June 22, 1977 (37 years old)

$$H^-(x,y,s) = \sum{}' \operatorname{sgn}(x+n)\,|x+n|^{-s} e^{2\pi i n y}$$

$$= H(x,y,s) - e^{-2\pi i y} H(1-x,-y,s) \qquad \text{for } 0 \leq x \leq 1$$

satisfies the functional equation

$$\pi^{-\left(\frac{s+1}{2}\right)} \Gamma\left(\frac{s+1}{2}\right) H^-(x,y,s) = i\,\pi^{-(1-\frac{s}{2})} \Gamma\left(1-\frac{s}{2}\right) H^-(y,-x,1-s)\, e^{-2\pi i x y}$$

Consider $x=0$, $y=\frac{1}{3}$.

$$H^-(0,\tfrac{1}{3},s) = \sum_{n\geq 1} n^{-s}\omega^{n} - \sum_{n \geq 1} n^{-s}\omega^{-n}$$

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad \omega - \omega^{-1} = i\sqrt{3}$$

So

$$H^-(0,\tfrac{1}{3},s) = i\sqrt{3}\, L(s,\chi) \qquad \text{where}$$

$$L(s,\chi) = \sum_{\substack{n\equiv 1\\ n\geq 1}} n^{-s} - \sum_{\substack{n\equiv 2\\ n\geq 1}} n^{-s} = \sum_{n\geq 1} 2\sin\left(\frac{2\pi n}{3}\right) n^{-s}$$

On the other hand

$$H^-(\tfrac{1}{3},0,s) = \sum_{n\geq 1}\left(\frac{1}{3}+n\right)^{-s} - \sum_{n\geq 1}\left(\frac{2}{3}+n\right)^{-s}$$

$$= 3^{s} L(s,\chi)$$

So the functional equation becomes

$$\pi^{-(1-\frac{s}{2})}\Gamma\left(1-\frac{s}{2}\right) i\sqrt{3}\, L(1-s,\chi) = i\,\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{1+s}{2}\right) 3^{s} L(s,\chi)$$

I can symmetrize this in 2 ways since

$$s - \frac{1}{2} = \frac{s}{2} - \frac{1-s}{2} = \left(\frac{1+s}{2}\right) - \left(1-\frac{s}{2}\right)$$

Recall

$$H^+(x,y,s) = \sum{}' |x+n|^{-s} e^{2\pi i n y}$$

$$= H(x,y,s) + e^{-2\pi i y} H(1-x,-y,s) \qquad 0 \le x \le 1$$

satisfies the functional equation

$$\pi^{-s/2}\Gamma(s/2)\, H^+(x,y,s) = e^{-2\pi i x y}\, \pi^{-\frac{(1-s)}{2}}\, \Gamma\left(\tfrac{1-s}{2}\right) H^+(y,-x,1-s)$$

If $x=0$, $y=\frac{1}{3}$

$$H^+(0,\tfrac{1}{3},s) = \sum_{n\ge 1} n^{-s}(\omega^n+\omega^{-n}) = \sum_{n\ge 1} 2\cos\left(\tfrac{2\pi n}{3}\right) n^{-s}$$

$$= 2\sum_{n\equiv 0}{}' n^{-s} - \sum_{n\not\equiv 0}{}' n^{-s} = -(1-3^{-s})\zeta(s) + 2\cdot 3^{-s}\zeta(s)$$

$$= 3\cdot 3^{-s}\zeta(s) - \zeta(s) = (3^{1-s}-1)\zeta(s)$$

$$H^+(\tfrac{1}{3},0,s) = \sum_{n\ge 0}\left(\tfrac{1}{3}+n\right)^{-s} + \sum_{n\ge 0}\left(\tfrac{2}{3}+n\right)^{-s} =$$

$$= 3^{s}(1-3^{-s})\zeta(s)$$

The functional equation then becomes

$$\pi^{-\frac{(1-s)}{2}}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s)\left\{3^{s}-1\right\} = \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right) 3^{s}(1-3^{-s})\zeta(s)$$

which works.

New notation: If $\varepsilon = \pm 1$

$$H^{\varepsilon}(x,y,s) = H^{\pm}(x,y,s)$$

$$= H(x,y,s) + \varepsilon\, e^{-2\pi i y} H(1-x,-y,s)$$

for $0 \le x \le 1$.

Functional equation is

$$\boxed{H^{\varepsilon}(x,y,1-s) = e^{-2\pi i xy}\,\frac{\Gamma(s)}{(2\pi)^s}\left(e^{i\frac{\pi s}{2}}+\varepsilon e^{-i\frac{\pi s}{2}}\right) H^{\varepsilon}(y,-x,s)}$$

To relate this to the one on page 73 you use

$$\frac{\pi^{-s/2}\,\Gamma(s/2)}{\pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)} = \pi^{-s+\frac{1}{2}}\,\Gamma(s/2)\,\Gamma\left(1-\frac{1-s}{2}\right)\frac{\sin \pi\frac{1-s}{2}}{\pi} = \pi^{-s-\frac{1}{2}}\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\cos\frac{\pi s}{2}$$

$$= \pi^{-s-\frac{1}{2}}\,2^{1-s}\sqrt{\pi}\,\Gamma(s)\cos\frac{\pi s}{2} = \frac{\Gamma(s)}{(2\pi)^s}\,2\cos\frac{\pi s}{2}$$

$$\frac{\pi^{-\frac{(1+s)}{2}}\,\Gamma\left(\frac{1+s}{2}\right)}{\pi^{-\left(1-\frac{s}{2}\right)}\,\Gamma\left(1-\frac{s}{2}\right)} = \pi^{-s+\frac{1}{2}}\,\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)\frac{\sin\pi\frac{s}{2}}{\pi} = \pi^{-s-\frac{1}{2}}\,2^{1-s}\sqrt{\pi}\,\Gamma(s)\sin\frac{\pi s}{2}$$

$$= \frac{\Gamma(s)}{(2\pi)^s}\cdot 2\sin\frac{\pi s}{2}$$

$$H^{\varepsilon}\left(0,\frac{a}{p},s\right) = \sum_{n\geq 1}\left[\left(e^{2\pi i\frac{a}{p}}\right)^n + \varepsilon\left(e^{-2\pi i\frac{a}{p}}\right)^n\right] n^{-s}$$

$$H^{\varepsilon}\left(\frac{a}{p},0,s\right) = p^s\sum_{n\geq 0}(a+pn)^{-s} + \varepsilon(p-a+pn)^{-s} \qquad 0<a<p$$

Example: $p=5$. Take $\varepsilon=+1$. There are 2 values of a to consider, [redacted] since a and $p-a$ give the same series. One should be able to take suitable linear combinations of these to get Dirichlet L-functions.

Let [redacted] $\chi:(\mathbb{Z}/5\mathbb{Z})^* \longrightarrow \mathbb{C}^*$ be given by

$$\chi:\begin{cases}1\\2\\3\\4\end{cases}\longmapsto\begin{matrix}1\\i\\-i\\-1\end{matrix}\qquad\qquad \chi^{-1}:\begin{cases}1\\2\\3\\4\end{cases}\mapsto\begin{matrix}1\\-i\\i\\-1\end{matrix}$$

Then $$L(s,\chi) = \sum_{n\geq 1} \chi(n) n^{-s}$$
$$= \sum_{n\geq 0} (1+5n)^{-s} - (4+5n)^{-s} + i(2+5n)^{-s} - i(3+5n)^{-s}$$
$$= 5^{-s}\left\{H^-\left(\tfrac{1}{5},0,s\right) + iH\left(\tfrac{2}{5},0,s\right)\right\}$$

Also $L(s,\chi^{-1}) = 5^{-s}\left\{H^-\left(\tfrac{1}{5},0,s\right) - iH^-\left(\tfrac{2}{5},0,s\right)\right\}$.

Now the functional equation relates $L(s,\chi)$ to

$$H^-\left(0,\tfrac{1}{5},s\right) + iH^-\left(0,\tfrac{2}{5},s\right)$$ after $1-s \mapsto s$

$$\cancel{H^-\left(0,\tfrac{1}{5},s\right)} = \sum_{n\geq 1} 2\left[\sin\left(2\pi\tfrac{n}{5}\right) + i\sin\left(2\pi\tfrac{2n}{5}\right)\right] n^{-s}$$
$$= i\sum_{n\geq 1}\left[\omega^n + i(\omega^2)^n - i(\omega^3)^n - (\omega^4)^n\right] n^{-s}$$

where $\omega = e^{2\pi i/5}$.

Before we get mixed up with Gauss sums it would be preferrable to work out what happens generally. The point is that given the modulus p we have a space of dimension $\frac{p-1}{2}$ (say p odd) consisting of the functions
$$H^\varepsilon\left(0,\tfrac{a}{p},s\right) = \sum_{n\geq 1}\left[\left(e^{2\pi i a/p}\right)^n + \varepsilon\left(e^{-2\pi i a/p}\right)^n\right] n^{-s}$$

for $a = 1,2,\ldots,\left(\frac{p-1}{2}\right)$, (suppose $\varepsilon = -1$).

Actually it seems to be better to work with all the functions $H^\varepsilon(0, \frac{a}{p}, s)$ $\qquad \varepsilon = \pm 1$, [redacted] $a \in \mathbb{Z}/p\mathbb{Z}$

and to note that

$$H^\varepsilon(0, -\tfrac{a}{p}, s) = \varepsilon\, H(0, \tfrac{a}{p}, s).$$

The space [redacted] spanned by these $2p$ function $H^\varepsilon(0, \frac{a}{p}, s)$ is obviously the same as the space spanned by the functions $\sum_{n \geq 1} (e^{2\pi i a/p})^n n^{-s}$, $a \in \mathbb{Z}/p\mathbb{Z}$, which is p-diml. If p is odd, then the action $a \mapsto -a$ on $\mathbb{Z}/p\mathbb{Z}$ has one fixpoint, so the [redacted] $\varepsilon = +1$ eigenspace is $\frac{p+1}{2}$-diml, while the $\varepsilon = -1$ eigenspace is $\frac{p-1}{2}$-diml. If p is even, then $a \mapsto -a$ has 2 fixpts, so [redacted] the $\varepsilon = +1$ eigenspace is $\left(\frac{p}{2}+1\right)$-diml. and the $\varepsilon = -1$ eigenspace is $\left(\frac{p}{2}-1\right)$-dimensional.

So next consider the other side:

$$H^\varepsilon(\tfrac{a}{p}, 0, s) = \sum_{n \geq 0}{}' \left(\frac{a}{p} + n\right)^{-s} + \varepsilon \sum_{n \geq 0} \left(\frac{p-a}{p} + n\right)^{-s}$$

[redacted] Here we have the functions

$$p^{-s} H^\varepsilon(\tfrac{a}{p}, 0, s) = \sum_{n \geq 0}{}' (a + pn)^{-s} + \varepsilon (p - a + pn)^{-s}$$

which you get by [redacted] ε-symmetrizing on the p-diml. space of functions of the form

$$\sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$

where $\chi(n) = \chi(n')$ if $n \equiv n' \pmod{p}$.

Start again: Let V be the p-diml complex vector space consisting of complex functions [redacted] $\chi : \mathbb{Z}_1 \to \mathbb{C}$ such that $n \equiv n' \pmod{p} \Rightarrow \chi(n) = \chi(n')$. Thus $V = Map(\mathbb{Z}/p\mathbb{Z} \to \mathbb{C})$.

Put $V^{\varepsilon} = \{ \chi \in V \mid \chi(-n) = \varepsilon \chi(n) \}$ whence

$$V = V^{+} \oplus V^{-}$$

where
$$\dim(V^{+}) = \begin{cases} \frac{p+1}{2} & p \text{ odd} \\ \frac{p}{2}+1 & p \text{ even} \end{cases}$$

A basis for V^{+} consists of the functions

$$\delta_{a+p\mathbb{Z}} + \delta_{-a+p\mathbb{Z}}$$

for $0 \leq a \leq [\frac{p}{2}]$ and a basis for V^{-} consists of the functions

$$\delta_{a+p\mathbb{Z}} - \delta_{-a+p\mathbb{Z}}$$

for $0 < a < \frac{p}{2}$.

To each $\chi \in V$ we associate a Dirichlet series

$$\sum_{n \geq 1} \chi(n) n^{-s}$$

We have $\chi \in V^{+} \Leftrightarrow \chi(a) = \chi(p-a)$ for $0 < a < \frac{p}{2}$

and
$$\chi \in V^{-} \Leftrightarrow \begin{cases} \chi(a) = -\chi(p-a) \text{ for } 0 < a < \frac{p}{2} \\ \text{and } \chi(p) = 0 \text{ and} \\ \chi(\frac{p}{2}) = 0 \text{ if } p \text{ is even} \end{cases}$$

These criteria allow us to recognize when a Dirichlet series with the property that $\chi(n) = \chi(n')$ for $n \equiv n' \pmod{p}$ comes from a χ in V^{+} or in V^{-}.

To the basis element $\delta_{a+p\mathbb{Z}} - \delta_{-a+p\mathbb{Z}}$ of V^{-} corresponds the series

$$\sum_{n \geq 0} (a+pn)^{-s} - \sum_{n \geq 0} (p-a+pn)^{-s} = p^{-s} H^{-}\left(\frac{a}{p}, 0, s\right).$$

where I suppose $0 < a < \frac{p}{2}$. The formula remains valid for $a = 0, \frac{p}{2}$ provided $\sum_{n \geq 0}$ is replaced by $\sum'_{n \geq 0}$.

Look at $\delta_{a+p\mathbb{Z}} + \delta_{-a+p\mathbb{Z}}$ for $0 < a < \frac{p}{2}$. This gives the series

$$\sum_{n \geq 0}' (a+pn)^{-s} + \sum_{n \geq 0}' (p-a+pn)^{-s} = p^{-s} H^+\left(\frac{a}{p}, 0, s\right).$$

and this formula remains valid for $a = 0$ whence one gets $2p^{-s}\zeta(s)$; also for $a = \frac{p}{2}$ where p is even where one gets

$$p^{-s} \sum_{n \in \mathbb{Z}}' \left|\frac{1}{2}+n\right|^{-s} = 2^s p^{-s}\, 2 \sum_{n \geq 0} (2n+1)^{-s}$$

$$= 2^s p^{-s}\, 2\,(1-2^{-s})\,\zeta(s)$$

$$= 2p^{-s}(2^s - 1)\,\zeta(s).$$

Another basis for V is given by the functions

$$n \longmapsto e^{2\pi i \frac{an}{p}} \qquad a \in \mathbb{Z}/p\mathbb{Z}$$

This gives rise to the functions

$$n \longmapsto e^{2\pi i \frac{an}{p}} - e^{-2\pi i \frac{an}{p}}$$

in V^-, hence the series

$$H^-\left(0, \frac{a}{p}, s\right) = \sum_{n \geq 1} \left[\left(e^{2\pi i \frac{a}{p}}\right)^n - \left(e^{-2\pi i \frac{a}{p}}\right)^n\right] n^{-s}$$

come from functions in V^-.

Now we have the functional equation

$$H^{\varepsilon}\left(0, \frac{a}{p}, 1-s\right) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{i\frac{\pi s}{2}} + \varepsilon e^{-i\frac{\pi s}{2}}\right) H^{\varepsilon}\left(\frac{a}{p}, 0, s\right)$$

which gives rise to an endomorphism of V^- as follows. Start with the function

$$\delta_{a+p\mathbb{Z}} - \delta_{-a+p\mathbb{Z}}$$

which gives rise to $p^{-s} H^-\left(\frac{a}{p}, 0, s\right)$. Now multiple by $\frac{\Gamma(s)}{(2\pi)^s}\left(e^{i\pi s/2} - e^{-i\pi s/2}\right) p^s$ to get

$$H^-\left(0, \frac{a}{p}, 1-s\right).$$

Next change s to $1-s$ and you have the Dirichlet series belonging to the function

$$n \longmapsto e^{2\pi i \frac{an}{p}} - e^{-2\pi i \frac{an}{p}}.$$

So it is clear that the endomorphism of V^- just described is essentially the Fourier transform

$$\chi \longmapsto \left(n \mapsto \sum_{m \in \mathbb{Z}/p\mathbb{Z}} \chi(m) e^{2\pi i \frac{nm}{p}} \right)$$

As a check note

$$\delta_{a+p\mathbb{Z}} \longmapsto \left(n \longmapsto \sum_m \delta_{a+p\mathbb{Z}}(m) e^{2\pi i \frac{nm}{p}} = e^{2\pi i \frac{an}{p}} \right)$$

∎

Check also that the same situation holds for V^+.

Yes.

June 23, 1977

Yesterday I saw that if we took Dirichlet series of the form $\sum_{n\geq 1} a_n n^{-s}$ where a_n depends on n modulo p, then this forms a p-diml complex vector space isomorphic to the space of functions on $\mathbb{Z}/p\mathbb{Z}$. Moreover, provided we split this space into even and odd functions, we got an endomorphism given by the functional equation satisfied by this D-series, which could be identified with the Fourier transforms:

$$f \longmapsto \hat{f}(n) = \sum_{m\in\mathbb{Z}/p\mathbb{Z}} f(m)\, e^{2\pi i \frac{nm}{p}}$$

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^* \longrightarrow \mathbb{C}^*$ be a character and extend it by 0 to $\mathbb{Z}/p\mathbb{Z}$. Consider

$$\hat{\chi}(n) = \sum_{m\in(\mathbb{Z}/p\mathbb{Z})^*} \chi(m)\, \zeta^{nm} \qquad \zeta = e^{2\pi i/p}$$

Then

$$\hat{\chi}(n_1 n_2) = \sum_m \chi(m)\, \zeta^{n_1 n_2 m} = \sum_m \chi(n_2^{-1} m)\, \zeta^{n_2 m}$$

$$= \overline{\chi(n_2)}\, \hat{\chi}(n_1)$$

Hence

$$\hat{\chi}(n) = \overline{\chi}(n)\, \hat{\chi}(1)$$

so consequently one has the functional equation relating $L(s,\chi)$ and $L(1-s,\overline{\chi})$ with the constant $\hat{\chi}(1)$. Try to write it out carefully.

First note that

$$\chi(-n) = \chi(-1)\chi(n)$$

hence $\chi(-1)^2 = 1$ so $\chi(-1) = \pm 1$. Thus in the functional equation

$$\varepsilon = \chi(-1)$$

~~$L(s,\chi) = \sum_{n\geq 1} \chi(n)\, n^{-s} = \frac{1}{2}\sum_{n\geq 1} (\chi(n) + \varepsilon\chi(p-n))\, n^{-s}$~~

Functional equation reads

$$L(1-s,\hat{\chi}) = p^{s}\frac{\Gamma(s)}{(2\pi)^s}\left(e^{i\pi s/2} + \chi(-1)e^{-i\pi s/2}\right) L(s,\chi)$$

and

$$\| \quad \hat{\chi}(1)\, L(1-s,\overline{\chi}).$$

Rewrite this

$$\frac{\hat{\chi}(1)}{p^s} = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{i\pi s/2} + \chi(-1)e^{-i\pi s/2}\right)\frac{L(s,\chi)}{L(1-s,\overline{\chi})}$$

~~We saw on page 78 that the first factor on the right is of the form $\frac{g(s)}{g(1-s)}$, [illegible] g depending only $\chi(-1)$. So~~

~~$$\frac{\hat{\chi}(1)}{p^s}\frac{\hat{\overline{\chi}}(1)}{p^{1-s}} = \frac{g(s)}{g(1-s)}\frac{L(s,\chi)}{L(1-s,\overline{\chi})}\cdot\frac{g(1-s)}{g(s)}\frac{L(1-s,\overline{\chi})}{L(s,\chi)} = 1$$~~

~~So $\hat{\chi}(1)\hat{\overline{\chi}}(1) = p$. Note that~~

~~$$\hat{\overline{\chi}}(1) = \sum{}' \overline{\chi}(m)\, e^{2\pi i \frac{m}{p}} =$$~~

Compute

$$\frac{\hat{\chi}(1)}{p^s}\frac{\hat{\overline{\chi}}(1)}{p^{1-s}} = \frac{\Gamma(s)\Gamma(1-s)}{2\pi}\left(e^{i\pi s/2} + \varepsilon e^{-i\pi/2}\right)\underbrace{\left(e^{i\pi\frac{(1-s)}{2}} + \varepsilon e^{-i\pi\left(\frac{1-s}{2}\right)}\right)}_{\left(ie^{-\frac{i\pi s}{2}} - i\varepsilon e^{+\frac{i\pi s}{2}}\right)}\frac{\cancel{L(s,\chi)}}{L(1-s,\overline{\chi})}\frac{\cancel{L(1-s,\overline{\chi})}}{L(s,\chi)}$$

$$\left(e^{i\pi s/2}+\varepsilon e^{-i\pi s/2}\right)\left(ie^{-\frac{i\pi s}{2}}-i\varepsilon e^{\frac{i\pi s}{2}}\right) = -i\varepsilon\left(e^{i\pi s/2}+\varepsilon e^{-\frac{i\pi s}{2}}\right)\left(e^{\frac{i\pi s}{2}}-\varepsilon e^{-\frac{i\pi s}{2}}\right)$$
$$= -i\varepsilon\left(e^{i\pi s}-e^{-i\pi s}\right) = 2\varepsilon \sin(\pi s)$$

$$\therefore \quad \boxed{\hat{\chi}(1)\,\hat{\bar{\chi}}(1) = \varepsilon p \qquad \varepsilon = \chi(-1)}$$

But
$$\hat{\chi}(1) = \sum_m{}' \bar{\chi}(m) e^{2\pi i \frac{m}{p}} = \chi(-1)\sum_m \bar{\chi}(m)\, e^{-2\pi i\frac{m}{p}}$$
$$= \chi(-1)\, \overline{\hat{\chi}(1)}$$

Thus we get
$$\boxed{\left|\hat{\chi}(1)\right| = p^{1/2}}$$

Actually we have to be more careful in the calculation on page 84, because the proof that $\hat{\chi}(n) = \bar{\chi}(n)\,\hat{\chi}(1)$ works only when $(n,p)=1$.

So suppose $(n,p) = d > 1$, then we have a factorization

$$\begin{tikzcd}
(\mathbb{Z}/p\mathbb{Z})^* \arrow[r, "m \mapsto \zeta^{nm}"] \arrow[d] & \mathbb{C}^* \\
(\mathbb{Z}/d\mathbb{Z})^* \arrow[ur] &
\end{tikzcd}$$

and hence $\hat{\chi}(n) = \sum_m{}' \chi(m)\,\zeta^{nm} = 0$ provided χ is a primitive character, i.e. does not come from a character mod d for any $d \mid p$, $d \neq p$. Thus the formula

$$\boxed{\hat{\chi}(n) = \bar{\chi}(n)\,\hat{\chi}(1) \quad \text{is valid for all } n \text{ when } \chi \text{ is primitive}}$$

Direct proof for the absolute value is as follows:

$$|\hat{\chi}(n)|^2 = \sum_{m_1, m_2}{}' \chi(m_1)\overline{\chi(m_2)}\, \zeta^{n(m_1 - m_2)}$$

sum over all $n \in \mathbb{Z}/p\mathbb{Z}$ and use $\hat{\chi}(n) = \overline{\chi}(n)\,\hat{\chi}(1)$.
You find

$$\varphi(p)\, |\hat{\chi}(1)|^2 = \sum_{m}{}' |\chi(m)|^2 \cdot p = \varphi(p)\cdot p$$

hence $|\hat{\chi}(1)|^2 = p$ as claimed.

Quadratic Gaussian sums: Consider

$$\theta(t) = \sum{}' e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum{}' e^{-\pi n^2/t}$$

with $t = \varepsilon + i\frac{p}{q}$ as $\varepsilon \searrow 0$. One has

$$\sum e^{-\pi n^2(\varepsilon + i\frac{p}{q})} = \sum e^{-\pi n^2 \varepsilon - \pi i n^2 \frac{p}{q}}$$

Notice that $\pi(n+q)^2\frac{p}{q} = \pi(n^2 + 2nq + q^2)\frac{p}{q} = \pi n^2\frac{p}{q} + 2\pi np + \pi pq$
hence if either p or q is even, the factor

$$e^{-\pi i n^2 \frac{p}{q}}$$

is periodic in n of period q, hence we can write

$$\sum_{n \in \mathbb{Z}}{}' e^{-\pi n^2(\varepsilon + i\frac{p}{q})} = \sum_{r \in \mathbb{Z}/q\mathbb{Z}} e^{-\pi i r^2 \frac{p}{q}} \sum_{m \in \mathbb{Z}}{}' e^{-\pi (r + mq)^2 \varepsilon}$$

Notice that I haven't assumed p, q relatively prime. Now I know that

$$\sum_{m\in\mathbb{Z}} e^{-\pi(r+mq)^2\varepsilon} = \sum_{m\in\mathbb{Z}} e^{-\pi\left(\frac{r}{q}+m\right)^2(\varepsilon q^2)}$$

$$= \frac{1}{\sqrt{\varepsilon q^2}} \sum_{m\in\mathbb{Z}} e^{-\pi m^2 \frac{1}{\varepsilon q^2} + 2\pi i m \frac{r}{q}}$$

Thus as $\varepsilon \to 0$ one has the leading term

$$\sum e^{-\pi n^2\left(\varepsilon + i\frac{p}{q}\right)} \sim \frac{1}{\sqrt{\varepsilon}}\,\frac{1}{q} \sum_{r\in\mathbb{Z}/q\mathbb{Z}} e^{-\pi i r^2 \frac{p}{q}}$$

actually this is the whole asymptotic expansion.

where we are assuming that either p or q is even. Note that when the rational number $\frac{p}{q}$ is a 2-unit then this leading term is zero.

To see this put $p = 2p'$, $q = 2q'$ with $(p', q') = 1$, and p', q' both odd. Then because

$$\pi(r+q')^2\frac{p'}{q'} = \pi r^2 \frac{p'}{q'} + 2\pi r p' + \pi p' q'$$

one has

$$e^{-i\pi(r+q')^2\frac{p'}{q'}} = e^{-i\pi r^2 \frac{p'}{q'}}(-1)$$

so that summing as r ranges over $\mathbb{Z}/q\mathbb{Z}$ gives zero

Now if $t = \varepsilon + i\frac{p}{q}$, then

$$\frac{1}{t} = -i\frac{q}{p}\left(1 - i\frac{q}{p}\varepsilon + O(\varepsilon^2)\right) = -i\frac{q}{p} + \varepsilon\frac{q^2}{p^2} + O(\varepsilon^2)$$

$$\frac{1}{t^{1/2}} = e^{-i\pi/4}\frac{q^{1/2}}{p^{1/2}} + O(\varepsilon)$$

so we also have

$$\left(\varepsilon + i\frac{p}{q}\right)^{-1/2} \sum' e^{-\pi n^2/(\varepsilon + i p/q)}$$

$$\sim e^{-i\pi/4} \frac{q^{1/2}}{p^{1/2}} \sum e^{-\pi n^2\left(\varepsilon \frac{q^2}{p^2} - i\frac{q}{p}\right)}$$

$$\sim e^{-i\pi/4}\frac{q^{1/2}}{p^{1/2}} \frac{1}{\sqrt{\varepsilon \frac{q^2}{p^2}}} \frac{1}{p}\sum_{r\in\mathbb{Z}/p\mathbb{Z}} e^{+\pi i r^2 \frac{q}{p}}$$

So we obtain the formula

$$\frac{1}{q} \sum_{r\in \mathbb{Z}/q\mathbb{Z}} e^{-\pi i r^2 \frac{p}{q}} = e^{-i\pi/4}\frac{1}{p^{1/2}q^{1/2}}\sum_{r\in\mathbb{Z}/p\mathbb{Z}} e^{\pi i r^2 \frac{q}{p}}$$

or

$$\boxed{\frac{1}{q^{1/2}} \sum_{r\in \mathbb{Z}/q\mathbb{Z}} e^{-\pi i r^2 \frac{p}{q}} = e^{-i\pi/4}\frac{1}{p^{1/2}}\sum_{r\in\mathbb{Z}/p\mathbb{Z}} e^{\pi i r^2 \frac{q}{p}}}$$

where p, q are integers >0 not both odd

Take $p=2$. $\quad e^0 + e^{\pi i \frac{q}{2}} = 1 + i^q$

$$e^{+i\pi/4} = (1+i)/\sqrt{2}$$

So one gets

$$\boxed{\sum_{r\in\mathbb{Z}/q\mathbb{Z}} e^{-2\pi i r^2/q} = \frac{1+i^q}{1+i}\sqrt{q}}$$

$$\sum_{r\in\mathbb{Z}/q\mathbb{Z}} e^{2\pi i r^2/q} = \frac{1+(-i)^q}{1-i}\sqrt{q} = \begin{cases} (1+i)\sqrt{q} & q \equiv 0 \ (4) \\ \sqrt{q} & q \equiv 1 \\ 0 & q \equiv 2 \\ i\sqrt{q} & q \equiv 3 \end{cases}$$

June 24, 1977

Recall that I am looking at Dirichlet series of the form $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$, where $\chi: \mathbb{Z}/p\mathbb{Z}\to\mathbb{C}$ is a map satisfying $\chi(-n)=\varepsilon\chi(n)$, where $\varepsilon=\pm 1$. One has the functional equations

$$L(1-s,\hat{\chi}) = p^s g^{\varepsilon}(s)L(s,\chi)$$

$$g^{\varepsilon}(s)=\frac{\Gamma(s)}{(2\pi)^s}\left(e^{\frac{i\pi s}{2}}+\varepsilon e^{-\frac{i\pi s}{2}}\right)$$

$$\hat{\chi}(n)=\sum_{m\in\mathbb{Z}/p\mathbb{Z}}\chi(m)\,e^{2\pi i\frac{mn}{p}}$$

Now the interesting cases for number theory occurs when χ is a character of $(\mathbb{Z}/p\mathbb{Z})^*$ extended by zero to the rest of $(\mathbb{Z}/p\mathbb{Z})$.

Note that if $u\in(\mathbb{Z}/p\mathbb{Z})^*$ then when χ is a character

$$\hat{\chi}(un) = \sum_{m\in(\mathbb{Z}/p\mathbb{Z})^*}\chi(m)\,\zeta^{mun} \qquad \zeta=e^{2\pi i\frac{1}{p}}$$

$$=\sum_{m\in(\mathbb{Z}/p\mathbb{Z})^*}\chi(mu^{-1})\,\zeta^{mn} = \bar{\chi}(u)\,\hat{\chi}(n)$$

Claim that each orbit of $(\mathbb{Z}/p\mathbb{Z})^*$ on $(\mathbb{Z}/p\mathbb{Z})$ contains a unique divisor d of p. In effect given n associate to it the ideal $n\mathbb{Z}+p\mathbb{Z}/p\mathbb{Z} = d\mathbb{Z}/p\mathbb{Z}$ where $d=(n,p)$. As $\mathbb{Z}/p\mathbb{Z}$ modules one has

$$d\mathbb{Z}/p\mathbb{Z} \xleftarrow{\sim} \mathbb{Z}/(\tfrac{d}{p})\mathbb{Z}.$$

Since $(\mathbb{Z}/p\mathbb{Z})^*\longrightarrow(\mathbb{Z}/(\frac{d}{p})\mathbb{Z})^*$ is surjective, it follows that

$(\mathbb{Z}/p\mathbb{Z})^*$ acts transitively on the set of generators for the ideal $d\mathbb{Z}/p\mathbb{Z}$, i.e. on the set of $n \in \mathbb{Z}/p\mathbb{Z}$ with $(n,p) = d$.

Let d be a fixed divisor of p, and let

$$K = \operatorname{Ker} (\mathbb{Z}/p\mathbb{Z})^* \to (\mathbb{Z}/\tfrac{p}{d}\mathbb{Z})^*.$$

If χ is non-trivial on K, then choose $u \in K$ with $\chi(u) \neq 1$ and you find $\hat{\chi}(d) = \overline{\chi(u)}\,\hat{\chi}(d)$ so $\hat{\chi}(d) = 0$. If χ is trivial on K, then

$$\hat{\chi}(d) = \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^*} \chi(m)\, e^{2\pi i m \left(\frac{d}{p}\right)}$$

$$= (\operatorname{card} K) \sum_{\bar{m} \in (\mathbb{Z}/\frac{p}{d}\mathbb{Z})^*} \chi(\bar{m})\, e^{2\pi i \bar{m} \left(\frac{d}{p}\right)}$$

which involves a term $\hat{\chi}(1)$ but for a character with modulus $\frac{p}{d}$. This will be non-trivial for χ primitive on $(\mathbb{Z}/\frac{p}{d}\mathbb{Z})^*$.

Thus for χ non-primitive, $\hat{\chi}$ will not be a multiple of $\bar{\chi}$.

What seems to be happening is that one is studying the Fourier transform on the ring $(\mathbb{Z}/p\mathbb{Z})$ in analogy with the usual transform on $\mathbb{R}$. We've seen two types of Gaussian sums which perhaps correspond to the F-transforms of $|x|^{-s}$ and $e^{-\pi x^2}$.

Let's try calculating the F-transform of $e^{2\pi i r^2 \frac{a}{p}}$

on the group $\mathbb{Z}/p\mathbb{Z}$. Thus we take

$$\chi(n) = e^{\pi i n^2 \frac{2a}{p}}$$

which is a well-defined function on $\mathbb{Z}/p\mathbb{Z}$ provided

$\alpha = \frac{2a}{p}$ $\qquad e^{\pi i (2pn + p^2)\alpha} = 1 \qquad$ for all n

i.e. $\qquad (pn + \frac{p^2}{2})\alpha \in \mathbb{Z} \qquad$ for all n

i.e. $p\alpha \in \mathbb{Z}$ and $\frac{p^2}{2}\alpha \in \mathbb{Z}$ for all n. So if $\alpha = \frac{q}{p}$ with $q \in \mathbb{Z}$, then $\frac{p^2\alpha}{2} = \frac{qp}{2} \in \mathbb{Z}$ so that either p or q has to be even. Thus α can't be a 2-adic unit.

So put $\chi(n) = e^{\pi i n^2 \frac{q}{p}}$ where either p, q are even. Then

$$\hat{\chi}(n) = \sum_{m=0}^{p-1} e^{\pi i m^2 \frac{q}{p} + 2\pi i \frac{mn}{p}}$$

$$= \sum_{0 \le m < p} e^{\pi i \frac{q}{p}\left(m^2 + 2m\frac{n}{q} + \frac{n^2}{q^2}\right) - \pi i \frac{n^2}{pq}}$$

$$= e^{-\pi i \frac{n^2}{pq}} \sum_{0 \le m < p} e^{\pi i \frac{q}{p}\left(m + \frac{n}{q}\right)^2}$$

~~$= e^{-\pi i \frac{n^2}{pq}} \sum_{0 \le m < p} e^{\pi i \frac{1}{pq}(qm+n)^2}$~~

~~Since either p, q even, pq is even so $\tilde{n} \mapsto e^{\pi i \frac{1}{pq} n^2}$ is a function on ? ? ?~~

$$\hat{\chi}(qn) = e^{-\pi i n^2 \frac{q}{p}} \sum_{0 \le m < p} e^{\pi i \frac{q}{p}(m+n)^2}$$

$$= e^{-\pi i n^2 \frac{q}{p}} \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} \text{[redacted]} m^2}$$

Now if $(q,p) = 1$, [redacted] so one is even and the other odd, this gives [redacted]

$$\hat{\chi}(n) = e^{-\pi i n^2 \frac{a^2 q}{p}} \cdot \text{constant} \qquad \text{where} \quad aq \equiv 1 \pmod{p}$$

so it should be possible to develop completely the Fourier transform of these quadratic functions. Nice project.

[redacted]

The following idea looks useful: Let x, y be rational numbers. Then consider the quadratic function

$$n \longmapsto xn^2 + yn$$

on $\mathbb{Z}$. We can form the sum

$$f(x,y) = \frac{1}{p} \sum_{n=0}^{p-1} e^{2i\pi(xn^2+yn)}$$

where p is large enough in the sense of divisibility so that xp [redacted] and yp are integers. The above function [redacted] does not depend upon [redacted] the choice of p. One can maybe generalize to non-rational x, y by taking the [redacted] average value

$$\lim_{p \to \infty} \frac{1}{2p+1} \sum_{n=-p}^{p} e^{2\pi i(xn^2+yn)},$$

but we will stick to x, y rational.

Consider $y=0$ first. Write $x = \frac{q}{p}$ in lowest terms and put

$$G(q,p) = \sum_{n \in \mathbb{Z}/p\mathbb{Z}}{}' e^{2\pi i n^2 \frac{q}{p}}$$

where p, q are [redacted] relatively prime.

Suppose $p = p_1 p_2$ with $(p_1, p_2) = 1$ whence we have an isomorphism

$$\mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/p_1p_2\mathbb{Z}$$
$$(n_1\ ,\ n_2) \longmapsto p_2 n_1 + p_1 n_2$$

$$e^{2\pi i (p_2 n_1 + p_1 n_2)^2 \frac{q}{p_1 p_2}} = e^{2\pi i \frac{q p_2}{p_1} n_1^2} \, e^{2\pi i \frac{p_1 q}{p_2} n_2^2}$$

so we see
$$G(q, p_1 p_2) = G(q p_2, p_1)\, G(q p_1, p_2)$$

If $\zeta = e^{2\pi i/p}$, then $G(q,p) = \sum_{n \in \mathbb{Z}/p\mathbb{Z}}{}' \zeta^{q n^2}$.

Suppose p is an odd prime. Then [redacted] as n goes from 1 to $p-1$, n^2 runs over the quadratic residues mod p hitting each one twice. Hence

$$G(q,p) = 1 + 2 \sum_{\left(\frac{n}{p}\right) = \left(\frac{q}{p}\right)}{}' \zeta^n = 2\Big(\tfrac{1}{2} + \sum_{\left(\frac{n}{p}\right) = \left(\frac{q}{p}\right)}{}' \zeta^n\Big)$$

But $\sum_{\left(\frac{n}{p}\right)=1} \zeta^{n} + \sum_{\left(\frac{n}{p}\right)=-1} \zeta^{n} + 1 = 0$, so

$$\frac{1}{2} + \sum_{\left(\frac{n}{p}\right)=1} \zeta^{n} = -\left(\frac{1}{2} + \sum_{\left(\frac{n}{p}\right)=-1} \zeta^{n}\right)$$

Hence we see that

$$G(q,p) = \left(\frac{q}{p}\right) G(1,p) = \begin{cases} \left(\frac{q}{p}\right)\sqrt{p} & p \equiv 1 \ (4) \\ \left(\frac{q}{p}\right) i\sqrt{p} & p \equiv 3 \ (4) \end{cases}$$

because on page 89 we found

$$G(1,p) = \frac{1+(-i)^{p}}{1+(-i)}\sqrt{p} = \begin{cases} (1+i)\sqrt{p} & p \equiv 0 \ (4) \\ \sqrt{p} & p \equiv 1 \ (4) \\ 0 & p \equiv 2 \ (4) \\ i\sqrt{p} & p \equiv 3 \ (4) \end{cases}$$

$$G(a,p) = \sum_{n \in \mathbb{Z}/p\mathbb{Z}}' e^{2\pi i \frac{a}{p} n^{2}} = \sum_{n \in \mathbb{Z}/p\mathbb{Z}}' \zeta^{a n^{2}}$$

Consider now $p = q^{r}$ ~~where q is prime~~ and write n running from 0 to $q^{r}-1$ in the form

$$n = x + q^{r-1} y \qquad \begin{matrix} 0 \le y < q-1 \\ 0 \le x < q^{r-1}-1 \end{matrix}$$

Then $\qquad n^{2} = x^{2} + 2q^{r-1}xy + \left(q^{r-1}\right)^{2} y^{2}$

$$\zeta^{an^{2}} = \zeta^{ax^{2} + 2aq^{r-1}xy} \qquad \begin{matrix} r \geq 2 \text{ so that} \\ 2(r-1) \geq r \end{matrix}$$

$$G(a,q^{r}) = \sum_{x=0}^{q^{r-1}-1}{}' \zeta^{ax^{2}} \sum_{y=0}^{q-1}{}' \left(\zeta^{q^{r-1}}\right)^{2axy}$$

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But $\zeta^{q^{r-1}} = e^{2\pi i/q}$ and $\sum_{y=0}^{q-1} \zeta^{cy} = \begin{cases} q & c \equiv 0 \ (q) \\ 0 & c \not\equiv 0 \ (q) \end{cases}$

$$\therefore \quad G(a, q^r) = \sum_{x=0}^{q^{r-1}-1} \zeta^{ax^2} \begin{cases} q & 2ax \equiv 0 \quad (q) \\ 0 & 2ax \not\equiv 0 \quad (q) \end{cases}$$

Suppose q an odd prime. By assumption $(a,q) = 1$. Put $x = q x'$ and we get

$$G(a, q^r) = q \sum_{x'=0}^{q^{r-2}-1} \left(\zeta^{q^2}\right)^{a(x')^2} = q\, G(a, q^{r-2})$$

~~If $q=2$, one has~~

$$\cancel{G(a, 2^r) = 2 \sum_{x=0}^{q^r-1} \zeta^{ax^2}}$$

~~which doesn't help any. But if $q=4$, then $2x \equiv 0 \ (4)$ means $x = 2x'$ where $0 \le x' \le 2^{2r-2}-1$, so that~~

$$\cancel{G(a, 2^{2r}) = 4 \sum_{x'=0}^{2^{2r-2}-1} e^{2\pi i \left(\frac{a}{2^{2r}}\right) 2^2 (x')^2} = 4\, G(a, 2^{2r-2})}$$

Repeat: suppose $p = bq$. Then by division by b:

$$n = x + by \qquad 0 \le x < b, \ 0 \le y < q$$

describes the integers $0 \le n < bq$, so

$$G(a, bq) = \sum_{0 \le x < b} e^{a \frac{2\pi i}{bq} x^2} \sum_{0 \le y < q} e^{\frac{2\pi i}{q} a (2xy + by^2)}$$

Suppose q divides b, i.e. $p = cq^2$. Then we get

$$G(a, bq) = \sum_{0 \le x < b} e^{2\pi i \frac{x^2}{bq} a} \begin{cases} 0 & 2xa \not\equiv 0 \pmod q \\ q & 2xa \equiv 0 \pmod q \end{cases}$$

As $(a,q)=1$, if q is odd, then the x's that count are of the form qx' with $0 \le x' < \frac{b}{q} = c$, so we get

$$G(a, cq^2) = q\, G(a,c) \qquad \text{if } q \text{ odd and } (a,q)=1$$

~~Suppose $q=4$ and $(a,q)=1$. Then $2xa \equiv 0 \mod 4$ $\Rightarrow$ $x = 2x'$, so~~

$$\cancel{G(a,4b) = 4 \sum_{0 \le x'}{}'}$$

~~Suppose $p = 2^r$. Taking $q=2$~~

$$\cancel{n = x + 2^{r-1}y \qquad n^2 \equiv x^2 \pmod{2^r}}$$

~~so~~

$$\cancel{G(a,2^r) = 2 \sum_{0 \le x < 2^{r-1}}{}' e^{2\pi i \frac{a}{2^r} x^2}}$$

But $1 + 2\mathbb{Z}/2^r\mathbb{Z} \simeq \{\pm 1\} \times$ cyclic group of order 2^{r-2}, so as n runs over $1 + 2\mathbb{Z}/2^r\mathbb{Z}$, n^2 runs over $1 + 8\mathbb{Z}/2^r\mathbb{Z}$ covering each number 4 times. So if we write

$$G(a,2^r) = \sum_{x \in 2\mathbb{Z}/2^r\mathbb{Z}}{}' + \sum_{x \in 1+2\mathbb{Z}/2^r\mathbb{Z}}{}'$$

The latter is:

$$\sum_{x \in 1+2\mathbb{Z}/2^r\mathbb{Z}} e^{2\pi i \frac{a}{2^r} x^2} = 4 e^{2\pi i \frac{a}{2^r}} \sum_{w \in 8\mathbb{Z}/2^r\mathbb{Z}}{}' e^{2\pi i \frac{a}{2^r} w}$$

$$= 4 e^{2\pi i \frac{a}{2^r}} \sum_{w \in \mathbb{Z}/2^{r-3}\mathbb{Z}}{}' e^{2\pi i \frac{aw}{2^{r-3}}} = \begin{cases} 4 e^{2\pi i \frac{a}{2^r}} & r = 3 \\ 0 & r > 3. \end{cases}$$

Now

$$\sum_{x \in 2\mathbb{Z}/2^r\mathbb{Z}} e^{2\pi i \frac{a}{2^r} x^2} = \sum_{x \in \mathbb{Z}/2^{r-1}\mathbb{Z}} e^{2\pi i \frac{a}{2^{r-2}} x^2} = 2 \sum_{x \in \mathbb{Z}/2^{r-2}\mathbb{Z}} e^{2\pi i \frac{a}{2^{r-2}} x^2} = 2G(a, 2^{r-2}).$$

So
$$G(a, 2^r) = 2\,G(a, 2^{r-2}) \qquad \text{if } r \geq 4$$
$$G(a, 8) = 2\,G(a, 2) + 4e^{\pi i \frac{a}{4}} \qquad \text{if } r = 3$$

$$G(a, 8) = 4e^{\pi i \frac{a}{4}}$$
$$G(a, 4) = 2\left(1 + e^{i\frac{\pi}{2}a}\right)$$
$$G(a, 2) = 0$$

Return to top of page 97 and take $q = 4$, $p = 16c$, $b = \frac{p}{q} = 4c$
Now $2xa \equiv 0 \ (4) \iff x \equiv 0 \ (2)$ so
$$G(a, 16c) = 4 \sum_{0 \leq x' < 2c} e^{2\pi i \frac{4x'^2}{16c} a} = 4 \sum_{0 \leq x' < 2c} e^{2\pi i \frac{ax'^2}{4c}}$$
$$= 2 \sum_{0 \leq x' < 4c} e^{2\pi i \frac{ax'^2}{4c}} = 2\,G(a, 4c)$$

Law of Quadratic Reciprocity: Suppose p, q are odd primes. We showed using the Θ function formula that
$$\frac{1}{\sqrt{q}} \sum_{r \in \mathbb{Z}/q} e^{-2\pi i r^2 \frac{p}{q}} = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{2p}} \sum_{r \in \mathbb{Z}/2p} e^{\pi i r^2 \frac{q}{2p}}$$
$$\frac{1}{\sqrt{q}}\, \overline{G(p, q)} = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{2p}} \frac{1}{2} \sum_{r \in \mathbb{Z}/4p} e^{2\pi i r^2 \frac{q}{4p}}$$
$$\shortparallel$$
$$\frac{1}{\sqrt{q}} \left(\frac{p}{q}\right) \overline{G(1, q)} = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{2p}} \frac{1}{2}\, G(q, 4p)$$

$$G(q,4p) = G(4q,p)\,G(pq,4)$$
$$= \left(\frac{q}{p}\right) G(1,p)\left(1+e^{i\frac{\pi}{2}pq}\right)2$$

$$\left(\frac{p}{q}\right)\frac{1-i^{q}}{1-i} = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}\;\frac{1+(-i)^{p}}{1+(-i)}\left(1+e^{i\frac{\pi}{2}pq}\right)\left(\frac{q}{p}\right)$$

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \begin{Bmatrix} 1 & p\equiv 1 \\ i & p\equiv 3\end{Bmatrix}\begin{Bmatrix} 1 & q\equiv 1 \\ +i & q\equiv 3\end{Bmatrix}\begin{Bmatrix} 1 & pq\equiv 1 \\ -i & pq\equiv 3\end{Bmatrix}$$
$$= \begin{Bmatrix} -1 & p\equiv 3,\ q\equiv 3 \\ 1 & \text{otherwise}\end{Bmatrix} = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

which is the law of quadratic reciprocity.

The next thing is to study the Fourier transform of these Gaussian functions. General remark is that if a function is periodic then its Fourier transform is supported on those characters which have the same periods. Thus if I take the transform

$$\hat{f}(n) = \sum_{m\in \mathbb{Z}/p} f(m)\,e^{2\pi i\frac{mn}{p}}$$

and if $f(m+d) = f(m)$ for some d dividing p, then $\hat{f}(n) = 0$ for $n \not\equiv 0 \pmod{p/d}$. In fact one has the following. If $d|p$ we can describe $0 \le m < p$ as $m = r + jd$, $0 \le r < d$, $0 \le j < p/d$

$$\cancel{\hat{f}(n) = \sum_{m\in\mathbb{Z}/d} f(m)\,e^{2\pi i\frac{m}{d}\frac{dn}{p}}\,\frac{p}{d}}$$

$$\hat{f}(n) = \sum_{0 \le r < d}{}' f(r) \sum_{0 \le j < \frac{p}{d}}{}' e^{2\pi i (r + jd)\frac{n}{p}}$$

[redacted]

$$\sum_{0 \le j < \frac{p}{d}}{}' e^{2\pi i j \frac{n}{p/d}} = \begin{cases} 0 & n \not\equiv 0 \quad (p/d) \\ p/d & n \equiv 0 \quad (p/d) \end{cases}$$

Hence

$$\hat{f}(n) = \begin{cases} 0 & n \not\equiv 0 \quad \bmod \left(\frac{p}{d}\right) \\ \frac{p}{d} \sum_{0 \le r < d} f(r)\, e^{2\pi i \frac{r}{d} \cdot \left(\frac{n}{p/d}\right)} & \text{if } n \equiv 0 \quad \bmod\left(\frac{p}{d}\right). \end{cases}$$

so we see that to compute the F-transform of f we can always restrict to the case where p = smallest period > 0 of f.

Next consider the function $f(n) = e^{\pi i \frac{q}{p} n^2}$ where $(q, p) = 1$. I will suppose this is periodic of period p, hence either q or p must be even.

$$\hat{f}(n) = \sum_{m \in \mathbb{Z}/p} e^{\pi i \frac{q}{p} m^2 + 2\pi i \frac{mn}{p}}$$

Choose a such that $aq \equiv 1 \mod p$.

$$\hat{f}(n) = \sum_{m \in \mathbb{Z}/p} e^{\pi i \frac{a}{p} q^2 m^2 + 2\pi i \frac{a}{p} q m n}$$

$$\hat{f}(n) = \sum_{m \in \mathbb{Z}/p} e^{\pi i \frac{a}{p}(qm + n)^2 - \pi i \frac{a}{p} n^2}$$

Note that if q is even and p is odd, I can suppose a even, [redacted] e.g. [redacted] replace a by $a + p$.

since $(q,p)=1$, as m ranges over $\mathbb{Z}/p$, so does $qm+n$, hence

$$\hat{f}(n) = e^{-\pi i \frac{a}{p} n^2} \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{a}{p} m^2}$$

so $$\hat{f}(n) = \text{const.}\ e^{-\pi i \frac{a}{p} n^2}$$ or

$$\hat{f}(n) = \hat{f}(0)\, e^{-\pi i \frac{a}{p} n^2}.$$

Note that this shows

$$\hat{f}(0) = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{+\pi i \frac{a}{p} m^2} = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} m^2}$$

which also follows by replacing m by qm in the former

Now one knows

$$\hat{\hat{f}}(n) = f(-n) \cdot c$$

where c can be worked out by taking $f = \delta$:

$$\hat{\delta}(n) = 1 \quad \text{all } n$$

$$\hat{\hat{\delta}}(n) = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{mn}{p}} = \begin{cases} 0 & n \not\equiv 0 \\ p & n \equiv 0 \end{cases}$$

So compute

$$p = \hat{\hat{f}}(0) = \hat{f}(0) \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{-\pi i \frac{a}{p} m^2} = \Big| \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} m^2} \Big|^2$$

Let us, instead of Lang, study the Gauss sums

$$g(q,p) = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} n^2}$$

under the assumptions $(p,q)=1$, not both p,q odd.

First suppose p odd. Then q is even, say $q=2a$, whence $g(q,p) = G(a,p) = \left(\frac{a}{p}\right) G(1,p) = \left(\frac{q}{p}\right)\left(\frac{2}{p}\right) G(1,p)$ $\left(\frac{q}{p}\right) g(4,p)$. Now

$$g(4,p) = \sum_{m\in \mathbb{Z}/p} e^{\pi i \frac{4}{p} m^2}$$

we ought to be able to evaluate using reciprocity:

$$\frac{1}{2}\sum_{m\in\mathbb{Z}/4\mathbb{Z}} e^{-\pi i \frac{p}{4} m^2} = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{p}} \sum_{m\in\mathbb{Z}/p} e^{\pi i \frac{4}{p} m^2}$$

$$\shortparallel$$

$$\frac{1}{2}\left[\cancel{1} + e^{-\pi i \frac{p}{4}} + \cancel{e^{-\pi i p}} + e^{-\pi i \frac{p}{4}}\right]$$

so

$$\frac{1}{\sqrt{p}} g(4,p) = e^{-\pi i/4 (p-1)} = i^{\frac{1-p}{2}} = \begin{cases} 1 & p\equiv 1 \ (8) \\ -i & p\equiv 3 \ (8) \\ -1 & p\equiv 5 \ (8) \\ i & p\equiv 7 \ (8) \end{cases}$$

Thus

$$\boxed{\sum_{m\in\mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} m^2} = \underset{\substack{\uparrow \\ \text{Jacobi symbol}}}{\left(\frac{q}{p}\right)} i^{\frac{1-p}{2}} \sqrt{p} \qquad \begin{matrix} p \text{ odd} \\ q \text{ even} \\ (p,q)=1 \end{matrix}}$$

Suppose now that q is odd and p is even. Then

$$g(q,p) = \sum_{m\in\mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} m^2} = e^{i\frac{\pi}{4}} \frac{\sqrt{p}}{\sqrt{q}} \sum_{m\in\mathbb{Z}/q\mathbb{Z}} e^{-\pi i \frac{p}{q} m^2}$$

$$= e^{i\frac{\pi}{4}} \sqrt{p} \left(\frac{p}{q}\right) i^{\frac{q-1}{2}} = \left(\frac{p}{q}\right) i^{q/2} \sqrt{p}$$

So
$$\boxed{\sum_{m\in \mathbb{Z}/p\mathbb{Z}} e^{\pi i \frac{q}{p} m^2} = \left(\frac{p}{q}\right) i^{\frac{q}{2}}\sqrt{p} \qquad \begin{matrix} p \text{ even} \\ q \text{ odd} \\ (p,q)=1 \end{matrix}}$$

One can calculate directly that $g(q,2) = 1+i^q$, $g(q,4) = 2e^{i\frac{\pi}{4}q}$ and $g(q,2^r) = 2g(q,2^{r-2})$ for $r>2$.

$$\left(\frac{2}{q}\right) i^{\frac{q}{2}}\sqrt{2} = \begin{Bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{Bmatrix} \begin{Bmatrix} e^{i\pi/4} \\ e^{3i\pi/4} \\ e^{5i\pi/4} \\ e^{7i\pi/4} \end{Bmatrix} \sqrt{2} = \begin{cases} 1+i & = 1+i^1 \\ -(-1+i) & = 1+i^3 \\ -(-1-i) & = 1+i^5 \\ +1-i & = 1+i^7 \end{cases}$$

and the rest is clear, so it checks.

~~Suppose p odd and put $q = 2a$~~ By Gauss, one knows the Galois group of $\mathbb{Z}[\zeta]/\mathbb{Z}$, $\zeta = e^{2\pi i/p}$ is $(\mathbb{Z}/p\mathbb{Z})^*$, i.e. for each a prime to p we have an autom. $\sigma_a : \zeta \mapsto \zeta^a$ of $\mathbb{Z}[\zeta]$. Clearly

$$\sigma_a\Big(\sum_{m\in\mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{1}{p} m^2}\Big) = \sum_{m\in\mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{a}{p} m^2}$$

i.e. $\sigma_a(G(1,p)) = G(a,p)$. But suppose p odd, whence

$$G(1,p) = \begin{matrix} \sqrt{p} & p\equiv 1 \quad (4) \\ i\sqrt{p} & p\equiv 3 \quad (4) \end{matrix}$$

These quantities are quadratic over $\mathbb{Z}$, so σ_a can only change its sign. Hence one has a formula of the form