^{-n+1} \cdots q^{-1} a_0 = q^{-\frac{n(n+1)}{2}} a_0$$

so $\quad f(x) = \sum_n q^{-\frac{n(n+1)}{2}} x^n \quad$ except for the fact that for $|q|<1$ this doesn't converge. Curious:

Put

$$f(x) = \sum' q^{\frac{n(n-1)}{2}} x^n$$

so that

$$f(qx) = \sum' q^{\frac{n(n-1)}{2}+n} x^n = \sum' q^{\frac{n(n+1)}{2}} x^n$$

$$= \sum' q^{\frac{(n-1)n}{2}} x^{n-1} = x^{-1} f(x)$$

Now $f(x)$ is defined and analytic for all $x \neq 0$. Look at $g(x) = f(x)^{-1}$. Then

$$\frac{g(qx)}{g(x)} = \frac{f(x)}{f(qx)} = x$$

yet $g(x)$ doesn't have a Laurent series expansion. This means I guess that $f(x)$ has lots of zeroes.

Look at the zeroes of $f(x)$; if λ is a zero then so is $q\lambda$ and $q^{-1}\lambda$. So let us look at the function with the zeroes $q^n\lambda$ for $n \geq 1$

$$h_1(x) = \prod_{n \geq 1} \left(1 - \frac{q^n \lambda}{x}\right)$$

and the function with the zeros $q^{-n}\lambda$ $\quad n \geq 0$ which is

$$h_2(x) = \prod_{n\geq 0}\left(1 - \frac{q^n x}{\lambda}\right)$$

Then

$$h_1(qx) = \left(1 - \frac{\lambda}{x}\right) h_1(x)$$

$$h_2(qx) = \left(1 - \frac{x}{\lambda}\right)^{-1} h_2(x)$$

So

$$h_1(qx)\,h_2(qx) = \frac{\frac{\lambda}{x}\left(\frac{x}{\lambda} - 1\right)}{\left(1 - \frac{x}{\lambda}\right)} h_1(x) h_2(x) = \left(-\frac{\lambda}{x}\right) h_1(x) h_2(x)$$

Thus if we take $\lambda = -1$, $h_1 h_2$ will satisfy the same functional equation as $f(x)$. So we have

$$f(x) = \sum_{n\in\mathbb{Z}}' q^{\frac{n(n-1)}{2}} x^n = g(x) \prod_{n\geq 0}(1 + q^n x) \prod_{n\geq 1}(1 + q^n x^{-1})$$

where $g(x)$ satisfies: $g(qx) = g(x)$.

Next note that $\frac{n(n-1)}{2}$ is symmetric under $n \mapsto 1-n$, hence

$$x f(x^{-1}) = \sum' q^{\frac{n(n-1)}{2}} x^{1-n} = f(x)$$

which shows that $f(-1) = 0$. Thus $g(x)$ is an entire function which is bounded, so $g(x)$ is a constant in x, given by

$$g = \frac{\sum_{n\geq 0}' q^{\frac{n(n+1)}{2}}}{\prod_{n\geq 1}(1 + q^n)^2}.$$

So g is some power series in q with leading term 1, whose exact determination as an infinite product is possible but involved (according to Bellman's book)

Review: I've been considering q-difference equations

$$\frac{f(qx)}{f(x)} = \tilde{R}(x)$$

where $\tilde{R}(x)$ is a rational function of x. I am **interested** in solutions with nice analytic properties as functions of x. For example, if $\tilde{R}(x) = 1$, then a solution meromorphic on $\mathbb{C} - \{0\}$ is the same thing as a doubly-periodic function on $\mathbb{C}$ with periods 1 and τ where $e^{2\pi i \tau} = q$. This means that the only solutions holomorphic off zero are constants.

Thus the solutions of the homogeneous equation $\frac{f(qx)}{f(x)} = 1$ can be understood in terms of elliptic functions. The next point is that if we split $\tilde{R}$ into factors of degree 1, then it suffices to solve the separate equations — this is the principle used in finding particular solutions. So we end up with the following cases

$$\tilde{R}(x) = \text{constant}$$

$$x, \ \frac{1}{x}$$

$$1+\lambda x, \ (1+\lambda x)^{-1}$$

Consider the latter cases. If we assume a soln. of the form $\quad f(x) = \sum' a_n x^n$

then
$$f(qx) = \sum a_n q^n x^n = (1+\lambda x)\sum a_n x^n$$
$$= \sum{}' (a_n + \lambda a_{n-1}) x^n$$

leads to the recursion formula
$$a_n q^n = a_n + \lambda a_{n-1}$$
$$a_n(q^n - 1) = \lambda a_{n-1}.$$

Hence $a_{-1} = 0$, [redacted] $a_n = 0$ for $n \leq -1$. If $a_0 = 1$ we get
$$a_n = \frac{\lambda}{q^n - 1} a_{n-1} = \cdots = \frac{\lambda^n}{(q^n-1)\cdots(q-1)}.$$

So
$$f(x) = \boxed{\sum_{n\geq 0} \frac{(-\lambda)^n}{(1-q)\cdots(1-q^n)} x^n = \prod_{j\geq 0} \frac{1}{(1+\lambda q^j x)}}$$

On the other hand if $\frac{f(qx)}{f(x)} = (1+\lambda x)^{-1}$, then
$$(1+\lambda x) f(qx) = (1+\lambda x)\sum a_n q^n x^n = \sum (a_n q^n + \lambda a_{n-1} q^{n-1}) x^n$$
$$= f(x) = \sum a_n x^n$$

so
$$a_n q^n + \lambda a_{n-1} q^{n-1} = a_n$$
$$\lambda a_{n-1} q^{n-1} = a_n(1-q^n)$$

Again $a_{-1} = a_{-2} = \cdots = 0$ [redacted] and if $a_0 = 1$, then
$$a_n = \frac{\lambda q^{n-1}}{1-q^n} a_{n-1} = \cdots = \frac{\lambda^n q^{\frac{n(n-1)}{2}}}{(1-q)\cdots(1-q^n)}$$

so
$$f(x) = \boxed{\sum_{n\geq 0} \frac{q^{\frac{n(n-1)}{2}} \lambda^n}{(1-q)\cdots(1-q^n)} x^n = \prod_{j\geq 0} (1+\lambda q^j x)}$$

Next thing to look at is those q-difference equations giving rise to simple recursion formulas for the coefficients. Thus consider

$$\frac{f(qx)}{f(x)} = \frac{1+\lambda x}{1+\mu x}$$

$$a_n q^n + \mu a_{n-1} q^{n-1} = a_n + \lambda a_{n-1}$$

$$a_{n-1}(\mu q^{n-1} - \lambda) = a_n(1-q^n)$$

As before these relation allow us to set $a_0 = 1$, $a_{-1} = a_{-2} = \cdots 0$ and

$$a_n = \frac{(\mu q^{n-1} - \lambda) \cdots (\mu - \lambda)}{(1-q^n) \cdots (1-q)}$$

$$= \frac{(1-\frac{\mu}{\lambda})(1-\frac{\mu}{\lambda}q) \cdots (1-\frac{\mu}{\lambda}q^{n-1})}{(1-q) \cdots (1-q^n)} (-\lambda)^n$$

$$= \frac{[\alpha]\,[\alpha+1] \cdots [\alpha+n-1]}{[1]\,[2] \cdots [n]} (-\lambda)^n$$

where $\boxed{q^\alpha = \frac{\mu}{\lambda}}$ so we get the "binomial-type series"

$$f(x) = \sum_{n \geq 0}{}' \frac{[\alpha] \cdots [\alpha+n-1]}{[1] \cdots [n]} (-\lambda)^n x^n$$

$$= \prod_{j \geq 0} \left(\frac{1+\mu q^j x}{1+\lambda q^j x} \right)$$

Notice that if $\frac{\lambda}{\mu} = q^m$ for some integer m, then we can put $a_m = 1$, $a_{m+1} = a_{m+2} = \cdots = 0$ and grind out a series solution holomorphic at ∞.

July 17, 1977

Recall that if $y_a(x) = \sum_{n\geq 0} \frac{[a]\cdots[a+n-1]}{[1]\cdots[n]} x^n$, then

$$\theta y_a = [a] y_{a+1} \quad \text{and} \quad y_{a+1} - y_a = q^a x\, y_{a+1}$$

so that $\frac{y_a(qx)}{y_a(x)} = \frac{1-x}{1-q^a x}$ and so we get the first of the following formulas

$$\boxed{\begin{aligned} \prod_{j\geq 0} \frac{1-q^{a+j}x}{1-q^j x} &= \sum_{n\geq 0} \frac{[a]\cdots\cdots[a+n-1]}{[1]\cdots\cdots[n]} x^n \\ \prod_{j\geq 0} \frac{1}{1-q^j x} &= \sum_{n\geq 0} \frac{x^n}{(1-q)\cdots(1-q^n)} \\ \prod_{j\geq 0} (1+q^j x) &= \sum_{n\geq 0} \frac{q^{n(n-1)/2} x^n}{(1-q)\cdots(1-q^n)} \end{aligned}}$$

The second is obtained by letting $a \to +\infty$ in the first, and the third by substituting: $x \mapsto -xq^{-a}$ in the first and letting $a \longrightarrow -\infty$.

$$\prod_{j\geq 0} \frac{1-q^j x}{1-q^{j-a}x} = \sum_{n\geq 0} \frac{(q^{-a}-1)\cdots(q^{-a}-q^{n-1})}{(1-q)\cdots(1-q^n)} x^n$$

$$\boxed{\prod_{j\geq 0} \frac{1-q^j x}{1-q^{j+a}x} = \sum_{n\geq 0} \frac{[a]\cdots[a-n+1]}{[1]\cdots[n]} q^{n(n-1)/2} (-x)^n}$$

Hahn denotes the last function by $(1-x)_a$ since as $q \to 1$ it converges to $(1-x)^a$

Consider a 2nd order DE with a regular singular point at $x=0$:

$$\left(x^2 p(x)\frac{d^2}{dx^2} + x q(x)\frac{d}{dx} + r(x)\right)y = 0$$

where p, q, r are analytic at 0 and $p(0)=1$. I want to assume that the recursion relation for the coefficients of a series solution is of the two term type. This will be the case if p, q, r are linear in x in which case the DE has the form

$$\left(x^2\frac{d^2}{dx^2} + c_1 x\frac{d}{dx} + c_2\right)y = \left(c_3 x^3\frac{d^2}{dx^2} + c_4 x^2\frac{d}{dx} + c_5 x\right)y$$

If we have a solution $y = x^{\mu}\sum_{n\geq 0}' a_n x^n$, $a_0 = 1$, then

$$\sum_{n\geq 0} a_n\left[(\mu+n)(\mu+n-1) + c_1(\mu+n) + c_2\right]x^n$$

$$= \sum_{n\geq 0}' a_n\left[c_3(\mu+n)(\mu+n-1) + c_4(\mu+n) + c_5\right]x^{n+1}$$

giving the indicial equation

$$\mu(\mu-1) + c_1\mu + c_2 = 0$$

and the recurrence formula

$$a_n = \frac{c_3(\mu+n-1)(\mu+n-2) + c_4(\mu+n-1) + c_5}{(\mu+n)(\mu+n-1) + c_1(\mu+n) + c_2}\, a_{n-1}.$$

Let me simplify things by requiring that $\mu = 0$ be a root of the indicial equation so that $c_2 = 0$. Then the recurrence formula is

$$a_n = \frac{c_3(n-1)(n-2) + c_4(n-1) + c_5}{(c_1 + n - 1)\, n}$$

Case 1: $c_3 = c_4 = 0,\ c_5 \neq 0$ By scaling x, we can suppose $c_5 = 1$. Then we (next p. $186\frac{1}{2}$) get the D.E.

$$\left(x\frac{d^2}{dx^2} + c_1\frac{d}{dx} - 1\right)y = 0$$

which is essentially Bessel's DE (start with Bessel's DE

$$\left(\left(z\frac{d}{dz}\right)^2 - z^2 - n^2\right)u = 0$$

and put $x = \left(\frac{z}{2}\right)^2$. $\left(dx = \frac{z}{2}dz \qquad x\frac{d}{dx} = \frac{z}{2}\frac{d}{dz}\right.$ $\left(z\frac{d}{dz}\right)^2 = 4\left(x\frac{d}{dx}\right)^2$, $\left.z^2 = 4x\right)$ and you get the DE

$$\left(\left(x\frac{d}{dx}\right)^2 + x - \frac{n^2}{4}\right)u = 0.$$

Now put $u = x^{n/2}v$ and you get

$$\left(\left(x\frac{d}{dx} + \frac{n}{2}\right)^2 - x - \frac{n^2}{4}\right)v = 0 \qquad \text{or}$$

$$\left(\left(x\frac{d}{dx}\right)^2 + nx\frac{d}{dx} + \frac{n^2}{4} - x - \frac{n^2}{4}\right)v = 0$$

or

$$\left(x\frac{d^2}{dx^2} + (1+n)\frac{d}{dx} - 1\right)v = 0. \quad)$$

Case 2: $c_3 = 0$, $c_4 \neq 0$; by scaling can suppose $c_4 = 1$, whence we have the confluent hypergeometric DE

$$\left(x\frac{d^2}{dx^2} + (c_1 - x)\frac{d}{dx} - c_5\right)y = 0$$

Case 3: $c_3 \neq 0$; whence by scaling we can arrange $c_3 = 1$, and we get the hypergeometric DE.

$$\left(x(1-x)\frac{d^2}{dx^2} + (c_1 - c_4 x)\frac{d}{dx} - c_5\right)y = 0.$$

Now

$$c+[n] = c+\frac{q^{n}-1}{q-1} = -\frac{q^{-\gamma}-1}{q-1}+\frac{q^{n}-1}{q-1}$$

$$= \frac{q^{n}-q^{-\gamma}}{q-1} = q^{-\gamma}[\gamma+n]$$

provided $c = -\frac{q^{-\gamma}-1}{q-1} = \frac{q^{-\gamma}-1}{1-q} = q^{-\gamma}[\gamma]$. γ exists provided $c \neq \frac{1}{q-1}$. Now the numerator can be written

$$c_4[n][n-1]+c_5[n]+c_6$$

$$= \alpha_0 q^{2n}+\alpha_1 q^{n}+\alpha_2 \qquad \alpha_0 = \frac{c_4}{q(q-1)^2}$$

$$= \alpha_0\left(q^{n}-r_1\right)\left(q^{n}-r_2\right)$$

$$= \alpha_0\left(q^{n}-q^{-\alpha}\right)\left(q^{n}-q^{-\beta}\right)$$

$$= \text{const}\cdot[\alpha+n][\beta+n]$$

provided $c_4 \neq 0$ and $r_1, r_2 \neq 0$. Hence the series we obtain is a Heine hypergeometric series:

$$\sum \frac{[\alpha]\cdots[\alpha+n-1][\beta]\cdots[\beta+n-1]}{[\gamma]\cdots[\gamma+n-1][1]\cdots[n]}(\lambda x)^{n}$$

July 18, 1977:

We saw that DE's of the form with $c_1 \neq 0$

(1) $$\left(c_1 x^2 \frac{d^2}{dx^2} + c_2 x \frac{d}{dx} + c_3\right) y = \left(c_4 x^3 \frac{d^2}{dx^2} + c_5 x^2 \frac{d}{dx} + c_6 x\right) y$$

have nice series solutions around $x=0$. If we suppose $c_3 = 0$ (replace y by $x^\mu y$ where μ is a root of the indicial equation), then we get DE's of the form

(2) $$\left((a_1 x^2 + a_2 x + a_3)\frac{d^2}{dx^2} + (a_4 x + a_5)\frac{d}{dx} + a_6\right) y = 0$$

with $a_3 = 0$, $a_2 \neq 0$. Conversely suppose given a DE of the second type, we look at the quadratic factor $(a_1 x^2 + a_2 x + a_3)$. By translating and scaling we can bring it into one of the forms:

1

x

x^2

$x(1-x)$

The cases $x(1-x)$ and x belong to the type (1). Let's look at the other types:

Take x^2. Then we have to have $a_5 = 0$ or else $x=0$ is an irregular singular point. So we have an equation of the form

$$\left(\left(x\frac{d}{dx}\right)^2 + b_1\left(x\frac{d}{dx}\right) + b_2\right) y = 0$$

which has solutions $y = x^\mu$.

Take the case of 1. By an affine transformation we can make $(a_4 x + a_5) = 2x$. (Assume $a_4 \neq 0$; otherwise the DE has constant coefficients, hence solutions $y = e^{\lambda x}$).

So we get the DE

$$\left(\frac{d^2}{dx^2} + 2x\frac{d}{dx} + b\right)y = 0$$

If $y = e^{-x^2/2}u$, then

$$e^{x^2/2}\left(\frac{d}{dx} + 2x\right)\left(\frac{d}{dx}\right)e^{-x^2/2}u = \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right) = \frac{d^2}{dx^2} - x^2 + 1$$

so we end up with the Hermite D.E.

Hahn considers q-difference equations of the form

$$(a_1x^2 + a_2x + a_3)\Theta^2 y + (a_4x + a_5)\Theta y + a_6 y = 0$$

but the analysis is quite different because the origin, as the unique fixpt of $x \mapsto qx$, must remain fixed under any x change. Possibilities for the leading factor are:

deg 0: 1

deg 1: x, $x-1$

deg 2: x^2, $x(1-x)$, $(1-x)(1-ax)$, $(1-x)^2$

[redacted] He also considers solutions expanded in series about $x=1$.

First consider series about $x=0$ with 2-term recursion relations. Thus I consider

$$\left(c_1 x^2\Theta^2 + c_2 x\Theta + c_3\right)y = \left(c_4 x^3\Theta^2 + c_5 x^2\Theta + c_6 x\right)y$$

with $c_3 = 0$, $c_1 = 1$, $c_2 = c$. This leads to a recursion fmla

$$a_n = \frac{\left(c_4[n-1][n-2] + c_5[n-1] + c_6\right)}{\left([n][n-1] + c[n]\right)}\, a_{n-1}$$