May 27, 1977

Still need to understand spectral measure.

To fix the ideas consider $u'' + (\lambda - q)u = 0$ on $0 \le x \le b$ with real boundary conditions given at both ends, say $u(0) = u(b) = 0$ to fix the ideas. One then gets a self-adjoint extension $\tilde{L}$ of $L = -\frac{d^2}{dx^2} + q$ on $L^2((0,b), dx)$. The eigenvalues λ are simple, let them be $\lambda_1 < \lambda_2 < \cdots$ and let u_{λ_j} be a normalized eigenfunction. Then

$$\tilde{L} \text{ belongs to the kernel } \sum \lambda_j \, u_{\lambda_j}(x) \, \bar{u}_{\lambda_j}(y)$$

so if

$$\tilde{L} = \int \lambda \, dE_\lambda$$

as in the spectral thm,

$$E_\lambda \Leftrightarrow \sum_{\lambda_j \le \lambda} u_{\lambda_j}(x) \, \bar{u}_{\lambda_j}(y) .$$

The Green's operator or resolvent of $\tilde{L}$ is

$$G_\lambda = (\lambda - L)^{-1} \Leftrightarrow \sum_j \frac{u_{\lambda_j}(x) \, \bar{u}_{\lambda_j}(y)}{\lambda - \lambda_j}$$

Let $\varphi_\lambda(x) = \varphi(x, \lambda)$ denote a non-zero solution of the DE satisfying the 0-boundary condition selected in some way as to be holomorphic in λ. Then λ is an eigenvalue $\Leftrightarrow$ φ_λ satisfies the b-boundary condition. Therefore we can take

$$u_{\lambda_j}(x) = \frac{\varphi_{\lambda_j}(x)}{\|\varphi_{\lambda_j}\|}$$

The expansion formula become

$$f = \sum_j u_{\lambda_j}(f, u_{\lambda_j}) = \sum \varphi_{\lambda_j}(f, \varphi_{\lambda_j}) \frac{1}{\|\varphi_{\lambda_j}\|^2}$$

or $$f(x) = \int \varphi_\lambda(x)\,(f, \varphi_\lambda)\, d\mu(\lambda)$$

where $$d\mu(\lambda) = \sum \frac{\delta(\lambda - \lambda_j)}{\|\varphi_{\lambda_j}\|^2}.$$ We get a Hilbert space isom:

$$L^2((0,b), dx) \xrightarrow{\sim} L^2(\mathbb{R}, d\mu)$$

$$f \longmapsto (\lambda \mapsto (f, \varphi_\lambda))$$

$$\left(x \mapsto \int g(\lambda)\varphi_\lambda(x)\,d\mu(\lambda)\right) \longleftarrow g(\lambda)$$

in which $\tilde{L}$ corresponds to multiplication by λ. Such an isomorphism is determined by a cyclic vector for $\tilde{L}$, namely the f corresponding to $g = 1$. So we get the cyclic vector

$$v = \int \varphi_\lambda(x)\,d\mu(\lambda) = \sum_j \varphi_{\lambda_j}(x) \frac{1}{\|\varphi_{\lambda_j}\|^2}$$

Since

$$G_\lambda \longleftrightarrow G_\lambda(x,y) = \int \frac{\varphi_\alpha(x)\overline{\varphi_\alpha(y)}}{\lambda - \alpha}\, d\mu(\alpha)$$

one has

$$G_\lambda v = \int G_\lambda \varphi_\alpha\, d\mu(\alpha) = \int \frac{\varphi_\alpha}{\lambda - \alpha}\, d\mu(\alpha)$$

so $$(G_\lambda v, v) = \left(\sum \frac{\varphi_{\lambda_j}}{\lambda - \lambda_j} \frac{1}{\|\varphi_{\lambda_j}\|^2}, \sum \frac{\varphi_{\lambda_j}}{\|\varphi_{\lambda_j}\|^2} \right)$$

$$= \sum \frac{1}{\lambda - \lambda_j} \frac{1}{\|\varphi_{\lambda_j}\|^2}$$

$$\therefore \quad \boxed{(G_\lambda v, v) = \int \frac{d\mu(\alpha)}{\lambda - \alpha} \qquad v = \int \varphi_\alpha \, d\mu(\alpha)}$$

Next suppose $\psi(x, \lambda)$ is a linearly independent solution to $\varphi(x, \lambda)$ chosen so that $\begin{vmatrix} \varphi & \psi \\ \varphi' & \psi' \end{vmatrix} = -1$.

For example, if $\begin{array}{l} \varphi(0, \lambda) = 0 \\ \varphi'(0, \lambda) = 1 \end{array}$ take $\begin{array}{l} \psi(0, \lambda) = 1 \\ \psi'(0, \lambda) = -1. \end{array}$

Let $m(\lambda)$ be such that

$$u(x, \lambda) = m(\lambda)\varphi(x, \lambda) + \psi(x, \lambda)$$

satisfies the b-boundary condition. This defines $m(\lambda)$ for $\lambda \neq \lambda_j$ and $m(\lambda_j) = \infty$. Compute the G-function

$$G_\lambda(x, y) = \begin{cases} a\, \varphi_\lambda(x) & x \le y \\ b\, u(x, \lambda) & x \ge y \end{cases}$$

$$a\varphi_\lambda(y) - b\, u_\lambda(y) = 0$$

$$a\varphi_\lambda'(y) - b\, u_\lambda'(y) = 1$$

$$a = \frac{\begin{vmatrix} 0 & -m\varphi - \psi \\ 1 & -m\varphi' - \psi' \end{vmatrix}}{\begin{vmatrix} \varphi & -m\varphi - \psi \\ \varphi' & -m\varphi' - \psi' \end{vmatrix}}(y) = (m\varphi + \psi)(y) = u_\lambda(y)$$

$b = \varphi(y)$. Thus

$$G_\lambda(x,y) = \begin{cases} \varphi_\lambda(x)\, u_\lambda(y) & x \le y \\ \varphi_\lambda(y)\, u_\lambda(x) & x \ge y \end{cases}$$

So

$$\varphi_\lambda(x)\big(m(\lambda)\,\varphi_\lambda(y) + \psi_\lambda(y)\big) = \sum_j \frac{\varphi_{\lambda_j}(x)\,\overline{\varphi_{\lambda_j}}(y)}{\lambda-\lambda_j}\,\frac{1}{\|\varphi_{\lambda_j}\|^2}$$

whence

$$\operatorname*{res}_{\lambda=\lambda_j}\big(m(\lambda)\big)\cdot\varphi_{\lambda_j}(x)\,\varphi_{\lambda_j}(y) = \varphi_{\lambda_j}(x)\,\overline{\varphi_{\lambda_j}}(y)\,\frac{1}{\|\varphi_{\lambda_j}\|^2}$$

So at least if $\overline{\varphi_\lambda} = \varphi_\lambda$ for λ real, one has

$$\operatorname*{res}_{\lambda=\lambda_j} m(\lambda) = \frac{1}{\|\varphi_{\lambda_j}\|^2}$$

hence

$$m(\lambda) = \int \frac{d\mu(\alpha)}{\lambda-\alpha} = (G_\lambda v, v)$$

<u>Example</u>:
$$u'' + \lambda u = 0 \qquad \text{on} \quad 0 \le x \le b$$
$$u(0) = 0 \qquad u'(b) = k\,u(b) \qquad k \in \mathbb{R}$$

$$u = \sin(\sqrt{\lambda}\,x)$$

$$u'(b) = \sqrt{\lambda}\cos(\sqrt{\lambda}\,b) = k\sin(\sqrt{\lambda}\,b)$$

λ eigenvalue $\Longleftrightarrow$ $\dfrac{\sqrt{\lambda}}{k} = \tan(\sqrt{\lambda}\,b)$ and $\lambda \neq 0$

For large λ the $\tan(\sqrt{\lambda}\,b)$ has to be large so $\sqrt{\lambda}\,b$ will be slightly less $(k>0)$ that $(n+\frac{1}{2})\pi$. Thus we have

$$\sqrt{\lambda_n} \sim \left(n+\tfrac{1}{2}\right)\frac{\pi}{b}$$

Notice also $\quad \dfrac{\tan x}{x} = \dfrac{\sin x}{x}\cos x = \left(1-\dfrac{x^2}{6}\right)\left(1-\dfrac{x^2}{2}\right) = 1-\dfrac{2x^2}{3} + O(x^4)$

hence if $b=1$ and $\dfrac{1}{k} = 1+\varepsilon$ $\quad \varepsilon$ small >0, then

$$1+\varepsilon = \frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}} = 1 - \frac{2\lambda}{3} + O(\lambda^2)$$

forces λ to be slightly negative, hence with these boundary conditions there are non-real values for $\sqrt{\lambda}$

<u>March 28, 1977:</u> Whittaker's function $W_{k,m}$ is a solution of the (so-called) confluent hypergeometric DE.

$$(1) \qquad \frac{d^2W}{dz^2} + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m^2}{z^2}\right\}W = 0 .$$

Put $\quad W = z^{1/2}u. \qquad W'' = \left(\tfrac{1}{2}z^{-1/2}u + z^{1/2}u'\right)'$

$$= -\tfrac{1}{4}z^{-3/2}u + 2\cdot\tfrac{1}{2}z^{-1/2}u' + z^{1/2}u''.$$

$$\cancel{z^{1/2}}\left(u'' + \frac{1}{z}u' - \cancel{\frac{1}{4}z^{-2}}\right) + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{\cancel{\frac{1}{4}}-m^2}{z^2}\right\}\cancel{z^{1/2}}u = 0$$

or

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left\{-\frac{1}{4} + \frac{k}{z} - \frac{m^2}{z^2}\right\}u = 0$$

or

$$\left(z\frac{d}{dz}\right)^2 u + \left\{-\frac{1}{4}z^2 + kz - m^2\right\}u = 0$$

Changing $z \mapsto 2z$ this becomes

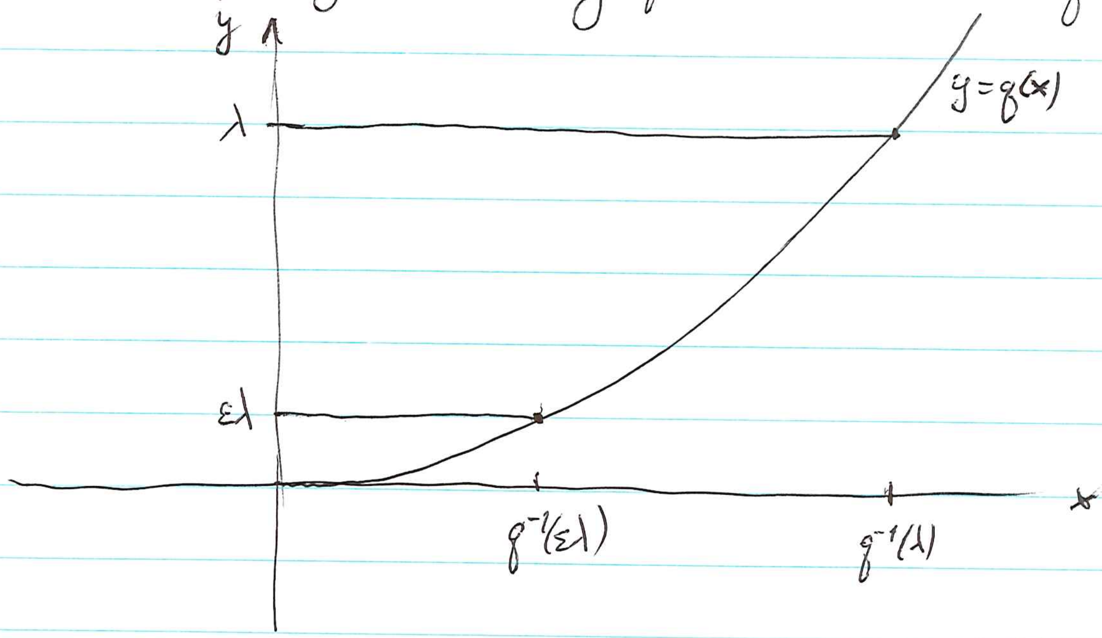
$$\left(z\frac{d}{dz}\right)^2 u + \{-z^2 + 2kz - m^2\}u = 0$$

which for $k=0$ is Bessel's DE with the imaginary argument. Thus

$$W = z^{1/2} J_m\left(\frac{iz}{2}\right) \quad \text{satisfies Whittaker's DE with } k=0.$$

<u>May 29, 1977</u>: Distribution of eigenvalues:

Start by trying to prove $\int_0^{g^{-1}(\lambda)} \sqrt{\lambda - g}\, dx \sim \sqrt{\lambda}\, g^{-1}(\lambda)$, if g is a rapidly increasing function with $g(0)=0$.



We have

$$\sqrt{\lambda}\, g^{-1}(\lambda) \geq \int_0^{g^{-1}(\lambda)} \sqrt{\lambda - g}\, dx \geq \int_0^{g^{-1}(\varepsilon\lambda)} \sqrt{\lambda - \varepsilon\lambda}\, dx = \sqrt{\lambda}\sqrt{1-\varepsilon}\, g^{-1}(\varepsilon\lambda)$$

$$1 \geq \frac{1}{\sqrt{\lambda}\, g^{-1}(\lambda)} \int_0^{g^{-1}(\lambda)} \sqrt{\lambda - g}\, dx \geq \sqrt{1-\varepsilon}\, \frac{g^{-1}(\varepsilon\lambda)}{g^{-1}(\lambda)}$$

Now when might it be true that

$$\lim_{\lambda\to\infty} \frac{g^{-1}(\varepsilon\lambda)}{g^{-1}(\lambda)} = 1 \quad ?$$

$\lambda = g(x) = x^n.$ $\qquad g^{-1}(\lambda) = \lambda^{1/n}$ $\quad$ so

$$\frac{g^{-1}(\varepsilon\lambda)}{g^{-1}(\lambda)} = \varepsilon^{1/n} \qquad \text{NO}$$

$\lambda = g(x) = e^x$ $\qquad g^{-1}(\lambda) = \log\lambda$

$$\frac{g^{-1}(\varepsilon\lambda)}{g^{-1}(\lambda)} = \frac{\log(\varepsilon\lambda)}{\log\lambda} = 1 + \frac{\log\varepsilon}{\log\lambda} \longrightarrow 1 \qquad \text{YES.}$$

Exact answers in these cases:

$y = g(x) = x^n$

$x = y^{1/n}$

$dx = \frac{1}{n} y^{1/n-1}\,dy$

$$\int_0^{\lambda^{1/n}} \sqrt{\lambda - x^n}\,dx = \int_0^{\lambda} \sqrt{\lambda - y}\;\frac{1}{n}\,y^{1/n-1}\,dy$$

$y = \lambda z$

$$= \int_0^1 \lambda^{1/2}(1-z)^{1/2}\,\frac{1}{n}\,\lambda^{1/n-1} z^{1/n-1}\,\lambda\,dz$$

$$= \frac{\lambda^{1/2+1/n}}{n}\int_0^1 (1-z)^{1/2} z^{1/n-1}\,dz = \frac{\lambda^{1/2+1/n}}{n}\;\frac{\Gamma(3/2)\,\Gamma(1/n)}{\Gamma(3/2+1/n)}$$

$$= \frac{\lambda^{1/2+1/n}}{n}\;\frac{\frac{1}{2}\sqrt{\pi}\;\Gamma(1/n)}{\left(\frac{1}{2}+\frac{1}{n}\right)\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

Take $y = e^{2x}$. Crude estimate for $\int_0^{\log(\lambda^{1/2})} \sqrt{\lambda - e^{2x}}\,dx$ is

$\lambda^{1/2}\log(\lambda^{1/2})$

Exact answer is
(see p. 84 April 9)

$$\lambda^{1/2}\log\left(\lambda^{1/2} + \sqrt{\lambda-1}\right) - \lambda^{1/2} = \lambda^{1/2}\log(2\lambda^{1/2}) - \lambda^{1/2} + O(\lambda^{-1/2})$$

$$= \lambda^{1/2}\log(\lambda^{1/2}) + (\log 2 - 1)\lambda^{1/2} + O(\lambda^{-1/2}).$$

May 30, 1977

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & x \\ x & -i\lambda \end{pmatrix} u \qquad \tilde{u} = \begin{pmatrix} u_1 - u_2 \\ iu_1 + iu_2 \end{pmatrix}$$

$$\frac{d\tilde{u}}{dx} = \begin{pmatrix} i\lambda u_1 + xu_2 - (xu_1 - i\lambda u_2) \\ i(i\lambda u_1 + xu_2) + i(xu_1 - i\lambda u_2) \end{pmatrix} = \begin{pmatrix} -x & \lambda \\ -\lambda & x \end{pmatrix} \tilde{u}$$

$$\left(\frac{d}{dx} + x\right)\tilde{u}_1 = \lambda \tilde{u}_2 \qquad \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right)\tilde{u}_1 = -\lambda^2 \tilde{u}_1$$

$$\left(\frac{d}{dx} - x\right)\tilde{u}_2 = -\lambda \tilde{u}_1 \qquad \left[\frac{d^2}{dx^2} + (\lambda^2 + 1 - x^2)\right]\tilde{u}_1 = 0$$

$$\tilde{u}_1 = e^{-x^2/2} v$$

$$\left(e^{-x^2/2} v\right)'' = \left(e^{-x^2/2}(-xv + v')\right)' = e^{-x^2/2}\left(x^2 v - xv' - v - xv' + v''\right)$$

$$e^{-x^2/2}\left[\left(x^2 v - v - 2xv' + v''\right) + (\lambda^2 + 1 - x^2)v\right] = 0$$

$$v'' - 2xv' + \lambda^2 v = 0 \qquad v = \int e^{tx}\phi(t)\,dt$$

$$t^2\phi - 2\left(-\frac{d}{dt}\right)(t\phi) + \lambda^2\phi = 0 \qquad v' = \int e^{tx}(t\phi(t))\,dt$$

$$\left[t^2 + (\lambda^2 + 2)\right]\phi + 2t\phi' = 0 \qquad xv' = \int x e^{tx}(t\phi)\,dt$$

$$= -\int e^{tx}(t\phi)'\,dt$$

$$\frac{\phi'}{\phi} = -\frac{t^2 + (\lambda^2 + 2)}{2t} = -\frac{t}{2} - \left(\frac{\lambda^2}{2} + 1\right)\frac{1}{t}$$

$$\log\phi = -\frac{t^2}{4} - \left(\frac{\lambda^2}{2} + 1\right)\log t$$

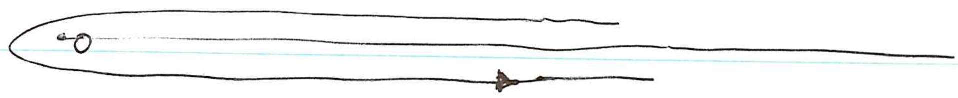
$$\phi = e^{-t^2/4}\, t^{-\left(\frac{\lambda^2}{2} + 1\right)}$$

$$v = \int_C e^{tx} e^{-t^2/4}\, t^{-\lambda^2/2}\, \frac{dt}{t}$$

can change $x \mapsto -x$ without changing the DE so we get the solution

$$V = \int_C e^{-t^2/4 - xt}\, t^{-\frac{\lambda^2}{2}} \frac{dt}{t}$$

where C is



This solution decays at $x \to +\infty$. It vanishes for $\frac{\lambda^2}{2}$ an integer ≤ 0. If $\frac{\lambda^2}{2}$ is a integer ≥ 0, then V is a polynomial in x.

Simplify by putting $2t$ in for t and dropping $2^{-\lambda^2/2}$

$$\tilde{u}_1(x,\lambda) = e^{-x^2/2} \int_C e^{-t^2 - 2xt}\, t^{-\frac{\lambda^2}{2}} \frac{dt}{t}$$

$$\tilde{u}_1(0,\lambda) = \int_C e^{-t^2} t^{-\lambda^2/2} \frac{dt}{t} = \left(e^{-\pi i \lambda^2} - 1\right) \int_0^\infty e^{-t} t^{-\lambda^2/4} \frac{dt}{2t}$$

$$= \frac{1}{2}\left(e^{-\pi i \lambda^2} - 1\right) \Gamma\left(-\frac{\lambda^2}{4}\right).$$

$$\tilde{u}_2 = \frac{1}{\lambda}\left(\frac{d}{dx} + x\right)\tilde{u}_1 = -\frac{2}{\lambda} e^{-x^2/2} \int_C e^{-t^2 - 2xt}\, t^{-\frac{\lambda^2}{2}+1} \frac{dt}{t} \qquad \lambda \neq 0$$

$$\tilde{u}_2(0,\lambda) = \frac{1}{2}\left(-\frac{2}{\lambda}\right)\left(e^{-\pi i \lambda^2} - 1\right) \Gamma\left(-\frac{\lambda^2}{4} + \frac{1}{2}\right)$$

Now $\quad \Gamma\left(-\frac{\lambda^2}{4}\right)$ has simple poles at $\quad \frac{\lambda^2}{2} = 0, 2, 4, \ldots$

$\quad \Gamma\left(-\frac{\lambda^2}{4} + \frac{1}{2}\right)$ has simple poles at $\quad \frac{\lambda^2}{2} = 1, 3, 5, \ldots\ldots$

Hence $\tilde{u}_1, \tilde{u}_2$ vanish identically when $\frac{\lambda^2}{2} = -1, -2, -3, \ldots$ so you want to multiply by

$$\Gamma\left(\frac{\lambda^2}{2} + 1\right)$$

UGH.

$$\Gamma(s)\zeta(s) = \sum_1^\infty n^{-s}\int_0^\infty e^{-t}\, t^s \frac{dt}{t} = \sum_1^\infty \int_0^\infty e^{-nt}\, t^s\frac{dt}{t}$$

$$= \int_0^\infty \frac{e^{-t}}{1-e^{-t}}\, t^s \frac{dt}{t} = \int_0^\infty \frac{1}{e^t-1}\, t^s\frac{dt}{t}$$

all abs. convergence for $Re(s) > 1$. Hence

$$(e^{2\pi i s}-1)\Gamma(s)\zeta(s) = \int_c \frac{1}{e^t-1}\, t^s\frac{dt}{t}$$

and this holds for all s. Observe that the integral is an entire function of s. $(e^{2\pi i s}-1)\Gamma(s)$ is entire with simple zeroes at $s=1,2,\dots$, hence we see that $\zeta(s)$ has at most a simple pole at $s=1$, since the contour integral vanishes for $s=2,3,\dots$. In fact the contour integral for $s=0$ has value $2\pi i$, so ζ has residue 1 at $s=1$.

Can write the above as

$$\zeta(s) = \frac{\frac{1}{2\pi i}\int_c \frac{1}{e^t-1}\, t^s\frac{dt}{t}}{\frac{1}{2\pi i}\int_c e^{-t}\, t^s\frac{dt}{t}}$$

If $s=-n$, then $\frac{1}{2\pi i}\int_c e^{-t}t^{-n}\frac{dt}{t} = \frac{(-1)^n}{n!}$. Since

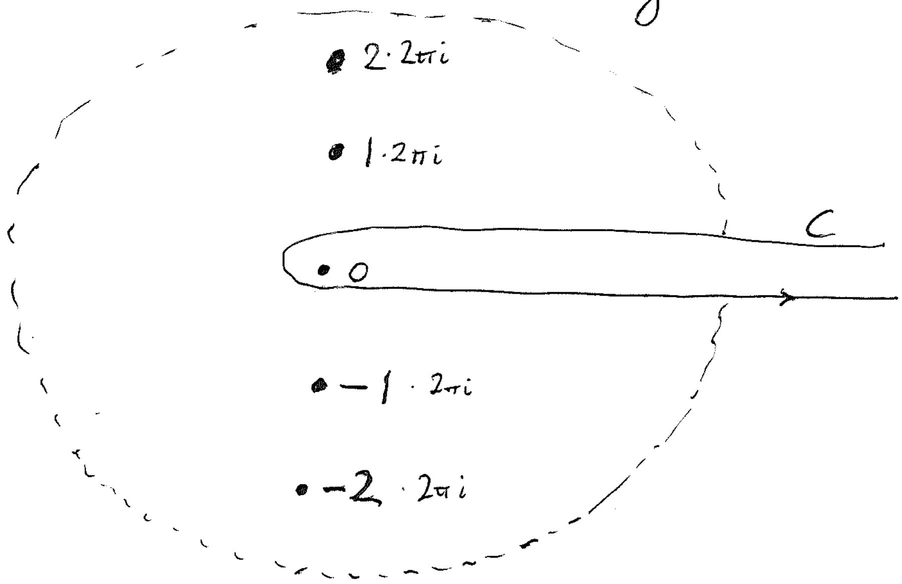
$$\frac{t}{e^t-1} = 1 + \sum_{m=1}^\infty \frac{B_m}{m!}t^m$$

$$\frac{1}{2\pi i}\int_c \frac{t}{e^t-1}\, t^{-n-1}\frac{dt}{t} = \frac{B_{n+1}}{(n+1)!}$$

so
$$\boxed{\zeta(-n) = \frac{(-1)^n B_{n+1}}{n+1}}$$

which agrees with $\zeta(-2) = \zeta(-4) = \cdots = 0$ and $B_3 = B_5 = \cdots 0$

Now use contour integration:



If $\mathrm{Re}(s) < 0$, then the integral over the dotted line should vanish in the limit as $\frac{1}{e^t - 1}$ is bounded horizontally and periodic vertically. Hence by residues

$$\int_C \frac{1}{e^t - 1} t^{s-1}\,dt = -2\pi i \sum_{n=1}^{\infty} (2\pi i n)^{s-1} - 2\pi i \sum_{n=1}^{\infty} (-2\pi i n)^{s-1}$$

$$= (-2\pi i)\left[(2\pi i)^{s-1} + (-2\pi i)^{s-1}\right] \zeta(1-s)$$

$$= \left[-(2\pi i)^s + (-2\pi i)^s\right] \zeta(1-s)$$

$$= (2\pi)^s \left(e^{-i\frac{\pi}{2}s} - e^{i\frac{\pi}{2}s}\right) \zeta(1-s)$$

NO

I should be more careful to use the right branch of t^s. $(2\pi i n)^{s-1}$ should be $(2\pi n)^{s-1} e^{i\frac{\pi}{2}(s-1)}$

$(-2\pi i n)^{s-1}$ ——— $(2\pi n)^{s-1} e^{i\frac{3\pi}{2}(s-1)}$

So it should be

$$\int_c \frac{1}{e^t - 1} t^{s-1}\,dt = (2\pi)^s \left(e^{i\frac{3\pi}{2}s} - e^{i\frac{\pi}{2}s}\right)\zeta(1-s)$$

$$\shortparallel$$

$$\left(e^{2\pi i s} - 1\right)\Gamma(s)\zeta(s).$$

Thus $$\frac{\zeta(1-s)}{\zeta(s)} = \frac{\cancel{e^{\pi i s}}\left(e^{\pi i s} - e^{-\pi i s}\right)/2i\,\Gamma(s)}{(2\pi)^s\,\cancel{e^{i\pi s}}\left(e^{i\pi\frac{s}{2}} - e^{-i\pi s/2}\right)/2i} = \frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})\,\cancel{\Gamma(s)}}{(2\pi)^s\,\cancel{\Gamma(s)}\,\Gamma(1-s)}$$

$$= \frac{\Gamma(s/2)\,\cancel{\Gamma(1-s/2)}}{(2\pi)^s\,\pi^{-1/2}\,\Gamma\left(\frac{1-s}{2}\right)\cancel{\Gamma\left(\frac{1-s}{2}+\frac{1}{2}\right)}\,\cancel{2}^{1-s-1}}$$

$$= \frac{\pi^{-s/2}\,\Gamma(s/2)}{\pi^{s-1/2}\,\Gamma(1-s/2)}$$

which is the <u>functional equation</u>.

Curiosity: $\int_c t^s \frac{dt}{t}$ is convergent for $Re(s)<0$

Compute it:

$$\int_0^{2\pi} e^{i\theta s}\,\frac{\cancel{e^{i\theta}}\,i\,d\theta}{\cancel{e^{i\theta}}} = i\,\frac{e^{i\theta s}}{is}\Bigg]_0^{2\pi} = \frac{e^{2\pi i s}-1}{s}$$

$$\left(e^{2\pi i s}-1\right)\int_1^\infty t^s\frac{dt}{t} = \left(e^{2\pi i s}-1\right)\frac{t^s}{s}\Bigg]_1^\infty = -\frac{e^{2\pi i s}-1}{s}$$

Thus $$\boxed{\int_c t^s \frac{dt}{t} = 0}$$

May 31, 1977

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u \qquad \tilde{u} = \begin{pmatrix} u_1 - u_2 \\ iu_1 + iu_2 \end{pmatrix}$$ is a good substitution when p is real

$$\tilde{u} = \begin{pmatrix} 1 & -1 \\ i & +i \end{pmatrix} u$$

$$\overbrace{\begin{vmatrix} 1 & -1 \\ i & +i \end{vmatrix}}^{2i} \cdot \begin{vmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{vmatrix} = \begin{vmatrix} \tilde{u}_1^+ & \tilde{u}_1^- \\ \tilde{u}_2^+ & \tilde{u}_2^- \end{vmatrix}$$

Consider
$$\sum_{n=1}^{\infty} (x+n)^{-s} = H(x,s)$$

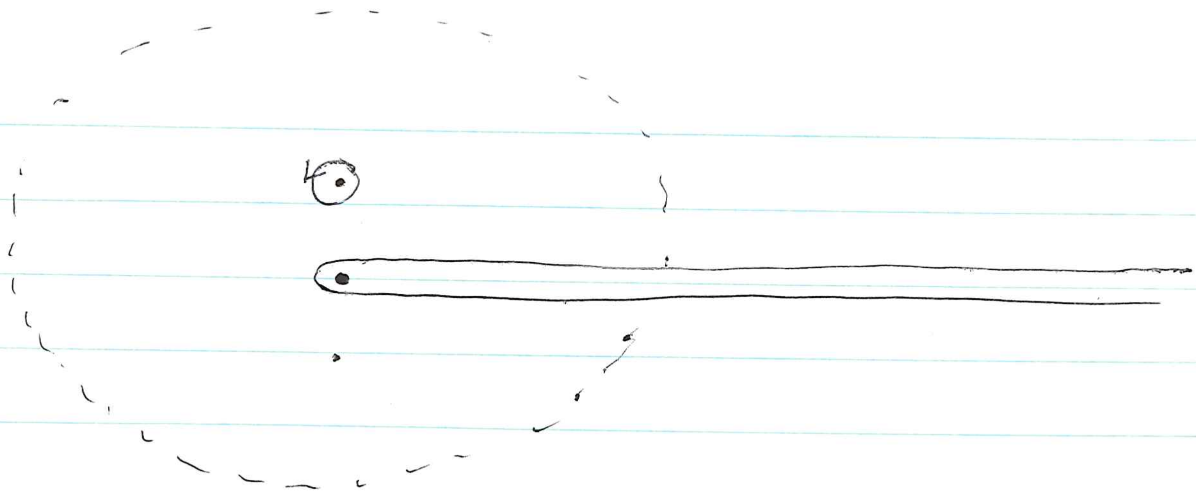
If $Re(s) > 1$ this converges and defines an analytic function of x for $x \neq -1, -2, \ldots$ which is single-valued provided x is [redacted] off $x \leq -1$. If $Re(x+1) > 0$, one has
$$\sum_1^{\infty} (x+n)^{-s} \Gamma(s) = \sum_1^{\infty} \text{[redacted]} \int_0^{\infty} e^{-(x+n)t} t^s \frac{dt}{t}$$

$$= \int_0^{\infty} \frac{e^{-xt}}{e^t - 1} t^s \frac{dt}{t}$$

So
$$\sum_1^{\infty} (x+n)^{-s} \Gamma(s) \left(e^{2\pi i s} - 1\right) = \int_c \frac{e^{-xt}}{e^t - 1} t^s \frac{dt}{t}$$

This shows that for x fixed with $Re(x) > -1$, $H(x,s)$ is a meromorphic function of s having only a simple pole at $s=1$ with residue 1. It also allows one to determine [redacted] $H(x,s)$ when s is [redacted] $0, -1, -2, -3, \ldots$ as some kind of Bernoulli polynomials

Try the contour integration:



Suppose $\mathrm{Re}(s)<0$. If $0<x<1$ then it seems that

$$\frac{e^{-xt}}{e^t-1}$$

is bounded on the circle$_n$ independent of radius provided the circle has radius $m+\frac{1}{2}$ so as to miss the zeroes of the denominator. So it should be legitimate to do a residue calculation.

$$-\int_C \frac{e^{-xt}}{e^t-1}t^s\frac{dt}{t} = \sum_{n=1}^{\infty} e^{-2\pi i n x}\left(2\pi e^{i\pi/2}\right)^s n^{s-1}$$

$$+\sum_{n=1}^{\infty} e^{2\pi i n x}(2\pi i)\left(2\pi e^{i\frac{3\pi}{2}}\right)^{s-1} n^{s-1}$$

$$=(2\pi)^s\left\{e^{i\frac{\pi}{2}s}\sum_1^\infty \frac{e^{-2\pi i n x}}{n^{1-s}} - e^{i\frac{3\pi}{2}s}\sum_1^\infty\frac{e^{2\pi i n x}}{n^{1-s}}\right\}$$

$$\int_C \frac{e^{-xt}}{e^t-1}t^s\frac{dt}{t} = (2\pi)^s e^{i\pi s}\left\{e^{i\frac{\pi}{2}s}\sum_1^\infty\frac{e^{2\pi i n x}}{n^{1-s}} - e^{-\frac{i\pi s}{2}}\sum_1^\infty\frac{e^{-2\pi i n x}}{n^{1-s}}\right\}$$

It seems clear that we want to consider the function

$$\sum_{n=1}^{\infty}(x+n)^{-s}e^{2\pi i n y}$$

which makes sense for $Im(y) \geq 0$, $Re(s) > 1$; in fact if $Im(y) > 0$ it makes sense for all s.

Then

$$\sum_{n=1}^{\infty} (x+n)^{-s} e^{2\pi i n y}\, \Gamma(s) = \sum_{1}^{\infty} \int_0^{\infty} e^{-xt - nt + n(2\pi i y)}\, t^s \frac{dt}{t}$$

$$= \int_0^{\infty} \frac{e^{-xt}}{e^{t-2\pi i y} - 1}\, t^s \frac{dt}{t}$$

~~Assuming~~ [redacted] ~~$-1 < x < 0$ residue evaluation should be possible as before~~

So

$$\left(\sum_{n=1}^{\infty} (x+n)^{-s} e^{2\pi i n y} \right) \Gamma(s) \left(e^{2\pi i s} - 1\right) = \int_C \frac{e^{-xt}}{e^{t-2\pi i y} - 1}\, t^s \frac{dt}{t}$$

The integral makes sense [redacted] as long as the denominator doesn't vanish on the contour; i.e.

$$t - 2\pi i y = 2\pi i n \qquad n \in \mathbb{Z}$$

$$(*) \qquad \text{or} \quad \frac{t}{2\pi i} - n = y.$$

I guess this means the function $\sum_{1}^{\infty} (x+n)^{-s} e^{2\pi i n y}$ has an analytic extension in y for [redacted] all y, but it probably won't be single-valued as y crosses the half-lines $(*)$.

If $-1 < x < 0$ it should be possible to replace the contour integral by the sum over the [redacted] residues:

$$\int_C \frac{e^{-xt}}{e^{t-2\pi i y} - 1}\, t^s \frac{dt}{t} = -\sum_{n \in \mathbb{Z}} e^{-x(2\pi i)(n+y)} \, (2\pi)^s\, i \left(i(n+y)\right)^{s-1}$$

Here $i = e^{i\frac{\pi}{2}}$ so we get

$$-(2\pi)^s e^{\frac{i\pi}{2}s} e^{-2\pi i x y} \sum_{n\in\mathbb{Z}} e^{-2\pi i n x} \cdot (n+y)^{s-1}$$

So we get the identity

$$(1) \qquad \sum_{n=1}^{\infty} \frac{e^{+2\pi i n y}}{(x+n)^s} \cdot \Gamma(s)\left(e^{2\pi i s}-1\right) = -(2\pi)^s e^{\frac{i\pi s}{2}} e^{-2\pi i x y} \sum_{n\in\mathbb{Z}} \frac{e^{-2\pi i n x}}{(y+n)^{1-s}}$$

valid for $-1<x<0$, $\mathrm{Im}(y)>0$. By taking limits it holds if $-1<x<0$ and $y \in \mathbb{R}-\mathbb{Z}$.

<u>June 1, 1977.</u> In formula (1) let $x \to 0$:

$$\left(\sum_{n=1}^{\infty} \frac{e^{2\pi i n y}}{n^s}\right) \cdot \frac{\Gamma(s)\left(e^{2\pi i s}-1\right)}{\left(2\pi e^{i\pi/2}\right)^s} = \sum_{n\in\mathbb{Z}} \frac{1}{(y+n)^{1-s}}$$

This should hold for $\mathrm{Im}(y) > 0$. One has to be careful that these series have to be analytically continued from the convergence region.

Better formulas might result if I work with

$$\tilde{H}(x,y,s) = \sum_{n\geq 0} (x+n)^{-s} e^{2\pi i n y}$$

$$\tilde{H}(x,y,s)\,\Gamma(s)\left(e^{2\pi i s}-1\right) = \int_C \sum_{n\geq 0} e^{-(x+n)t} e^{2\pi i n y}\, t^s \frac{dt}{t}$$

$$= \int_C \frac{e^{-xt}}{1-e^{-t+2\pi i y}} t^s \frac{dt}{t} = \int_C e^{-2\pi i y} \frac{e^{(1-x)t}}{e^{t-2\pi i y}-1} t^s \frac{dt}{t}$$

if $0<x<1$

$$= -2\pi i \sum_{n\in\mathbb{Z}} e^{-2\pi i y} e^{(1-x)(2\pi i(y+n))} (2\pi i(y+n))^{s-1}$$

$$= -(2\pi)^s e^{\frac{i\pi}{2}s} e^{-2\pi i x y} \sum_{n\in\mathbb{Z}} e^{-2\pi i n x} (y+n)^{s-1}$$

which is the same as before. This is no contradiction because now $0<x<1$.

Next

$$\sum_{n\in\mathbb{Z}} e^{-2\pi i n x}(y+n)^{s-1} = \sum_{n\geq 0} e^{-2\pi i n x}(y+n)^{s-1} + \sum_{n\geq 0} e^{+2\pi i(1+n)x}(y-1-n)^{s-1}$$

Now the branch of $(y+n)^{s-1}$ is determined as if $\operatorname{Im}(y)>0$, hence $(y-1-n) = e^{i\pi}(1-y+n)$ if say $0<y<1$. Thus

$$e^{2\pi i(1+n)x}(y-1-n)^{s-1} = e^{2\pi i x} e^{2\pi i n x} e^{i\pi(s-1)}(1-y+n)^{s-1}$$

So

$$\sum_{n\in\mathbb{Z}} e^{-2\pi i n x}(y+n)^{s-1} = \tilde{H}(y,-x,1-s) - e^{i\pi s} e^{2\pi i x} \tilde{H}(1-y,x,1-s)$$

which gives the formula

$$\tilde{H}(x,y,s)\Gamma(s)(e^{2\pi i s}-1) = (-1)(2\pi)^s e^{\frac{i\pi}{2}s} e^{-2\pi i x y}\left\{\tilde{H}(y,-x,1-s) - e^{i\pi s}e^{2\pi i x}\tilde{H}(1-y,x,1-s)\right\}$$

valid for all s but $0<x<1$, $0<y<1$.

Compute some Fourier transforms.

$$\int_0^\infty e^{ix\xi}x^{-s}\,dx = \Gamma(-s+1)(-i\xi)^{s-1} \qquad \text{if } \begin{array}{l} Re(s)<1 \\ Im(\xi)>0. \end{array}$$

Presumably this formula [redacted] should hold for ξ real $\neq 0$. Now if $\xi>0$, then $-i\xi$ should have argument $-\frac{\pi}{2}$ and if $\xi<0$, $-i\xi$ should have argument $+\frac{\pi}{2}$, hence

$$(-i\xi)^{s-1} = \begin{cases} e^{-i\frac{\pi}{2}(s-1)}\xi^{s-1} & \xi>0 \\ e^{+i\frac{\pi}{2}(s-1)}(-\xi)^{s-1} & \xi<0 \end{cases}$$

Now

$$\int_{-\infty}^{0} e^{ix\xi}e^{-i\pi s}(-x)^{-s}\,dx = e^{-i\pi s}\int_0^\infty e^{-ix\xi}x^{-s}\,dx$$

$$= e^{-i\pi s}\Gamma(-s+1)(i\xi)^{s-1} \qquad Im(\xi)<0$$

$$\arg(i\xi) = \frac{\pi}{2} \quad \text{[redacted] if } \xi>0$$
$$= -\frac{\pi}{2} \quad \text{if } \xi<0$$

$$(i\xi)^{s-1} = \begin{cases} e^{i\frac{\pi}{2}(s-1)}\xi^{s-1} & \xi>0 \\ e^{-i\frac{\pi}{2}(s-1)}(-\xi)^{s-1} & \xi<0 \end{cases}$$

June 2, 1977:

$$-\frac{\pi}{2} < \arg(x+n) < \frac{3\pi}{2}$$

$$G(x,y,s) = \sum_{n\in\mathbb{Z}} (x+n)^{-s} e^{2\pi i n y}$$

$$= \sum_{n\geq 0} (x+n)^{-s} e^{2\pi i n y} + \sum_{n\geq 0} (x-1-n)^{-s} e^{2\pi i(-1-n)y}$$

NOTE: H now is $\sum_{n\geq 0}$.

$$= H(x,y,s) + \underset{\substack{\| \\ (e^{+i\pi})^{-s}}}{(-1)^{-s}} \sum_{n\geq 0} (1-x+n)^{-s} e^{-2\pi i y} e^{-2\pi i n y}$$

1) $$\boxed{G(x,y,s) = H(x,y,s) + e^{-i\pi s} e^{-2\pi i y} H(1-x,-y,s)}$$

Next

$$H(x,y,s)\,\Gamma(s) = \sum_{n\geq 0} \int_0^\infty e^{-(x+n)t} e^{2\pi i n y} t^s \frac{dt}{t}$$

$$= \int_0^\infty e^{-xt} \frac{1}{1-e^{-t+2\pi i y}} t^s \frac{dt}{t}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = \frac{2\pi i}{e^{i\pi s} - e^{-i\pi s}} = \frac{2\pi i e^{i\pi s}}{e^{2\pi i s} - 1}$$

$$H(x,y,s) = \frac{1}{\Gamma(s)(e^{2\pi i s}-1)} \int_C e^{-xt} (1-e^{-t+2\pi i y})^{-1} t^s \frac{dt}{t}$$

$$= \frac{\Gamma(1-s) e^{-i\pi s}}{2\pi i} \int_C \quad ''$$

Now $\dfrac{1}{1-e^{-t+2\pi i y}}$ has poles at $-t+2\pi i y = -2\pi i n$ or $t = 2\pi i(y+n)$ and

$$\frac{d}{dt}\left(1-e^{-t+2\pi i y}\right)\Big|_{2\pi i(y+n)} = e^{-t+2\pi i y}\Big|_{t=2\pi i(y+n)} = 1 \quad \text{so the residue is } 1$$

$$H(x,y,s) = (-1)\,\Gamma(1-s)\,e^{-i\pi s}\sum_n e^{-x(2\pi i)(y+n)}\,\underline{(2\pi i(y+n))^{s-1}}$$

$$(-1)e^{-i\pi s}\; e^{\frac{i\pi}{2}(s-1)} = e^{-\frac{i\pi}{2}(s-1)} \qquad\qquad (2\pi e^{\frac{i\pi}{2}})^{s-1}(y+n)^{s-1}$$

2) $$\boxed{H(x,y,s) = \Gamma(1-s)\left(2\pi e^{-\frac{i\pi}{2}}\right)^{(s-1)} e^{-2\pi i x y}\, G(y,-x,1-s)}$$

$\boxed{\text{if } 0<x<1}$ *

3) $$\boxed{\begin{aligned} G(x,y+1,s) &= G(x,y,s)\\ H(x,y+1,s) &= H(x,y,s)\\ G(x+1,y,s) &= e^{-2\pi i y}\,G(x,y,s)\end{aligned}}$$

[redacted]

What seems to be important to understand is the natural domains of analyticity of these functions. So recall that the etale space of holomorphic fns. in x,y,s is Hausdorff and that each of the functions G, H has a complete continuation, defined to be a component of this etale space.

Start with $H(x,y,z) = \sum_{n\geq 0}' (x+n)^{-s} e^{2\pi i n y}$. This series converges for $\mathrm{Im}(y)>0$ and all x,s x not an integer. Also it converges for y real and $\mathrm{Re}(s)>1$. For fixed y,s it is multiple-valued in x but single-valued provided x doesn't cross the lines $-n+i\mathbb{R}_-$, $n=0,1,2,\ldots$

The contour integral

$$(+) \qquad H(x,y,s) = \frac{\Gamma(1-s)e^{-i\pi s}}{2\pi i}\int_C e^{-xt}\frac{1}{1-e^{-t+2\pi i y}}t^s\frac{dt}{t}$$

shows that for any $\operatorname{Re}(x) > 0$ and s except $1, 2, 3, 4, \ldots$ that $H(x,y,s)$ extends analytically for all $y \in \mathbb{C} - \mathbb{Z}$. It is singled valued if y doesn't cross the lines $n + i\mathbb{R}_-$. The same should work for any fixed x by removing a finite number of terms in the series defining $H(x,y,s)$. Also

$$H(x,y,s)\,\Gamma(s) = \int_0^\infty e^{-xt}\frac{1}{1-e^{-t+2\pi i y}}t^s\frac{dt}{t}$$

is good for $\operatorname{Re}(s) > 0$ provided the denominator stays nice, i.e. $y \notin n + i\mathbb{R}_-$ any $n \in \mathbb{Z}$.

So I conclude that $H(x,y,s)$ is a multi-valued holomorphic function defined for $y \notin \mathbb{Z}$ and $x \notin \mathbb{Z}_{\leq 0}$. Furthermore for s integral ≤ 0 the contour integral $(+)$ becomes a circle around zero:

$$H(x,y,s) = +\frac{\Gamma(1-s)e^{-i\pi s}}{2\pi i}\oint \frac{e^{-xt}}{1-e^{-t+2\pi i y}}t^s\frac{dt}{t}$$

which shows that $H(x,y,s)$ is a polynomial in x and a rational function of $e^{2\pi i y}$. For example:

$$H(x,y,0) = + \frac{1}{1-e^{2\pi i y}}$$

Derivative formulas:

$$H(x,y,s) = \sum_{n\geq 0} (x+n)^{-s} e^{2\pi i n y}$$

$$\frac{\partial}{\partial x} H(x,y,s) = \sum_{n\geq 0} (-s)(x+n)^{-s-1} e^{2\pi i n y} = -s\,H(x,y,s+1)$$

$$\frac{\partial}{\partial y} H(x,y,s) = \sum_{n\geq 0} (x+n)^{-s} (2\pi i n) e^{2\pi i n y}$$

$$= 2\pi i \left\{ \sum_{n\geq 0} (x+n)^{-s} \left[(n+x) - x \right] e^{2\pi i n y} \right\}$$

$$\boxed{\frac{\partial}{\partial y} H(x,y,s) = 2\pi i \left\{ H(x,y,s-1) - x\,H(x,y,s) \right\}}$$

$$\boxed{\frac{\partial}{\partial x} H(x,y,s) = -s\,H(x,y,s+1)}$$

$$e^{2\pi i x y}\left(\frac{\partial}{\partial y} + 2\pi i x\right) H(x,y,s) = 2\pi i\, H(x,y,s-1)\, e^{2\pi i x y}$$

$$\parallel$$

$$\frac{\partial}{\partial y}\left(e^{2\pi i x y} H(x,y,s)\right).$$

So

$$e^{2\pi i x y} H(x,y,s) = \int^{y} 2\pi i\, e^{2\pi i x \hat{y}} H(x,\hat{y},s-1)\, d\hat{y}$$

where $\hat{y}$ is an integration variable. In particular for

$s = 1$, we should have

$$H(x,y,1) = 2\pi i \int^{y} e^{2\pi i x(\hat{y}-y)} \frac{1}{1-e^{2\pi i \hat{y}}}\, d\hat{y}$$

Now we have

$$H(x,y,1) = \int_0^\infty e^{-xt} \frac{1}{1-e^{-t+2\pi i y}}\, dt$$

Put

$$-t + 2\pi i y = 2\pi i \hat{y} \qquad \text{or} \qquad \hat{y} = y - \frac{t}{2\pi i}$$

and we get

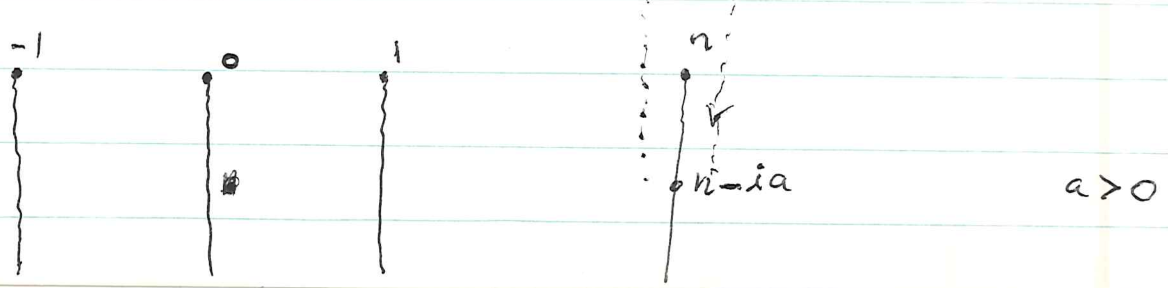
$$H(x,y,1) = \int_{y}^{y+i\infty} e^{2\pi i x(\hat{y}-y)} \frac{1}{1-e^{2\pi i \hat{y}}} (-2\pi i\, d\hat{y})$$

or

$$\cancel{H(x,y,1) = 2\pi i \int_{i\infty}^{y} \frac{e^{2\pi i x(\hat{y}-y)}}{1-e^{2\pi i \hat{y}}}\, d\hat{y}}$$

$$\boxed{H(x,y,1) = 2\pi i \int_{y+i\infty}^{y} \frac{e^{2\pi i x(\hat{y}-y)}}{1-e^{2\pi i \hat{y}}}\, d\hat{y}}$$

For this integral to converge one needs $Re(x) > 0$.
This formula shows that $H(x,y,1)$ is ~~not~~ single-valued when one cuts the y plane along $y = n + i\mathbb{R}_-$



$$H(x,(n-ia)^+,1) - H(x,(n-ia)^-,1) = 2\pi i \oint_{\hat{y}=n} \frac{e^{2\pi i x(\hat{y}-y)}}{1-e^{2\pi i \hat{y}}}\, d\hat{y}$$

$$= (2\pi i)(-2\pi i)\,\frac{e^{2\pi i x(n-y)}}{-2\pi i} = 2\pi i\, e^{2\pi i n x - 2\pi i x y}$$

since

$$\left(\frac{\partial}{\partial x}\right)^n H(x,y,s) = (-1)^n s(s+1)\cdots(s+n-1)\, H(x,y,s+n)$$

$$(-1)^{n-1}(n-1)!\; H(x,y,n) = \left(\frac{\partial}{\partial x}\right)^{n-1} H(x,y,1)$$

$$= (2\pi i)^n \int_{i\infty}^{y} \frac{e^{2\pi i x(\hat{y}-y)}}{1-e^{2\pi i \hat{y}}}\,(\hat{y}-y)^{n-1}\, d\hat{y}$$

so

$$\boxed{H(x,y,n) = \frac{(2\pi i)^n}{(n-1)!}\int_{i\infty}^{y} \frac{e^{2\pi i x(\hat{y}-y)}}{1-e^{2\pi i \hat{y}}}\,(y-\hat{y})^{n-1}\, d\hat{y}}$$

The differentiation formulas on page 96 should also hold for G

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} G(x,y,s) &= -s\, G(x,y,s+1) \\ \left(\frac{1}{2\pi i}\frac{\partial}{\partial y} + x\right) G(x,y,s) &= G(x,y,s-1) \end{aligned}}$$

Now

$$G(x,y,0) = H(x,y,0) + e^{-2\pi i y} H(1-x,-y,0)$$
$$= \frac{1}{1-e^{2\pi i y}} + e^{-2\pi i y}\frac{1}{1-e^{-2\pi i y}} = 0.$$

Therefore

$$\boxed{G(x,y,s) = 0 \qquad s = 0,-1,-2,\dots}$$

Also we should have

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial y} + x\right) G(x,y,1) = 0$$

so $$G(x,y,1) = c(x)\, e^{-2\pi i x y}.$$

$$G(x,y,1) = H(x,y,1) - e^{-2\pi i y} H(1-x,-y,1)$$
$$= \int_0^\infty e^{-xt}\frac{dt}{1-e^{-t+2\pi i y}} - e^{-2\pi i y}\int_0^\infty e^{-(1-x)t}\frac{dt}{1-e^{-t-2\pi i y}}$$

$$\parallel$$

$$-\int_0^\infty e^{xt}\frac{dt}{e^{t+2\pi i y}-1}$$

$$\parallel$$

$$-\int_0^{-\infty} e^{-xt}\frac{-dt}{e^{-t+2\pi i y}-1}$$

$$\parallel$$

$$+\int_{-\infty}^0 e^{-xt}\frac{dt}{1-e^{-t+2\pi i y}}$$

$$G(x,y,1) = \int_{-\infty}^{\infty}\frac{e^{-xt}}{1-e^{-t+2\pi i y}}\,dt \qquad \text{for } 0<\operatorname{Re}(x)<1$$

If we put $\quad 2\pi i\hat{y} = -t + 2\pi i y \quad$ i.e. $\hat{y} = y - \frac{t}{2\pi i}$

$$G(x,y,1) = \int_{y-i\infty}^{y+i\infty} \frac{e^{+x(2\pi i)(\hat{y}-y)}}{1-e^{2\pi i\hat{y}}}(-2\pi i\, d\hat{y})$$

$$= \left(2\pi i \int_{y-i\infty}^{y+i\infty} \frac{e^{2\pi i x\hat{y}}}{e^{2\pi i\hat{y}}-1}\, d\hat{y}\right) e^{-2\pi i x y}$$

Better supposing $\quad 0 < \mathrm{Re}(x) < 1, \quad 0 < \mathrm{Re}(y) < 1 \quad$ use residues

$$G(x,y,1) = \int_{-\infty}^{\infty} \frac{e^{-xt}}{1-e^{-t+2\pi i y}}\, dt$$

$$-t + 2\pi i y = -2\pi i n$$
$$2\pi i(y+n) = t$$
$n \geq 0$ if $0 < \mathrm{Re}(y) < 1$.

$$= 2\pi i \sum_{n=0}^{\infty} e^{-x\, 2\pi i(y+n)}$$

$$\boxed{G(x,y,1) = \frac{2\pi i\, e^{-2\pi i x y}}{1-e^{-2\pi i x}}}$$

if $\quad 0 < x < 1 \quad$ (used in derivation)
and $\quad 0 < \mathrm{Re}(y) < 1$

This formula can't be valid for all y because $G(x,y,1)$ is periodic in y. However we know that $e^{+2\pi i x y}G(x,y,s)$ is periodic in x, hence maybe it is correct for all x.