$$G(a,p) \equiv \chi'(a) G(1,p)$$

where $\chi : (\mathbb{Z}/p\mathbb{Z})^* \longrightarrow \{\pm 1\}$ is a quadratic character. Now we know χ vanishes on squares, hence $\chi(a) = \left(\frac{a}{p}\right)$ or it is trivial, assuming p prime. Since $G(-1,p) = \overline{G(1,p)}$ one sees $\chi(-1) = -1$ in the case $p \equiv 3 \ (4)$, so $\chi(a) = \left(\frac{a}{p}\right)$ in this case. Actually Galois theory shows the action is non-trivial, p an odd prime.

This Galois argument works for p a power of an odd prime $p = q^r$. If r is even, then $p \equiv 1 \ (4)$ and $G(1,p) \in \mathbb{Z}$ so χ is trivial, and also $\left(\frac{a}{p}\right)$ is trivial. If r is odd, then Galois theory shows χ is non-trivial, hence it must coincide with $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)^r = \left(\frac{a}{q}\right)$, because $(\mathbb{Z}/p\mathbb{Z})^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \times (1 + q\mathbb{Z}/q^r\mathbb{Z})^*$ has a unique quotient cyclic of order 2. Thus one gets the formula

$$G(a,p) = \left(\frac{a}{p}\right) G(1,p)$$

for ~~p an~~ odd prime power, and hence for any odd p.

Consider p odd and the function

$$f(m) = e^{\pi i \frac{1}{p}(m^2+m)} = e^{2\pi i \frac{1}{p}\frac{m(m+1)}{2}} = \zeta^{\frac{m(m+1)}{2}}$$

on $\mathbb{Z}/p\mathbb{Z}$. (Note that if a is odd, then $(m+p)^2 + a(m+p) = m^2 + 2mp + p^2 + am + ap \equiv m^2 + am \pmod{2p}$). One has

$$\hat{f}(n) = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{1}{p}(m^2 + m + 2mn)} = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{1}{p}[(m+n)^2 + (m+n) - n^2 - n]}$$

$$= e^{-\frac{\pi i}{p}(n^2+n)} \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{1}{p}(m^2+m)}$$

So again I can conclude

$$\left| \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{1}{p}(m^2+m)} \right| = \sqrt{p}$$

More generally let us consider

$$f(m) = e^{\pi i \frac{q}{p}(m^2+rm)} = \zeta^{q \frac{m^2+rm}{2}}$$

where [redacted] r is odd. Then

$$\hat{f}(n) = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p}(m^2+rm) + 2\pi i \frac{mn}{p}(qa)} \qquad aq \equiv 1 \ (p)$$

$$\cancel{= \sum e^{\pi i \frac{q}{p}(a^2m^2+arm) + 2\pi i \frac{amn}{p} aq}}$$

$$\cancel{= \sum e^{\pi i \frac{q}{p}[a^2m^2+arm+2a^2mn]}}$$

$$\cancel{= \sum e^{\pi i \frac{q}{p}[(am+an)^2 + r(am+an) - a^2n^2 - ran]}}$$

$$= \sum e^{\pi i \frac{q}{p}[m^2+rm+2amn]}$$

$$= \sum e^{\pi i \frac{q}{p}[(m+an)^2 + r(m+an) - a^2n^2 - ran]}$$

$$= e^{-\pi i \frac{a}{p}(n^2+qrn)} \sum e^{\pi i \frac{q}{p}(m^2+rm)}$$

Actually there is nothing new here because once you've established that $f(m) = e^{\pi i \frac{q}{p}(m^2+rm)} = \zeta^{q\frac{m^2+rm}{2}}$

has period p, (for any q) then you can replace q by a congruent even integer. For example, if $q \equiv 2q'$ (mod p)

$$\sum_{m\in\mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p}(m^2+rm)} = \sum e^{\pi i \frac{2q'}{p}m^2 + 2\pi i \frac{q' r m}{p}}$$

$$= e^{-\pi i \frac{a}{p}(q'r)^2} \cdot \sum_{m\in\mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{2q'}{p}m^2}$$

where a is an even multiplicative inverse to $2q'$ mod p.

Let's see what Gaussian sums arise when we look at $i\mathbb{Q}$ limits of modified Θ-functions:

$$\sum e^{-\pi(x+n)^2 t + 2\pi i n y} = \frac{e^{-2\pi i x y}}{\sqrt{t}} \sum e^{-\pi(y+n)^2/t - 2\pi i n x}$$

Let $t = \varepsilon - i\frac{q}{p}$ and let $\varepsilon \to 0$.

$$\sum e^{-\pi(x+n)^2(\varepsilon - i\frac{q}{p}) + 2\pi i n y} = \sum_n e^{-\pi(x+n)^2\varepsilon} \underbrace{e^{+\pi i (x+n)^2\frac{q}{p} + 2\pi i n y}}_{\text{periodic in } = n}$$

Let N be a period:

$$= \sum_{r=0}^{N-1}\left(\sum_m e^{-\pi(x+r+mN)^2\varepsilon}\right) e^{\pi i (x+r^2)\frac{q}{p} + 2\pi i r y}$$

Now $$\sum_m e^{-\pi(\frac{x+r}{N}+m)^2 N^2\varepsilon} = \frac{1}{\sqrt{N^2\varepsilon}} \sum_n e^{-\pi n^2 \frac{1}{N^2\varepsilon}} e^{-2\pi i n\left(\frac{x+r}{N}\right)}$$

$$\sim \frac{1}{N\sqrt{\varepsilon}} \quad \text{as } \varepsilon \to 0$$

Consequently the asymptotic expansion is

$$\frac{1}{\sqrt{\varepsilon}}\;\frac{1}{N}\sum_{r\in\mathbb{Z}/N\mathbb{Z}} e^{\pi i (x+r)^2\frac{q}{p} + 2\pi i r y} .$$

Consider the differentiated Θ-function:

$$\sum (x+n)e^{-\pi(x+n)^2 t + 2\pi i n y} = \frac{e^{-2\pi i x y}\, i}{t^{3/2}}\sum (y+n) e^{-\pi(y+n)^2/t - 2\pi i n x}$$

Put $t = \varepsilon - i\frac{q}{p}$ and suppose x rational

$$\sum (x+n)\, e^{-\pi(x+n)^2\varepsilon}\; e^{+\pi i (x+n)^2\frac{q}{p}}\; e^{2\pi i n y}$$

$$= \sum (x+n)\, e^{-\pi(x+n)^2\varepsilon}\, e^{2\pi i n y}\; \underbrace{e^{\pi i (x+n)^2\frac{q}{p}}}_{\text{periodic, say of period } N}$$

$$= \sum_{r=0}^{N}\left(\sum_m (x+r+mN)e^{-\pi(x+r+mN)^2\varepsilon}\, e^{2\pi i (r+mN)y}\right) e^{\pi i (x+r)^2\frac{q}{p}}$$

$$= \sum_{r=0}^{N}\left(\sum_m N\left(\frac{x+r}{N}+m\right)e^{-\pi\left(\frac{x+r}{N}+m\right)^2\varepsilon N^2}\, e^{2\pi i m (Ny)}\right) e^{2\pi i r y + \pi i (x+r)^2\frac{q}{p}}$$

$$\Vert$$

$$\left(N\frac{i\, e^{-2\pi i \left(\frac{x+r}{N}\right) N y}}{(\varepsilon N^2)^{3/2}}\sum (Ny+n)\, e^{-\pi(Ny+n)^2/\varepsilon N^2}\, e^{-2\pi i n\left(\frac{x+r}{N}\right)}\right)$$

So this will have asymptotic expansion 0 if y is irrational and also if y is rational because of the $Ny+N$ factor.

The following might be useful. Take

$$\theta(0,y,t) = {\sum}' e^{-\pi n^2 t} e^{2\pi i n y}$$

where y is irrational. Put $t = \varepsilon - i\frac{q}{p}$

$$\sum_n e^{-\pi n^2 \varepsilon} e^{2\pi i n y} \underbrace{e^{\pi i n^2 \frac{q}{p}}}_{\text{periodic of period } p \text{ assuming } pq \text{ is even.}}$$

$$= \sum_{r=0}^{p-1} \left({\sum_m}' e^{-\pi(r+mp)^2\varepsilon} e^{2\pi i (r+mp) y} \right) e^{\pi i r^2 \frac{p}{q}}$$

The term inside parentheses is

$$\sum_m e^{-\pi\left(\frac{r}{p}+m\right)^2 (p^2\varepsilon)} e^{2\pi i \left(\frac{r}{p}+m\right)(py)}$$

$$= \frac{1}{p\sqrt{\varepsilon}} \sum_m e^{-\pi(py+m)^2 \frac{1}{p^2\varepsilon}} e^{-2\pi i m \frac{r}{p}} \cancel{e^{-2\pi i \left(\frac{r}{p}\right)(py)}} \cancel{e^{2\pi i r y}}$$

leading term

$$\approx \frac{1}{p\sqrt{\varepsilon}} e^{-\pi\left(y+\frac{m}{p}\right)^2 \frac{1}{\varepsilon}} e^{-2\pi i \left(\frac{m}{p}\right) r}$$

where m is chosen to minimize $y+\frac{m}{p}$.

So $\theta\left(0, y, \varepsilon - i\frac{q}{p}\right) \sim \frac{1}{\sqrt{\varepsilon}} e^{-\pi\left(y+\frac{m}{p}\right)^2 \frac{1}{\varepsilon}} \frac{1}{p} {\sum_{r=0}^{p-1}}' e^{-2\pi i \left(\frac{m}{p}\right) r} e^{\pi i r^2 \frac{p}{q}}$

So the point is that the asymptotic behavior depends on how well y can be approximated by rational numbers with denominator p.

June 27, 1977:

Observation: We've looked at two kinds of Gaussian sums:

$$\sum_{m\in(\mathbb{Z}/p\mathbb{Z})^*}' \chi(m)\,\zeta^m \qquad \zeta = \exp\left(\frac{2\pi i}{p}\right)$$

where χ is a character of $(\mathbb{Z}/p\mathbb{Z})^*$, and

$$\sum_{m\in(\mathbb{Z}/p\mathbb{Z})}' e^{\pi i \frac{a}{p} m^2}$$

where either a, p is even and $(a,p)=1$. If p is an odd prime, then

$$\sum_{m\in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{m^2}{p}} = 1 + 2\sum_r' \zeta^r - \overbrace{\left(1+\sum_r + \sum_n\right)}^{0}$$

$$= \sum_r' \zeta^r - \sum_n \zeta^n = \sum_m \left(\frac{m}{p}\right)\zeta^m$$

so in this case we see the Gaussian sum belonging to the Legendre character $m \mapsto \left(\frac{m}{p}\right)$ coincides with the quadratic Gaussian sum $G(2,p)$.

Schur's proof of the formula for Gaussian sums is based on an analysis of the Fourier transform.

$$(Ff)(n) = \sum_m' f(m)\, e^{2\pi i \frac{mn}{p}}$$

i.e. of the matrix $\left(e^{2\pi i \frac{mn}{p}}\right) = \left(\zeta^{mn}\right)$. One has

$$tr(F) = \sum_n' \zeta^{n^2} = \sum_{n\in\mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{n^2}{p}} = \text{sum of eigenvalues}$$

Denote this S. One knows that

$$(F^2 f)(n) = \sum_m (Ff)(m)\, \zeta^{mn} = \sum_{m,l} f(l)\, \zeta^{lm+mn}$$

$$= \sum_l f(l) \sum_m \zeta^{(l+n)m} = \sum_l f(l) \begin{cases} p & l=-n \\ 0 & l \neq -n \end{cases}$$

$$= p f(-n)$$

Hence $(F^4 f)(n) = p^2 f(n)$ so the eigenvalues of F are $i^a\sqrt{p}$, $a = 0,1,2,3$. Let m_a = multiplicity of $i^a\sqrt{p}$. Then

$$m_0 + m_1 + m_2 + m_3 = p$$

$$m_0 + m_1 i - m_2 - i m_3 = \frac{S}{\sqrt{p}}$$

$$m_0 - m_1 + m_2 - m_3 = \begin{cases} 1 & p \text{ odd} \\ 2 & p \text{ even} \end{cases}$$

The last comes from the fact that the trace of F^2 is p times the number of $n \bmod p$ with $n \equiv -n$ or $2n \equiv 0$. Also one can compute

$$|S|^2 = S\bar{S} = \sum_{m,n} \zeta^{m^2-n^2} = \sum_{m,n} \zeta^{(m+n)(m-n)} \qquad n \mapsto n+m$$

$$= \sum_{m,n} \zeta^{(n+2m)(-n)} = \sum_n \zeta^{-n^2} \sum_m \zeta^{-2mn}$$

$$= \sum_n \zeta^{-n^2} \begin{cases} p & 2m \equiv 0 \\ 0 & 2m \not\equiv 0 \end{cases}$$

$$= \begin{cases} p & p \text{ odd} \\ \left(1 + e^{2\pi i \frac{-p^2/4}{p}}\right) p & p \text{ even} \end{cases}$$

$$1 + e^{-\pi i p} = \begin{cases} 2 & p \equiv 0\ (4) \\ 0 & p \equiv 2\ (4). \end{cases}$$

Suppose $p \equiv 2\ (4)$. Then $|S|^2 = 0 \Rightarrow S = 0 \Rightarrow$

$m_0 = m_2$, $m_1 = m_3$. $\qquad 2m_0 - 2m_1 = 2 \qquad m_0 - m_1 = 1$

$\qquad 2m_0 + 2m_1 = p \qquad m_0 + m_1 = \frac{p}{2}$

so we see that

$$\begin{cases} m_0 = m_2 = \dfrac{p+2}{4} \\ m_1 = m_3 = \dfrac{p-2}{4} \end{cases} \qquad \text{if } p \equiv 2 \pmod 4$$

In general one has

$$(m_0 - m_2)^2 + (m_1 - m_3)^2 = \frac{|S|^2}{p}$$

Suppose $p \equiv 0\ (4)$. Then $(m_0 - m_2)^2 + (m_1 - m_3)^2 = 2$

$\Rightarrow |m_0 - m_2| = 1$ and $|m_1 - m_3| = 1$. From

$$(m_0 + m_2) + (m_1 + m_3) = p$$
$$(m_0 + m_2) - (m_1 + m_3) = 2$$

one gets

$$m_0 + m_2 = \frac{p}{2} + 1 \qquad \text{dim of even fns.}$$
$$m_1 + m_3 = \frac{p}{2} - 1 \qquad \text{dim. of odd fns.}$$

Another ingredient in Schur's proof is to calculate the determinant of F.

$$\det(F) = (\sqrt{p})^{m_0} (i\sqrt{p})^{m_1} (-\sqrt{p})^{m_2} (-i\sqrt{p})^{m_3}$$
$$= p^{\frac{p}{2}} e^{i\left(m_2 \pi + (m_1 - m_3)\frac{\pi}{2}\right)}$$

$$\det(F) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{p-1} \\ 1 & \zeta^2 & & & \\ & & & & \\ 1 & \zeta^{p-1} & & & \zeta^{(p-1)^2} \end{vmatrix} = \prod_{p > k > j \geq 0} (\zeta^k - \zeta^j)$$

vander Monde

Now $\quad \gamma^k - \gamma^j = e^{2\pi i \frac{k}{p}} - e^{2\pi i \frac{j}{p}} = e^{\pi i \frac{k+j}{p}}\left(e^{\pi i \frac{k-j}{p}} - e^{-\pi i \frac{k-j}{p}}\right)$

$$= e^{\pi i \frac{k+j}{p}}\, 2i \sin\left(\frac{k-j}{p}\pi\right)$$

For $0 \le j < k < p$ the sin is > 0. Also

$$\sum_{0\le j<k<p} (j+k) = \sum_{0\le j<k<p} j + \sum_{0\le k<j<p} j = \sum_{0\le j<p} j \sum_{\substack{0\le k<p\\ k\ne j}} 1$$

$$= \frac{p(p-1)}{2}(p-1).$$

$$\sum_{0\le j<k<p} 1 = 1+2+\cdots+p-1 = \frac{p(p-1)}{2}.$$

Thus

$$\frac{\det(F)}{|\det(F)|} = e^{\pi i \frac{1}{p} p \frac{(p-1)^2}{2}} \; e^{i\frac{\pi}{2}\cdot\frac{p(p-1)}{2}}$$

$$= e^{i\left[\pi\frac{(p-1)^2}{2} + \pi\frac{p(p-1)}{4}\right]}$$

so

$$m_2 + \frac{m_1 - m_3}{2} \equiv \frac{(p-1)^2}{2} + \frac{p(p-1)}{4} \pmod 2$$

better: $\quad 2m_2 + m_1 - m_3 \equiv (p-1)^2 + \frac{p(p-1)}{2} \pmod 4$

Doesn't seem to give anything more if $p \equiv 0 \ (4)$.

Suppose p odd. Then $(m_0-m_2)^2 + (m_1-m_3)^2 = 1$ so

$$\frac{S}{\sqrt{p}} = (m_0-m_2) + (m_1-m_3)i = \pm 1 \text{ or } \pm i$$

$$= \nu\mu \qquad \nu = \pm 1 \qquad \mu = 1 \text{ or } i.$$

From

$$(m_0+m_2) + (m_1+m_3) = p$$

$$(m_0+m_2) - (m_1+m_3) = 1$$

we get

$$m_0 + m_2 = \frac{p+1}{2}$$
$$m_1 + m_3 = \frac{p-1}{2}$$

$$v\mu = \left(2m_0 - \frac{p+1}{2}\right) + \left(2m_1 - \frac{p-1}{2}\right)i =$$

So $p \equiv 1\ (4) \implies$ the even number $2m_1 - \frac{p-1}{2} = 0$

$$\implies m_1 = \frac{p-1}{4} \text{ and } m_3 = \frac{p-1}{4} \implies \mu = 1$$

$$\implies m_0 - m_2 = v$$
$$m_0 + m_2 = \frac{p+1}{2}$$

$$2m_2 = \frac{p+1}{2} - v$$

From the determinant $2m_2 \equiv (p-1)^2 + \frac{p(p-1)}{2} \equiv \frac{p-1}{2} \mod (4)$

So $\frac{p+1}{2} - v \equiv \frac{p-1}{2} \mod 4$

$$\implies v \equiv 1 \pmod 4 \text{ so } v = +1.$$

This is sort of stupid. What we should determine is the different multiplicities.

$p \equiv 1 \pmod 4$ $\quad \frac{S}{\sqrt{p}} = (m_0 - m_2) + (m_1 - m_3)i = 1$ so

$$m_0 = \frac{p+3}{4}, \quad m_2 = \frac{p-1}{4}, \quad m_1 = m_3 = \frac{p-1}{4}$$

$p \equiv 3 \pmod 4$. $\quad \frac{S}{\sqrt{p}} = (m_0 - m_2) + (m_1 - m_3)i = i$, so

$$m_0 = m_2 = \frac{p+1}{4}, \quad m_1 = \frac{p+1}{4}, \quad m_3 = \frac{p-3}{4}$$

$p \equiv 2 \pmod 4$, $\quad S = 0$, so

$$m_0 = m_2 = \frac{p+2}{4} \qquad m_1 = m_3 = \frac{p-2}{4}$$

$p \equiv 0 \pmod 4$, $\quad \frac{S}{\sqrt{p}} = 1 + i \quad$ so

$$m_0 = \frac{p+4}{4} \quad m_2 = \frac{p}{4} \qquad m_1 = \frac{p}{4} \qquad m_3 = \frac{p-4}{4}$$

But these formulas for the multiplicities of the eigenvalues to be useful have to be improved to actual eigenvectors.

Suppose p odd, let q be even $\not\equiv 0 \bmod p$ and let a be even such that $aq \equiv 1 \bmod p$. Then we have seen that

$$\sum_m e^{\pi i \frac{q}{p} m^2 + 2\pi i \frac{mn}{p}} = \sum_m e^{\pi i \frac{q}{p}(m^2 + 2man + a^2n^2) - \pi i \frac{q}{p} a^2 n^2}$$

$$= e^{-\pi i \frac{a}{p} n^2} \cdot \sum_m e^{\pi i \frac{q}{p} m^2}$$

Suppose p is an odd prime $\equiv 1 \bmod 4$. Then we can choose q so that $q^2 \equiv -1$, whence $-a \equiv q$. Then $e^{\pi i \frac{q}{p} n^2}$ is an eigenvector for the Fourier transform with eigenvalue

$$\sum_m e^{\pi i \frac{q}{p} m^2} = \left(\frac{2q}{p}\right)\sqrt{p} = \sqrt{p}.$$

In effect q being a square root of -1 in $(\mathbb{Z}/p\mathbb{Z})^*$ is a residue when $p \equiv 1$ (8) and a non-residue when $p \equiv 5$ (8), and 2 is a residue when $p \equiv 1$ (8) and a non-residue when $p \equiv 5$ (8).

Now the functions $e^{\pi i \frac{q}{p} n^2 + 2\pi i \frac{xn}{p}} = f_x$ as $x \in \mathbb{Z}/p\mathbb{Z}$ form a basis, and

$$\sum_m e^{\pi i \frac{q}{p} m^2 + 2\pi i \frac{xm}{p} + 2\pi i \frac{mn}{p}} = e^{\pi i \frac{q}{p}(n+x)^2}\sqrt{p}$$

$$= \sqrt{p}\, e^{\pi i \frac{q}{p} x^2}\, e^{\pi i \frac{q}{p} n^2 + 2\pi i \frac{qx}{p} n}$$

i.e.

$$\hat{f}_x = \sqrt{p}\, e^{\pi i \frac{q}{p} x^2}\, \hat{f}_{qx}$$

Thus on the basis f_x, the Fourier transform is given by a monomial matrix associated to the permutation of $\mathbb{Z}/p\mathbb{Z}$ given by multiplying by q. Except for the fixpt 0, the orbits are cyclic of order 4, so one sees at once that the multiplicities are $m_0 = \frac{p-1}{4} + 1$ and the rest are $\frac{p-1}{4}$. This picture of the Fourier transform is reminiscent of the Weyl picture where the transform is $\hat{f}(z) = f(iz)$, here $q = \sqrt{-1}$.

Odd functions in this basis:

$$s_x = e^{\pi i \frac{q}{p} n^2} \sin\left(2\pi \frac{xn}{p}\right) = e^{\pi i \frac{q}{p} n^2} \frac{1}{2i}\left(e^{2\pi i \frac{xn}{p}} - e^{-2\pi i \frac{xn}{p}}\right)$$

One has

$$\hat{s}_x = e^{\pi i \frac{q}{p} x^2} s_{qx}$$

hence $\hat{\hat{s}}_x = -s_x$ as it must.

There doesn't seem to be any finite Gaussian analogue of the differentiated Θ-fns:

$$\sum n e^{-\pi n^2 t + 2\pi i n y}$$

June 28, 1977

Recall that the Hermite-Weber DE

$$\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)u=\left(\frac{d^2}{dx^2}-x^2+1\right)u=2su$$

has the solutions

$$e^{-x^2/2}\int e^{-t^2\pm 2xt}\,t^s\,\frac{dt}{t}$$

and

$$e^{x^2/2}\int e^{-t^2\pm 2ixt}\,t^{1-s}\,\frac{dt}{t}$$

(see June 4)

Now let us consider the system

$$\frac{d}{dx}u=\begin{pmatrix} i(\lambda+x) & 1 \\ 1 & -i(\lambda+x)\end{pmatrix}u$$

which corresponds to $p(x)=e^{ix^2}$ in the standard form. We can write this

$$\left[\frac{d}{dx}-i(\lambda+x)\right]u_1=u_2$$

$$\left[\frac{d}{dx}+i(\lambda+x)\right]u_2=u_1$$

so

$$\left(\frac{d}{dx}+i(\lambda+x)\right)\left(\frac{d}{dx}-i(\lambda+x)\right)u_1$$

$$=\left(\frac{d^2}{dx^2}+(\lambda+x)^2-i\right)u_1=u_1$$

Put $\lambda+x=e^{i\pi/4}z$. $(dx)^2=e^{i\pi/2}(dz)^2=i(dz)^2$

$$\left(\frac{1}{i}\frac{d^2}{dz^2} + iz^2 - i\right) u_1 = u_1 \qquad \text{or}$$

$$\left(\frac{d^2}{dz^2} - z^2 + 1\right) u_1 = i u_1$$

So we get solutions of the form

$$u_1^{\pm} = e^{-z^2/2} \int e^{-t^2 \pm 2zt}\, t^{i/2} \frac{dt}{t}$$

Take the standard contour C and ask what happens to these solutions as $x \longmapsto +\infty$. Then z goes to ∞ along the line $\arg(z) = -\frac{\pi}{4}$. Thus

$$e^{-z^2/2} = e^{i\frac{(\lambda+x)^2}{2}}$$

oscillates rapidly. On the other hand

$$\int e^{-t^2 - 2zt}\, t^{i/2} \frac{dt}{t}$$

decays as $z \to \infty$ in any sector $|\arg(z)| \leq \frac{\pi}{2} - \varepsilon$. So u_1^+ decays as $x \to +\infty$. Similarly u_1^- decays as $x \to -\infty$. [redacted] ?

Now it is known that for the potential $g = -x^2$ the Schroedinger DE is in the limit point case, hence

$$\left[\frac{d^2}{dx^2} + (\lambda + x)^2\right] u = (1+i) u$$

has exactly one ℓ^2 solution on $(0, \infty)$, [redacted] namely u_1^+, so we know u_1^- is not ℓ^2 as $x \to +\infty$, but still we should determine its behavior. ?

Correction: According to Watson's lemma one the asymptotic expansion

$$\int_0^\infty e^{-t^2-2zt}\, t^s \frac{dt}{t} \sim \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_0^\infty e^{-2zt}\, t^{2n+s} \frac{dt}{t}$$

$$\sim \sum_{n\geq 0} \frac{(-1)^n}{n!} \frac{\Gamma(2n+s)}{(2z)^{2n+s}}$$

so $u_1^+ = \dfrac{e^{-z^2/2}}{\Gamma(s)} \displaystyle\int_0^\infty e^{-t^2-2zt}\, t^s \frac{dt}{t} \sim e^{-z^2/2} \left\{ \frac{1}{(2z)^s} + c_1 \frac{1}{(2z)^{s+2}} + \cdots \right\}$

This does not decay as $z \to \infty$ along $\arg(z) = -\pi/4$, but rather for $s = i/2$ it oscillates.

$$z^{-i/2} = e^{-i/2(\log|z| - i\pi/4)} = e^{-\frac{1}{2}\frac{\pi}{4} - i/2 \log|z|}$$

More rigorous procedure is to replace t by $\frac{u}{z}$ which will shift the contour harmlessly to $\arg u = \arg z = -\frac{\pi}{4}$

$$\frac{\Gamma(1-s)}{2\pi i e^{i\pi s}} \int_C e^{-\frac{u^2}{z^2} - 2u} \left(\frac{u}{z}\right)^s \frac{du}{u} \qquad \text{etc.}$$

So we see that it seems hard to find the asymptotic behavior of

$$e^{-z^2/2} \int_C e^{-t^2+2zt}\, t^s \frac{dt}{t}$$

as $z \to \infty$ with $|\arg z| \leq \frac{\pi}{2} - \varepsilon$. However we also have the solutions obtained by changing z to iz

and s to $1-s$, namely

$$e^{z^2/2}\int_C e^{-t^2-2izt}\, t^{1-s}\,\frac{dt}{t}$$

If $\mathrm{Im}(z)<0$, then $\mathrm{Re}(2iz)>0$, so we can apply Watson's lemma to this

$$\sim e^{z^2/2}\int_C \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\, e^{-(2iz)t}\, t^{2n+1-s}\,\frac{dt}{t}$$

$$\sim e^{z^2/2}\sum_{n\geq 0}\frac{(-1)^n}{n!}\,\frac{\Gamma(2n+1-s)}{(2iz)^{2n+1-s}}\left(e^{2\pi i(1-s)}-1\right)$$

It would have been cleaner to use $\dfrac{e^{z^2/2}}{\Gamma(1-s)}\displaystyle\int_0^{\infty} e^{-t^2-2izt}\, t^{1-s}\frac{dt}{t}$ for then the expansion would be

$$\sim e^{z^2/2}\sum_{n\geq 0}\frac{(-1)^n}{n!}\,\frac{\Gamma(2n+1-s)}{\Gamma(1-s)}\,\frac{1}{(2iz)^{2n+1-s}}$$

Therefore it seems that the two solutions have the asymptotic behaviors (leading terms)

$$e^{-z^2/2}(2z)^{-s}\,, \qquad e^{z^2/2}(2z)^{s-1}$$

in the sector $-\pi/2<\arg z<0$. If this is true, then as $x\to\infty$ we have for $s=i/2$ one oscillatory solution and one solution which decays, i.e. is in l^2.

June 29, 1977.

Hermite - Weber DE:

$$\left(\frac{d}{dz} - z\right)\left(\frac{d}{dz} + z\right) u = 2s\,u$$

$$\left(\frac{d^2}{dz^2} - z^2\right) u = 2(s-\tfrac{1}{2})u$$

is symmetric under $(z,s) \longmapsto (-z,s)$ and $(z,s) \mapsto (iz, 1-s)$.

Contour integral solutions: If

$$u_s = e^{-z^2/2} \int e^{-t^2-2zt}\, t^s \frac{dt}{t}$$

then

$$\begin{cases} \left(\dfrac{d}{dz} + z\right) u_s = -2u_{s+1} \\ \left(\dfrac{d}{dz} - z\right) u_{s+1} = -s\, u_s \end{cases}$$

so u_s satisfies the H-W DE.

Series solutions: If one puts $u(z) = e^{-z^2/2} v(z^2)$, then $v(x)$ satisfies the confluent hypergeometric DE

$$\left[x \frac{d^2}{dx^2} + (\tfrac{1}{2} - x)\frac{d}{dx} - \frac{s}{2}\right] v = 0$$

which has the series solutions

$$F(\tfrac{s}{2}, \tfrac{1}{2}, x) = \sum_{n \geq 0} \frac{(\frac{s}{2})(\frac{s}{2}+1)\cdots(\frac{s}{2}+n-1)}{n!\,(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2n-1}{2})} (z^2)^n$$

$$x^{1/2} F(\tfrac{s+1}{2}, \tfrac{3}{2}, x) = \sum_{n \geq 0} \frac{(\frac{s+1}{2}) \cdots (\frac{s+1}{2}+n-1)}{n!\,(\frac{3}{2})(\frac{5}{2}) \cdots (\frac{2n+1}{2})} z^{2n+1}$$

hence the Weber DE has the solutions

$$h_s^{ev}(z) = e^{-z^2/2} \sum_{n\geq 0} \text{[redacted]} \frac{\Gamma(\frac{s}{2}+n)}{\Gamma(\frac{s}{2})} \frac{(2z)^{2n}}{(2n)!} = 1 + O(z^2)$$

$$h_s^{odd}(z) = e^{-z^2/2} \sum_{n\geq 0} \frac{\Gamma(\frac{s+1}{2}+n)}{\Gamma(\frac{s+1}{2})} \frac{(2z)^{2n+1}}{(2n+1)!} = 2z + O(z^3)$$

Consider the solution dying at $z = +\infty$:

$$h_s^+(z) = \frac{e^{-z^2/2}}{\Gamma(s)} \int_0^\infty e^{-t^2-2zt} t^s \frac{dt}{t} = \frac{\Gamma(1-s)}{2\pi i e^{i\pi s}} e^{-z^2/2} \int_C \cdots$$

One [redacted] can obtain the series expansion by expanding e^{-2zt}, and one finds

$$h_s^+(z) = \frac{\sqrt{\pi}\, 2^{-s}}{\Gamma(\frac{s+1}{2})} h_s^{ev}(z) - \frac{\sqrt{\pi}\, 2^{-s}}{\Gamma(\frac{s}{2})} h_s^{od}(z)$$

Using the symmetry $(z,s) \mapsto (iz, 1-s)$ we have

$$h_{1-s}^{ev}(iz) = h_s^{ev}(z)$$

$$h_{1-s}^{odd}(iz) = i\, h_s^{odd}(z).$$

Asymptotic Expansion

$$h_s^+(z) \sim e^{-z^2/2} \sum \frac{(-1)^n}{n!} \frac{\Gamma(s+2n)}{\Gamma(s)} \frac{1}{(2z)^{s+2n}} \sim e^{-z^2/2} (2z)^{-s}$$

valid as $z \to \infty$ in the sector [redacted] $|\arg(z)| \leq \frac{\pi}{2} - \varepsilon$ for any $\varepsilon > 0$. (actually for $|\arg(z)| \leq \frac{3\pi}{4} - \varepsilon$, see p. 123)

$$h_s^+(z) = \frac{1}{2\Gamma(s)}\left[\Gamma\left(\tfrac{s}{2}\right) h_s^e(z) - \Gamma\left(\tfrac{s+1}{2}\right) h_s^o(z)\right]$$

$$h_{1-s}^+(iz) = \frac{1}{2\Gamma(1-s)}\left[\Gamma\left(\tfrac{1-s}{2}\right) h_s^e(z) - i\Gamma\left(1-\tfrac{s}{2}\right) h_s^o(z)\right]$$

$$\frac{1}{4\Gamma(s)\Gamma(1-s)}\begin{vmatrix} \Gamma\left(\frac{s}{2}\right) & -\Gamma\left(\frac{s+1}{2}\right) \\ \Gamma\left(\frac{1-s}{2}\right) & -i\Gamma\left(1-\frac{s}{2}\right)\end{vmatrix} = \frac{1}{4\frac{\pi}{\sin \pi s}}\left[-i\frac{\pi}{\sin\frac{\pi s}{2}} + \frac{\pi}{\sin\frac{\pi(1-s)}{2}}\right]$$

$$= \frac{1}{2}\left[-i\cos\frac{\pi s}{2} + \sin\frac{\pi s}{2}\right] = -\frac{i}{2}e^{i\frac{\pi}{2}s}$$

so the solutions $h_s^+(z)$ and $h_{1-s}^+(iz)$ are everywhere linearly independent.

$$h_s^e(z) = 2ie^{-i\frac{\pi s}{2}}\begin{vmatrix} h_s^+(z) & -\frac{\Gamma\left(\frac{s+1}{2}\right)}{2\Gamma(s)} \\ h_{1-s}^+(iz) & -i\frac{\Gamma\left(1-\frac{s}{2}\right)}{2\Gamma(1-s)}\end{vmatrix}$$

$$= 2e^{-i\frac{\pi s}{2}}\left(\frac{\Gamma\left(1-\frac{s}{2}\right)}{2\Gamma(1-s)}h_s^+(z) + i\frac{\Gamma\left(\frac{s+1}{2}\right)}{2\Gamma(s)}h_{1-s}^+(iz)\right)$$

$$= 2e^{-i\frac{\pi s}{2}}\left(\frac{\sqrt{\pi}\,2^{s-1}}{\Gamma\left(\frac{1-s}{2}\right)}h_s^+(z) + i\frac{\sqrt{\pi}\,2^{-s}}{\Gamma\left(\frac{s}{2}\right)}h_{1-s}^+(iz)\right)$$

$$= \sqrt{\pi}e^{-i\frac{\pi s}{2}}\left(\frac{2^s h_s^+(z)}{\Gamma\left(\frac{1-s}{2}\right)} + i\frac{2^{1-s}h_{1-s}^+(iz)}{\Gamma\left(\frac{s}{2}\right)}\right)$$

$$h_s^o(z) = 2ie^{-i\frac{\pi s}{2}}\begin{vmatrix} \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma(s)} & h_s^+(z) \\ \frac{\Gamma\left(\frac{1-s}{2}\right)}{2\Gamma(1-s)} & h_{1-s}^+(iz)\end{vmatrix} = 2ie^{-\frac{i\pi s}{2}}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma(s)}h_{1-s}^+(iz) - \frac{\Gamma\left(\frac{1-s}{2}\right)}{2\Gamma(1-s)}h_s^+(z)\right]$$

$$h_s^o(z) = \sqrt{\pi}\, e^{-\frac{i\pi s}{2}}\left(-i\,\frac{2^{s} h_s^+(z)}{\Gamma(1-\frac{s}{2})} + i\,\frac{2^{1-s} h_{1-s}^+(iz)}{\Gamma(\frac{s+1}{2})} \right)$$

Suppose we move the integration path in the integral defining $h_s^+(z)$ to the ray $\arg(t)=\theta$ where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$, so that by Cauchy's thm. we get the same function. Then $Re(zt)>0$ for

$$-\frac{\pi}{2} < \arg(z)+\theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2}-\theta < \arg(z) < \frac{\pi}{2}-\theta$$

so we see that the asymptotic expansion given on page 121 for $h_s^+(z)$ has to be valid in the just-described sector for any θ, hence valid for $|\arg(z)|<\frac{3\pi}{4}$. ~~i.e. off the negative real axis~~

So it seems that the rays $\arg(z)=\pm\frac{3\pi}{4}$ are Stokes lines for the asymptotic expansion of $h_s^+(z)$. Now I am interested in the asymptotic expansion of $h_s^+(z)$ on the ray $\arg(z)=\frac{3\pi}{4}$.

Determine a, b such that

$$h_s^+(-z) = a\, h_s^+(z) + b\, h_{1-s}^+(iz)$$

$$\frac{\sqrt{\pi}\,2^{-s}}{\Gamma(\frac{s+1}{2})} = \frac{\Gamma(\frac{s}{2})}{2\Gamma(s)} = a\,\frac{\Gamma(\frac{s}{2})}{2\Gamma(s)} + b\,\frac{\Gamma(\frac{1-s}{2})}{2\Gamma(1-s)} = \frac{a\sqrt{\pi}\,2^{-s}}{\Gamma(\frac{s+1}{2})} + b\,\frac{\sqrt{\pi}\,2^{s-1}}{\Gamma(1-\frac{s}{2})}$$

$$\frac{\sqrt{\pi}\,2^{-s}}{\Gamma(\frac{s}{2})} = \cancel{i}\,\frac{\Gamma(\frac{s+1}{2})}{2\Gamma(s)} = -\cancel{i}a\,\frac{\Gamma(\frac{s+1}{2})}{2\Gamma(s)} - \cancel{i}ib\,\frac{\Gamma(\frac{2-s}{2})}{2\Gamma(1-s)} = -a\,\frac{\sqrt{\pi}\,2^{-s}}{\Gamma(\frac{s}{2})} - ib\,\frac{\sqrt{\pi}\,2^{s-1}}{\Gamma(\frac{1-s}{2})}$$

$$\begin{vmatrix} \dfrac{2^{-s}}{\Gamma(\frac{s+1}{2})} & \dfrac{2^{s-1}}{\Gamma(1-\frac{s}{2})} \\ -\dfrac{2^{-s}}{\Gamma(\frac{s}{2})} & -i\dfrac{2^{s-1}}{\Gamma(\frac{1-s}{2})} \end{vmatrix} = \frac{1}{2}\left(-i\frac{\sin\frac{\pi}{2}(1-s)}{\pi} + \frac{\sin\frac{\pi s}{2}}{\pi}\right) = \frac{1}{2\pi i} e^{\frac{i\pi s}{2}}$$

$$a = 2\pi i\, e^{-i\frac{\pi s}{2}} \begin{vmatrix} \dfrac{2^{-s}}{\Gamma(\frac{s+1}{2})} & \dfrac{2^{s-1}}{\Gamma(1-\frac{s}{2})} \\ \dfrac{2^{-s}}{\Gamma(\frac{s}{2})} & -i\dfrac{2^{s-1}}{\Gamma(\frac{1-s}{2})} \end{vmatrix} = \cancel{\pi} i e^{-\frac{i\pi s}{2}}\left(-i\frac{\cos\frac{\pi}{2}s}{\cancel{\pi}} - \frac{\sin\frac{\pi}{2}s}{\cancel{\pi}}\right)$$

$$= e^{-i\pi s}$$

$$b = 2\pi i\, e^{-i\frac{\pi s}{2}} \begin{vmatrix} \dfrac{2^{-s}}{\Gamma(\frac{s+1}{2})} & \dfrac{2^{-s}}{\Gamma(\frac{s+1}{2})} \\ -\dfrac{2^{-s}}{\Gamma(\frac{s}{2})} & \dfrac{2^{-s}}{\Gamma(\frac{s}{2})} \end{vmatrix}$$

$$= 2\pi i\, e^{-i\frac{\pi s}{2}}\, 2^{-2s} \frac{2}{2^{1-s}\sqrt{\pi}\,\Gamma(s)} = \frac{i\sqrt{\pi}\, e^{-i\frac{\pi s}{2}}\, 2^{1-s}}{\Gamma(s)}$$

So

$$\boxed{h_s^+(-z) = e^{-i\pi s} h_s^+(z) + \frac{i\sqrt{\pi}\, e^{-i\frac{\pi}{2}s}\, 2^{1-s}}{\Gamma(s)} h_{1-s}^+(iz)}$$

or

$$\boxed{h_s^+(z) - e^{i\pi s} h_s^+(-z) = -\frac{i\sqrt{\pi}\, e^{+i\frac{\pi}{2}s}}{2^{s-1}\Gamma(s)} h_{1-s}^+(iz)}$$

There are two ways of checking the value of a as follows. The first method uses the fact that if

A is the contour:

then as $\mathrm{Im}(t) > 0$ on the contour, as $\mathrm{Im}(z) \to -\infty$ the term e^{-2zt} goes to zero, hence

$$\frac{e^{-z^2/2}}{\Gamma(s)} \int_A e^{-t^2-2zt}\, t^s \frac{dt}{t}$$

should be a multiple of $h^+_{1-s}(iz)$. This is not convincing because of the $e^{-z^2/2}$ in front, so write the above as

$$\frac{e^{z^2/2}}{\Gamma(s)} \int_A e^{-(t+z)^2}\, t^{s-1}\, dt = \frac{e^{z^2/2}}{\Gamma(s)} \int_{\infty}^{-\infty} e^{-u^2} (u-z)^{s-1}\, du$$

and this clearly dies as z goes to ∞ along $\arg(z) = -\frac{\pi}{4}$.
Thus the function

$$\frac{e^{-z^2/2}}{\Gamma(s)} \left[\int_0^\infty e^{-t^2-2zt}\, t^s \frac{dt}{t} + \underbrace{\int_{-\infty}^0 e^{-t^2-2zt}\, \overset{e^{i\pi s}(-t)^s}{t^s} \frac{dt}{t}}_{\int_\infty^0 e^{-t^2+2zt}\, e^{i\pi s}\, t^s \frac{dt}{t}} \right]$$

$$= h^+_s(z) - e^{i\pi s} h^+_s(-z)$$

is a multiple of $h^+_{1-s}(iz)$.

The second way of checking a is to use the asymptotic expansions for $h^+_s(z)$ and $h^+_s(-z)$ along the ray $\arg(z) = -\frac{\pi}{2}$.

In the last formula on p. 124 put $s \mapsto 1-s$, $z \mapsto -iz$

$$h^+_{1-s}(-iz) = e^{i\pi(1-s)}\, h^+_{1-s}(iz) - \frac{i\sqrt{\pi}\, e^{i\frac{\pi}{2}(1-s)}}{2^{-s}\,\Gamma(1-s)}\, h^+_s(z)$$

or

$$\boxed{\begin{pmatrix} h^+_s(-z) \\ h^+_{1-s}(-iz) \end{pmatrix} = \begin{pmatrix} e^{-i\pi s} & \dfrac{i\sqrt{\pi}\, e^{-i\frac{\pi}{2}s}\, 2^{1-s}}{\Gamma(s)} \\ \dfrac{\sqrt{\pi}\, e^{-i\frac{\pi}{2}s}\, 2^{s}}{\Gamma(1-s)} & -e^{-i\pi s} \end{pmatrix} \begin{pmatrix} h^+_s(z) \\ h^+_{1-s}(iz) \end{pmatrix}}$$

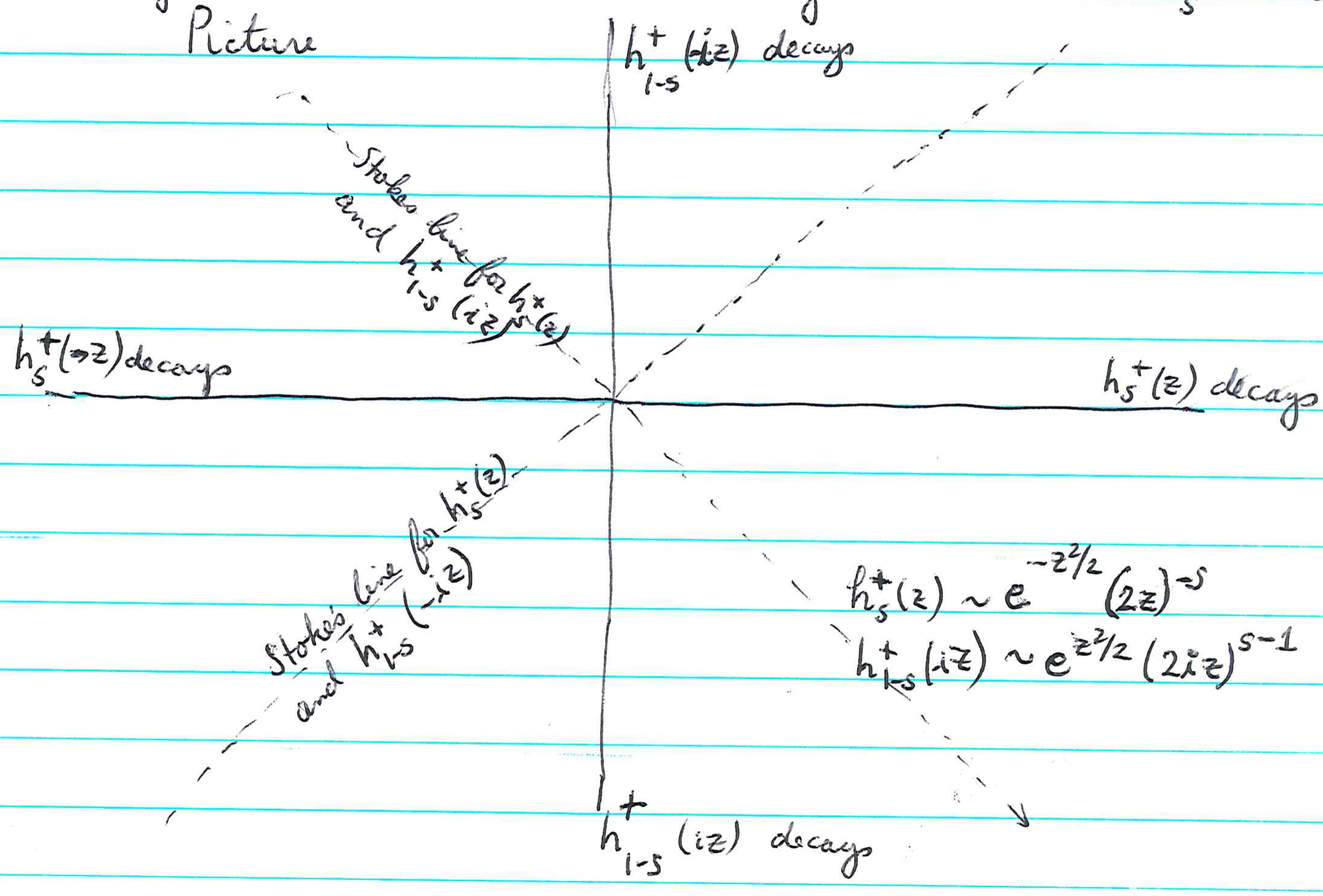
As a check compute the determinant

$$-e^{-2\pi i s} - i\pi e^{-i\pi s}\, 2\, \frac{\sin \pi s}{\pi} = -e^{-2\pi i s} - e^{-i\pi s}\left(e^{i\pi s} - e^{-i\pi s}\right)$$

$$= -1$$

Since the trace is zero, the char. poly is $X^2 = 1$, so the eigenvalues are ± 1. The eigenvectors are h^{ev}_s and h^{od}_s.

Picture



On the ray $\arg(z) = \frac{3\pi}{4}$ to get asymptotic expansion of $h_s^+(z)$ one uses

$$h_s^+(z) = \underbrace{e^{-i\pi s} h_s^+(-z)}_{\sim\, e^{-z^2/2}(-z)^{-s}} + \underbrace{\frac{i\sqrt{\pi}\, e^{-i\pi\frac{s}{2}}\, 2^{1-s}}{\Gamma(s)}\, h_{1-s}^+(-iz)}_{\sim\, e^{z^2/2}(-2iz)^{s-1}}$$

and when s is purely imaginary, this means that $h_s^+(z)$ is not ℓ^2 on the ray $\arg(z) = \frac{3\pi}{4}$.

July 2, 1977

$$\frac{d}{dx}(u) = \begin{pmatrix} i(\lambda+x) & \rho \\ \rho & -i(\lambda+x) \end{pmatrix} u \qquad \rho \text{ real constant}$$

$$\left[\frac{d}{dx} - i(\lambda+x)\right](u_1) = \rho u_2 \qquad \left(\frac{d}{dz} + z\right) u_1 = \rho e^{i\pi/4} u_2$$

$$\left[\frac{d}{dx} + i(\lambda+x)\right](u_2) = \rho u_1 \qquad \left(\frac{d}{dz} - z\right) u_2 = \rho e^{i\pi/4} u_1$$

where $(\lambda+x) = e^{\pi i/4} z$.

$$\left(\frac{d^2}{dz^2} - z^2 + 1\right) u_1 = (i\rho^2) u_1 \qquad s = \frac{i\rho^2}{2}$$

For u_1 we will take either of the solutions $h_s(z)$, $h_{1-s}(iz)$ (we drop the $+$) whose asymptotic behavior on the ray $\arg(z) = -\pi/4$ we understand.

$$\left(\frac{d}{dz} + z\right) h_s(z) = \frac{e^{-z^2/2}}{\Gamma(s)} \int_0^\infty e^{-t^2-2zt} (-2t) t^s \frac{dt}{t} = -2s h_{s+1}(z)$$

forgot $\Gamma(s)$ in bottom

$$\left(\frac{d}{dz}-z\right)h_{s+1}(z) = \frac{e^{-z^2/2}}{\Gamma(s+1)}\int_0^\infty e^{-t^2-2zt}(-2t-2z)\,t^s\,dt$$

$$= \frac{e^{-z^2/2}}{\Gamma(s+1)}\int_0^\infty e^{-t^2-2zt}\left(-\frac{d}{dt}\right)t^s\,dt = -h_s(z)$$

$$\left(\frac{1}{i}\frac{d}{dz}-iz\right)h_{s+1}(iz) = -h_s(iz)$$

various s factors have to be changed.

$$\left(\frac{d}{dz}+z\right)h_{1-s}(iz) = i\cancel{s}\,h_{-s}(iz).$$

If $u_1 = h_s(z)$, then $\rho e^{\pi i/4}u_2 = \left(\frac{d}{dz}+z\right)h_s(z) = -2h_{s+1}$

$$u_2 = \left(-\frac{2}{\rho}\right)e^{-\pi i/4}h_{s+1}(z)$$

If $u_1 = h_{1-s}(iz)$, then $\rho e^{\pi i/4}u_2 = \left(\frac{d}{dz}+z\right)h_{1-s}(iz) = is\,h_{-s}(iz)$, or

$$u_2 = i\left(\frac{\rho}{2}\right)e^{-\pi i/4}h_{-s}(iz).$$

Paradox: $\dfrac{u_1}{u_2} = \dfrac{h_s(z)}{\left(-\frac{2}{\rho}\right)e^{-\pi i/4}h_{s+1}(z)} \sim \left(-\frac{\rho}{2}\right)e^{\pi i/4}\dfrac{(2z)^{-s}}{(2z)^{-(s+1)}}$

tends to ∞ as $z\to\infty$ along $\arg(z) = -\pi/4$. I expected $\frac{u_1}{u_2}$ to be on S^1, but it is necessary to start with solutions $u = (u_1, u_2)$ having boundary values on S^1 at same point.

Two solutions which are independent are

$$\begin{pmatrix} h_s(z) \\ -\frac{2}{\rho}e^{-\pi i/4}h_{s+1}(z)\end{pmatrix} \qquad \begin{pmatrix} h_{1-s}(iz) \\ i\left(\frac{\rho}{2}\right)e^{-\pi i/4}h_{-s}(iz)\end{pmatrix}$$

$$\sim \begin{pmatrix} \left(\frac{2}{\rho}\right)e^{-\pi i/4}h_{1-s}(iz) \\ h_{-s}(iz)\end{pmatrix}$$

Consider now the ratio of two solutions:

$$\frac{u_1}{u_2} = \frac{a\,h_s(z) + \frac{2}{\rho} e^{-\pi i/4} h_{1-s}(iz)}{a\left(-\frac{2}{\rho}\right) e^{-\pi i/4} h_{s+1}(z) + h_{-s}(iz)}$$

and suppose a chosen so that this is on the unit circle for one and hence all $z = e^{-i\pi/4}(\lambda + x)$. Now let $x \to +\infty$ and use asymptotic expansions. Since

$$h_{s+1}(z) \sim e^{-z^2/2}(2z)^{-s-1} \qquad h_{1-s}(iz) \sim e^{z^2/2}(2iz)^{s-1}$$

and $s = i\frac{\rho^2}{2} \in i\mathbb{R}_{>0}$, these cross-terms die, so

$$\frac{u_1}{u_2} \sim a\,\frac{e^{-z^2/2}(2z)^{-s}}{e^{z^2/2}(2iz)^{s}} \qquad z^2 = -i(\lambda+x)^2$$

$$\sim a\,e^{i(\lambda+x)^2}\, 2^{-2s}\, \frac{\cancel{\left(e^{-i\pi/4}\right)^{-s}}\,(\lambda+x)^{-s}}{\cancel{\left(e^{+i\pi/4}\right)^{s}}\,(\lambda+x)^{s}}$$

$$\sim a\,2^{-2s} e^{i(\lambda+x)^2}(\lambda+x)^{-2s}$$

Since s is purely imaginary we see that $|a| = 1$. Consequently the fractional linear transformation described at the top of this page preserves the unit circle. Also if $e^{2i\phi} = \frac{u_1}{u_2}$, then

$$\phi \sim \text{const} + \frac{(\lambda+x)^2}{2} - \frac{\cancel{2}s\log(\lambda+x)}{\cancel{2}i}$$

$$\sim \text{const.} + \frac{(\lambda+x)^2}{2} - \frac{\rho^2}{2}\log(\lambda+x)$$

Interesting point: WKBJ tends to be OK in this example. In effect consider the original system:

$$(*)\qquad \frac{d}{dx}v = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} v \qquad p(x) = e^{ix^2}.$$

We put

$$v = \begin{pmatrix} e^{-\frac{ix^2}{2}} & 0 \\ 0 & e^{\frac{ix^2}{2}} \end{pmatrix} u$$

in order to transform this into

$$\frac{d}{dx}u = \begin{pmatrix} i(\lambda+x) & 1 \\ 1 & -i(\lambda+x) \end{pmatrix} u$$

and found

$$\frac{u_1}{u_2} \approx (\text{const})\, e^{i(\lambda+x)^2} (\lambda+x)^{-i} \qquad x \to +\infty$$

hence

$$\frac{v_1}{v_2} = e^{-ix^2}\frac{u_1}{u_2} \sim (\text{const})\, e^{2i\lambda x} (\lambda+x)^{-i} \qquad x \to +\infty$$

which is consistent, it seems, with the view that the angle parameter for the system $(*)$ above should be asymptotic to

$$\int (\lambda^2 - |p|^2)^{1/2}.$$