January 5, 1977. Statistical Mechanics.

In classical mechanics the states of the system under consideration form a manifold M called phase space. The time evolution of the system is given by a vector field on M. Hamilton's method of describing the vector field is as follows. On M there is a closed non-degenerate 2-form Ω and a function H called the Hamiltonian. Then using ω, the form dH can be converted into a vector field X_H determined by

$$i(X_H)\Omega = dH.$$

In statistical mechanics, one considers a large number n of identical copies of the system. The new system has phase space M^n and Hamiltonian $H_n = \sum_{i=1}^{n} H \circ pr_i$.

Let f be a real function on M (so called "observable"). Then I want to compute the average value of f granted the n-fold system has total energy E. This means I want to integrate the function $x \mapsto \frac{1}{n}\sum_{i=1}^{n} f(x_i)$ over the hypersurface $H_n^{-1}(E)$ with respect to a suitable volume on this hypersurface.

To describe this volume, let ω denote the volume on M obtained from the symplectic form Ω.

Then assuming $dH \neq 0$ at the points we are working, we can divide ω by dH obtaining a $(r-1)$-form $\frac{\omega}{dH}$ such that $dH \frac{\omega}{dH} = \omega$; this form is unique up to adding an element $dH \cdot \eta$, hence its restriction to $H^{-1}(t)$ is well-defined; denote it

$$\frac{\omega}{dH}\Big|_{H^{-1}(t)}.$$

We have the formula:

1) $$\int_M f(x)\,\omega = \int_{t \in \mathbb{R}} dt \int_{H^{-1}(t)} f(x) \frac{\omega}{dH}\Big|_{H^{-1}(t)}.$$

General formula: If $\varphi : X \to Y$ is a submersion and ω is a volume on X, ν a volume on Y, then

2) $$\boxed{\int_X f\omega = \int_{y \in Y} \nu \int_{\varphi^{-1}(y)} f \frac{\omega}{\varphi^*(\nu)}\Big|_{\varphi^{-1}(y)}}.$$

Next let us apply this to the hypersurface $H_n^{-1}(t)$, and to the function $\bar{f}(x) = \frac{1}{n} \sum' f(x_i)$. What I want to compute is

3) $$\frac{\displaystyle\int_{H_n^{-1}(t)} \bar{f}(x) \frac{\omega_n}{dH_n}\Big|_{H_n^{-1}(t)}}{\displaystyle\int_{H_n^{-1}(t)} \frac{\omega_n}{dH_n}\Big|_{H_n^{-1}(t)}}$$

Look at the denominator first:

$$v_n(t) = \int_{H_n^{-1}(t)} \frac{\omega_n}{dH_n}\Big|_{H_n^{-1}(t)}$$

Compute the Laplace transform of $v_n(t)$

$$\hat{v}_n(s) = \int_0^\infty e^{-st} v_n(t)\,dt = \int_0^\infty dt \int_{x \in H_n^{-1}(t)} e^{-sH_n(x)} \frac{\omega_n}{dH_n}\Big|_{H_n^{-1}(t)}$$

$$= \int_{M^n} e^{-sH_n(x)} \omega_n \qquad\qquad e^{-sH_n(x)} = \prod_{i=1}^n e^{-sH(x_i)}$$

4)
$$= \left(\int_M e^{-sH(x)} \omega \right)^n$$

$$= \left(\hat{v}(s) \right)^n$$

where $v(t) = v_1(t)$. Here I am assuming $H(x) \geq 0$ on M. From 4) we see that

$$v_n(t) = (v * \cdots * v)(t) \qquad n\text{-times.}$$

Next we look at the numerator of 3). One has by symmetry in the x_i that

$$\int_{H_n^{-1}(t)} \bar{f}(x) \frac{\omega_n}{dH_n}\Big|_{H_n^{-1}(t)} = \int_{H_n^{-1}(t)} f(x_1) \frac{\omega_n}{dH_n}\Big|_{H_n^{-1}(t)}.$$

Use the map $H_n^{-1}(t) \to M$, $x \mapsto x_1$, and the formula 2) with measure ω on M, to see this is

$$\int_M f(x_1)\,\omega \int_{H_{n-1}^{-1}(t - H(x_1))} \frac{\omega_{n-1}}{dH_{n-1}} = \int_M f(x_1)\, v_{n-1}(t - H(x_1))\,\omega$$

Thus the quantity[3] I am interested in is

$$\int_{x\in M} f(x)\,\frac{v_{n-1}(t-H(x))}{v_n(t)}\,\omega = \int \frac{v_{n-1}(t-h)}{v_n(t)}\,dh \int_{H^{-1}(h)} f(x)\,\frac{\omega}{dH}\Big|$$

I want to take a suitable limit as $n \to \infty$; this means I want to vary t as the limit is taken.

For example let us consider the harmonic oscillator. Here $M = \mathbb{R}^2$ with coords p, q, $\omega = dp\,dq$ and $H = \frac{1}{2}(p^2+q^2)$. Then

$$\hat{v}(s) = \int_{\mathbb{R}^2} e^{-\frac{s}{2}r^2}\, r\,dr\,d\theta = 2\pi\left[-\frac{1}{s}e^{-\frac{s}{2}r^2}\right]_0^\infty$$

$$= \frac{2\pi}{s}$$

So $\quad \hat{v}_n(s) = \left(\hat{v}(s)\right)^n = \frac{(2\pi)^n}{s^n}$

Recall $\quad \int_0^\infty e^{-st}\,t^{n-1}\,dt = \frac{\Gamma(n)}{s^n}$. Then

$$\int_0^\infty e^{-st}\,v_n(t)\,dt = \frac{(2\pi)^n}{s^n} = \frac{(2\pi)^n}{\Gamma(n)}\,\frac{\Gamma(n)}{s^n}$$

so $\quad v_n(t) = \frac{(2\pi)^n}{\Gamma(n)}\,t^{n-1} = (2\pi)^n\,\frac{t^{n-1}}{(n-1)!}$

So

$$\frac{v_{n-1}(t-h)}{v_n(t)} = \frac{(2\pi)^{n-1}}{(2\pi)^{n}}\,\frac{(n-1)!}{(n-2)!}\,\frac{(t-h)^{n-2}}{t^{n-1}}$$

$$= (2\pi)^{-1}(n-1)\,\frac{1}{t}\left(1-\frac{h}{t}\right)^{n-2}$$

Now it is clear that if I want this to converge as $n \to \infty$, I want to put

$$t = n\tau$$

whence I get

$$4) \quad \lim_{n\to\infty} \frac{v_{n-1}(n\tau - h)}{v_n(n\tau)} = \lim \left((2\pi)^{-1} \frac{n-1}{n\tau} \left(1 - \frac{h}{\tau} \cdot \frac{1}{n}\right)^{n-2} \right)$$

$$= \frac{1}{2\pi\tau} e^{-\frac{h}{\tau}}$$

Question: In general does it follow that

$$\lim_{n\to\infty} \frac{v_{n-1}(n\tau - h)}{v_n(n\tau)}$$

exists? Maybe it should be $\dfrac{e^{-h/\tau}}{\tau\, v(h)}$.

We are going to do some heuristic calculations using the method of steepest descent. I'll begin with deriving Stirling's formula.

We start with

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n \, dt = \int_0^\infty e^{-t + n \log t} \, dt$$

Put

$$y = -t + n \log t$$

To locate where the integrand is maximum:

$$y' = -1 + \frac{n}{t} = 0 \implies t = n$$

$$y'' = -\frac{n}{t^2}$$

Find Taylor series of y around $t=n$.

$$y = (-n + n\log n) + 0\cdot(t-n) + \frac{1}{2}\left(-\frac{n}{n^2}\right)(t-n)^2 + \cdots\cdots$$

Replacing y by its 2nd degree Taylor poly. we get

$$n! \doteq e^{-n+n\log n}\int_0^\infty e^{-\frac{1}{2n}(t-n)^2}\,dt$$

$$= n^n e^{-n}\int_{-n}^\infty e^{-\frac{t^2}{2n}}\,dt$$

$$\doteq n^n e^{-n}\int_{-\infty}^\infty e^{-\frac{t^2}{2n}}\frac{dt}{\sqrt{2n}}\sqrt{2n} = n^n e^{-n}\sqrt{2n}\int_{-\infty}^\infty e^{-x^2}\,dx.$$

$$= n^n e^n\sqrt{2\pi n}$$

which is the Stirling approximation.

I want to try the same thing for $v_n(n\tau)$; we use the inversion formula for the Laplace transform:

$$v_n(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\,\hat{v}(s)^n\,ds$$

(where a is a sufficiently large real number to be in the analyticity region of $\hat{v}$). We want to use steepest descent, so rewrite this

$$v_n(n\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{n(s\tau + \log\hat{v}(s))}\,ds$$

$$y(s) = s\tau + \log\hat{v}(s)$$

$$y'(s) = \tau + \frac{\hat{v}'}{\hat{v}}(s) \qquad y'' = \frac{\hat{v}\hat{v}'' - \hat{v}'^2}{\hat{v}^2}(s).$$

We want a unique root $s = \alpha(\tau)$ of the equation:

$$y'(s) = \tau + \frac{\hat{v}'}{\hat{v}}(s) = 0.$$

Recall

$$\hat{v}(s) = \int_0^\infty e^{-st} v(t)\,dt > 0 \qquad v(t) = \int_{H^{-1}(t)} \frac{d\omega}{dH}\Big|_{H^{-1}(t)}$$

and

$$\hat{v}'(s) = -\int_0^\infty e^{-st}\, t\, v(t)\,dt < 0$$

So

$$(\tau\hat{v} - \hat{v}')(s) = \int_0^\infty e^{-st}(\tau - t)v(t)\,dt.$$

Now it is clear that in reasonable examples from 0 to ∞ $\int_0^\infty t\,v(t)\,dt = \infty$, hence as s increases, the preceding integral will go from $-\infty$ to 0, but ~~[illegible]~~ it should be positive provided τ is. The reason is that $e^{-st}dt$ will emphasize the low end of $v(t)$. Another reason is that the function $\frac{\hat{v}'}{\hat{v}}$ is increasing: By Cauchy-Schwarz

$$\left(\int 1\cdot t\, e^{-st} v(t)\,dt\right)^2 \leq \int 1\cdot e^{-st} v(t)\,dt \cdot \int t^2 e^{-st} v(t)\,dt$$

$$\left(-\hat{v}'(s)\right)^2 \leq \hat{v}(s)\cdot\hat{v}''(s)$$

so

$$\frac{d}{ds}\left(\frac{\hat{v}'}{\hat{v}}\right) = \frac{\hat{v}\hat{v}'' - \hat{v}'^2}{\hat{v}^2} \geq 0.$$

Also we expect it to have ~~[illegible]~~ value $-\infty$ at $s=0$ and value 0 at $s = \infty$.

Continuing now with steepest descent:

$$y(s) = \underbrace{\alpha(\tau)\tau + \log \hat{v}(\alpha(\tau))}_{\mu} + \frac{1}{2}\underbrace{\frac{\hat{v}\hat{v}'' - \hat{v}'^2}{\hat{v}^2}(\alpha(\tau))}_{\lambda}(s - \alpha(\tau))^2$$

so

$$v_n(n\tau) \doteq \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{n\left[\mu + \frac{1}{2}\lambda (s-\alpha(\tau))^2\right]}\, ds$$

$$= e^{n\mu}\, \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\frac{n\lambda}{2}s^2}\, ds \qquad s = ix$$

$$= e^{n\mu}\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{n\lambda}{2}x^2}\, dx \sqrt{\frac{n\lambda}{2}} \cdot \sqrt{\frac{2}{n\lambda}}$$

$$= e^{n\mu}\, \frac{1}{\sqrt{2\pi n\lambda}}$$

So we "get" the asymptotic formula

$$v_n(n\tau) \doteq e^{n\left[\alpha(\tau)\tau + \log \hat{v}(\alpha(\tau))\right]} \left(2\pi n\, \frac{\hat{v}\hat{v}'' - \hat{v}'^2}{\hat{v}^2}(\alpha(\tau))\right)^{-1/2}$$

Compare with $v_{n-1}(n\tau - h) = v_{n-1}((n-1)\tau + (\tau - h))$

$$v_{n-1}\left((n-1)\left(\tau + \underbrace{\frac{\tau-h}{n-1}}_{\varepsilon}\right)\right) =$$

$$= e^{(n-1)\left[\alpha(\tau+\varepsilon)(\tau+\varepsilon) + \log \hat{v}(\alpha(\tau+\varepsilon))\right]} \left(2\pi(n-1)\, y''(\alpha(\tau+\varepsilon))\right)^{-1/2}$$

so

$$\frac{v_{n-1}(n\tau-h)}{v_n(n\tau)} = e^{\frac{\tau-h}{\tau-h}\overset{1/\varepsilon}{(n-1)}\left[\alpha(\tau+\varepsilon)(\tau+\varepsilon) + \log \hat{v}(\alpha(\tau+\varepsilon)) - \alpha(\tau)\tau - \log \hat{v}(\alpha(\tau))\right]} \cdot e^{-\alpha(\tau)\tau - \log \hat{v}(\alpha(\tau))}$$

$$\cdot \left(2\pi(n-1)\, y''(\alpha(\tau+\varepsilon))\right)^{-1/2} \left(2\pi n\, y''(\alpha(\tau))\right)^{1/2}$$

$$\to e^{(\tau-h)\frac{d}{d\tau}\left[\alpha(\tau)\tau + \log \hat{v}(\alpha(\tau))\right]} \cdot e^{-\alpha(\tau)\tau - \log \hat{v}(\alpha(\tau))}$$

$$= e^{\tau-h\left[\cancel{\alpha'(\tau)\tau} + \alpha(\tau) + \cancel{\frac{\hat{v}'(\alpha(\tau))}{\hat{v}(\alpha(\tau))}\cdot \alpha'(\tau)}\right]} \cdot e^{-\alpha(\tau)\tau - \log \hat{v}(\alpha(\tau))}$$

$\therefore$

$$\lim_{n\to\infty} \frac{v_{n-1}(n\tau-h)}{v_n(n\tau)} = e^{[\tau-h-\tau]\alpha(\tau) - \log \hat{v}(\alpha(\tau))}$$

5)

$$= \hat{v}(\alpha(\tau))^{-1} e^{-h\alpha(\tau)}$$

Check: $v(\tau) = 2\pi$, $\hat{v}(s) = \frac{2\pi}{s}$ so

$$\tau + \frac{d}{ds}\log\left(\frac{2\pi}{s}\right) = \tau - \frac{1}{s} = 0 \implies \alpha(\tau) = \frac{1}{\tau}$$

Thus we get $\frac{1}{2\pi\tau} e^{-\frac{h}{\tau}}$ as on page 5, 4).

Another check: We should have

$$\int_0^\infty \lim_{n\to\infty} \frac{v_{n-1}(n\tau-h)}{v_n(n\tau)} \cdot v(h)\,dh = \lim_{n\to\infty} \frac{\int_0^\infty v_{n-1}(n\tau-h)v(h)\,dh}{v_n(n\tau)} = 1$$

which is indeed the case as

$$\int_0^\infty e^{-h\alpha(\tau)} v(h)\,dh = \hat{v}(\alpha(\tau)).$$

Let us summarize the above calculation. We suppose given a classical mechanical system with phase space M and Hamiltonian H. Then we consider n copies of this system running independently, which I can think of as n-independent particles in the given system. Supposing this system of n particles has total energy $n\tau$, i.e. the average energy per particle is τ, then I can calculate the average value of a function f on M, which is the integral

$$\frac{\displaystyle\int_{H_n^{-1}(n\tau)} \frac{1}{n}\sum f(x_i)\ \frac{\omega_n^{\blacksquare}}{dH_n}\Big|_{H_n^{-1}(n\tau)}}{\displaystyle\int_{H_n^{-1}(n\tau)} \frac{\omega_n^{\blacksquare}}{dH_n}\Big|_{H_n^{-1}(n\tau)}}$$

The result is that as $n \to \infty$ this approaches

$$\int_{x\in M} f(x)\,\frac{e^{-\alpha(\tau)H(x)}}{\hat{v}(\alpha(\tau))}\,\omega = \int_0^\infty \frac{e^{-\alpha(\tau)E}}{\hat{v}(\alpha(\tau))}\,dE \int_{x\in H^{-1}(E)} f(x)\,\frac{\omega}{dH}\Big|_{H^{-1}(E)}$$

where the following notation is used.

$$v(t) = \int_{H^{-1}(t)} \frac{d\omega}{dH}\Big|_{H^{-1}(t)}$$

$$\hat{v}(s) = \int_0^\infty e^{-st}\,v(t) = \int_M e^{-sH}\,\omega$$

and $\alpha(\tau)$ is the unique (?) value of s such that

$$\tau + \frac{\hat{v}'}{\hat{v}}(s) = 0$$

i.e. such that $$\tau\int_M e^{-sH}\,\omega = \int_M e^{-sH} H\,\omega$$

It is more convenient to invert the function $\alpha(\tau)$, taking s to be the independent variable and defining τ to be

$$\tau = -\frac{\hat{v}'}{\hat{v}}(s) = \frac{\int_M e^{-sH} H\omega}{\int_M e^{-sH}\omega}$$

Then the basic formula derived above says the average value of f is

$$\bar{f} = \frac{\int_M f e^{-sH}\omega}{\int_M e^{-sH}\omega}$$

where s is such that the average energy per particle is τ. $\therefore \bar{H} = \tau$.

Now what remains is to understand why s is essentially the inverse of temperature. Note that as s increases τ drops to 0.

Entropy-extremal derivation of the Maxwell-Bolyman distribution. The <u>entropy</u> of a measure $\rho\omega$ on M is defined to be

$$S = -\int_M \rho \log \rho\, \omega$$

We consider the extremal problem of maximizing entropy with ρ subject to the conditions that it be a probability measure

$$\int \rho\,\omega = 1$$

and that the average energy be given

$$\int H\rho\,\omega = \mathcal{E}.$$

Use Lagrange multipliers

$$F(\rho) = +\int \rho \log \rho\,\omega + \lambda\left(\int H\rho\,\omega - \mathcal{E}\right) + \mu\left(\int \rho\,\omega - 1\right)$$

Replace ρ by $\rho + t\varepsilon$ and differentiate with resp. to t + then set $t=0$.

$$\frac{d}{dt}\,(\rho + t\varepsilon)\log(\rho + t\varepsilon)\Big|_{t=0} = \varepsilon \log(\rho) + \rho\,\frac{1}{\rho}\,\varepsilon$$

$$= \varepsilon(1 + \log \rho).$$

Thus the variation = zero conditions are

$$\int \varepsilon\,[1 + \log \rho + \lambda H + \mu]\,\omega = 0$$

$$\int H\rho\,\omega = \mathcal{E}$$

$$\int \rho\,\omega = 1$$

Since the first is to hold for any function ε, one has

$$1 + \log \rho + \lambda H + \mu = 0$$

or $\rho = e^{-1-\mu-\lambda H} = C e^{-\lambda H}$

where C, λ are constants to be determined by the second two conditions.

$$\int C e^{-\lambda H}\omega = 1 \Rightarrow C = \frac{1}{\int e^{-\lambda H}\omega}$$

and so

$$\rho = \frac{e^{-\lambda H}}{\int e^{-\lambda H}\omega}$$

where λ is chosen so that the average energy is τ:

$$\frac{\int H e^{-\lambda H}\omega}{\int e^{-\lambda H}\omega} = \tau .$$

January 8, 1977. Statistical mechanics

Let M be the set of states of a classical system, e.g. a particle in a force field. Take a gas made up of a large number n of such particles, whence a state of the gas is a point in M^n. Let H be the energy function on M, and let H_n be the energy on M^n. If the particles do not interact $H_n(x) = \sum_{i=1}^{n} H(x_i)$. Now we can fix the average energy τ of a particle of the gas, that is, look at $H_n^{-1}(n\tau)$ and compute the probability that the i-th particle lies in a certain region of M. This probability is evidently the ratio

$$f_n(x)\,\omega = \frac{\text{volume}\,\{x' \in M^{n-1} \mid H_{n-1}(x') = n\tau - H(x)\}}{\text{volume}\,\{x \in M^n \mid H_n(x) = n\tau\}} \cdot \omega$$

and I know that this approaches the Maxwell-Bolzman distribution as $n \to \infty$.

Fix r particles say the first thru r-th at positions $x_1, \dots, x_r \in M$ and ask the same question about the probability. It is

$$\frac{\text{volume}\,\{x' \in M^{n-r} \mid H_{n-r}(x') = n\tau - H(x_1) - \dots - H(x_r)\}}{\text{volume}\,\{x \in M^n \mid H_n(x) = n\tau\}}$$

Thus we want $\lim\limits_{n\to\infty} \dfrac{v_{n-r}(n\tau - h)}{v_n(n\tau)}$

But if we replace n by mr and M by M^r

$$M^{mr} = (M^r)^m$$

$$H_{mr}(x) = H_r(x_1) + \dots + H_r(x_m).$$

This becomes $\lim\limits_{m\to\infty} \dfrac{w_{m-1}(m(r\tau)-h)}{w_m(m(r\tau))}$

where $w(t) = v_r(t)$. So we know this limit is

$$\frac{e^{-sh}}{\int_{M^r} e^{-sH_r}\,\omega_r}$$

s chosen so that

$$\frac{\int_{M^r} H_r e^{-sH_r}\,\omega_r}{\int e^{-sH_r}\,\omega_r} = r\tau$$

~~But in any case we arrive at the MB distribution~~

But this, by $H_r = H + \dots + H$ and symmetry, means

$$\frac{\int H e^{-sH}\,\omega}{\int e^{-sH}\,\omega} = \tau$$

Thus the distribution on M^r we get is the product of the MB distributions on each M. Another proof which we do for $r=2$.

$$\frac{v_{n-2}(n\tau-h)}{v_n(n\tau)} = \frac{v_{n-2}((n-1)\tau+\tau-h)}{v_{n-1}((n-1)\tau)} \cdot \frac{v_{n-1}(n\tau-\tau)}{v_n(n\tau)}$$

$$\rightarrow \frac{e^{+s(\tau-h)}}{\int e^{-sH}\omega} \cdot \frac{e^{-s\tau}}{\int e^{-sH}\omega} = \frac{e^{-sh}}{\left(\int e^{-sH}\omega\right)^2}.$$

Now the real problem is to introduce interactions between the gas molecules.

So this means that $H_n(x)$ will be more than just $H(x_1)+\cdots+H(x_n)$, however, it still should be symmetric in the x_i. We can ask the same questions about whether

$$\lim_{n\to\infty} \frac{\text{volume}\,\{x' \in M^{n-1} \mid H_n(x,x') = n\tau\}}{\text{volume}\,\{H_n^{-1}(n\tau)\}}$$

exists. Assuming it does and also for each r-tuple we therefore get a sequence of distributions on M^n $n \geq 0$ which are symmetric. Then we can try to relate this to Dobrushin's definition of equilibrium state which will be some sort of measure on M^∞ maybe.

Gibbs' procedure: Instead of forming the probability distribution of M^k

$$\frac{\mathrm{vol}\,\{x' \in M^{n-k} \,/\, H_n(x,x') = t\}}{\mathrm{vol}\,\{x \in M^n \,/\, H_n(x) = t\}}$$

and letting $n, t \to \infty$ suitably so the limit exists, Gibbs forms

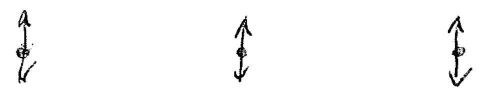
$$\frac{\displaystyle\int_{x' \in M^{n-k}} e^{-s H_n(x,x')}}{\displaystyle\int_{x \in M^n} e^{-s H_n(x)}}$$

and lets n go to ∞. What is the justification, physical or mathematical, for this?

Possibility: The second limit might exist with s fixed and so avoid the problem of how to make t go to infinity.

January 10, 1977.

Goal: To find a nice example for thermal equilibrium. Idea. Suppose we consider a [redacted] bunch of particles on the x-axis which are constrained to move vertically



Let y_i be the vertical displacement of the i-th particle. Suppose these particles attract each other by a force depending on the distance. [redacted]

~~So the vertical force on the i-th particle is~~

$$\cancel{\sum_{j\neq i} \text{[redacted]} \; F_{ji}\left(\left((x_j-x_i)^2+(y_j-y_i)^2\right)^{1/2}\right)}$$

~~Assuming the displacements [redacted] y_i are small one can approximate.~~ ~~and suppose that the forces are quadratic.~~

~~Thus the [redacted] force on the i-th particle is~~

$$\cancel{\sum_{j\neq i} a_{ji}(y_j-y_i)^2 \qquad a_{ji}>0}$$

~~Let $\vec{F}(y_i, y_j)$ be the force exerted by y_j on y_i.~~

$$\cancel{\vec{F}(y_i,y_j) = F(|y_i-y_j|).}$$

If the displacements y_i are small, then the force exerted by the j-th particle on the i-th particle will be proportional to $y_j - y_i$, hence the total force on the i-th particle is

$$\sum_{j \neq i}' a_{ij} (y_j - y_i)$$

with $a_{ij} \geq 0$. Hence the differential equations system of motion is

$$\frac{d^2 y_i}{dt^2} = \sum_{j \neq i}' a_{ij} (y_j - y_i)$$

and this is the equations of motion for the Hamiltonian

$$H = \sum \tfrac{1}{2} (\dot{y}_i)^2 + \sum_{i \neq j} \tfrac{1}{2} a_{ij} (y_i - y_j)^2$$

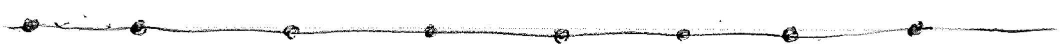
where a_{ij} is a symmetric matrix of ≥ 0 numbers.

Observe that the form $\sum_{i \neq j} a_{ij} (y_i - y_j)^2$ is ≥ 0, and if it is zero, then $y_i = y_j$ when $a_{ij} > 0$. Hence if we assume the ~~relation~~ relation $a_{ij} > 0$ for pairs i, j connects the set of indices, then $y_i = y_j$ for all i, j.

January 11, 1977

I want to see if I can understand temperature flow in a bar.

The model will the discrete analogue of a vibrating string. So we have particles at the integer points in $\mathbb{R}$



which can move vertically, say. Let y_ν be the displacement of the ν-th particle. The equations of motion are

$$\frac{d^2 y_\nu}{dt^2} = +\gamma\left(y_{\nu+1} - 2y_\nu + y_{\nu-1}\right)$$

Solve this by using eigenfunctions

$$y_\nu(t) = e^{i\lambda t} x^\nu$$

$$-\lambda^2 e^{i\lambda t} x^\nu = \gamma\left(x - 2 + x^{-1}\right) e^{i\lambda t} x^\nu$$

or

$$-\lambda^2 = \gamma\left(x - 2 + x^{-1}\right)$$

where γ is the coupling constant giving the interaction between the particles. Replace x by $e^{i\theta}$, then

$$e^{i\theta} - 2 + e^{-i\theta} = 2\cos\theta - 2$$

so

$$\lambda^2 = 2\gamma(1 - \cos\theta) = 4\gamma \sin^2\frac{\theta}{2}$$

or $$\boxed{\lambda = \pm 2\sqrt{\gamma}\,\sin\frac{\theta}{2}}$$

Here's the problem. All these solutions

$$e^{i\lambda t}\, e^{i\nu\theta} = e^{i(\lambda t + \nu\theta)}$$

represent waves travelling maybe at the same speed. NO, the above wave has velocity

$$\frac{\Delta\nu}{\Delta t} = -\frac{\lambda}{\theta} = \pm\sqrt{\gamma}\,\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} .$$

Solve the special case where the ends $\nu = \pm n$ are fixed. Thus we have the boundary conditions $y_{\pm n}(t) = 0$ for all t, which means something for possible θ.

$$\sin \nu\theta = 0 \quad \text{when} \quad \nu = \pm n$$

means $n\theta = j\pi$ so we get the functions

$$\sin\left(\nu \frac{j\pi}{n}\right) \qquad j = 1, \dots, n-1$$

not an eigenfunction

$$\cos\left(\nu \frac{j\pi}{n}\right) - (-1)^j \qquad j = 1, \dots, n-1, n$$

It will be simpler maybe to take the endpoints to be fixed to be $\nu=0$ and $\nu=n$. Then the eigenfunctions are

$$\sin \nu\left(\frac{j\pi}{n}\right) \qquad j=1,\dots,n-1$$

and with time variation they become

$$e^{\pm i\left(2\sqrt{\gamma}\,\sin\left(\frac{j\pi}{2n}\right)\right)t}\,\sin\left(\nu\left(\frac{j\pi}{n}\right)\right).$$

NO

These are the eigenfunctions, note there are $2(n-1)$ of them.

Suppose we work on the interval $-n\leq \nu \leq n$ and require periodic behavior at the ends. Then the functions

$$e^{i(j\nu)\frac{2\pi}{2n}} = e^{i(j\nu)\frac{\pi}{n}} \qquad j=0,\dots,2n-1$$

form a basis. We can expand any function $f(\nu)$

$$f(\nu) = \sum_{j=0}^{2n-1} a_j\, e^{i(j\nu)\frac{\pi}{n}}$$

where $$a_j = \frac{1}{2n}\sum_{\nu=-n}^{n-1} f(\nu)\, e^{-i(j\nu)\frac{\pi}{n}}$$

So the δ_0 function: $\delta_0(\nu) = \begin{cases} 1 & \nu=0 \\ 0 & \nu\neq 0\end{cases}$ has the expansion

$$\boxed{\delta_0(\nu) = \frac{1}{2n}\sum_{j=0}^{2n-1} e^{i(j\nu)\frac{\pi}{n}}}$$

I want the solution with initial value $\delta_0(\nu)$ and 0 initial velocity. To the ν-eigenfunction $e^{i\nu\theta}$ we have two values of λ as on page 8. ?

We want to solve

$$\frac{d^2 y_\nu}{dt^2} = \gamma\left(y_{\nu+1} - 2y_\nu + y_{\nu-1}\right)$$

subject to the boundary conditions $y_n(t) = y_{-n}(t) = 0$.
Expand y_ν in a Fourier series in t with typical term

$$e^{i\lambda t} a(\nu).$$

This solves the DE iff

$$-\lambda^2 = \gamma\left(a(\nu+1) - 2a(\nu) + a(\nu-1)\right).$$

But this has solution

$$a(\nu) = c_1 e^{i\nu\theta_1} + c_2 e^{i\nu\theta_2}$$

where θ_i are roots of

$$-\lambda^2 = \gamma\left(e^{i\nu\theta} - 2 + e^{-i\nu\theta}\right) = -\gamma(2 - 2\cos\theta)$$

or $\lambda = \pm\sqrt{\gamma}\cdot 2\sin\frac{\theta}{2}$ $\qquad \therefore\ \theta_2 = -\theta_1 = -\theta.$

Now the boundary conditions imply

$$a(n) = c_1 e^{in\theta} + c_2 e^{-in\theta} = 0$$

$$a(-n) = c_1 e^{-in\theta} + c_2 e^{in\theta} = 0$$

$$e^{2in\theta} - e^{-2in\theta} = 0$$

$$e^{4in\theta} = 1 \qquad 4in\theta = 2\pi i j$$

$$\theta = \frac{\pi j}{2n}$$

Conversely if θ has this form then

$$c_2 = -c_1 e^{2in\theta} = -c_1 e^{i\pi j} = \begin{cases} -c_1 & j \text{ even} \\ +c_1 & j \text{ odd} \end{cases}$$

So our eigenfunctions are

j even: $\quad e^{+i\nu\theta} - e^{i\nu\theta} \sim \sin\left(\nu \frac{\pi j}{2n}\right)$

j odd: $\quad e^{i\nu\theta} + e^{i\nu\theta} \sim \cos\left(\nu \frac{\pi j}{2n}\right)$

and j should run between 1 and $2n-1$, giving us the $2n-1$ eigenfunctions required.

Another way of interpreting the above is that we take the functions $e^{i\nu\theta}$ $\quad \theta = \frac{j\pi}{2n} \quad 0<j<4n$ and replace them by

$$e^{i\nu\theta} - e^{i(2n-\nu)\theta}$$

Hence starting with the δ function

$$\delta(\nu) = \frac{1}{4n} \sum_{j=0}^{4n-1} e^{i \nu j \frac{\pi}{2n}}$$

we get the expansion

$$\delta(\nu) - \delta(2n-\nu) = \frac{1}{4n} \sum_{j=0}^{4n-1} e^{i\nu \frac{j\pi}{2n}} - e^{i(2n-\nu)\frac{j\pi}{2n}}$$

$$= \frac{1}{4n} \sum_{j=0}^{4n-1} e^{i\nu \frac{j\pi}{2n}} - e^{ij\pi} e^{-i\nu \frac{j\pi}{2n}}$$

$$= \frac{1}{2n} \sum_{\substack{j=0 \\ j \text{ odd}}}^{4n-1} \cos\left(\nu \frac{j\pi}{2n}\right)$$

$$= \frac{1}{n} \sum_{\substack{j=0 \\ j \text{ odd}}}^{2n-1} \cos\left(\nu \frac{j\pi}{2n}\right)$$

January 12, 1977

To find solution of

$$\frac{d^2 y_\nu}{dt^2} = \gamma\left(y_{\nu+1} - 2y_\nu + y_{\nu-1}\right)$$

$$y_n(t) = y_{-n}(t) = 0$$

$$y_\nu(0) = 0 \qquad \text{all } \nu$$

$$\frac{dy_\nu}{dt}(0) = \begin{cases} 0 & \nu \neq 0 \qquad -n \leq \nu \leq +n \\ 1 & \nu = 0 \end{cases}$$

I expanded $y_\nu(t)$ as a Fourier series or integral in terms of $e^{i\lambda t} a_\lambda(\nu)$. For $e^{i\lambda t} a(\nu)$ to be a solution of the DE with the first boundary conditions means

$$a(\nu) = \text{const.}\left(e^{i\nu\theta_j} - e^{i(2n-\nu)\theta_j}\right)$$

where $\theta_j = j\frac{\pi}{2n}$

and $\lambda = \pm\lambda_j, \quad \lambda_j = \sqrt{\gamma}\; 2\sin\frac{\theta_j}{2}$.

Since

$$\delta_{4n\mathbb{Z}}(\nu) = \frac{1}{4n}\sum_{j\in\mathbb{Z}/4n\mathbb{Z}} e^{i\nu j\frac{\pi}{2n}}$$

one has

$$\delta_{4n\mathbb{Z}}(\nu) - \delta_{2n+4n\mathbb{Z}}(\nu) = \frac{1}{4n}\sum_{j\in\mathbb{Z}/4n\mathbb{Z}} e^{i\nu\theta_j} - e^{i(2n-\nu)\theta_j}$$

So the solution we seek is clearly

$$y_\nu(t) = \frac{1}{4n}\sum_{j=0}^{4n-1} \frac{\sin(\lambda_j t)}{\lambda_j}\left(e^{i\nu\theta_j} - e^{i(2n-\nu)\theta_j}\right)$$

Since $2n\theta_j = 2n\frac{j\pi}{2n} = j\pi$ $\quad e^{i2n\theta_j} = e^{ij\pi} = (-1)^j$,

so

$$e^{i\nu\theta_j} - e^{i(2n-\nu)\theta_j} = e^{i\nu\theta_j} - (-1)^j e^{-i\nu\theta_j}$$

$$= 2i\sin(\nu\theta_j) \qquad j \text{ even}$$

$$= 2\cos(\nu\theta_j) \qquad j \text{ odd.}$$

So taking real part

$$y_\nu(t) = \frac{1}{2n}\sum_{\substack{j=0\\ j\text{ odd}}}^{4n-1} \frac{\sin(\lambda_j t)}{\lambda_j}\cos(\nu\theta_j)$$

Next note that $\quad \lambda_{4n-j} = \sqrt{\gamma}\,2\sin\left(\frac{1}{2}(4n-j)\frac{\pi}{2n}\right)$

$$= \sqrt{\gamma}\cdot 2\cdot\sin\left(\pi - \tfrac{1}{2}\theta_j\right) = \sqrt{\gamma}\,2\sin\left(\tfrac{1}{2}\theta_j\right)$$

$$= \lambda_j$$

$$\cos(\nu\theta_{4n-j}) = \cos\left(\nu(4n-j)\frac{\pi}{2n}\right) = \cos\left(\cancel{\nu 2\pi} - \nu\theta_j\right)$$

$$= \cos(\nu\theta_j)$$

Thus

$$\boxed{y_\nu(t) = \frac{1}{n}\sum_{\substack{j=0\\ j\text{ odd}}}^{2n-1} \frac{\sin(\lambda_j t)}{\lambda_j}\cos(\nu\theta_j)}$$

Introduce some changes in the preceding. The first thing to do is to view ν as giving the subdivisions of a fixed interval. So put

$$x = \frac{\pi \nu}{2n}$$

so that $-n \leq \nu \leq n$ corresponds to making $2n$ equal subdivisions of $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. In this notation our x-eigenfunctions are $\qquad \nu\theta_j = \nu \frac{j\pi}{2n} = jx$

$$e^{i\nu\theta_j} - e^{i(2n-\nu)\theta_j} = e^{ijx} - e^{ij(\pi - x)}$$

$$= e^{ijx} - (-1)^j e^{-ijx}$$

$$= \begin{cases} 2\cos(jx) & j \text{ odd} \\ 2i\sin(jx) & j \text{ even} \end{cases}$$

and again j runs over $0 < j < 2n$.

Second change is to suppose each particle is a harmonic oscillator in its own right and that the interaction is small. Thus we have the DE

$$\frac{d^2 y_\nu}{dt^2} = -k^2 y_\nu + \gamma^2 (y_{\nu+1} - 2y_\nu + y_{\nu-1})$$

where we replace γ by γ^2. Now eigenfunctions

$$e^{-i\lambda_j t} e^{-i\nu\theta_j}$$

have

$$-\lambda_j^2 = -k^2 + \gamma^2(2\cos\theta_j - 2)$$

$$\lambda_j^2 = k^2 + 4\gamma^2 \sin^2\left(\frac{\theta_j}{2}\right) = k^2 + 4\gamma^2 \sin^2\left(\frac{j\pi}{4n}\right)$$

Let's solve the heat equation:

$$\frac{du_\nu}{dt} = \alpha^2\left(u_{\nu+1} - 2u_\nu + u_{\nu-1}\right)$$

with boundary conditions

$$u_n(t) = u_{-n}(t) = 0.$$

$$u_\nu(0) = \delta_{4\mathbb{Z}}(\nu) - \delta_{2n+4n\mathbb{Z}}(\nu)$$

$$= \frac{1}{4n}\sum_{j\in\mathbb{Z}/4\mathbb{Z}} e^{i(\nu j)\frac{\pi}{2n}} - e^{i(2n-\nu)j\frac{\pi}{2n}}$$

$$= \frac{1}{n}\sum_{\substack{j=0\\ j\text{ odd}}}^{2n-1} \cos\Big(\nu \overbrace{\tfrac{j\pi}{2n}}^{\theta_j}\Big)$$

solution is

$$u_\nu(t) = \frac{1}{n}\sum_{\substack{j=0\\ j\text{ odd}}}^{2n-1} e^{-\mu_j t} \cos(\nu\theta_j) .$$

where

$$-\mu_j = \alpha^2\left(e^{i\theta_j} - 2 + e^{-i\theta_j}\right) = -\alpha^2(2 - 2\cos\theta_j)$$

$$\mu_j = \left(\alpha\cdot 2\cdot \sin\theta_j/2\right)^2$$

January 15, 1977

It will be simpler to use periodic boundary conditions: $y_{\nu+2n}(t) = y_\nu(t)$. This corresponds to having oscillators in a circular ring.

$$\frac{d^2 y_\nu}{dt^2} = -k^2 y_\nu + \gamma^2 (y_{\nu+1} - 2y_\nu + y_{\nu-1})$$

Eigenfunctions $e^{i\lambda t} e^{i\nu\theta}$ where

$$-\lambda^2 = -k^2 + \gamma^2 (e^{i\nu\theta} - 2 + e^{i\nu\theta})$$

$$\boxed{\lambda^2 = k^2 + \gamma^2 \left(2\sin\frac{\theta}{2}\right)^2}$$

and the boundary condition is

$$e^{i2n\theta} = 1$$

or $\theta = j\frac{\pi}{n} = \frac{j}{n}\pi$. If we view our points on the unit circle, then the eigen-functions become

$$e^{i\lambda t} \underbrace{e^{ijx}}_{z^j} \qquad x = \nu\frac{\pi}{n}$$

so that the ν-th point has angle $\frac{\nu\pi}{n}$.

So the general solution of DE + boundary condititions is

$$\boxed{\begin{gathered} y_\nu(t) = \sum_{j=0}^{2n-1} \left(a_j e^{it\lambda_j} + b_j e^{-it\lambda_j}\right) e^{i\nu j\frac{\pi}{n}} \\ \lambda_j = \sqrt{k^2 + \gamma^2 \left(2\sin\frac{j\pi}{2n}\right)^2} \end{gathered}}$$

The fundamental problem now is to introduce a burst of heat at $\nu=0$ and $t=0$ and to get the system to approach equilibrium according to the laws of temperature flow. This involves some sort of mathematical approximation.

Summary of ideas to try:

1) The introduced heat should be random. Perhaps this means we have an external force applied to the 0-th particle which is a function of a point in a probability space. Instead of external force maybe initial conditions could be random in the same way. Then the motion of the system is random, in particular, the energy of the ν-th particle might be some averages.

2) Assume $\left|\frac{\gamma}{k}\right| << 1$. This means we have $2n$ harmonic oscillators which are weakly coupled. We can then think of an eigenfunction

$$e^{it\lambda_j} e^{i\nu\theta} = e^{itk}\left(e^{it(\lambda_j - k) + i\nu\theta}\right)$$

as a precessing oscillator of frequency k, with precessing frequency $\lambda_j - k = \frac{\gamma^2}{2k}\left(2\sin\frac{j\pi}{2n}\right)^2$ to first order

3) The real problem to be solved by the approximation you seek is how to make

$$\lim_{t\to\infty} e^{ita} = 0 \qquad \text{for} \quad a \in \mathbb{R} - \{0\}$$

January 16, 1977

Our vibrating system has the normal modes

$$e^{\pm it\lambda_j} e^{i\nu\theta_j}$$

where λ_j is close to k, if we assume $\frac{\gamma}{k}$ small. Thus the general solution can be written

$$y_\nu = a_\nu(t)e^{itk} + b_\nu(t)e^{-itk}$$

where a_ν, b_ν vary slowly in t. Let us compute the energy of the ν-th particle as a function of time

First suppose we consider a single harmonic oscillator

$$y(t) = ae^{itk} + be^{-itk}$$

but write it in the form

$$y(t) = \mathrm{Re}(ae^{itk}) = \frac{ae^{itk} + \bar{a}e^{-itk}}{2}$$

Then

$$y'(t) = Re\left(ik\, a e^{tk}\right) = -k\, Im\left(a e^{tk}\right)$$

so the kinetic energy

$$\frac{1}{2} y'(t)^2 + \frac{k^2}{2} y(t)^2 = \frac{k^2}{2}\left(Re\left(a e^{itk}\right)^2 + Im\left(a e^{itk}\right)^2\right)$$

$$= \frac{k^2}{2}\left|a e^{itk}\right|^2 = \frac{k^2}{2}|a|^2$$

So if

$$y_\nu(t) = Re\left(a_\nu(t) e^{itk}\right)$$

then

$$y_\nu'(t) = -k\, Im\left(a_\nu(t) e^{itk}\right) + Re\left(a_\nu'(t) e^{itk}\right)$$

$$y_\nu'^2 = k^2\, Im\left(a_\nu e^{itk}\right)^2 - 2k\, Im\left(a_\nu(t) e^{itk}\right) Re\left(a_\nu'(t) e^{itk}\right) + Re\left(a_\nu'(t) e^{itk}\right)^2$$

$$\frac{1}{2}(y_\nu')^2 + \frac{k^2}{2} y_\nu^2 = \frac{k^2}{2}|a_\nu(t)|^2 + Re\left(a_\nu'(t) e^{itk}\right)\left[Re\left(a_\nu'(t) e^{itk}\right) - 2k\, Im\left(a_\nu e^{itk}\right)\right]$$

The last term will be negligible if k is large

So suppose we consider the general solution of the circular discrete string

$$y_\nu(t) = Re\left(\sum_{j=0}^{2n-1}{}' a_j e^{it\lambda_j} e^{i\nu\theta_j}\right)$$

$$\theta_j = j\frac{\pi}{n} \qquad \lambda_j = +\sqrt{k^2 + \gamma^2\left(2 \sin \tfrac{1}{2}\theta_j\right)^2}$$

Then we find that the energy of ν-th particle is

$$\frac{k^2}{2}\left|\sum_{j=0}^{2n-1} a_j e^{it(\lambda_j - k)} e^{i\nu\theta_j}\right|^2$$

up to an error divisible by δ, hence negligible.

Idea: Suppose a_j $j=0,\dots,2n-1$ are random variables which are independent and of mean 0. Then the expected value for the energy of the ν-th particle at time t is

$$\frac{k^2}{2}\sum_{j,j'=0}^{2n-1} E\left(a_j \overline{a_{j'}}\right) e^{it(\lambda_j - \lambda_{j'})} e^{i\nu(\theta_j - \theta_{j'})}$$

$$= \frac{k^2}{2}\sum_j E(|a_j|^2) \qquad \text{as} \quad E(a_j \overline{a_{j'}}) = E(a_j)\overline{E(a_{j'})} = 0$$

for $j \neq j'$.

So I have proved:

<u>Prop</u>: Assume we have an ensemble of the circular vibrating string systems such that the **different** normal modes are independently distributed of mean **zero**. Then the average energy of the ν-th particle is independent of ν.

Application of $\frac{\gamma}{k} \ll 1$ tells us that the energy of the ν-th particle for the solution

$$y_\nu(t) = \mathrm{Re}\left(\sum_{j=0}^{2n-1} a_j e^{it\lambda_j} e^{i\nu\theta_j}\right)$$

is

$$E_\nu(t) = \frac{k^2}{2} \left| \sum_j a_j e^{it(\lambda_j - k)} e^{i\nu\theta_j} \right|^2$$

$$= \frac{k^2}{2} \sum_{j,j'} (a_j \bar{a}_{j'}) e^{it(\lambda_j - \lambda_{j'}) + i\nu(\theta_j - \theta_{j'})}$$

Now if we apply the principle

$$\text{"}\lim_{t\to\infty}\text{"} \; e^{ita} = \begin{cases} 0 & a \neq 0 \\ 1 & a = 0 \end{cases} \qquad a \text{ real}$$

Then we get

$$\text{"}\lim_{t\to\infty}\text{"} \; E_\nu(t) = \frac{k^2}{2} \sum_j |a_j|^2 + \frac{k^2}{2} \sum_{j \neq 0, n} a_j \bar{a}_{2n-j} e^{2i\nu\theta_j}$$

This follows because $\lambda_j = \lambda_{j'} \Rightarrow \cos(\theta_j) = \cos(\theta_{j'})$
and $\theta_j = j\frac{\pi}{n}$, $0 \leq j < 2n \Rightarrow 0 \leq \theta_j < 2\pi \Longrightarrow \theta_{j'} = 2\pi - \theta_j$
$\Rightarrow \frac{j'\pi}{n} = \frac{2\pi n - j\pi}{n} = (2n-j)\frac{\pi}{n} \Rightarrow j' = 2n-j$

January 17, 1977

Go back to periodic string $\quad y_{\nu+2n}(t) = y_\nu(t)$.
Eigenfunctions are

$$e^{it\lambda}\, e^{i\nu\theta}$$

where $\qquad \lambda^2 = k^2 + \gamma^2\left(2\sin\frac{\theta}{2}\right)^2$

and $\qquad e^{2in\theta} = 1 \implies \theta = j\frac{\pi}{n} \quad$ some $j \in \mathbb{Z}$.

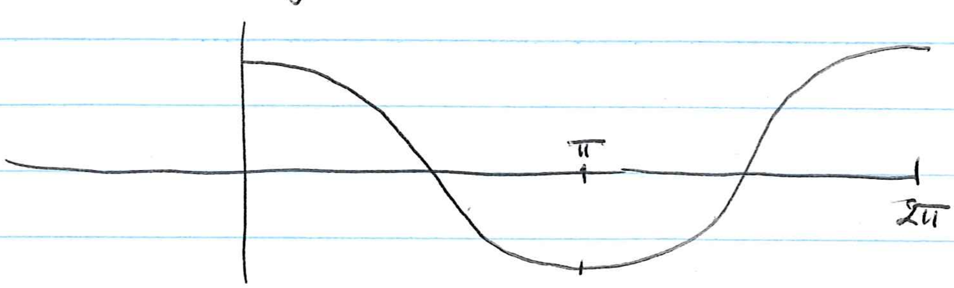
Put $\theta_j = \frac{j\pi}{n}$ and let $\lambda_j = +\sqrt{k^2+\gamma^2\left(2\sin\frac{\theta_j}{2}\right)^2}$. Then the [redacted] eigenfunctions are

$$e^{it\lambda_j} e^{i\nu\theta_j}, \quad e^{-it\lambda_j} e^{i\nu\theta_j}$$

where $\quad j = 0, 1, .., 2n-1$. Observe there are $4n$ eigenfunctions as there should be. Therefore [redacted] the general solution of the DE is

$$y_\nu(t) = \sum_{j=0}^{2n-1} \left(a_j e^{it\lambda_j} + b_j e^{-it\lambda_j}\right) e^{i\nu\theta_j}$$

Next point is to understand when $\lambda_j = \lambda_{j'} \iff$ $\cos\theta_j = \cos\theta_{j'} \iff \theta_{j'} = \pm\theta_j + 2k\pi$. Now for $0 \le j < 2n$, $0 < \theta_j < 2\pi$,



hence $\cos(\theta_j) = \cos(\theta_{j'}) \iff \theta_j = \theta_{j'}$ or $\theta_{j'} = 2\pi - \theta_j$
$\iff j' = j$ or $2n - j$.

$$\sum_{j=0}^{2n-1} b_j e^{-it\lambda_j} e^{i\nu\theta_j} = \sum_{j'=2n}^{1} b_{2n-j'} e^{-it\lambda_{j'}} e^{i\nu(2\pi-\theta_{j'})}$$

$$= \sum_{j=1}^{2n} b_{2n-j} e^{-it\lambda_j} e^{-i\nu\theta_j}$$

Hence if I replace b_j by b_{2n-j}, then my general solution becomes

$$y_\nu(t) = \sum_{j=0}^{2n-1} a_j e^{it\lambda_j + i\nu\theta} + b_j e^{-it\lambda_j - i\nu\theta}$$

in which case $y_\nu(t)$ is real $\Leftrightarrow b_j = \bar{a}_j$. Hence the general real solution is

$$y_\nu(t) = Re\left(\sum_{j=0}^{2n-1} a_j e^{it\lambda_j + i\nu\theta}\right) \quad \text{up to } \tfrac{1}{2}$$

where the a_j are arb. complex numbers. Thus still we have $2n$ arb. constants. Since

$$y_\nu(t) = Re\left(\left(\sum a_j e^{it(\lambda_j - k) + i\nu\theta}\right) e^{itk}\right)$$

the energy of the ν-th particle, assuming k is very large is

$$E_\nu(t) = \frac{k^2}{2} \left| \sum_{j=0}^{2n-1} a_j e^{it(\lambda_j - k) + i\nu\theta} \right|^2$$

$$= \frac{k^2}{2} \sum_{j,j'} a_j \bar{a}_{j'} e^{it(\lambda_j - \lambda_{j'}) + i\nu(\theta_j - \theta_{j'})}$$

So if we use the principle

$$\underset{t\to\infty}{\text{"lim"}}\ e^{ita} = \begin{cases} 0 & a \neq 0 \\ 1 & a = 0 \end{cases}$$

we get

$$\underset{t\to\infty}{\text{"lim"}}\ E_\nu(t) = \frac{k^2}{2}\left(\sum_{j=0}^{2n-1} |a_j|^2 + \sum_{\substack{j=0 \\ j\neq 0,n}}^{2n-1} a_j \bar{a}_{2n-j}\, e^{2i\omega\theta_j}\right)$$

The second term represents an "interference" pattern between the two normal modes associated to j and $2n-j$ which have the same frequencies.

Continuous circular string has DE

$$\frac{\partial^2 y}{\partial t^2} = -k^2 y^2 + \delta^2 \frac{\partial^2 y}{\partial x^2}$$

$$y = e^{it\lambda + iqx} \qquad\qquad y(x+2\pi) = y(x)$$

$$-\lambda^2 = -k^2 \mp \delta^2 q^2 \qquad e^{2\pi i q} = 1 \Rightarrow q = j \in \mathbb{Z}$$

eigenfunctions

$$y = e^{\pm it\lambda_j + ijx} \qquad\qquad j \in \mathbb{Z}.$$

$$\lambda_j = \sqrt{k^2 + \delta^2 j^2}$$

General solution

$$y = \sum_{j \in \mathbb{Z}}' a_j e^{it\lambda_j + ijx} + b_j e^{-it\lambda_j + ijx}$$

Note $\lambda_j = \lambda_{j'} \Leftrightarrow j = \pm j'$. Thus

$$y = \sum_j' a_j e^{it\lambda_j + ijx} + b_{-j} e^{-it\lambda_j - ijx}$$

is real $\Leftrightarrow$ $b_{-j} = \overline{a_j}$. So the general real solution can be written

$$y = \mathrm{Re}\left(\left(\sum_j' a_j e^{-it(\lambda_j - k) + ijx}\right) e^{itk}\right)$$

and the energy of the ~~particle~~ particle at x is

$$E_x(t) = \frac{k^2}{2}\left|\sum_j' a_j e^{-it(\lambda_j - k) + ijx}\right|$$

$$= \frac{k^2}{2} \sum_{j,j'} a_j \overline{a_{j'}}\, e^{it(\lambda_j - \lambda_{j'})} e^{i(j-j')x}$$

So

$$\underset{t\to\infty}{\text{"lim"}} E_x(t) = \frac{k^2}{2}\left(\sum_j |a_j|^2 + \sum_{j\neq 0} a_j \overline{a}_{-j} e^{2ijx}\right)$$

So it appears again that the energy distribution does not approach a constant because of interference.

Remaining question appears to be this. Suppose we apply a random force to the 0-th particle. What happens? ~~I suppose~~ the system at rest for $t<0$ and the force is applied in a short time around $t=0$, so short that the energy transferred by the coupling can be neglected. Then afterward the 0-th particle ~~is in~~ a random position.

I need to change variables from y_ν, y'_ν $\nu \in \mathbb{Z}/2n\mathbb{Z}$ to $a_j \in \mathbb{C}$ $j \in \mathbb{Z}/2n\mathbb{Z}$. The formulas relating them are:

$$y_\nu(0) = \mathrm{Re}\Big(\sum_j a_j e^{i\nu\theta_j}\Big)$$

$$= \frac{1}{2}\sum_j a_j e^{i\nu\theta_j} + \bar{a}_j e^{-i\nu\theta_j}$$

$$y_\nu(0) = \sum_j \left(\frac{a_j + \bar{a}_{-j}}{2}\right) e^{i\nu\theta_j}$$

$$y'_\nu(0) = \mathrm{Re}\Big(\sum i\lambda_j a_j e^{i\nu\theta_j}\Big)$$

$$= \frac{1}{2}\sum i\lambda_j a_j e^{i\nu\theta_j} + -i\lambda_j \bar{a}_j e^{-i\nu\theta_j}$$

$$= \frac{1}{2}\sum (i\lambda_j a_j - i\lambda_j \bar{a}_{-j}) e^{+i\nu\theta_j} \qquad \text{use } \lambda_j = \lambda_{-j}$$

$$y'_\nu(0) = \sum_j i\lambda_j \left(\frac{a_j - \bar{a}_{-j}}{2}\right) e^{i\nu\theta_j}$$

So Fourier inversion gives

$$\frac{a_j + \bar{a}_{-j}}{2} = \frac{1}{2n}\sum_\nu y_\nu(0) e^{-i\nu\theta_j}$$

$$i\lambda_j\left(\frac{a_j - \bar{a}_{-j}}{2}\right) = \frac{1}{2n}\sum_\nu y'_\nu(0) e^{-i\nu\theta_j}$$

Now suppose the initial values are concentrated at $\nu = 0$ and $y_\nu(0)$, $y'_\nu(0)$ are random

$$\frac{a_j + \bar{a}_{-j}}{2} = X \qquad \left(= \tfrac{1}{2n}\, y_\nu(0)\right)$$

$$i\lambda_j \frac{a_j - \bar{a}_{-j}}{2} = Y \qquad \left(= \tfrac{1}{2n}\, y'_\nu(0)\right)$$

so

$$a_j = X + \frac{1}{i\lambda_j} Y = X - \frac{i}{\lambda_j} Y$$

$$\bar{a}_{-j} = X - \frac{1}{i\lambda_j} Y = X + \frac{i}{\lambda_j} Y$$

~~Therefore if X, Y are independently distributed~~

Thus $a_{-j} = a_j$. So all a_j are essentially equal and

$$\underset{t\to\infty}{\text{"lim"}}\ E_\nu(t) = \frac{k^2}{2}\Big(\sum |a_j|^2 + \sum_{j\neq 0,n} a_j \bar{a}_{-j} e^{2i\nu\theta_j}\Big)$$

~~So this calculation shows the limiting energy distribution~~

$$2 + \sum_{\substack{j\in\mathbb{Z}/2n\mathbb{Z}\\ j\neq 0,n}} e^{2i\nu\theta_j} = \sum_{j\in\mathbb{Z}/2n\mathbb{Z}} e^{2i\nu\theta_j} = 2n\,\delta_0(2\nu) = \begin{cases} 0 & \nu \neq 0,n \\ 2n & \nu = 0,n \end{cases}$$

It therefore appears that the limiting energy distribution is not constant in ν. ~~Suppose~~ Thus

$$\text{mean } |a_j|^2 = \text{mean}(X^2) + \frac{1}{\lambda_j^2} \text{ mean}(Y^2) = \varepsilon$$

Then

$$\text{"}\lim_{t\to\infty}\text{"} \; E_\nu(t) = \frac{k^2}{2}\left(2n\cdot\varepsilon + \varepsilon\begin{Bmatrix} -2 & \nu\neq 0, n \\ 2n-2 & \nu = 0, n \end{Bmatrix}\right)$$

Maybe a possibility exists because this is constant for $\nu \neq 0, n$.

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Suppose instead of periodic boundary conditions we introduce a phase rotation:

(1) $$y_{\nu+2n}(t) = e^{i\alpha} y_\nu(t)$$

where α is a fixed real number. Then the eigenfunctions are

$$e^{it\lambda} e^{-i\nu\theta}$$

where $$e^{-i(\nu+2n)\theta} = e^{i\alpha} e^{i\nu\theta}$$

$$e^{i2n\theta} = e^{i\alpha}$$

or $$2n\theta = \text{[redacted]}\,\alpha + j2\pi$$

(2) $$\theta_j = \text{[redacted]}\,\frac{\alpha}{2n} + j\frac{\pi}{n}$$

Again we have the relation

(3) $$\lambda_j^2 = k^2 + \gamma^2 \left(2\sin\frac{\theta_j}{2}\right)^2$$

Now $$\lambda_j = \lambda_{j'} \implies \cos\theta_j = \cos\theta_{j'}$$

$$\implies \theta_j = \pm\theta_{j'} \mod 2\pi\mathbb{Z}$$

$\theta_j = +\theta_{j'}$ $(2\pi\mathbb{Z}) \implies (j-j')\frac{\pi}{n} \in 2\pi\mathbb{Z} \implies j-j' \in 2n\mathbb{Z}$ impossible unless $j=j'$.

$\theta_j = -\theta_{j'}$ $(2\pi\mathbb{Z}) \iff \frac{\alpha}{n} + (j+j')\frac{\pi}{n} \in 2\pi\mathbb{Z}$. This will not be possible if α is chosen generically. In fact putting $\alpha = \pi\alpha_0$ we have $\alpha_0 + (j+j') \in 2n\mathbb{Z}$, so as long as α_0 is not integral [redacted] there are no solutions of this equation.

Hence $j \neq j'$ in $\mathbb{Z}/2n\mathbb{Z} \implies \lambda_j \neq \lambda_{j'}$ as long as $e^{i\alpha} \neq \pm 1$.

So the general solution of the DE + boundary condition (1) is

$$y_\nu(t) = \sum_{j\in\mathbb{Z}/2n\mathbb{Z}}{}' a_j e^{it\lambda_j + i\nu\theta_j} + b_j e^{-it\lambda_j + i\nu\theta_j}$$

where a_j, b_j are $4n$ complex numbers, and

$$\theta_j = \frac{\alpha}{2n} + j\frac{\pi}{n}$$

$$\lambda_j = +\sqrt{k^2 + \gamma^2\left(2\sin\frac{\theta_j}{2}\right)^2}$$

Note this solution is complex. This raises an interesting problem as to how the condition (1) is to be interpreted on the motion level.

Solution I seek should be

$$y_\nu(t) = \mathrm{Re}\sum_{j=0}^{2n-1}{}' a_j e^{i\lambda_j t} e^{i\nu\theta_j}$$

where λ_j, θ_j are related as before but where θ_j is close to $\frac{\alpha}{2n} + j\frac{\pi}{n}$.

$$y_\nu'(t) = -\mathrm{Im}\sum_{j=0}^{2n-1}{}' \lambda_j a_j e^{i\lambda_j t} e^{i\nu\theta_j}$$

$$\sim -k\,\mathrm{Im}\sum_{j=0}^{2n-1} a_j e^{i\lambda_j t} e^{i\nu\theta_j}$$

So put $z_\nu(t) = \sum a_j e^{i\lambda_j t} e^{i\nu\theta_j}$. Then

$$z_\nu(t) \sim y_\nu(t) - \frac{i}{k} y_\nu'(t)$$

So that $\quad z_{\nu+2n} = e^{i\alpha} z_\nu \quad$ comes to

$$y_{\nu+2n} - \frac{i}{k} y'_{\nu+2n} = (\cos\alpha + i\sin\alpha)\left(y_\nu - \frac{i}{k} y'_\nu\right)$$

or

$$\left[\begin{aligned} y_{\nu+2n}(t) &= (\cos\alpha)\, y_\nu + (\sin\alpha)\frac{y'_\nu}{k} \\ \frac{1}{k} y'_{\nu+2n}(t) &= (-\sin\alpha)\, y_\nu + (\cos\alpha)\frac{y'_\nu}{k} \end{aligned}\right.$$

Now we want to use these boundary conditions exactly with the DE. So if $e^{it\lambda}e^{i\nu\theta}$ is to be an eigenfunction then

$$\underbrace{e^{it\lambda} e^{i\nu\theta}}\, e^{i2n\theta} = (\cos\alpha) e^{it\lambda} e^{i\nu\theta} + (\sin\alpha)\frac{i\lambda}{k} e^{it\lambda} e^{i\nu\theta}$$

$$\frac{i\lambda}{k} e^{it\lambda} e^{i\nu\theta} e^{i2n\theta} = (-\sin\alpha)\, e^{it\lambda + i\nu\theta} + (\cos\alpha)\frac{i\lambda}{k} e^{it\lambda} e^{i\nu\theta}$$

Thus

$$\cancel{e^{2in\theta}\begin{pmatrix} 1 \\ \frac{i\lambda}{k} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ \frac{i\lambda}{k} \end{pmatrix}}$$

$$e^{2in\theta} = \cos(2n\theta) + i\sin(2n\theta) = \cos\alpha + i\frac{\lambda}{k}\sin\alpha$$

which is impossible. So we must seek eigenfunctions of the form $e^{it\lambda}(c_1 e^{i\nu\theta_1} + c_2 e^{i\nu\theta_2})$. The D.E. forces $\cos\theta_2 = \cos\theta_1$ hence $\theta_2 = -\theta_1 = -\theta$ say. So we want then solutions of the form $e^{it\lambda}(c_1 e^{i\nu\theta} + c_2 e^{-i\nu\theta})$. The boundary condition becomes too complicated.

Here's the way: If $y_\nu(t)$ is a solution of the DE

$$\frac{d^2 y_\nu}{dt^2} = -k^2 y_\nu + \gamma^2 (y_{\nu+1} - 2y_\nu + y_{\nu-1})$$

then so is $y_\nu - \frac{i}{k} y_\nu' = z_\nu$; but z_ν satisfies the boundary condition

$$z_{\nu+2n}(t) = e^{i\alpha} z_\nu(t)$$

hence we have for some $(a_j, b_j) \in \mathbb{C}^{4n}$

$$z_\nu(t) = \sum_{j=0}^{2n-1} a_j e^{it\lambda_j + i\nu\theta_j} + b_j e^{-it\lambda_j + i\nu\theta_j}$$

with θ_j, λ_j as on page 29 (2), (3). But this gives twice as many constants,

$$y_\nu(t) = \mathrm{Re} \sum_{j=0}^{2n-1} e^{it\lambda_j} (a_j e^{i\nu\theta_j} + \bar{b}_j e^{-i\nu\theta_j})$$

$$y_\nu'(t) = -\mathrm{Im} \sum_{j=0}^{2n-1} \lambda_j e^{it\lambda_j} (a_j e^{i\nu\theta_j} + \bar{b}_j e^{-i\nu\theta_j})$$

too hard.

Another Example: Try the boundary conditions $y_0(t) = y_n(t) = 0$. Then the eigenfunctions are evidently

$$e^{\pm it\lambda_j} \sin(\nu\theta_j) \qquad \theta_j = \frac{j\pi}{n} \qquad j = 1, \dots, n-1$$

and

$$\lambda_j = +\sqrt{k^2 + \gamma^2(2 - 2\cos\theta_j)}$$

As $1 \le j \le n-1 \Rightarrow 0 < \theta_j < \pi$ and cos is 1-1 on this interval, so $\lambda_j = \lambda_{j'} \Rightarrow j = j'$. The general real solution of the DE is

$$y_\nu(t) = \mathrm{Re} \sum_{j=1}^{n-1} a_j e^{it\lambda_j} \sin(\nu\theta_j)$$

so the energy of the ν-th particle ignoring γ is

$$E_\nu(t) = \frac{k^2}{2} \left| \sum_{j=1}^{n-1} a_j e^{it(\lambda_j - k)} \sin(\nu\theta_j) \right|^2$$

$$= \frac{k^2}{2} \sum_{j,j'=1}^{n-1} a_j \overline{a_{j'}}\, e^{it(\lambda_j - \lambda_{j'})} \sin(\nu\theta_j) \sin(\nu\theta_{j'})$$

and since the λ_j are distinct we get

$$\boxed{\lim_{t\to\infty} E_\nu(t) = \frac{k^2}{2} \sum_{j=1}^{n-1} |a_j|^2 \sin(\nu\theta_j)^2}$$

Calculate now the a_j for a δ function

$$\delta_{m+2n\mathbb{Z}}(\nu) = \frac{1}{2n} \sum_{j \in \mathbb{Z}/2n\mathbb{Z}} e^{i(\nu - m)\theta_j} \qquad \theta_j = j\frac{2\pi}{2n} = j\frac{\pi}{n}$$

$$\delta_{m+2n\mathbb{Z}} - \delta_{-m+2n\mathbb{Z}} = \frac{1}{2n} \sum_j \left(e^{i(\nu - m)\theta_j} - e^{-i(\nu+m)\theta_j} \right)$$

$$= \frac{1}{2n} \sum_j e^{i(\nu - m)\theta_j} - e^{-i(\nu+m)\theta_j}$$

$$= \frac{2}{2n} \sum_j i e^{-im\theta_j} \left(\frac{e^{i\nu\theta_j} - e^{-i\nu\theta_j}}{2i} \right)$$

$$= \frac{1}{n} \sum_j i e^{-im\theta_j} \sin(\nu\theta_j)$$

$$= \frac{1}{n} \sum_{j \in \mathbb{Z}/2\mathbb{Z}n}{}' \sin(m\theta_j) \sin(\nu\theta_j)$$

$$= \frac{2}{n} \sum_{j=1}^{n-1}{}' \sin(m\theta_j) \sin(\nu\theta_j)$$

so $\quad \mathrm{Re}\left(\sum_{j=1}^{n-1} e^{-it\lambda_j} \frac{2}{n} \sin(m\theta_j) \sin(\nu\theta_j) \right)$

is the solution of the motion with $y_\nu(0) = \delta_m(\nu)$, $y_\nu'(0) = 0$.
So the energy function in this case:

$$\text{"}\lim_{t\to\infty}\text{"}\; E_\nu(t) = \frac{k^2}{2} \sum_{j=0}^{n-1}{}' \frac{4}{n^2} \sin^2(m\theta_j) \sin^2(\nu\theta_j)$$

$$= \frac{k^2}{2} \sum_{j=0}^{n-1}{}' \frac{1}{n^2} \left(1 - \cos 2m\theta_j\right)\left(1 - \cos 2\nu\theta_j\right)$$

Now $2\theta_j = \frac{2j\pi}{n}$ as j goes from 0 to $n-1$ runs over all angles $0, \frac{1}{n}2\pi, \ldots, \frac{n-1}{n}2\pi$, i.e. a full cycle so average of cos over this are zero (note that as $0<\nu, m<n$, $m.2\theta_j$, $\nu.2\theta_j$ still vary so the average has to be 0.) Thus the above becomes

$$\frac{k^2}{2}\left(\frac{1}{n} + \frac{1}{n^2}\sum_{j=0}^{n-1}{}' \cos(2m\theta_j)(\cos 2\nu\theta_j)\right)$$

But we have by a calculation like at the bottom of p. 33

$$\delta_{m+2n\mathbb{Z}}(\nu) + \delta_{-m+2n\mathbb{Z}}(\nu) = \frac{1}{n} \sum_{j \in \mathbb{Z}/2n\mathbb{Z}} \cos(m\theta_j) \cos(\nu\theta_j)$$

$$= \frac{2}{n} \sum_{j=0}^{n-1}{}' \cos(m\theta_j) \cos(\nu\theta_j)$$

Hence

$$\underset{t\to\infty}{\text{"lim"}} \; E_\nu(t) = \frac{k^2}{2n}\left(1+\tfrac{1}{2}\delta_{2m+2n\mathbb{Z}}(2\nu)+\tfrac{1}{2}\delta_{-2m+2n\mathbb{Z}}(2\nu)\right)$$

$$= \frac{k^2}{2n}\left(1+\tfrac{1}{2}\delta_{m+n\mathbb{Z}}(\nu)+\tfrac{1}{2}\delta_{-m+n\mathbb{Z}}(\nu)\right)$$

So we have focusing again at the points $\nu=m$ and $\nu=n-m$. As a check we compute the total energy, assuming $m\neq n-m$. (unnecessary)

$$\frac{k^2}{2n}\left((n-1)-2+1\tfrac{1}{2}+1\tfrac{1}{2}\right)=\frac{k^2}{2}$$

as it should be.

Now let's take the limit as $n\to\infty$ so as to get a continuous string with fixed ends on the interval $0\leq x\leq\pi$. Here x corresponds to $\frac{\nu\pi}{n}$ so that $\nu\theta_j=\nu\frac{j\pi}{n}=jx$. We have to ~~replace~~ change γ with n so that the limit of our DE is a PDE

$$\frac{\partial^2 y}{\partial t^2} = -k^2 y + \gamma^2\frac{\partial^2 y}{\partial x^2}$$

Thus we ought to replace γ by $\dfrac{\gamma}{\frac{\pi}{n}}$ so

that $\left(\dfrac{\gamma}{\frac{\pi}{n}}\right)^2\left(y_{\nu+1}-2y_\nu+y_{\nu-1}\right)\longrightarrow \gamma^2\frac{\partial^2 y}{dx^2}$

Then $\quad \lambda_j^2 = k^2 + \gamma^2\left(2\sin\frac{\theta_j}{2}\right)^2 \quad$ becomes

$$\lambda_j^2 = \lim_{n\to\infty}\left(k^2 + \frac{\gamma^2}{\left(\frac{\pi}{n}\right)^2}\left(2\sin\frac{j\pi}{n}\right)^2\right) = k^2 + \gamma^2 j^2$$

. The limit of the **solution** $(*)$ is to be found next. Clearly we have to multiply by n or $n\cdot$const. to get a good limit. Recall Fourier sine series:

$$f(x) = \sum_{j=1}^{\infty} a_j \sin jx \qquad \int_0^{\pi} f(x)\sin jx\,dx = a_j\frac{\pi}{2}$$

$$\therefore\ \delta_{d+\mathbb{Z}}(x) = \frac{2}{\pi}\sum_{j=1}^{\infty}\sin(jd)\sin(jx)$$

Thus

$$y(t,x) = \mathrm{Re}\left(\frac{2}{\pi}\sum_{j=1}^{\infty} e^{i\lambda_j t}\sin(jd)\sin(jx)\right)$$

is the solution with $\quad y(0,x) = \delta(x), \ \frac{\partial y}{\partial t}(0,x) = 0.$

$$E(t,x) = \frac{k^2}{2}\left(\frac{2}{\pi}\right)^2\sum_{j,j'=1}^{\infty} e^{i(\lambda_j-\lambda_{j'})t}\sin(jd)\sin(jx)\sin(j'x)\sin(j'd)$$

As $\lambda_j = \lambda_{j'} \Rightarrow j^2 = j'^2 \Rightarrow j = j'$ as $j, j' \geq 1$. Thus

$$\text{"}\lim_{t\to\infty}\text{"}\ E(t,x) = \frac{k^2}{2}\left(\frac{2}{\pi}\right)^2\sum_{j=1}^{\infty}\sin^2(jd)\sin^2(jx)$$

Unfortunately this energy is infinite because in replacing

$\delta_m(\nu)$ by $\frac{n}{\pi}\cdot\delta_m(\nu)$ we multiply the energy at any x by $\frac{n^2}{\pi^2}$.

Check the periodic example:

$$y_\nu(t) = \mathrm{Re}\left(\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} a_j e^{it\lambda_j + i\nu\theta_j}\right) \qquad \theta_j = j\cdot\frac{2\pi}{2n} = j\frac{\pi}{n}$$

$$\text{"lim"}\; E_\nu(t) = \frac{k^2}{2}\left(\sum_j |a_j|^2 + \sum_{j\neq 0,n} a_j \bar{a}_{-j} e^{i2\nu\theta_j}\right)$$

For a δ-function initial condition

$$\delta_0(\nu) = \frac{1}{2n}\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} e^{i\nu\theta_j} \qquad \text{all } a_j = \frac{1}{2n}$$

$$\text{"lim"}\; E_\nu(t) = \frac{k^2}{2}\left(\frac{1}{2n}\right)^2\left(2n + \sum_{j\neq 0,n} e^{i\nu 2\theta_j}\right)$$

But

$$\sum_{j\neq 0,n} e^{i(\nu 2)\theta_j} = \sum_j e^{-i(2\nu)\theta_j} - 2 = 2n\,\delta_0(2\nu) - 2.$$

Thus

$$\text{"lim"}\; E_\nu(t) = \frac{k^2}{2}\,\frac{1}{4n^2}\left(2n - 2 + 2n\,\delta_{2n\mathbb{Z}}(2\nu)\right)$$

~~Note that the deviation of this from constant average energy doesn't improve with n.~~

$$= \frac{k^2}{4n}\left(1 - \frac{1}{n} + \delta_{n\mathbb{Z}}(\nu)\right)$$

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Try reflecting boundaries, i.e. free ends for the string. This means that there is no coupling force on the first particle except from the 2nd, and this can be expressed by requiring $y_0(t) = y_1(t)$ for all t. Similarly $y_n = y_{n+1}$ and we might as well then extend to all ν by $y_{\nu+2n}(t) = y_\nu(t)$, $y_{-\nu}(t) = y_{1-\nu}(t)$. This gives symmetry around $\nu = n + \frac{1}{2}$ also for:

$$y_{n+1-\nu} = y_{1-(\nu-n)} = y_{\nu-n} = y_{\nu+n}$$

If $e^{it\lambda}(ae^{i\nu\theta} + be^{-i\nu\theta})$ is an eigenfunction, then we must have $e^{2in\theta} = 1$ so $2n\theta = 2j\pi \Rightarrow \theta = \theta_j = j\frac{\pi}{n}$. And

$$ae^{i\nu\theta} + be^{-i\nu\theta} = ae^{i\theta}e^{-i\nu\theta} + be^{-i\theta}e^{i\nu\theta}$$

for all $\nu \Rightarrow a = be^{-i\theta}$, $b = ae^{i\theta}$

$$ae^{i\nu\theta} + be^{-i\nu\theta} = a\left(e^{i\nu\theta} + e^{i(1-\nu)\theta}\right)$$

$$= 2ae^{i\frac{1}{2}\theta}\left(\frac{e^{i(\nu-\frac{1}{2})\theta} + e^{i(\frac{1}{2}-\nu)\theta}}{2}\right)$$

$$= \text{const}\left(\cos(\nu - \tfrac{1}{2})\theta\right).$$

Thus we get eigenfunctions

$$e^{it\lambda_j}\cos(\nu - \tfrac{1}{2})\theta_j.$$

We started with the bases $e^{i\nu\theta_j}$ $j \in \mathbb{Z}/2n\mathbb{Z}$ for the $2n\mathbb{Z}$-periodic functions and applied the symmetrization operator $f(\nu) \mapsto \frac{1}{2}(f(\nu) + f(1-\nu))$. This tells us the $\cos(\nu - \frac{1}{2})\theta_j$

form a basis for the vector space of functions $\ni$ $f(\nu+2n)=f(\nu)$ and $f(\nu)=f(1-\nu)$. However clearly $\cos(\nu-\frac{1}{2})\theta_n = \cos(\nu-\frac{1}{2})\pi = 0$ identically and $\cos(\nu-\frac{1}{2})\theta_j = \cos(\nu-\frac{1}{2})\theta_{-j}$. Hence the functions $\cos(\nu-\frac{1}{2})\theta_j$, $j=0,1,..,n-1$ span the vector space in question; as this space has dimension n because $f(1),..,f(n)$ can be prescribed arbitrarily, we see our eigenfunctions are

(1) $$e^{it\lambda_j}\cos(\nu-\tfrac{1}{2})\theta_j \qquad j=0,..,n-1.$$

Note that $\lambda_j=\lambda_{j'} \Leftrightarrow \cos\theta_j=\cos\theta_{j'}$, since $0\le\theta_j<\pi$, this means the λ_j are distinct.

So the general solution of the motion is

(2) $$\boxed{y_\nu(t) = \mathrm{Re}\left(\sum_{j=0}^{n-1} a_j e^{it\lambda_j} \cos(\nu-\tfrac{1}{2})\theta_j\right)}$$

from which we get, as the λ_j are distinct

(3) $$\boxed{\text{"}\lim\text{"}\; E_\nu(t) = \frac{k^2}{2}\sum_{j=0}^{n-1}|a_j|^2\cos^2(\nu-\tfrac{1}{2})\theta_j}$$

To find δ function:

$$\delta_{m+2n\mathbb{Z}}(\nu) = \frac{1}{2n}\sum_{j\in\mathbb{Z}/2n\mathbb{Z}}{}' e^{i(\nu-m)\theta_j} = \frac{1}{2n}\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} e^{-im\theta_j}e^{i\nu\theta_j}$$

$$\cancel{\frac{1}{2}\left(\delta_{m+2n\mathbb{Z}}(\nu)+\delta_{m+2n\mathbb{Z}}(1-\nu)\right) = \frac{1}{2n}\sum_j{}' e^{-im\theta_j}\underbrace{\cos(\nu-\tfrac{1}{2})\theta_j}_{0 \text{ for } j=n}}$$

$\cancel{1-\nu\equiv m \Leftrightarrow \nu\equiv 1-m}$

$$\cancel{=\frac{1}{2n}\left(1+\sum_{j=1}^{n-1}\left(e^{-im\theta_j}+e^{-im\theta_{-j}}\right)\cos(\nu-\tfrac{1}{2})\theta_j\right)}$$

$$e^{i\nu\theta} + e^{i(1-\nu)\theta} = e^{i\frac{1}{2}\theta}\left(e^{i(\nu-\frac{1}{2})\theta} + e^{i(\frac{1}{2}-\nu)\theta}\right)$$

$$= 2e^{i\frac{1}{2}\theta}\cos(\nu-\tfrac{1}{2})\theta$$

$$\delta_{m+2n\mathbb{Z}}(\nu) + \delta_{m+2n\mathbb{Z}}(1-\nu) = \frac{1}{2n}\sum_j e^{-im\theta_j}\, 2e^{i\frac{1}{2}\theta_j}\cos(\nu-\tfrac{1}{2})\theta_j$$

$$= \frac{1}{n}\sum_j e^{-i(m-\frac{1}{2})\theta_j}\cos(\nu-\tfrac{1}{2})\theta_j$$

$$= \frac{1}{n}\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} \cos(m-\tfrac{1}{2})\theta_j\ \cos(\nu-\tfrac{1}{2})\theta_j$$

$$= \frac{1}{n}\left(1 + 2\sum_{j=1}^{n-1}\cos(m-\tfrac{1}{2})\theta_j\ \cos(\nu-\tfrac{1}{2})\theta_j\right)$$

Thus $\quad a_0 = \frac{1}{n} \qquad a_j = \frac{2}{n}\cos(m-\tfrac{1}{2})\theta_j \quad$ for $j = 1, \ldots, n-1$.

So

$$\text{"}\lim\text{"}\ E_\nu(t) = \frac{k^2}{2n^2}\left(1 + \sum_{j=1}^{n-1} 4\cos^2(m-\tfrac{1}{2})\theta_j\ \cos^2(\nu-\tfrac{1}{2})\theta_j\right)$$

$$= \frac{k^2}{2n^2}\left(-1 + \sum_{j=-n+1}^{n} 2\cos^2(m-\tfrac{1}{2})\theta_j\ \cos^2(\nu-\tfrac{1}{2})\theta_j\right)$$

$$= \frac{k^2}{4n^2}\left(-2 + \sum_{j=-n+1}^{n}\left(1+\cos(2m-1)\theta_j\right)\left(1+\cos(2\nu-1)\theta_j\right)\right)$$

since $(2m-1)\theta_j = (2m-1)\frac{2\pi}{2n}$ is never an integral multiple of 2π one has
$$\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} \cos(2m-1)\theta_j = 0$$

and similarly with m replaced by ν. Also if $p, q \in \mathbb{Z}$

$$\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} \cos(p\theta_j)\cos(q\theta_j) = \frac{1}{4}\sum_j e^{i(p+q)\theta_j} + e^{i(p-q)\theta_j} + e^{i(-p+q)\theta_j} + e^{i(-p-q)\theta_j}$$

$$= \frac{2n}{4}\left(\delta_{2n\mathbb{Z}}(p+q) + \delta_{2n\mathbb{Z}}(-p-q) + \delta_{2n\mathbb{Z}}(p-q) + \delta_{2n\mathbb{Z}}(q-p)\right)$$

$$= n\left(\delta_{2n\mathbb{Z}}(p+q) + \delta_{2n\mathbb{Z}}(p-q)\right).$$

So $$\sum_{j\in\mathbb{Z}/2n\mathbb{Z}} \cos((2m-1)\theta_j)\cos((2\nu-1)\theta_j) = n\left[\delta_{2n\mathbb{Z}}(2(m+\nu)-2) + \delta_{2n\mathbb{Z}}(2(m-\nu))\right]$$

$$= n\left[\delta_{n\mathbb{Z}}(\nu-(1-m)) + \delta_{n\mathbb{Z}}(\nu-m)\right]$$

$$= n\left[\delta_{1-m+n\mathbb{Z}}(\nu) + \delta_{m+n\mathbb{Z}}(\nu)\right].$$

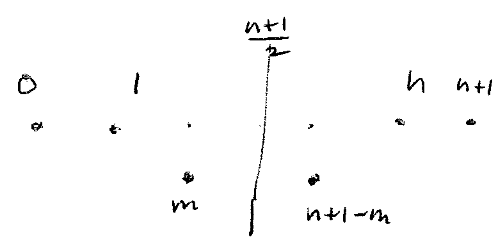
So finally we get

$$\text{"lim"}\, E_\nu(t) = \frac{k^2}{4n^2}\left(-2+2n + n\,\delta_{1-m+n\mathbb{Z}}(\nu) + n\,\delta_{m+n\mathbb{Z}}(\nu)\right)$$

$$= \frac{k^2}{2n}\left(1 - \frac{1}{n} + \frac{1}{2}\delta_{\underset{(n+1-m)+n\mathbb{Z}}{1-m+n\mathbb{Z}}}(\nu) + \frac{1}{2}\delta_{m+n\mathbb{Z}}(\nu)\right)$$

As a check we compute total energy:

$$\frac{k^2}{2n}\left[n\left(1-\frac{1}{n}\right) + \frac{1}{2} + \frac{1}{2}\right] = \frac{k^2}{2}$$



Summary:

1) $y_{\nu+2n}(t) = y_\nu(t)$. Here with initial impulse at $\nu=0$

$$\text{"lim"}\ E_\nu(t) = \frac{k^2}{4n}\left(1-\frac{1}{n}+\delta_{n\mathbb{Z}}(\nu)\right)$$ (p. 37, 27–28)

2) $y_0(t) = y_n(t) = 0$. Initial impulse at $\nu=m$, $0<m<n$. (p. 35)

$$\text{"lim"}\ E_\nu(t) = \frac{k^2}{2n}\left(1+\frac{1}{2}\delta_{m+n\mathbb{Z}}(\nu)+\frac{1}{2}\delta_{n-m+n\mathbb{Z}}(\nu)\right)$$

3) $y_0(t)=y_1(t)$, $y_n(t)=y_{n+1}(t)$. Initial impulse at $\nu=m$, $1\le m\le n$

$$\text{"lim"}\ E_\nu(t) = \frac{k^2}{2n}\left(1-\frac{1}{n}+\frac{1}{2}\delta_{m+n\mathbb{Z}}(\nu)+\frac{1}{2}\delta_{n+1-m+n\mathbb{Z}}(\nu)\right)$$ (p. 41)

4) $y_{\nu+n}(t) = y_\nu(t)$ n odd. Initial impulse at 0.

$$\text{"lim"}\ E_\nu(t) = \frac{k^2}{2n}\left(1-\frac{1}{n}+\delta_{n\mathbb{Z}}(\nu)\right)$$

Consider next the general setup. We have the D.E.

(1) $$\frac{d^2y}{dt^2} = -k^2y \text{ [redacted] } \gamma^2 By$$

where $y = (\cdot\cdot\, y_\nu \cdots)$ is a vector of functions of t and B is a symmetric matrix. If

(2) $$L(y,\dot y) = \text{[redacted] } \tfrac{1}{2}\Big(\sum{}' \dot y_\nu^2 - k^2 \sum y_\nu^2 - \gamma^2 \sum B_{\nu\nu'} y_\nu y_{\nu'}\Big)$$

then

$$\frac{\partial L}{\partial \dot y_\nu} = y_\nu \qquad \frac{\partial L}{\partial y_\nu} = -k v_\nu - \gamma^2 \sum_{\nu'} B_{\nu\nu'} y_{\nu'}$$

so that the system (1) is the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y_\nu}\right) = \frac{\partial L}{\partial y_\nu} .$$

The general solution of (1) will be given by

(3) $$y = \mathrm{Re}\left(\sum c_j e^{it\lambda_j} v_j\right)$$

where

$$\lambda_j^2 v_j = (k^2 + \gamma^2 B)\, v_j$$

i.e. v_j is a full set of [redacted] independent eigenvectors for B and if $Bv_j =$ [redacted] $\varepsilon_j v_j$, then

(4) [redacted] $$\lambda_j = +\sqrt{k^2 + \gamma^2 \varepsilon_j}$$

This is well-defined if [redacted] $(\gamma/k) \ll 1$.

The energy of the ν-th particle

$$y_\nu(t) = \mathrm{Re}\left(\sum_j{}' a_j e^{it\lambda_j} v_{j\nu}\right)$$

is essentially (ignoring γ/k)

$$E_\nu(t) = \frac{k^2}{2}\left|\sum_j{}' a_j e^{it(\lambda_j - k)} v_{j\nu}\right|^2$$

$$= \frac{k^2}{2}\sum_{j,j'} e^{it(\lambda_j - \lambda_{j'})} v_{j\nu}\overline{v_{j'\nu}}\, a_j\overline{a_{j'}}$$

so

unnecessary

$$\text{"lim"}\, E_\nu(t) = \frac{k^2}{2}\sum_{\lambda_j = \lambda_{j'}}{}' a_j\overline{a}_{j'} v_{j\nu}\overline{v_{j'\nu}}$$

In the nice case where the eigenvalues of B are distinct, hence the λ_j are distinct by 4), this becomes

$$\text{"lim"}\, E_\nu(t) = \frac{k^2}{2}\sum_j{}' |a_j|^2 v_{j\nu}^2$$

We suppose the v_j form an orthonormal basis for the vectors $(\cdots, \alpha_\nu, \ldots)$ we are considering. Then one has the orthogonal expansion

$$f = \sum_j v_j \langle f, v_j\rangle$$

hence

$$\delta_\mu = \sum_j{}' \overline{v_{j\mu}}\, v_j \qquad \overline{v_{j\mu}} = v_{j\mu}$$

so for the solution $y = \mathrm{Re}\left(\sum{}' v_{j\mu} e^{it\lambda_j} v_j\right)$ which has $\quad y(0) = \delta_\mu \,, \quad y'(0) = 0 \,,$ we have

$$\text{"lim"}\ E_\nu(t) = \frac{k^2}{2} \sum_j v_{j\mu}^2 v_{j\nu}^2$$

and in general

$$\text{"lim"}\ E_\nu(t) = \frac{k^2}{2} \sum_{\lambda_j = \lambda_{j'}} v_{j\mu} v_{j'\mu} v_{j\nu} v_{j'\nu}$$

This gives the average energy transfer from the μ-th to the ν-th positions.

Since $\{v_j\}$ is orthonormal one has

$$\sum_j v_{j\mu} v_{j\nu} = \delta_{\mu\nu} .$$

So we have the curious problem of understanding the function which associates to the orthogonal matrix $\{v_{j\mu}\}$ the matrix $\sum_j v_{j\mu}^2 v_{j\nu}^2$ which is symmetric and

$$\sum_\mu \sum_j v_{j\mu}^2 v_{j\nu}^2 = \sum_j v_{j\nu}^2 \Big(\sum_\mu v_{j\mu}^2 \Big) = \sum_j v_{j\nu}^2 = 1$$

hence it is a stochastic + symmetric matrix.

$$\{v_{j\mu}\} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$v_{11}^2 v_{11}^2 + v_{21}^2 v_{21}^2 = \cos^4 + \sin^4 = \left(\frac{1+\cos 2\theta}{2}\right)^2 + \left(\frac{1-\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} + \frac{\cos^2 2\theta}{2} = \frac{1}{2} + \frac{1+\cos 4\theta}{4}$$

$$v_{11}^2 v_{12}^2 + v_{21}^2 v_{22}^2 = \cos^2 \sin^2 + \sin^2 \cos^2\theta = \tfrac{1}{2}(\sin 2\theta)^2 = \frac{1-\cos 4\theta}{2 \cdot 2}$$

So it seems to send

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \longmapsto \begin{pmatrix} \frac{3}{4} + \frac{\cos 4\theta}{4} & \frac{1}{4} - \frac{\cos 4\theta}{4} \\ \frac{1}{4} - \frac{\cos 4\theta}{4} & \frac{3}{4} + \frac{\cos 4\theta}{4} \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} \frac{1}{2} + \frac{\cos^2 2\theta}{2} & \frac{\sin^2 2\theta}{2} \\ \frac{\sin^2 2\theta}{2} & \frac{1}{2} + \frac{\cos^2 2\theta}{2} \end{pmatrix}$$

There should be a complex version which should associate to a unitary matrix $(v_{j\nu})$ the symmetric stochastic matrix

$$\sum_j |v_{j\mu}|^2 |v_{j\nu}|^2$$

In fact note that $|v_{j\nu}|^2$ is a doubly-stochastic matrix if $(v_{j\nu})$ is unitary because [redacted] columns (also rows) of a unitary matrix are [redacted] of unit length.

So we have a map

$$(*) \qquad \begin{aligned} T \backslash U(n) / T &\longrightarrow \text{doubly-stochastic } n\times n\text{-matrices} \\ (\alpha_{ij}) &\longmapsto (|\alpha_{ij}|^2) \end{aligned}$$

Now the set of doubly-stochastic matrices is a convex body with non-empty interior in the vector space over $\mathbb{R}$ consisting of matrices (a_{ij}) with $\sum_j a_{ij} = 0 = \sum_i a_{ij}$ $\forall i, j$. This

vector space evidently has dim $n^2 - (n + n-1) = (n-1)^2$.
So the dimension of the doubly-stochastic matrices is $(n-1)^2$.
But $U(n)/T$ has real dim $= 2[(n-1)+(n-2)+\cdots+1] = n^2-n$
and so $T\backslash U(n)/T$ should have dim $= n^2-n-n-1 = (n-1)^2$.

Question: Is the map (*) an isomorphism?

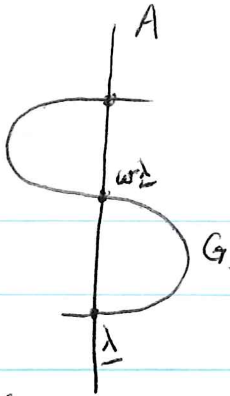
Try $n=2$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ doubly-stoch $\Rightarrow$ $b=c$ $d=a$ $a+b=1$

$$\begin{pmatrix} \sqrt{a} & -\sqrt{b} \\ \sqrt{b} & \sqrt{a} \end{pmatrix}$$

works. This matrix is orthogonal, but the orthogonal group has dim $\frac{n(n-1)}{2} < (n-1)^2$ once $\frac{n}{2} < n-1 \Leftrightarrow n>2$.

January 23, 1977.

If we let $U(n)$ acts on hermitian matrices, then $U(n)/T$ can be identified with the orbit of a diagonal matrix with distinct eigenvalues. Thus I can think of $U(n)/T$ as the space of hermitian matrices B with a given set $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ of eigenvalues. The intersection points with the subspace of diagonal matrices are indexed by the permutation matrices.



Now the orbit $U(n)\underline{\lambda}$ breaks up into N orbits where N is the strictly upper triangular subgroup of GL_n. Each N orbit is T-invariant and contains a unique T-fixpt corresponding to an element of the Weyl gp.

Reformulate: It seems more natural to put things in a more complex, as opposed to real, form.

So let's suppose we have a generalized vibration problem whose solutions are of the form

$$y(t) = \sum_{j=1}^{n} a_j e^{it\omega_j} v_j$$

where y $(=(y_\nu))$ is an n-vector, v_j is an orthonormal basis for $\mathbb{C}^n$, and the a_j are n arbitrary complex coefficients. The "intensity" of the vibration is the function of position and time given by

$$t, \nu \longmapsto |y_\nu(t)| = \Big| \sum_j a_j e^{it\omega_j} v_{j\nu} \Big|$$

$$= \sum_{j,j'} a_j \overline{a_{j'}}\, e^{it(\omega_j - \omega_{j'})} v_{j\nu} \overline{v_{j'\nu}}$$

The "time average" of [redacted] the intensity is

$$\text{"lim"}\ |y_\nu(t)| = \sum_{j=1}^{n} |a_j|^2 |v_{j\mu}|^2$$

assuming the frequencies ω_j are distinct. Thus the double-stochastic matrix $P = |v_{j\nu}|^2$ relates intensities at the different positions to the intensities of the different normal modes of vibrations. The symmetric stochastic matrix

$$P^*P = \sum_j |v_{j\mu}|^2 |v_{j\nu}|^2$$

describes how intensities [redacted] change if one starts at time 0 with random intensities at the different positions and then takes the time average.

<u>Example 1.</u> $\nu \in \mathbb{Z}/n\mathbb{Z}$ $\qquad v_{j\nu} = \dfrac{e^{i\nu\theta_j}}{\sqrt{n}}$ $\qquad \theta_j = \dfrac{2\pi}{n}$
$j \in \mathbb{Z}/n\mathbb{Z}$. Then

$$\frac{1}{n} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} e^{i\nu j \frac{2\pi}{n}} = \delta_{n\mathbb{Z}}(\nu)$$

so this family is orthonormal. Then

$$P_{j\nu} = |v_{j\nu}|^2 = \frac{1}{n}$$

for all j, ν and $P^*P = P$.

Example 2. Suppose $\nu \in G$ a finite abelian group and $j \in \hat{G}$ the dual group and $v_{j\nu} = cj(\nu)$, where c is a constant to make v_j of norm 1, i.e.

$$\sum_{\nu} c^2 |v_{j\nu}|^2 = c^2 |G| = c^2 n = 1 \quad \Rightarrow c = \frac{1}{\sqrt{n}}$$

where $n = |G|$. Then $P_{j\nu} = \frac{1}{n}$ so we have the same situation as in the preceding example.

Example 3. Let G be a finite group, let ν run over the conjugacy classes in G, and let $j \in \hat{G}$ = the set of irreducible characters of G. The orthogonality relations for characters say that

$$\sum_{g \in G} \chi_j(g) \overline{\chi_{j'}(g)} = \sum_{\nu} h_\nu \chi_j(\nu) \overline{\chi_{j'}(\nu)} = n \delta_{jj'}$$

where $h_\nu = \text{card}(\nu)$, $n = |G|$. Thus

$$v_j = \sqrt{\frac{h_\nu}{n}} \chi_j$$

will be an orthonormal basis, so

$$P_{j\nu} = |v_{j\nu}|^2 = \frac{h_\nu}{n} |\chi_j(\nu)|^2$$

Jan. 25, 1977

The fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is $\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$. Its Fourier transform is

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} e^{ix\xi} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\left(\frac{x}{2\sqrt{t}} - i\sqrt{t}\,\xi\right)^2 - t\xi^2} dx$$

$$= \frac{e^{-t\xi^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{2\sqrt{t}}\right)^2} \frac{dx}{2\sqrt{t}} = e^{-t\xi^2}$$

Hence
$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t\xi^2} e^{-ix\xi} d\xi$$

Let's return to the problem with the boundary conditions $y_0(t) = y_n(t) = 0$. The general motion associated to

$$\frac{d^2 y_\nu}{dt^2} = -k^2 y_\nu + \gamma^2 (y_{\nu+1} - 2y_\nu + y_{\nu-1})$$

is then
$$y_\nu(t) = \mathrm{Re}\left(\sum_{j=1}^{n-1} a_j e^{it\lambda_j} \sin(\nu\theta_j) \right)$$

where
$$\lambda_j = \left(k^2 + \gamma^2 \left(2\sin\frac{\theta_j}{2}\right)^2\right)^{1/2}$$

$$= k\left(1 + \frac{\gamma^2}{2k^2}\left(2\sin\frac{\theta_j}{2}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{\gamma^2}{k^2}\right)^2\left(2\sin\frac{\theta_j}{2}\right)^2 + \cdots\right)$$

Suppose now we let $k, \gamma \to \infty$ in such a way that $\frac{\gamma^2}{k} \to \alpha$ converges. Then $\lambda_j - k \to \frac{\alpha}{2}\left(2\sin\frac{\theta_j}{2}\right)^2$. Hence the leading term of the energy of the ν-th particle approaches

$$\frac{1}{k^2} E_\nu(t) \longrightarrow \frac{1}{2}\left| \sum_{j=1}^{n-1} a_j e^{it\frac{\alpha}{2}\left(\sin\frac{\theta_j}{2}\right)^2} \sin(\nu\theta_j)\right|^2$$

But the general solution of the heat equation:

$$\frac{du_\nu}{dt} = c\,(u_{\nu+1} - 2u_\nu + u_{\nu-1})$$

is u

January 27, 1977

[redacted] Recall the general solution of

$$\frac{d^2 y_\nu}{dt^2} = -k^2 y_\nu + \gamma^2 (y_{\nu+1} - 2y_\nu + y_{\nu-1})$$

$$y_0(t) = y_n(t) = 0$$

is
$$y_\nu(t) = \mathrm{Re}\left(\sum_{j=1}^{n-1} a_j e^{it\lambda_j} \sin(\nu\theta_j)\right) \qquad \theta_j = j\frac{\pi}{n}$$

$$\lambda_j = \sqrt{k^2 + \gamma^2 \left(2 \sin \frac{\theta_j}{2}\right)^2}$$

Let v_j be vectors in a Hilbert space. For example in the space of functions $\nu \mapsto f(\nu)$ [redacted] with $f(\nu + 2n) = f(\nu)$ [redacted] $f(-\nu) = -f(\nu)$ one can take $v_j = a_j \sin(\nu\theta_j)$. These functions are orthogonal since

$$[redacted] \sum_{\nu=1}^{n-1} \sin\left(\nu j \frac{\pi}{n}\right) \sin\left(\nu j' \frac{\pi}{n}\right)$$

$$= \frac{1}{2} \sum_{\nu \in \mathbb{Z}/2n\mathbb{Z}} \frac{e^{i\nu(j+j')\pi/n} - e^{i\nu(j-j')\pi/n} - e^{i\nu(j'-j)\pi/n} + e^{-i\nu(j+j')\pi/n}}{-4}$$

$$= -\frac{1}{8}\left[2n\,\delta_{2n\mathbb{Z}}(j+j') - 2n\,\delta_{2n\mathbb{Z}}(j-j') - 2n\,\delta_{2n\mathbb{Z}}(j-j') + 2n\,\delta_n(j+j')\right]$$

$$= \frac{n}{2}\left[\delta_{2n\mathbb{Z}}(j-j') - \delta_{2n\mathbb{Z}}(j+j')\right]$$

Thus
$$\boxed{\sum_{\nu=1}^{n-1} \sin\left(\nu j \frac{\pi}{n}\right) \sin\left(\nu j' \frac{\pi}{n}\right) = \frac{n}{2}\delta_{jj'} \quad \text{if } 1 \le j, j' \le n-1}$$

[redacted] In general let v_j be a set of vectors in a Hilbert space, and λ_j be a set of frequencies. Then what can I say about the statistical properties of the "process"

$$(*) \qquad f(t) = \sum_j e^{-it\lambda_j} v_j .$$

<u>Idea</u>: Think of the path [redacted] $t \longmapsto (e^{it\lambda_1}, \ldots, e^{it\lambda_m})$ in the torus as being everywhere [redacted] dense. Then take the image of Lebesgue measure on the torus [redacted] under the map

$$(\zeta_1, \ldots, \zeta_m) \longmapsto \sum_{j=1}^{m} \zeta_j v_j .$$

This gives us a measure on Hilbert space describing all the statistical properties of (*) we could be interested in.

[redacted] ______________________________

New approach: Take as starting point the fact that states are probability measures on phase space. Now modify the flow so that all states approach the Maxwell-Boltzman distribution as $t \to \infty$.

January 28, 1977

Let's find out about equilibrium states for the infinite discrete string. One has the [redacted] Hamiltonian

$$H = \frac{1}{2}\sum \dot{y}_\nu^2 + \frac{k^2}{2}\sum y_\nu^2 + [redacted]\ \frac{\varepsilon}{2}\sum (y_\nu - y_{\nu-1})^2$$

where $\nu \in \mathbb{Z}$. Here ε is the coupling constant which I now [redacted] should think of as possibly being negative, i.e. the particles repel each other. [redacted] The partition function is [redacted]

$$\int e^{-\beta H}$$

When ε is zero, this is [redacted] an n-fold product of the one-particle partition function

$$\int_{q=-\infty}^{\infty} dq \int_{p=-\infty}^{\infty} e^{-\beta\left(\frac{1}{2}p^2 + \frac{k^2}{2}q^2\right)} dp = [redacted]$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\beta}}\,\frac{\sqrt{2\pi}}{\sqrt{\beta k^2}} = \frac{2\pi}{\beta k}$$

So the MB distribution for one [redacted] particle is

$$\frac{\beta k}{2\pi}\, e^{-\beta\left(\frac{1}{2}p^2 + \frac{k^2}{2}q^2\right)}\, dp\, dq \qquad \begin{array}{l} p = \dot{y}_\nu \\ q = y_\nu . \end{array}$$

So what does the interaction do? On the phase space for n-particles it gives a certain Gaussian measure.