June 3, 1977

$$G(x,y,s) = H(x,y,s) + e^{-i\pi s} e^{-2\pi i y} H(1-x,-y,s)$$

$$e^{-i\pi s} e^{-2\pi i y} H(1-x,-y,s)\,\Gamma(s) = e^{-i\pi s} e^{-2\pi i y} \int_0^\infty \frac{e^{-(1-x)t}}{1-e^{-t-2\pi i y}}\, t^s \frac{dt}{t}$$

$$= e^{-i\pi s} \int_0^\infty \frac{e^{xt}}{e^{t+2\pi i y}-1}\, t^s \frac{dt}{t}$$

$$= \int_{-\infty}^0 \frac{e^{-xt}}{1-e^{-t+2\pi i y}}\, \underbrace{e^{-i\pi s}(-t)^s}_{t^s \text{ if } \arg t = -\pi} \frac{dt}{t}$$

Thus

$$\boxed{G(x,y,s) = \frac{1}{\Gamma(s)} \int_{-\infty}^{\infty} \frac{e^{-xt}}{1-e^{-t+2\pi i y}}\, t^s \frac{dt}{t} \quad \text{where } \arg(t) = -\pi \text{ if } t<0, \quad 0<\mathrm{Re}(x)<1}$$

Use contour integration on this integral, assuming $0<\mathrm{Re}(y)<1$.
and $0<x<1$, $\quad -t+2\pi i y = 2\pi i n \quad t = 2\pi i(y-n) \quad n \geq 1$.

$$G(x,y,s) = (-1)\frac{2\pi i}{\Gamma(s)} \sum_{n\geq 1} e^{-x 2\pi i(y-n)} \Big(\underbrace{2\pi i(y-n)}_{\text{has arg. } -\frac{\pi}{2}}\Big)^{s-1}$$

$$= \frac{\left(2\pi e^{-i\pi/2}\right)^s}{\Gamma(s)} \sum_{n\geq 0} e^{-2\pi i x y + 2\pi i (n+1) x} (-y+n+1)^{s-1}$$

$$\boxed{G(x,y,s) = \frac{\left(2\pi e^{-i\pi/2}\right)^s}{\Gamma(s)} e^{-2\pi i x y} e^{2\pi i x} H(1-y, x, 1-s)}$$

$\cancel{0<\mathrm{Re}(x)<1}$	$0<\mathrm{Re}(y)<1$	$\mathrm{Im}(x)>0$

The point is that once the formula is established for $0<x<1$, $0<\mathrm{Re}(y)<1$, then it has to hold in the region $\mathrm{Im}(x)>0$.

Philosophy is that the functions H, G are defined for $0<x<1$ and $0<y<1$ first and then analytically continued to mult-valued functions. There's still a problem because we have not figured out how to to define G for $\mathrm{Re}(y)$ integral. What is clear is that somehow we can't have both G periodic in y and analytic.

<u>Example</u>: We ~~define~~ try to define $G(x,y,1)$ by

$$(+) \qquad G(x,y,1) = \sum_{n\in\mathbb{Z}}{}' (x+n)^{-1} e^{2\pi i n y}$$

however this series doesn't converge absolutely so it is necessary to use some device. Methods possible:

1) Eisenstein summation: $\lim_{N\to\infty} \sum_{n=-N}^{N}{}'$

2) Break up into $H(x,y,1) - e^{-2\pi i y} H(1-x,-y,1)$ and define $H(x,y,1)$ by analytic continuation from $\mathrm{Im}(y>0)$, or from $\mathrm{Re}(s)>1$.

3) Theory of distributions: $G(x,y,1)$ as defined by (+) is a well-defined distribution on the y-line periodic in y defined for all $x\in\mathbb{C}-\mathbb{Z}$. It makes

sense to say that $G(x,y,1)$ is a function for $y \notin \mathbb{Z}$. Thus we can say easily that

$$G(x,y,1) = \frac{2\pi i \, e^{-2\pi i x y}}{1 - e^{-2\pi i x}}$$

for all $x \in \mathbb{C} - \mathbb{Z}$ and for all $0 < y < 1$. Then by periodicity we know $G(x,y,1)$ for all real non-integral y.

θ-transf. formula

$$\frac{1}{\sqrt{t}} \sum e^{-\frac{\pi}{t}(x-n)^2} = \sum_n e^{-\pi n^2 t} \, e^{2\pi i n x}$$

Replace t by t^2 and x by tx

$$\frac{1}{t} \sum e^{-\frac{\pi}{t^2}(t^2x^2 - 2txn + n^2)} = \sum_n e^{-\pi n^2 t^2 + 2\pi i n t x}$$

or

$$\frac{e^{-\pi x^2}}{t} \sum_n e^{-\pi n^2/t^2 + 2\pi n x/t} = \sum_n e^{-\pi n^2 t^2 + 2\pi i n x t}$$

Hence if I put

$$\varphi(t,x) = \sum_n e^{-\pi n^2 t^2 + 2\pi i n x t}$$

we have

$$\varphi(t,x) = \frac{e^{-\pi x^2}}{t} \varphi\left(\frac{1}{t}, \frac{x}{i}\right)$$

Recall that

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \int_0^\infty \Big(\sum_n{}' e^{-\pi n^2 t^2} - 1\Big) t^s \frac{dt}{t}$$

Look at

$$\int_0^\infty \Big(\sum_n e^{-\pi(nt-x)^2} - e^{-\pi x^2}\Big) t^s \frac{dt}{t}$$

which reduces to ξ when $x = 0$. So this means we want to look at integrals

$$\int_0^\infty e^{-\pi n^2 t^2 + 2\pi n x t}\, t^s \frac{dt}{t}$$

i.e.

$$\int_0^\infty e^{-at^2 + bt}\, t^s \frac{dt}{t} \qquad a > 0$$

which we looked at previously in connection with the Hermite polynomials.

Recall that $H_n(x)$ is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

hence

$$\sum_{n\geq 0} \frac{y^n}{n!} H_n(x) = e^{x^2} \sum_{n\geq 0} \frac{(-y)^n}{n!} \frac{d^n}{dx^n} e^{-x^2}$$

$$= e^{x^2} e^{-(x-y)^2} = e^{-y^2+2xy}$$

Thus

$$\frac{1}{2\pi i} \oint e^{-t^2+2xt}\, t^{-n} \frac{dt}{t} = \frac{1}{n!} H_n(x)$$

$n \geq 0$. Calculation shows that a contour integral with appropriate endpts of the form

$$\int e^{-t^2+2xt}\, t^s \frac{dt}{t}$$

satisfies the Hermite DE

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} = 2s\, u$$

In particular

$$\int_0^\infty e^{-t^2+2xt}\, t^s \frac{dt}{t} \qquad Re(s) > 0$$

is "the" solution of this DE. vanishing at $x = -\infty$.

θ-relation is

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2 + 2\pi i x n t} = \frac{e^{-\pi x^2}}{t} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t^2 + 2\pi x n/t}$$

This should tells me that

$$\int_0^\infty \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2 t^2 + 2\pi i x n t}\, t^s \frac{dt}{t} \qquad \text{for } Re(s) > 1$$

has the analytic continuation

$$e^{-\pi x^2} \int_0^\infty \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-\pi n^2/t^2 + 2\pi x n/t}\, t^{s-1} \frac{dt}{t}$$

$$= e^{-\pi x^2} \int_0^\infty \sum_{n\neq 0} e^{-\pi n^2 t^2 + 2\pi x n t}\, t^{1-s} \frac{dt}{t} \qquad \text{for } Re(s) < 0$$

Now $\displaystyle\int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 t^2 + 2\pi i x n t}\, t^s \frac{dt}{t} = \zeta(s) \int_0^\infty e^{-\pi t^2 + 2\pi i x t}\, t^s \frac{dt}{t}$

$n \leq -1$

So
$$\int_0^\infty \sum_{n\neq 0} e^{-\pi n^2 t^2 + 2\pi i x n t}\, t^s \frac{dt}{t} =$$
$$\xi(s)\left[\int_0^\infty e^{-\pi t^2 + 2\pi i x t} t^s \frac{dt}{t} + \int_0^\infty e^{-\pi t^2 - 2\pi i x t} t^s \frac{dt}{t}\right]$$

Similarly
$$e^{-\pi x^2}\int_0^\infty \sum_{n\neq 0} e^{-\pi n^2 t^2 + 2\pi x n t}\, t^{1-s} \frac{dt}{t}$$
$$= \xi(1-s)\, e^{-\pi x^2}\left[\int_0^\infty e^{-\pi t^2 + 2\pi x t} t^{1-s} \frac{dt}{t} + \int_0^\infty e^{-\pi t^2 - 2\pi x t} t^{1-s} \frac{dt}{t}\right]$$

For some reason these two expressions should be equal. The reason probably is that both are solutions of the same DE with derivative zero at $x=0$, hence coincide up to a scalar multiple.

<u>June 4, 1977.</u>

The D.E.

1) $$-u'' + x^2 u = 2\left(\tfrac{1}{2} - s\right) u$$

has the solutions $$u = e^{-x^2/2}\int_P e^{-t^2+2tx}\, t^s \frac{dt}{t}$$

for different contours P. Observe that $s=-n$ corresponds to the eigenvalue $2(n+\frac{1}{2})$. However the D.E. is invariant under the substitution $x \mapsto ix$, $\frac{1}{2} - s \mapsto s - \frac{1}{2}$ (i.e. $s \mapsto 1-s$), hence it also has the solutions

$$e^{x^2/2}\int_{\mathcal{P}} e^{-t^2+2ixt}\, t^{1-s}\frac{dt}{t}$$

One also can change $x \mapsto -x$ without changing s.

Back to

$$\frac{d}{dx}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix} i\lambda & x\\ x & -i\lambda\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} \qquad \frac{d}{dx}\tilde{u} = \begin{pmatrix}-x & \lambda\\ -\lambda & x\end{pmatrix}\tilde{u} \qquad \tilde{u} = \begin{pmatrix}u_1-u_2\\ u_1+iu_2\end{pmatrix}$$

$$\left(\frac{d}{dx}+x\right)\tilde{u}_1 = \lambda\tilde{u}_2$$

$$\left(-\frac{d}{dx}+x\right)\tilde{u}_2 = \lambda\tilde{u}_1$$

$$\left(-\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)\tilde{u}_1 = \left(-\frac{d^2}{dx^2}+x^2-1\right)\tilde{u}_1 = \lambda^2\tilde{u}_1$$

eigenvalues are $\lambda^2+1 = 2n+1$, $n=0,1,\dots$

Put

$$h(x,s) = e^{-x^2/2}\int_{\mathcal{P}} e^{-t^2-2tx}\, t^s\,\frac{dt}{t}$$

$\mathcal{P}$ an appropriate contour. Then

$$\left(\frac{d}{dx}+x\right)h(x,s) = e^{-x^2/2}\int_{\mathcal{P}} e^{-t^2-2tx}(-2t)t^s\frac{dt}{t} = -2h(x,s+1)$$

$$\left(-\frac{d}{dx}+x\right)h(x,s) = \left(-\frac{d}{dx}-x+2x\right)h(x,s) = e^{-x^2/2}\int_{\mathcal{P}} e^{-t^2-2tx}(+2t+2x)t^s\frac{dt}{t}$$

$$= e^{-x^2/2} \int_p -\frac{d}{dt}\left(e^{-t^2-2tx}\right) t^{s-1}\, dt = e^{-x^2/2} \int_p e^{-t^2-2tx} (s-1) t^{s-2} dt$$

$$= (s-1)\, h(x, s-1).$$ Thus we have

$$\left(\frac{d}{dx} + x\right) h(x,s) = \lambda\left(-\frac{2}{\lambda} h(x, s+1)\right)$$

$$\left(-\frac{d}{dx} + x\right)\left(-\frac{2}{\lambda} h(x, s+1)\right) = -\frac{2}{\lambda}\, s\, h(x,s) = \lambda\, h(x,s)$$

provided $-\frac{2}{\lambda} s = \lambda$ i.e. $\boxed{s = -\frac{\lambda^2}{2}}$

Hence

$$\tilde{u} = \begin{pmatrix} h(x,s) \\ -\frac{2}{\lambda} h(x, s+1) \end{pmatrix}$$ is a solution of the original system.

So now we know that if we take two different contours and let $h_i(x,s)$ be the corresponding functions, then the Wronskian

$$\begin{vmatrix} h_1(x,s) & h_2(x,s) \\ -\frac{2}{\lambda} h_1(x, s+1) & -\frac{2}{\lambda} h_2(x,s) \end{vmatrix}$$

is independent of x.

Another version: Put

$$v(x,s) = \int_p e^{-t^2-2xt}\, t^s \frac{dt}{t}$$

whence $$\frac{d}{dx} v(x,s) = -2 v(x, s+1)$$

$$\left(-\frac{d}{dx} + 2x\right) v(x,s) = \int e^{-t^2-2xt} (+2t + 2x)\, t^s \frac{dt}{t}$$

$$= \int -\frac{d}{dt} e^{-t^2-2xt} \cdot t^s \frac{dt}{t} = (s-1)\, v(x, s-1)$$

and so v is a solution of

$$\left(-\frac{d}{dx}+2x\right)\frac{dv}{dx} = \left(-\frac{d}{dx}+2x\right)\left(-2v(x,s+1)\right)$$
$$= -2s\,v(x,s)$$

i.e.

$$\boxed{v'' - 2x v' = 2s v}$$

Then the Wronskian of two solutions

$$w = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} = \begin{vmatrix} v_1(x,s) & v_2(x,s) \\ -2v_1(x,s+1) & -2v_2(x,s+1) \end{vmatrix}$$

satisfies $w' = 2x\,w$ so $w = \text{const}.\, e^{x^2}$.

The thing I would like to see whether I can establish this fact about the Wronskian using the contour integrals directly.

Suppose $v_1(x,s) = \int_0^\infty e^{-t^2-2xt}\, t^s \frac{dt}{t}$ and

$$v_2(x,s) = \int_0^\infty e^{-t^2+2xt}\, t^s \frac{dt}{t}$$

Then the Wronskian is 2 times:

$$\int_0^\infty e^{-t^2-2xt} t^s \frac{dt}{t} \int_0^\infty e^{-t^2+2xt} t^{s+1} \frac{dt}{t} + \int_0^\infty e^{-t^2-2xt} t^{s+1} \frac{dt}{t} \int_0^\infty e^{-t^2+2xt} t^s \frac{dt}{t}$$

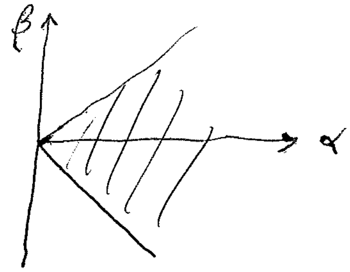
Introduce u for the dummy variable in the integral for v_2:

$$\int_0^\infty\int_0^\infty e^{-t^2-2xt-u^2+2xu}\left(t^{s-1}u^s + t^s u^{s-1}\right) dt\, du$$

For some reason this multiplied by e^{-x^2} is independent of x.

Change variables:

$$u = \alpha+\beta$$
$$t = \alpha-\beta$$



$$u+t = 2\alpha$$
$$u-t = 2\beta$$

$$dt\,du = (d\alpha - d\beta)(d\alpha + d\beta) = 2\,d\alpha\, d\beta$$

So the double integral becomes

$$\int_{-\infty}^{\infty} d\beta \cdot 2 \int_{|\beta|}^{\infty} d\alpha \; e^{-(\alpha-\beta)^2 - (\alpha+\beta)^2 + 2x(2\beta)} (\alpha-\beta)^{s-1} (\alpha+\beta)^{s-1}\, 2\alpha$$

$$= 2\int_{-\infty}^{\infty} d\beta \int_{|\beta|}^{\infty} 2\alpha\, d\alpha \; e^{-2\alpha^2 - 2\beta^2 + 4\beta x} (\alpha^2-\beta^2)^{s-1}$$

$$\gamma = \alpha^2 - \beta^2$$
$$d\gamma = 2\alpha\, d\alpha$$

$$= 2\int_{-\infty}^{\infty} e^{-2\beta^2 + 4\beta x} d\beta \int_0^{\infty} d\gamma\, e^{-2(\gamma+\beta^2)} \gamma^{s-1}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-4\beta^2 + 4\beta x} d\beta}_{e^{x^2}\frac{1}{2}\int_{-\infty}^{\infty} e^{-\beta^2 + 2\beta x - x^2} d\beta} \cdot \underbrace{\int_0^{\infty} e^{-2\gamma} \gamma^{s-1} 2\, d\gamma}_{\Gamma(s)\, 2^{-s}}$$

$$= \frac{\sqrt{\pi}}{2} e^{x^2} \Gamma(s)\, 2^{-s}.$$

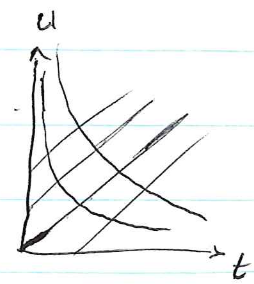
June 5, 1977

Duplication formula for Γ:

$$\Gamma(s/2) = \int_0^\infty e^{-t} t^{s/2} \frac{dt}{t} = 2\int_0^\infty e^{-t^2} t^s \frac{dt}{t}$$

$$\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{s+1}{2}\right) = 2\left[\int_0^\infty e^{-t^2} t^{s-1} dt \int_0^\infty e^{-u^2} u^{s} du + \int_0^\infty e^{-u^2} u^{s-1} ds \int_0^\infty e^{-t^2} t^s dt\right]$$

$$= 2\int_0^\infty\int_0^\infty e^{-t^2-u^2} (tu)^{s-1} (u+t)\,dt\,du$$



Now put $\begin{cases} x = tu & 0 < x < \infty \\ y = u - t & -\infty < y < \infty \end{cases}$

$$dx\,dy = (t\,du + u\,dt)(du - dt) = \underset{\uparrow \text{ positive}}{(u+t)}\,dt\,du$$

$$y^2 = u^2+t^2-2ut = u^2+t^2-2x$$

so

$$\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{s+1}{2}\right) = 2\int_{-\infty}^{\infty} dy \int_0^\infty dx\; e^{-y^2-2x} x^{s-1} = 2\int_{-\infty}^{\infty} e^{-y^2} dy \int_0^\infty e^{-2x} x^s \frac{dx}{x}$$

$$= 2\sqrt{\pi}\,\frac{\Gamma(s)}{2^s}$$

so we obtain Legendre's duplication formula

$$\boxed{\sqrt{\pi}\,\Gamma(s) = 2^{s-1}\,\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{s+1}{2}\right)}$$

Next thing to do is to try to do this same thing in other cases. Review the Bessel situation again:

$$K_s(r) = \int_0^\infty e^{-r\frac{t+t^{-1}}{2}} t^s \frac{dt}{t}$$

$$\frac{d}{dr} K_s(r) = \int_0^\infty e^{-r\frac{t+t^{-1}}{2}} \left(-\frac{t+t^{-1}}{2}\right) t^s \frac{dt}{t} = -\frac{1}{2} K_{s+1}(r) - \frac{1}{2} K_{s-1}(r)$$

$$s\,K_s(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} s\,t^{s-1}\,dt = \int_0^\infty \left(-\frac{d}{dt}\right)\left(e^{-r\left(\frac{t+t^{-1}}{2}\right)}\right) t^s\,dt$$

$$= \int_0^\infty e^{-r(t+t^{-1})/2}\,(+r/2)(t-t^{-1})\,t^s\,\frac{dt}{t}$$

$$\frac{s}{r}K_s(r) = +\frac{1}{2}K_{s+1}(r) - \frac{1}{2}K_{s-1}(r)$$

$$\begin{cases} \left(\dfrac{d}{dr} + \dfrac{s}{r}\right) K_s(r) = -K_{s-1}(r) \\ \left(\dfrac{d}{dr} - \dfrac{s}{r}\right) K_s(r) = -K_{s+1}(r) \end{cases}$$

$$\left(\frac{d}{dr} - \frac{s-1}{r}\right)\left(\frac{d}{dr} + \frac{s}{r}\right) K_s(r) = K_s(r)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s}{r^2} - \frac{s^2-s}{r^2}\right) K_s(r) = K_s(r)$$

$$\oplus \qquad \left[\left(r\frac{d}{dr}\right)^2 + (-s^2 - r^2)\right] K_s(r) = 0$$

which is obtained from Bessel's DE

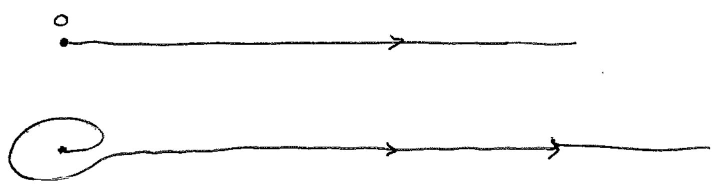
$$\left(z\frac{d}{dz}\right)^2 u + (z^2 - n^2)\,u = 0$$

by putting $z = ir$ and $n = \pm s$.

I will continue to work with the imaginary form $\oplus$ and now will discuss Hankel functions. These are solutions of the DE of the form

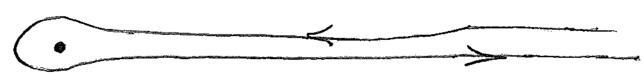
$$k(r,s) = \int_P e^{-r(t+t^{-1})/2}\, t^s\, \frac{dt}{t}$$

for different contours P. The contours have endpoints at either $t=0$ and $t=\infty$, or both these points. The contour integral converges or not depending on the argument of r. For example supposing $r>0$ if $t=\infty$ is an endpt of the contour we must approach $t=\infty$ so that $e^{-rt} \to 0$ i.e. such that $|\arg(t)| < \pi/2$. If $t=0$ is the other endpoint we must have $\arg(t) < \pi/2$ as $t \to 0$. But with these constraints we have several contours:



etc. What we get are linear combinations of $K_s(r)$ and the k function belonging to either of the contours

or



The conditions for convergence are

$$\left|\operatorname{Arg}(rt)\right| < \frac{\pi}{2} - \delta \qquad \text{as } t \to \infty$$

$$\left|\arg(rt^{-1})\right| < \frac{\pi}{2} - \delta \qquad \text{as } t \to 0$$

So if $\operatorname{Im}(r) > 0$ we want to use contours like:

Try analytic continuation: It's clear that as we move the argument of r from 0 to $\pi/2$ that the contour 

should be moved to

and then as $\arg(r)$ goes from $\pi/2$ to π we should move the contour to

similarly the contour

which gives a solution vanishing at $r = -\infty$ should as $\arg(r)$ goes from π to 0 be moved to

then

~~Consider the contour C and let's expand the size of the circle~~

$$\cancel{\int_0^{2\pi} e^{-r(Re^{i\theta}+R^{-1}e^{-i\theta})/2}(Re^{i\theta})^s\, i\,d\theta + (e^{2\pi i s}-1)\int_R^{\infty} e^{-r(t+t^{-1})/2}\, t^s \frac{dt}{t}}$$

So it clear that we get two solutions

$$K(r,s)=\int_0^\infty e^{-r(t+t^{-1})/2}\, t^s \frac{dt}{t} \qquad \text{defined for } \operatorname{Re}(r)>0$$

and

$$\int_{-\infty}^{0} e^{-r(t+t^{-1})/2}\, t^s \frac{dt}{t} \qquad \text{defined for } \operatorname{Re}(r)<0.$$

Notice that up to a factor $(-1)^{s-1}$ the second solution can be put in the form

$$(*) \qquad \int_0^\infty e^{r(t+t^{-1})/2}\, t^s \frac{dt}{t} \qquad \text{defined for } \operatorname{Re}(r)<0$$

So now what I want to do is compute the Wronskian of these two solutions (which maybe vanishes for no s because one knows these solutions are linearly independent). Notice that although we can analytically continue $(*)$ to $\operatorname{Re}(r)<0$ in at least two ways, they differ by a multiple of $K(r,s)$ so the Wronskian is well-defined.

From Courant-Hilbert

$$H^1_\lambda(z)=\frac{e^{-i\pi\lambda/2}}{\pi i}\int_{-\infty}^{\infty} e^{iz\cosh\eta-\lambda\eta}\, d\eta$$

Recall $z=ir$ so that $z=+i\infty \Longleftrightarrow r=+\infty$. Put $t=e^\eta$, $dt=e^\eta d\eta=t\,d\eta$

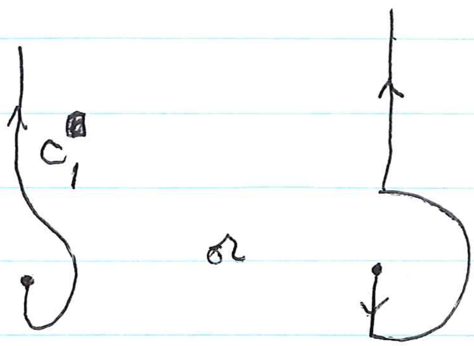
$$H^1_\lambda(ir)=\frac{e^{-i\pi\lambda/2}}{\pi i}\int_0^\infty e^{-r(t+t^{-1})/2}\, t^{-\lambda}\frac{dt}{t}$$

Finally use that the integral is symmetric in λ, so we get then that

$$K(r,s) = \frac{\pi i \, H^1_s(ir)}{e^{i\pi s/2}} = \pi i e^{-i\pi s/2} H^1_s(ir)$$

Thus $K(r,s)$ is essentially the function $H^1_s(ir)$. Now let us analytically continue $K(r,s)$ to $Re(z) > 0$ i.e. to $Im(r) < 0$ and we get

$$K(r,s) = \int_{C_1} e^{-r(t+t^{-1})/2} t^s \frac{dt}{t}$$



where C_1 is the contour:

Similarly the other solution of interest

$$\int_0^{-\infty} e^{-r(t+t^{-1})} t^s \frac{dt}{t}$$

analytically continues to $\int_{C_2}$ where C_2 :
so it must be essentially the same as $H^2_s(ir)$.

June 6, 1977

We know that
$$K(r,s) = \int_0^\infty e^{-r(t+t^{-1})/2}\, t^s \frac{dt}{t}$$

and $$K(-r,s) = \int_0^\infty e^{+r(t+t^{-1})/2}\, t^s \frac{dt}{t}$$ are solutions of

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 1 - \frac{s^2}{r^2}\right) u = 0$$

So I want to compute the Wronskian of these solutions which should be a function of s times $\frac{1}{r}$. The problem arises with the fact that the integral expressions above do not have a common region of convergence.

$$\begin{vmatrix} K(r,s) & K(-r,s) \\ \frac{d}{dr}K(r,s) & \frac{d}{dr}K(-r,s) \end{vmatrix} = \begin{vmatrix} K(r,s) & K(-r,s) \\ \frac{s}{r}K(r,s) - K(r,s+1) & \left(\frac{s}{-r}K(-r,s) - K(-r,s+1)\right)(-1) \end{vmatrix}$$

$$= \begin{vmatrix} K(r,s) & K(-r,s) \\ -K(r,s+1) & K(-r,s+1) \end{vmatrix} = K_s(r)K_{s+1}(-r) + K_s(-r)K_{s+1}(r)$$

Also $$\begin{vmatrix} K_s(r) & K_s(-r) \\ \frac{d}{dr}K_s(r) & \frac{d}{dr}K_s(-r) \end{vmatrix} = \begin{vmatrix} K_s(r) & K_s(-r) \\ -\frac{s}{r}K_s(r) - K_{s-1}(r) & \left(-\frac{s}{-r}K_s(r) - K_{s-1}(-r)\right)(-1) \end{vmatrix}$$

$$= \begin{vmatrix} K_s(r) & K_s(-r) \\ -K_{s-1}(r) & K_{s-1}(-r) \end{vmatrix} = K_s(r)K_{s-1}(-r) + K_s(-r)K_{s-1}(r)$$

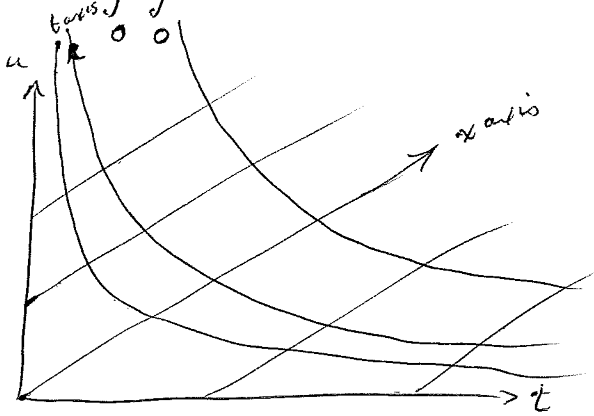
Hence we see that

$$s \longmapsto K_s(r)K_{s+1}(-r) + K_s(-r)K_{s+1}(r)$$

is periodic in s.

Let's proceed formally

$$K_s(r)K_{s+1}(-r) + K_s(-r)K_{s+1}(r) = \int_0^\infty\int_0^\infty e^{-r(t+t^{-1})/2 + r(u+u^{-1})/2}\left(t^{s-1}u^{s} + u^{s-1}t^{s}\right)\overbrace{dt\,du}$$



Put

$$x = tu$$
$$y = u - t$$

$$dx\,dy = (t\,du + u\,dt)(du - dt)$$
$$= (t+u)\,dt\,du$$

$$x^{s-1}dx\,dy = (t^{s-1}u^{s} + t^{s}u^{s-1})\,dt\,du$$

$$u^{-1} - t^{-1} = \frac{t-u}{tu} = -\frac{y}{x}$$

So the double integral becomes

$$\int_0^\infty dx \int_{-\infty}^\infty dy \; e^{+ry - r\frac{y}{x}}\, x^{s-1}\, dy$$

which formally is $\displaystyle \frac{1}{r}\int_0^\infty dx \int_{-\infty}^\infty e^{y-\frac{y}{x}}\, x^{s-1}\, dy$.

Suppose $r = ia$ with $a > 0$. Change x to $\frac{1}{x}$

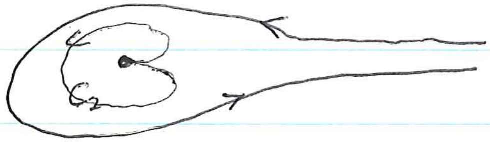
$$\int_0^\infty x^{-s}\frac{dx}{x}\int_{-\infty}^\infty e^{ia(1-x)y}\,dy = \int_0^\infty x^{-s}\frac{dx}{x}\;\frac{1}{a}\int_{-\infty}^\infty e^{i(1-x)y}\,dy$$

$$= \frac{2\pi}{a}\int_0^\infty x^{-s}\frac{dx}{x}\;\delta(1-x)$$

$$= \frac{2\pi}{a} = \frac{2\pi i}{r}$$

Compare $K_s(r)$ with the contour integral with contour C:

$$\int_C e^{-r(t+t^{-1})/2} t^s \frac{dt}{t} = \left(e^{2\pi i s}-1\right)\int_0^\infty + \int_{C_2'} e^{-r(t+t^{-1})/2} t^s \frac{dt}{t}$$



Now as t runs over C_2, t^{-1} runs over C backwards, so

$$\left(e^{2\pi i s}-1\right) K_s(r) = \int_C e^{-r(t+t^{-1})/2} t^s \frac{dt}{t} - \int_{C_2} e^{-r(t+t^{-1})/2} t^s \frac{dt}{t}$$

$$= \int_C e^{-r(t+t^{-1})/2} t^s \frac{dt}{t} - e^{2\pi i s}\int_C e^{-r(t+t^{-1})/2}\, t^{-s} \frac{dt}{t}$$

So far we've been assuming $r>0$. Put $t=\frac{u}{r}$ in the integrals:

$$\left(e^{2\pi i s}-1\right)K_s(r) = r^{-s}\int_C e^{-(u+r^2/u)/2} u^s \frac{du}{u} - e^{2\pi i s} r^s \int_C e^{-(u+r^2/u)/2} u^{-s}\frac{du}{u}$$

Observe that the integrals give entire functions of r^2 which we can expand in series if we want. I should have put $t=\frac{2u}{r}$:

$$\left(e^{2\pi i s}-1\right) K_s(r) = \left(\frac{r}{2}\right)^{-s}\int_C e^{-u-r^2/4u} u^s \frac{du}{u} - e^{2\pi i s}\left(\frac{r}{2}\right)^s \int_C e^{-u-r^2/4u} u^{-s}\frac{du}{u}$$

$$\int_C e^{-u-r^2/4u} u^s \frac{du}{u} = \int_C e^{-u} \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{-r^2}{4u}\right)^k u^s \frac{du}{u}$$

$$= \sum_{k=0}^\infty \frac{1}{k!}\left(-\frac{r^2}{4}\right)^k \int_C e^{-u} u^{s-k}\frac{du}{u}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{r^2}{4}\right)^k \Gamma(s-k)\left(e^{2\pi i s}-1\right)$$

Thus we get

$$K_s(r) = \left(\frac{r}{2}\right)^{-s}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{r^2}{4}\right)^k\Gamma(s-k) + \left(\frac{r}{2}\right)^{s}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{r^2}{4}\right)^k\Gamma(-s-k)$$

June 7, 1977:

$$\sum e^{-\pi n^2 t} e^{2\pi i n x} = \frac{1}{\sqrt{t}}\sum e^{-\pi(x-n)^2/t}$$

$$\int_0^\infty \sum_{n\neq 0} e^{-\pi n^2 t} e^{2\pi i n x} t^{s/2}\frac{dt}{t} = \sum_{n\neq 0} e^{2\pi i n x}\int_0^\infty e^{-\pi n^2 t} t^{s/2}\frac{dt}{t}$$

$$= \sum_{n\neq 0} e^{2\pi i n x}\,\Gamma(s/2)\left(\pi n^2\right)^{-s/2}$$

$$= \pi^{-s/2}\Gamma(s/2)\sum_{n\neq 0}|n|^{-s} e^{2\pi i n x}$$

This should analytically continue to

$$\int_0^\infty \frac{1}{\sqrt{t}}\sum_{n} e^{-\pi(x-n)^2/t}\, t^{s/2}\,\frac{dt}{t} = \int_0^\infty \sum_n e^{-\pi(x-n)^2 t}\, t^{(1-s)/2}\frac{dt}{t}$$

$$= \sum_{n\in\mathbb{Z}} \Gamma\left((1-s)/2\right)\left(\pi(x-n)^2\right)^{(s-1)/2}$$

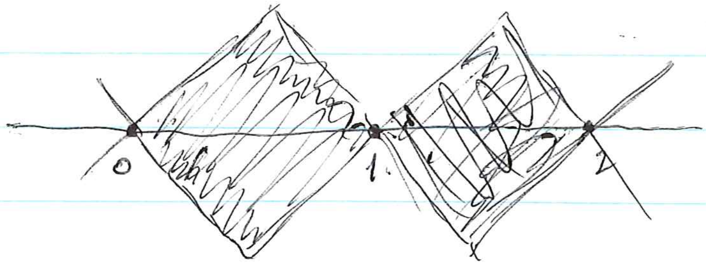
$$= \pi^{(s-1)/2}\,\Gamma\left((1-s)/2\right)\sum_{n\in\mathbb{Z}}\left[(x-n)^2\right]^{(s-1)/2}$$

The point to note is that if $\mathrm{Re}(x)\notin\mathbb{Z}$, then

$(x-n)^2 \notin \mathbb{R}_{\leq 0}$ so we can raise it to the exponent $(s-1)/2$. Actually one needs

$$\mathrm{Re}(x-n)^2 = (\mathrm{Re}(x)-n)^2 - (\mathrm{Im}(x))^2 > 0$$

for all n in order to make the above calculation.



So now to fill the symmetry out I need a Θ-function with two variables x, y. This means something like

$$\sum_n e^{-\pi(n+y)^2 t} e^{2\pi i n x} = \sum_n e^{-\pi n^2 t - 2\pi n y t - \pi y^2 t + 2\pi i n x}$$

$$= e^{-\pi y^2 t} \sum_n e^{-\pi n^2 t + 2\pi i n (x + i y t)}$$

$$= e^{-\pi y^2 t} \frac{1}{\sqrt{t}} \sum_n e^{-\frac{\pi}{t}(n + x + i y t)^2}$$

$$= \cancel{e^{-\pi y^2 t}} \frac{1}{\sqrt{t}} \sum_n e^{-\frac{\pi}{t}(n+x)^2 - \frac{2\pi}{t}(n+x) i y t + \cancel{\frac{\pi}{t} y^2 t^2}}$$

$$= \frac{1}{\sqrt{t}} \sum_n e^{-\frac{\pi}{t}(x+n)^2 - 2\pi i n y - 2\pi i x y}$$

$$\cancel{\sum_n e^{-\pi(y+n)^2 t} e^{2\pi i n x} = \frac{1}{\sqrt{t}} \sum_n e^{-\pi(x+n)^2/t} e^{-2\pi i y}}$$

$$\boxed{\sum_n e^{-\pi(y+n)^2 t} e^{2\pi i n x} = \frac{e^{-2\pi i x y}}{\sqrt{t}} \sum_n e^{-\pi(x+n)^2/t} e^{-2\pi i n y}}$$

$$\int_0^\infty \sum_n e^{-\pi(y+n)^2 t} e^{2\pi i n x}\, t^{s/2}\frac{dt}{t} = \sum_n \left((y+n)^2\right)^{-s/2} \pi^{-s/2}\Gamma(s/2) e^{2\pi i n x}$$

$$= \pi^{-s/2}\Gamma(s/2) \sum_n \left((y+n)^2\right)^{-s/2} e^{2\pi i n x}$$

here y is real say and not integral and

$$\left(\underset{\uparrow\ \text{positive}}{(y+n)^2}\right)^{-s/2} = |y+n|^{-s}$$

Thus we do get $|y+n|^{-s}$ as in Weil's book. On the other side one has

$$e^{-2\pi i x y} \int_0^\infty \frac{1}{\sqrt{t}} \sum_n e^{-\pi(x+n)^2/t} e^{-2\pi i n y}\, t^{s/2}\frac{dt}{t}$$

$$e^{-2\pi i x y} \int_0^\infty \sum e^{-\pi(x+n)^2 t} e^{-2\pi i n y}\, t^{1-s/2}\frac{dt}{t}$$

$$= e^{-2\pi i x y} \sum_n |x+n|^{(s-1)/2} e^{-2\pi i n y} \cdot \pi^{(s-1)/2}\Gamma\left((1-s/2)\right)$$

So the functional equation is

$$\pi^{-s/2}\Gamma(s/2) \sum_n{}' |y+n|^{-s} e^{+2\pi i n x} = e^{-2\pi i x y}\pi^{(s-1)/2}\Gamma\left((1-s)/2\right)\sum_n{}' |x+n|^{(s-1)/2} e^{-2\pi i n y}$$

where the prime means the 0 term is to be dropped if x or y is integral.

June 8, 1977:

New idea is that the important place to look is for generalizations of the Legendre duplication formula. The idea is that:

$$\sqrt{\pi}\, 2^{1-s}\, \Gamma(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{(s+1)}{2}\right)$$

replaced by $\zeta(2s)$ or something like the Gaussian integers ζ-function.

replace these by some sort of ζ-like functions of $x, -x$

Thus let's review the ζ, L function for $A = \mathbb{Z}[i]$. Since A is a PID with units $\pm 1, \pm i$ one has

$$\zeta_A(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-1} = \frac{1}{4}\sum_{(m,n)\neq 0} (m^2+n^2)^{-s}$$

$$\pi^{-s}\Gamma(s)\,\zeta_A(s) = \sum_{\substack{(m,n)\\ \neq 0}} \int_0^\infty e^{-\pi(m^2+n^2)t}\, t^s \frac{dt}{t} = \int_0^\infty \left[\theta(t)^2 - 1\right] t^s \frac{dt}{t}$$

Now $\zeta_A(s) = \zeta(s) L(s)$ where

$$L(s) = \prod_{\substack{p \text{ odd}\\ \text{prime}}} \left(1 - \left(\frac{-1}{p}\right)p^{-s}\right)^{-1}$$

$$\left(\frac{-1}{n}\right) = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd}\end{cases}$$

$$= \sum_{n\geq 1}' \left(\frac{-1}{n}\right) n^{-s} = \sum_{\substack{n \text{ odd}\\ \geq 1}}' (-1)^{\frac{n-1}{2}} n^{-s}$$

$$L(s) = \sum_{m \geq 0} (-1)^m (2m+1)^{-s}$$

From the functional equations satisfied by ζ, ζ_A we know

$$\frac{\pi^{-s}\,\Gamma(s)\,\zeta_A(s)}{\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)} = \pi^{-s/2}\,\frac{2^{s-1}}{\sqrt{\pi}}\,\Gamma\!\left(\frac{s+1}{2}\right) L(s)$$

is symmetric under $s \longmapsto 1-s$.

$$\Gamma\!\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}\,t^{s/2}\,\frac{dt}{t} = 2\int_0^\infty e^{-t^2}\,t^s\,\frac{dt}{t}$$

$$\pi^{-s/2}\,\Gamma\!\left(s/2\right) = 2\int_0^\infty e^{-\pi t^2}\,t^s\,\frac{dt}{t}$$

$$2^{s-1}\,\pi^{-(s+1)/2}\,\Gamma\!\left(\frac{s+1}{2}\right) L(s) = \sum_{n\geq 1}\left(\frac{-1}{n}\right) n^{-s}\,2^s \int_0^\infty e^{-\pi t^2}\,t^{s+1}\,\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty \sum_{n\geq 1}\left(\frac{-1}{n}\right) n\,e^{-\pi t^2}\left(\frac{2t}{n}\right)^{s+1}\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty \underbrace{\left[\sum_{n\geq 1}\left(\frac{-1}{n}\right) e^{-\pi n^2 t^2/4}\,(nt)\right]}_{\rho(t)} t^s\,\frac{dt}{t}$$

June 9, 1977

$$\cancel{\int_0^\infty e^{-\pi m^2 t^2 + 2\pi i m t x}\, t^{s-1}\, dt \int_0^\infty e^{-\pi n^2 u^2 + 2\pi i n u y}\, dny}$$

I want ~~to~~ see what be done with the proof of the duplication formulas. Start with typical quadratic integrand:

$$e^{-a^2t^2 + bt}$$

Then in a typical Wronskian we will end up with

$$e^{-a^2t^2 + bt - \tilde{a}^2u^2 + \tilde{b}u}\,(tu)^{s-1}(a'u + b't)\,dt\,du$$

If we try the same substitution

$$\alpha = tu$$

$$\beta = c_1 u - c_2 t,$$

then

$$d\alpha\, d\beta = (t\,du + u\,dt)(c_1\,du - c_2\,dt) = (c_1 u + c_2 t)\,dt\,du$$

so we want $\frac{c_1}{c_2} = \frac{a'}{b'}$. Actually by rescaling t, u we can arrange that $a' = b' = 1$. So suppose that $a' = b' = 1$, and take $c_1 = c_2 = 1$.

Unfortunately it seems that I have to have $a = \tilde{a}$ and $\tilde{b} = -b$.