May 13, 1977

$$Lu = -\frac{d^2u}{dx^2} + x^2 u = \lambda u$$

$$\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right) = \frac{d^2}{dx^2} + 1 - x^2 = 1 - L$$

$u = e^{-x^2/2}$ belongs to $\lambda = 1$.

$\left[\frac{d}{dx}-x,\ \frac{d}{dx}+x\right] = 2$ hence if we put

$$a = \frac{i}{\sqrt{2}}\left(\frac{d}{dx}+x\right) \qquad a^* = \frac{i}{\sqrt{2}}\left(\frac{d}{dx}-x\right)$$

$$[a,a^*] = 1 \qquad a^*a = \tfrac{1}{2}(L-1)$$

$$a \longleftrightarrow \frac{d}{dz} \qquad a^* \longleftrightarrow z \qquad a^*a \longleftrightarrow z\frac{d}{dz}$$

$$\therefore\ (a^*a)(a^{*n}\cdot 1) = n\,(a^{*n}\cdot 1)$$

So $\frac{1}{2}(L-1)$ has eigenvalues $n = 0, 1, 2, \cdots$ and L has the eigenvalues $2n+1$, $n = 0, 1, \cdots$

I want to see if I can produce the solution $u(x,\lambda)$ dying at $x = +\infty$. Set up an integral equation for the eigenfunctions.

$e^{x^2/2}$ killed by $\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right) = -(L+1)$. Set $u = ve^{+x^2/2}$ to find the general solution of

$$-(L+1)u = \frac{d^2u}{dx^2} - (x^2+1)u = 0$$

$$\left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}-x\right)\left(e^{x^2/2}v\right) = \left(\frac{d}{dx}+x\right)\left(e^{x^2/2}v'\right) = e^{x^2/2}\left(v''+2xv'\right) = 0$$

$$v'' = -2xv' \qquad v' = c_1 e^{-x^2}$$

$$v = c_1 \int^x e^{-x^2} + c_2$$

So
$$u = c_1 e^{x^3/2} \int^x e^{-x^2} + c_2 e^{x^2/2}$$

is the general solution of $\frac{d^2u}{dx^2} - (x^2+1)u = 0$. Inhomogeneous DE

$$e^{x^2/2}(v'' + 2xv') = \frac{d^2u}{dx^2} - (x^2+1)u = f$$

$$(e^{x^2} v')' = e^{x^2/2} f \qquad e^{x^2} v' = \int e^{x^2/2} f$$

$$v = \int e^{-x^2} \int e^{x^2/2} f$$

$$u = e^{x^2/2} \int e^{-x^2} \int e^{x^2/2} f$$

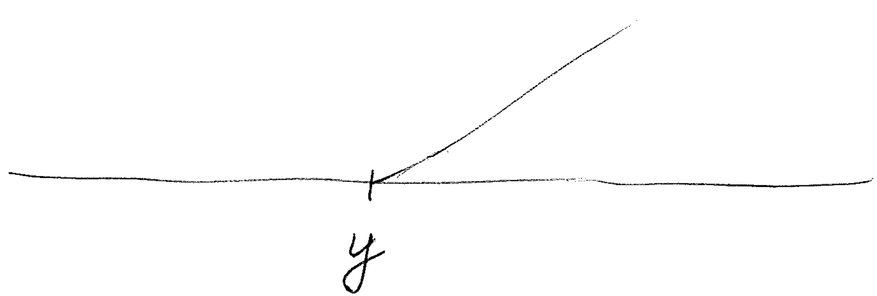
$$= e^{x^2/2} \int_l^x e^{-z^2} dz \int_m^z e^{y^2/2} f(y)\,dy$$

To choose l, m: $u = \iint f(y)\, e^{x^2/2 + y^2/2 - z^2}\, dz\,dy$. Take $l = m = -\infty$. Then we have $y \le z \le x$ and we get

$$u = \int_{-\infty}^x f(y)\,dy\, e^{\frac{x^2+y^2}{2}} \int_y^x e^{-z^2} dz = \int G_0(x,y) f(y)\,dy$$

where

$$G_0(x,y) = \begin{cases} e^{x^2/2 + y^2/2} \int_y^x e^{-z^2} dz & y \le x \\ 0 & y \ge x \end{cases}$$



As $x \longrightarrow +\infty$ $\quad \int_y^x e^{-z^2}dz \longrightarrow \int_y^\infty e^{-z^2}dz$ so

$$G_0(x,y) \sim e^{x^2/2 + y^2/2} \int_y^\infty e^{-z^2}dz \qquad x \to \infty$$

Put

$$E(x) = \int_x^\infty e^{-z^2}dz$$

so that $\frac{d^2u}{dx^2} - (x^2+1)u$ has the independent solutions

$$e^{x^2/2} \quad \text{and} \quad e^{x^2/2}E(x)$$

Then

$$\int_y^x e^{-z^2}dz = E(y) - E(x) \qquad \text{so}$$

$$G_0(x,y) = \begin{cases} 0 & x \le y \\ e^{x^2/2 + y^2/2}\big(E(y) - E(x)\big) & x \ge y \end{cases}$$

~~Subtract from this the solution~~

$$\cancel{e^{x^2/2 + y^2/2} E(y) \qquad \text{for all } x}$$

~~and you get a symmetric Green's function~~

$$\cancel{G(x,y) = \begin{cases} -e^{x^2/2 + y^2/2} E(y) & x \le y \\ -e^{x^2/2 + y^2/2} E(x) & x \ge y \end{cases}}$$

~~which decays in both directions~~

So if I subtract from G_0 the solution

$$\frac{e^{y^2/2}E(y)}{\sqrt{\pi}}\, e^{x^2/2} \underbrace{\left(\sqrt{\pi} - E(x)\right)}_{\int_{-\infty}^x e^{-z^2}dz}$$

which decays at $-\infty$ I get

$$G(x,y) = \frac{e^{\frac{x^2}{2}+\frac{y^2}{2}}}{\sqrt{\pi}}\left(\cancel{\sqrt{\pi}E(y)} - \sqrt{\pi}E(x) - \sqrt{\pi}E(y) + E(y)E(x)\right)$$

$$= \begin{cases} \dfrac{e^{\frac{x^2}{2}+\frac{y^2}{2}}}{\sqrt{\pi}} E(x)\left(E(y)-\sqrt{\pi}\right) & x \geq y \\[2ex] \dfrac{e^{\frac{x^2}{2}+\frac{y^2}{2}}}{\sqrt{\pi}} E(y)\left(E(x)-\sqrt{\pi}\right) & x \leq y \end{cases}$$

which is symmetric in x, y and decays as $x \to \pm\infty$

Correct approach from Titchmarsh: $u = e^{-\frac{x^2}{2}} v$ in

$$\frac{d^2u}{dx^2} + (\lambda - x^2)u = 0$$

$$u'' = \left(e^{-\frac{x^2}{2}}(v' - xv)\right)' = e^{-x^2/2}\left(v'' - xv' - v - xv' + x^2 v\right)$$

$$= e^{-x^2/2}\left(v'' - 2xv' + (x^2-1)v\right)$$

$$\boxed{v'' - 2xv' + (\lambda - 1)v = 0}$$

Now because x occurs to the first order in the coefficients we can try to solve this by Laplace's method.

$$v = \int_C e^{xz} g(z)\,dz \qquad xv' = \int e^{xz}_x z g(z)\,dz$$

$$= \int_C \frac{d}{dz}\left(e^{xz}\right) zg$$

$$z^2 g + 2\frac{d}{dz}(zg) + (\lambda - 1)g = 0$$

$$= \left[e^{xz} zg\right]_a^b - \int_C e^{xz}\frac{d}{dz}(zg)\,dz$$

$$2zg' + (z^2 + \lambda + 1)g = 0$$

$$\frac{g'}{g} + \left(\frac{z}{2} + \frac{\lambda+1}{2z}\right) = 0$$

$$\log g + \frac{z^2}{4} + \frac{\lambda+1}{2}\log z = 0$$

$$g = e^{-\frac{z^2}{4}} z^{-\frac{\lambda+1}{2}}$$

$$v = \int_a^b e^{xz - \frac{z^2}{4}} z^{-\frac{\lambda+1}{2}}\, dz.$$ Take C to be $[0,\infty]$.

This will work provided $z^{-\frac{\lambda}{2}-\frac{1}{2}} z$ vanishes at 0 i.e. $-\frac{\lambda}{2}+\frac{1}{2} > 0$ or $\lambda < 1$.

Change x to $-x$ in the above.

$$v = \int_0^\infty e^{-xz - z^2/4} z^{-\left(\frac{\lambda-1}{2}\right)} \frac{dz}{z}$$

$$v(0,\lambda) = \int_0^\infty e^{-z^2/4} z^{-\left(\frac{\lambda-1}{2}\right)} \frac{dz}{z}$$

$\frac{z^2}{4} = t \quad z = 2t^{1/2}$

$2\frac{dz}{z} = \frac{dt}{t}$

$$= \int_0^\infty e^{-t} \left(2t^{1/2}\right)^{-\left(\frac{\lambda-1}{2}\right)} \frac{1}{2}\frac{dt}{t}$$

$$= 2^{\frac{-\lambda-1}{2}} \int_0^\infty e^{-t} t^{(-\lambda+1)/4} \frac{dt}{t}$$

$$= 2^{-\left(\frac{\lambda+1}{2}\right)} \Gamma\left(\frac{1-\lambda}{4}\right)$$

$$v_x(0,\lambda) = -\int_0^\infty e^{-z^2/4} z^{\frac{-\lambda+3}{2}} \frac{dz}{z} = 2^{\frac{1-\lambda}{2}} \Gamma\left(\frac{3-\lambda}{4}\right)$$

May 14, 1977

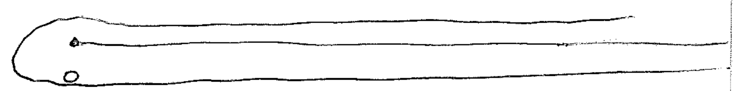
Recall the definition of the Γ-function:

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} \qquad \operatorname{Re}(s) > 0.$$

To get a global expression put

$$f(s) = \int_C e^{-t} t^s \frac{dt}{t}$$

where C is the contour



Clearly $f(s)$ is an entire function of s which vanishes for $s = 1, 2, \ldots$ as the integrand is analytic. One has

$$f(s) = \int_0^{2\pi} e^{-re^{i\theta}} r^s e^{i\theta s}\, i\, d\theta + \int_{-\infty}^{r} e^{-t} t^s \frac{dt}{t}$$

$$+ e^{2\pi i s} \int_r^\infty e^{-t} t^s \frac{dt}{t}$$

If $\operatorname{Re}(s) > 0$, then letting $r \to 0$ one gets

$$f(s) = \left(e^{2\pi i s} - 1\right) \Gamma(s).$$

A global expression for the Γ-function is therefore

$$\Gamma(s) = \frac{\int_C e^{-t} t^s \frac{dt}{t}}{e^{2\pi i s} - 1}$$

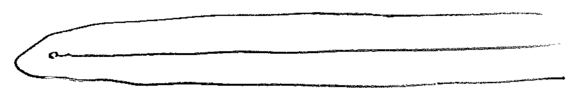
Back to $\dfrac{d^2u}{dx^2} + (\lambda - x^2)\,u = 0$

$$v'' - 2xv' + (\lambda - 1)\,v = 0 \qquad u = e^{-x^2/2}v$$

Let
$$v = \int_C e^{-xz - z^2/4}\, z^{\frac{1-\lambda}{2}} \frac{dz}{z}$$

and put $\boxed{\mu = \frac{1-\lambda}{2}}$. Then we get the solution

$$u(x,\mu) = e^{-x^2/2} \int_C e^{-xz - z^2/4}\, z^{\mu} \frac{dz}{z}$$

where here the contour C is: 

Note that because of the $e^{-x^2/2 - xz}$ and because $Re(z)$ is bounded below, $u(x,\lambda) \longrightarrow 0$ as $x \longrightarrow +\infty$. u vanishes identically for [redacted] $\mu = 1, 2, 3, \cdots$. If $Re(\mu) > 0$ we get as for Γ above:

$$u(x,\mu) = \left(e^{2\pi i \mu} - 1\right) \int_0^{\infty} e^{-x^2/2 - xz - z^2/4}\, z^{\mu} \frac{dz}{z}$$

This shows that the Euler integral on the right represents a meromorphic [redacted] function of μ [redacted] with at most simple poles at $\mu = 0, -1, -2, -3, \cdots$ which satisfies the DE. [redacted] Hence the Euler integral is the good function away from $\mu = 0, -1, -2, \cdots$ since it obviously is [redacted] non-vanishing at $1, 2, \cdots$. So it seems that the good global solution [redacted] might be

$$\frac{1}{\Gamma(\mu)} \int_0^{\infty} e^{-x^2/2 - xz - z^2/4}\, z^{\mu} \frac{dz}{z}$$

Put

$$v(x) = \int_0^\infty e^{-xz - z^2/4} z^\mu \frac{dz}{z}$$

This is meromorphic in μ with possibly simple poles at $\mu = 0, -1, -2, \ldots$

$$v(0) = \int_0^\infty e^{-z^2/4} z^\mu \frac{dz}{z} = \frac{1}{2}\int_0^\infty e^{-t/4}\, t^{\mu/2} \frac{dt}{t}$$ $z^2 = t$, $z = t^{1/2}$

$$= \frac{1}{2} 4^{\frac{\mu}{2}} \int_0^\infty e^{-t} t^{\mu/2} \frac{dt}{t}$$

$$v(0) = 2^{\mu-1}\Gamma\left(\frac{\mu}{2}\right)$$ bad at $\mu = 0, -2, -4$

$$\frac{dv}{dx}(0) = -\int_0^\infty e^{-z^2/4} z^{\mu+1} \frac{dz}{z} = 2^\mu \Gamma\left(\frac{\mu+1}{2}\right)$$ bad at $\mu = -1, -3, -5, \ldots$

Since Γ has no zeroes this implies $v(x,\mu)$ is never identically zero in x for any μ.

We have Legendre's formula:

$$\sqrt{\pi}\,\Gamma(\mu) = 2^{\mu-1}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)$$

hence

$$\frac{v(0)}{\Gamma(\mu)} = \frac{2^{\mu-1}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma(\mu)} = \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)}$$

$$\frac{v'(0)}{\Gamma(\mu)} = \frac{2^\mu\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma(\mu)} = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{\mu}{2}\right)}$$

These are entire functions of μ; the former vanishes at $\mu = -1, -3, -5, \ldots$, the latter vanishes at $\mu = 0, -2, -4, \ldots\ldots$ hence we conclude that

$$u(x,\mu) = \frac{e^{-x^2/2}}{\Gamma(\mu)} \int_0^\infty e^{-xz - z^2/4} z^\mu \frac{dz}{z}$$

is the solution of $\frac{d^2u}{dx^2} + (\lambda - x^2)u = 0 \qquad \frac{1-\lambda}{2} = \mu$

vanishing at $x = +\infty$. It is entire in μ because the initial values at $x=0$ are

$$u(0,\mu) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)} \qquad u'(0,\mu) = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{\mu}{2}\right)}$$

$$v(x,\mu) = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-xz - \frac{z^2}{4}} z^\mu \frac{dz}{z}$$

$$= \frac{1}{\Gamma(\mu)} \frac{1}{(e^{2\pi i\mu} - 1)} \int_c e^{-xz - \frac{z^2}{4}} z^\mu \frac{dz}{z}$$

But $\Gamma(\mu)\Gamma(1-\mu) = \frac{\pi}{\sin \pi\mu} = \frac{\pi\, 2i}{e^{i\pi\mu} - e^{-i\pi\mu}} = \frac{2\pi i e^{i\pi\mu}}{e^{2i\pi\mu} - 1}$

$$v(x,\mu) = \frac{\Gamma(1-\mu)}{2\pi i\, e^{i\pi\mu}} \int_c e^{-xz - \frac{z^2}{4}} z^{\mu-1}\, dz$$

$u(0,\mu) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{\mu+1}{2}\right)}$ Now $\frac{\pi}{\Gamma(z)} = \Gamma(1-z)\sin(\pi z)$

hence $\frac{1}{\Gamma(z)}$ as $z \to -\infty$ is an oscillatory function of rapidly increasing ~~[illegible]~~ amplitude. Now

$$\frac{\mu+1}{2} = \frac{\frac{1-\lambda}{2}+1}{2} = \frac{3-\lambda}{4} \to -\infty \quad \text{as} \quad \lambda \to +\infty$$

hence it is clear that $u(x,\lambda)$ is not well-behaved as $\lambda \to +\infty$. TOO BAD.

May 15, 1977

Suppose $S(x,\lambda) = \begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}$ is the solution matrix starting at $x=0$ for the DE

$$\frac{d^2u}{dx^2} + (\lambda^2 - q)u = 0$$

Then according to Weyl there is a ! meromorphic function $m(\lambda)$ such that

$$\chi(x,\lambda) = m(\lambda)\psi(x,\lambda) + \varphi(x,\lambda)$$

is square integrable on $[0,\infty)$. Now I believe one knows that the Fourier transform of $\varphi(x,\lambda), \psi(x,\lambda)$ with respect to λ have support in $[-x,x]$. $m(\lambda)$ has poles at those real λ such that ψ is square-integrable, hence we have

$$m(\lambda) = \frac{a(\lambda)}{b(\lambda)}$$

where $a(\lambda)$ is entire with zeroes where $\varphi(x,\lambda) \in L^2$ and $b(\lambda)$ is entire with zeroes where $\psi(x,\lambda) \in L^2$. Now suppose these eigenvalues grow sufficiently fast. What I want is for $a(\lambda)$ and $b(\lambda)$ to have nice Fourier transforms. Note that if we put

$$u(x,\lambda) = a(\lambda)\psi(x,\lambda) + b(\lambda)\varphi(x,\lambda)$$

then $a(\lambda) = u(0,\lambda)$, $b(\lambda) = u'(0,\lambda)$. So what I have is a sort of circular reasoning to the effect that if the eigenvalue distribution is sufficiently nice to give nice Fourier transforms at $x=0$, then it will give nice Fourier transforms at all x.

When p is real:

$$\begin{cases} \dfrac{du_1}{dx} - p u_2 = i\lambda u_1 \\ p u_1 - \dfrac{du_2}{dx} = i\lambda u_2 \end{cases}$$

$$\begin{cases} \dfrac{d}{dx}(u_1-u_2) + p(u_1-u_2) = i\lambda(u_1+u_2) \\ \dfrac{d}{dx}(u_1+u_2) - p(u_1+u_2) = i\lambda(u_1-u_2) \end{cases}$$

So put $\omega_1 = u_1+u_2 \qquad \omega_2 = u_1-u_2$ and you have

$$\left(\frac{d}{dx}-p\right)\omega_1 = i\lambda\omega_2$$

$$\left(\frac{d}{dx}+p\right)\omega_2 = i\lambda\omega_1$$

hence

$$\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right)\omega_1 = -\lambda^2\omega_1 \qquad \left(\frac{d}{dx}-p\right)\left(\frac{d}{dx}+p\right)\omega_2 = -\lambda^2\omega_2$$

or

$$\frac{d^2\omega_1}{dx^2} + (\lambda^2 - p' - p^2)\omega_1 = 0 \qquad \frac{d^2\omega_2}{dx^2} + (\lambda^2 + p' - p^2)\omega_2 = 0$$

In the case of $p=x$ we therefore get the eigenvalues $\lambda = \pm\sqrt{2n}\,, \quad n = 0, 1, 2, \cdots$

Observe that the system with p real is related to the second order DE

(1) $$\frac{d^2 w}{dx^2} + (\lambda^2 - q) w = 0$$

with $q = p' + p^2$. Now if w is a solution of [redacted]

[redacted] $$\frac{d^2 w}{dx^2} = q w$$

then $p = \frac{w'}{w}$ satisfies the Riccati equation

$$p' = \frac{w w'' - w'^2}{w^2} = q - p^2$$

May 16, 1977

From Progress in Optics, Vol III, E. Wolf editor, H. Gamo - Matrix Treatment of Partial Coherence

Suppose F, f are Fourier transforms:

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} dt \qquad f(t) = \int_{-W}^{W} F(\nu) e^{+2\pi i \nu t} d\nu$$

where $F(\nu)$ has support in $[-W, W]$. Example: $F(\nu) = 1$ for $\nu \in [-1, 1]$ and 0 outside

$$f(t) = \int_{-W}^{W} e^{2\pi i \nu t} d\nu = \left[\frac{e^{2\pi i \nu t}}{2\pi i t} \right]_{-W}^{W}$$

$$=$$ [redacted] $$\frac{\sin(2\pi W t)}{[redacted]\pi t}$$

We can expand $F(\nu)$ in a Fourier series

$$F(\nu) = \sum_n \alpha_n e^{-\pi i n \nu / W}$$

$$\alpha_n = \frac{1}{2W} \int_{-W}^{W} F(\nu) e^{\pi i n \nu / W} d\nu = \frac{1}{2W} f\left(\frac{n}{2W}\right)$$

$$f(t) = \frac{1}{2W} \sum_n f\left(\frac{n}{2W}\right) \int_{-W}^{W} e^{-\pi i n \nu / W + 2\pi i \nu t} d\nu$$

$$\left. \frac{e^{\pi i (2t - \frac{n}{W}) \nu}}{\pi i (2t - \frac{n}{W})} \right|_{-W}^{W}$$

$$\boxed{f(t) = \sum_n f\left(\frac{n}{2W}\right) \frac{\sin \pi (2Wt - n)}{\pi (2Wt - n)}}$$

Observe that the function $u_n(t) = \dfrac{\sin \pi (2Wt - n)}{\pi (2Wt - n)}$ has the transform

$$\begin{cases} \frac{1}{2W} e^{-\pi i n \nu / W} & \nu \in [-W, W] \\ 0 & \text{otherwise} \end{cases}$$

Moreover one has $u_n\left(\frac{m}{2W}\right) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$. By Parseval's formula the u_n form an orthonormal sequence in $L^2(\mathbb{R})$, because their transforms are orthonormal on $L^2([-W, W])$.

May 18, 1977

I've seen that if $\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -a \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then

$y = \frac{u_1}{u_2}$ satisfies the Riccati equation

$$y' = \frac{u_1'}{u_2} - \frac{u_1 u_2'}{u_2^2} = \frac{au_1 + bu_2}{u_2} + \frac{u_1(cu_1 + au_2)}{u_2^2}$$

$$= ay + b + cy^2 + ay$$

$$\boxed{y' = b + 2ay + cy^2}$$

Now I want to find the corresponding DE on the unit circle. Put

$w = -1 \Leftrightarrow y = \infty$

$w = 1 \Leftrightarrow y = 0$

$w = i \Leftrightarrow y = 1$

$$\boxed{w = \frac{1+iy}{1-iy} \qquad y = \frac{1}{i}\,\frac{w-1}{w+1}}$$

$$y' = \frac{1}{i}\,\frac{(w+1)(w') - (w-1)w'}{(w+1)^2} = \frac{2}{i}\,\frac{w'}{(w+1)^2}$$

$$\frac{2}{i}\,\frac{w'}{(w+1)^2} = b + \frac{2a}{i}\,\frac{w-1}{w+1} - c\left(\frac{w-1}{w+1}\right)^2$$

$$w' = \frac{ib}{2}\left[w^2+2w+1\right] + a\left[w^2-1\right] + \left(-\frac{ic}{2}\right)\left[w^2-2w+1\right]$$

$$w' = \left(a + \frac{i(b-c)}{2}\right)w^2 + i(b+c)w + \left(-a + \frac{i(b-c)}{2}\right)$$

$$\boxed{\begin{array}{c} w' = \alpha w^2 + i\beta w - \bar{\alpha} \\ \beta \text{ real} \end{array}}$$

Check: $(w\bar{w})' = (\alpha w^2 + i\beta w - \bar{\alpha})\bar{w} + w(\bar{\alpha}\bar{w}^2 - i\beta\bar{w} - \alpha)$

if $|w|=1$ $\quad = \alpha w + i\beta - \bar{\alpha}\bar{w} + \bar{\alpha}\bar{w} - i\beta - \alpha w = 0$

$$\frac{1}{2i}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -c & -a \end{pmatrix}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} a+bi & -a+bi \\ -c-ai & c-ai \end{pmatrix}$$

$$= \frac{1}{2i}\begin{pmatrix} \cancel{ai}-b-c-\cancel{ai} & -ia-b+c-ai \\ -ia+b-c-ia & \cancel{ia}+b+c-\cancel{ai} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ib+ic}{2} & -a+\frac{ib-ic}{2} \\ -a-\frac{ib-ic}{2} & -\frac{ib+ic}{2} \end{pmatrix}$$

Consider now the system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where p is a function of x, λ is a scalar. The associated Riccati DE is

$$\frac{dw}{dx} = \bar{p} + 2i\lambda w - p w^2$$

$$w = \frac{u_1}{u_2}$$

Put $\quad w = e^{i\theta} \qquad \frac{dw}{dx} = e^{i\theta} i \frac{d\theta}{dx} = w i \frac{d\theta}{dx}$

$$\frac{d\theta}{dx} = \frac{1}{iw}\frac{dw}{dx} = \frac{\bar{p}}{i} w^{-1} + 2\lambda - \frac{p}{i} w$$

$$= +(\overline{ip})e^{-i\theta} + 2\lambda + ipe^{i\theta}$$

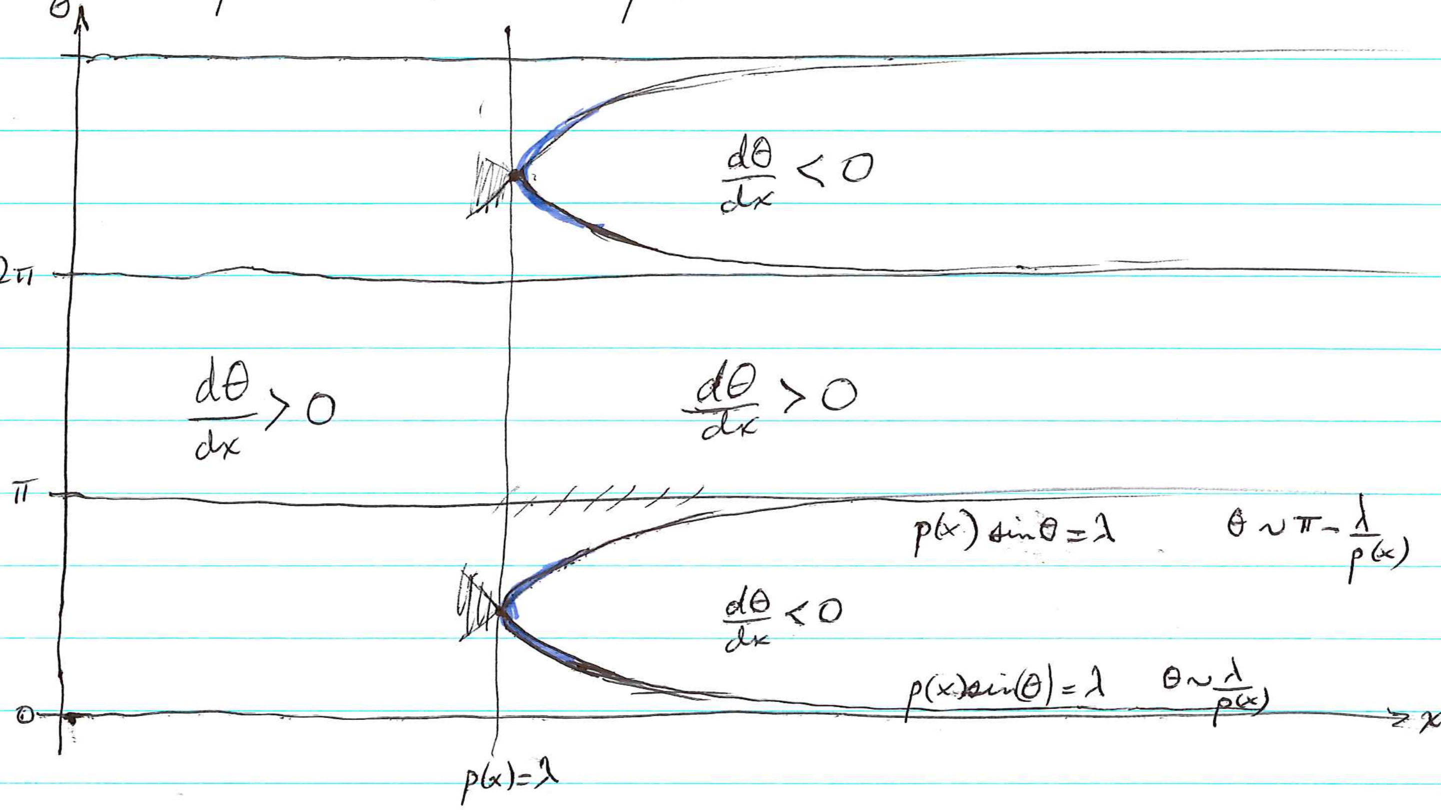
or

$$\boxed{\frac{d\theta}{dx} = 2\lambda + 2\,\mathrm{Re}(ipe^{-i\theta})}$$

Suppose to begin with that p is real so that

$$\frac{d\theta}{dx} = 2(\lambda - p\sin\theta)$$

Assume $p' > 0$ and $p(x) \nearrow +\infty$ as $x \to +\infty$.



Look at the possible initial values for θ on $p(x) = \lambda$, or any [redacted] larger x. Then most of the integral curves are asymptotic to $\theta = 2\pi n$ $n \in \mathbb{Z}$. In fact there is exactly one integral curve with

$$\frac{\pi}{2} < \theta(p^{-1}(x)) < \pi$$

which is asymptotic to $\theta = \pi$.

May 19, 1977

The system $\frac{d}{dx} u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$ has associated to it the Riccati DE

$$\frac{dw}{dx} = \bar{p} + 2i\lambda w + (-p) w^2 .$$

Put $w = e^{i\theta}$ and this becomes

$$\frac{d\theta}{dx} = 2\lambda + 2\,\mathrm{Re}\,(ip e^{i\theta})$$

Now write p in polar form

$$p = \rho e^{-i\alpha}$$

$$\mathrm{Re}\,(ipe^{i\theta}) = \mathrm{Re}\,(i\rho e^{i(\theta-\alpha)}) = -\rho \sin(\theta-\alpha).$$

$\therefore$

$$\boxed{\frac{d\theta}{dx} = 2(\lambda - \rho \sin(\theta-\alpha))}$$

If we put $\theta = \tilde{\theta} + \alpha$ this becomes

$$\frac{d\tilde{\theta}}{dx} = (2\lambda - \alpha') - 2\rho \sin(\tilde{\theta}) .$$

This last equation might be more suitable for an asymptotic analysis. If this is the case, then it makes sense to put in the original system

$$u = \begin{pmatrix} e^{+i\alpha/2}\tilde{u}_1 \\ e^{-i\alpha/2}\tilde{u}_2 \end{pmatrix} \qquad e^{+i\alpha/2}\left(\frac{d\tilde{u}_1}{dx} + i\frac{\alpha'}{2}\tilde{u}_1\right) = i\lambda e^{+i\alpha/2}\tilde{u}_1 + \bar{p}e^{-i\alpha/2}\tilde{u}_2$$

$$\boxed{\frac{d}{dx}\tilde{u} = \begin{pmatrix} i(\lambda - \frac{\alpha'}{2}) & \rho \\ \rho & -i(\lambda - \frac{\alpha'}{2}) \end{pmatrix} \tilde{u}}$$

The latter can be written in the "real" form:

$$\frac{d}{dx}\begin{pmatrix} \tilde{u}_1 - \tilde{u}_2 \\ i\tilde{u}_1 + i\tilde{u}_2 \end{pmatrix} = \begin{pmatrix} -\rho & \lambda - \frac{\alpha'}{2} \\ -\lambda + \frac{\alpha'}{2} & \rho \end{pmatrix}\begin{pmatrix} \tilde{u}_1 - \tilde{u}_2 \\ i\tilde{u}_1 + i\tilde{u}_2 \end{pmatrix}$$

or better:

$$\begin{cases} \frac{d}{dx}(\tilde{u}_1 - \tilde{u}_2) + \rho(\tilde{u}_1 - \tilde{u}_2) = \left(\lambda - \frac{\alpha'}{2}\right)(i\tilde{u}_1 + i\tilde{u}_2) \\ \frac{d}{dx}(i\tilde{u}_1 + i\tilde{u}_2) - \rho(i\tilde{u}_1 + i\tilde{u}_2) = -\left(\lambda - \frac{\alpha'}{2}\right)(\tilde{u}_1 - \tilde{u}_2) \end{cases}$$

I understand this system somewhat when α' is constant. We see that the effect of a constant α', i.e. a linear α, is to shift the spectrum away from the symmetrical situation $\lambda \leftrightarrow -\lambda$.

May 20, 1977: I've seen that the system

$$\frac{d}{dx}u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u \qquad p = \rho e^{-i\alpha} \quad \rho > 0$$

under the substitution $v = \begin{pmatrix} e^{-i\alpha/2} u_1 \\ e^{i\alpha/2} u_2 \end{pmatrix}$ becomes

$$\frac{d}{dx}v = \begin{pmatrix} i(\lambda - \alpha'/2) & \rho \\ \rho & -i(\lambda - \alpha'/2) \end{pmatrix} v$$

Hence if we now change independent variable

$$\rho\, dx = dy \quad \text{or} \quad y = \int_0^x \rho(\tilde{x})\, d\tilde{x}$$

then we get the system

$$\frac{dv}{dy} = \begin{pmatrix} ig & 1 \\ 1 & -ig \end{pmatrix} v \qquad \frac{dv_1}{dy} = igv_1 + v_2$$

$$\frac{dv_2}{dy} = -igv_2 + v_1$$

where $g = \frac{1}{\rho}\lambda - \frac{\alpha'}{2\rho}$. The associated Riccati DE is

$$\frac{dw}{dy} = 1 + 2igw - w^2 \qquad w = \frac{v_1}{v_2}$$

Notice that if w is known then we have the following DE for v_1, v_2.

$$\frac{dv_2}{dy} = -igv_2 + wv_2 = (w - ig)v_2$$

$$\frac{dv_1}{dy} = igv_1 + \frac{1}{w}v_1 = (w^{-1} + ig)v_1$$

Since λ real $\Rightarrow$ g real $\Rightarrow$ $|w| = 1$ (assuming $|w(0)| = 1$) one has $w^{-1} = \overline{w}$, hence

$$\frac{d\overline{v}_1}{dy} = (w - ig)\overline{v}_1$$

so that $v_2 = c\overline{v}_1$ for some c with $|c| = 1$. Also

$$v_1 = v_1(0)\, e^{\int_0^y w^{-1} + ig} \qquad v_2 = v_2(0)\, e^{\int_0^y w - ig}$$

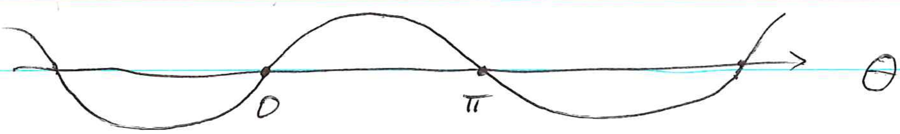
$$|v_1| = |v_2| = |v_1(0)|\, e^{\int_0^y \operatorname{Re} w} = |v_1(0)|\, e^{\int_0^y \cos\theta}$$

So the problem is to see if there are integral curves of

$$\frac{d\theta}{dy} = 2(g(y) - \cos\theta) \qquad g = \frac{\lambda}{\rho} - \frac{\alpha'}{2\rho}$$

such that $\int_0^y \cos\theta \, dy \searrow -\infty$ as $y \to +\infty$.

For example suppose $g(y) \to 0$ as $y \to \infty$. Plot $\sin\theta$:



Once y is sufficiently large that $|g(y)| < 1$, then it follows that around the peaks $\pi/2$, $3\pi/2$, etc θ is either increasing or decreasing. [redacted] $\exists\, \varepsilon(y) > 0$ tending to zero such that

$$|\theta(y)| < \pi - \varepsilon(y) \quad \text{implies } \theta(y) \text{ decreases to zero}$$

There should be exactly one integral curve approaching π. This is clear. Since then $\cos(\theta(y)) \to -1$ we have $\int_0^y \cos\theta \, dy \searrow -\infty$.

May 21, 1977

Symmetry $\lambda \mapsto -\lambda$ of the eigenvalues

The angle-equation belonging to the system

$$(1) \qquad \frac{d}{dx} u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

with p real is

$$(2) \qquad \frac{d\theta}{dx} = 2(\lambda - p \sin\theta)$$

Here $e^{i\theta} = \frac{u_1}{u_2}$. Note that the substitution $\theta \mapsto -\theta$, $\lambda \mapsto -\lambda$ leaves (2) invariant. This corresponds to the substitution $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$ $\quad \lambda \mapsto -\lambda$

which leaves (1) invariant. Consider a boundary condition at $x = 0$:

$$(3) \qquad \frac{u_1}{u_2} = e^{i\tau_0}.$$

If $e^{i\tau_0} = \pm 1$, then this boundary condition is invariant under the substitution $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$, hence we conclude that the spectrum belonging to (1) with the boundary condition

$$(4) \qquad \frac{u_1}{u_2} = +1 \quad \text{or} \quad -1$$

is symmetric under $\lambda \mapsto -\lambda$.

Thus we get $\lambda \mapsto -\lambda$ symmetry for the interval $[0, \infty)$ provided p real and we have the boundary condition (4).

Next suppose the interval [redacted] is $(-\infty,\infty)$ and that p is even:

$$p(x) = p(-x)$$

Let us consider the change $x \mapsto -x$, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$ and $\lambda \mapsto -\lambda$. Then (1) becomes

$$\frac{d}{d(-x)}\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} -i\lambda & \bar{p}(x) \\ p(x) & +i\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$$

$$-\frac{du_1}{dx} = -i\lambda u_1 - \bar{p}\, u_2$$

$$\frac{du_2}{dx} = p\, u_1 - i\lambda u_2$$

and so (1) is invariant. Similarly if p is odd

$$p(-x) = -p(x)$$

then $x \mapsto -x$, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\lambda \mapsto -\lambda$ applied to (1)

$$\frac{d}{d(-x)} u = \begin{pmatrix} -i\lambda & -\bar{p} \\ -p & +i\lambda \end{pmatrix} u$$

so (1) is invariant.

[redacted] Suppose p is [redacted] even and that $u^+(x,\lambda)$ is a solution of (1) on $(-\infty,\infty)$ which decays at $x = +\infty$. Then

$$u^-(x,\lambda) = \begin{pmatrix} u_1^+(-x,-\lambda) \\ -u_2^+(-x,-\lambda) \end{pmatrix}$$

is a solution decaying at $x = -\infty$. Form the Wronskian of these solutions

$$\omega(\lambda) = \begin{vmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{vmatrix} = \begin{vmatrix} u_1^+(x,\lambda) & u_1^+(-x,-\lambda) \\ u_2^+(x,\lambda) & -u_2^+(-x,-\lambda) \end{vmatrix}$$

which doesn't depend on x, hence

$$\omega(\lambda) = -\left(u_1^+(0,\lambda)\,u_2^+(0,-\lambda) + u_1^+(0,-\lambda)\,u_2^+(0,+\lambda)\right)$$

so $$\omega(\lambda) = \omega(-\lambda).$$

Assuming λ such that $u^+(x,\lambda)$ $u^+(x,-\lambda)$ are not identically zero in x, it follows that

$$\omega(\lambda) = 0 \iff \lambda \text{ eigenvalue for } (1)$$

hence λ has to be real. (I am tacitly assuming that for a given value of λ there is exactly one solution of (1) decaying toward ∞. This depends on properties of p to be elucidated.)

Notice that if p is odd then

$$\omega(\lambda) = \begin{vmatrix} u_1^+(0,\lambda) & u_1^+(0,-\lambda) \\ u_2^+(0,\lambda) & u_2^+(0,-\lambda) \end{vmatrix}$$

satisfies $\omega(\lambda) = -\omega(-\lambda)$. Hence maybe I can prove 0 is an eigenvalue in this case. This is clear because we know that if $u(x)$ is a solution with $\lambda = 0$ then so is $u(-x)$, hence since $u(x)$ and $u(-x)$ coincide at 0 we have $u(x) = u(-x)$ identically.

Thus if u decays at $+\infty$ it also decays at $-\infty$.

May 22, 1977.

Consider $\quad u'' + (\lambda^2 - q)u = 0 \quad$ and put

$$y = \frac{\lambda u}{u'}$$

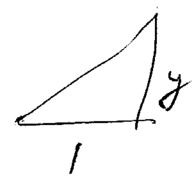
Then

$$\frac{dy}{dx} = \lambda - \frac{\lambda u}{(u')^2} u'' = \lambda + \lambda(\lambda^2 - q) \frac{u^2}{(u')^2}$$

$$\frac{dy}{dx} = \lambda + \left(\lambda - \frac{q}{\lambda}\right) y^2 = \lambda(1+y^2) - \frac{q}{\lambda} y^2$$

So put $\quad \tan\phi = y \qquad \phi = \arctan(y)$

$$\frac{d\phi}{dx} = \frac{y'}{1+y^2} = \lambda - \frac{q}{\lambda}\frac{y^2}{1+y^2}$$



or

1) $$\boxed{\frac{d\phi}{dx} = \lambda - \frac{q}{\lambda}\sin^2\phi}$$

Other possible substitutions are

$$\tan\phi = y = \frac{u}{u'}$$

$$\frac{d\phi}{dx} = \frac{1}{1+y^2}\left(1 + (\lambda^2 - q)y^2\right) \quad \text{or}$$

2) $$\boxed{\frac{d\phi}{dx} = \cos^2\phi + (\lambda^2 - q)\sin^2\phi = 1 + (\lambda^2 - q - 1)\sin^2\phi}$$

More generally I can put $\tan\phi = y = \frac{\alpha u}{u'}$ and get

$$\frac{d\phi}{dx} = \alpha \cos^2\phi + \frac{\lambda^2 - q}{\alpha} \sin^2\phi$$

3) $$\boxed{\frac{d\phi}{dx} = \left(\alpha \cos^2\phi + \frac{\lambda^2}{\alpha} \sin^2\phi\right) - \frac{q}{\alpha} \sin^2\phi}$$

Version 2) has the advantage of showing that increasing λ^2 leads to an increase in the turning rate, and also the advantage that y and ϕ are independent of λ.

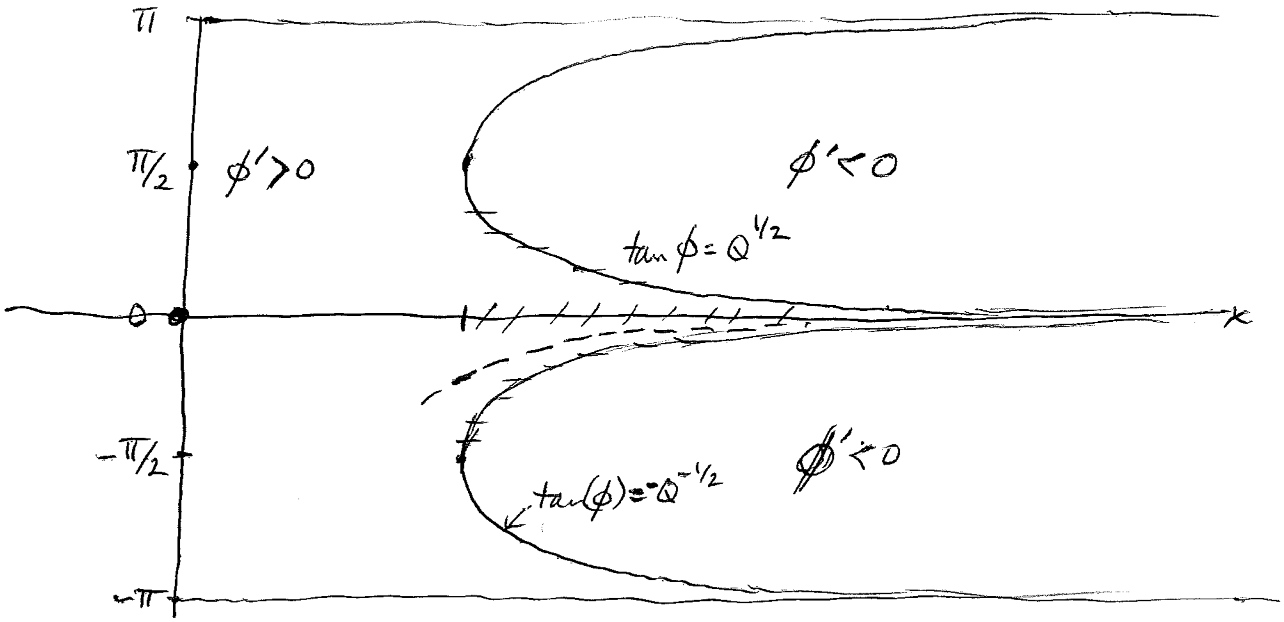
May 23, 1977 (Alice is 15)

Consider $u'' + (\lambda^2 - q)\,u = 0$. Put $Q = q - \lambda^2$.

If $\tan(\phi) = \dfrac{u}{u'}$, then

$$\frac{d\phi}{dx} = \cos^2\phi - Q\sin^2\phi$$

Suppose we look where $\frac{d\phi}{dx} < 0, = 0, > 0$.



The dotted curve represents the unstable solution belonging to the WKB approximant $Q^{-1/4}e^{-\int Q^{1/2}}$. Note that

$$u = Q^{-1/4}e^{-\int Q^{1/2}} \qquad u' = \left((Q^{-1/4})' - Q^{-1/4}Q^{1/2}\right)e^{-\int Q^{1/2}}$$

so

$$\cot(\phi) = \frac{u'}{u} = \frac{(Q^{-1/4})'}{Q^{-1/4}} - Q^{1/2} = -\frac{1}{4}\frac{Q'}{Q} - Q^{1/2}$$

and

$$\tan(\phi) = \frac{u}{u'} = -Q^{-1/2}\left[1 + \frac{1}{4}\frac{Q'}{Q^{3/2}}\right]^{-1} = -Q^{-1/2} + \frac{1}{4}\frac{Q'}{Q^2} + \cdots$$

May 25, 1977.

It is gradually appearing that the important theoretical object of study is Weyl meromorphic function $m_\infty(\lambda)$ which gives the initial values for the solution of the DE satisfying the boundary conditions at ∞. This function is meromorphic, i.e. a holomorphic map

$$m : \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C})$$

carrying $\mathbb{R}$ to $\mathbb{P}^1(\mathbb{R})$, the UHP into the UHP, etc. According to a theorem, due perhaps to Riesz + Herglotz, m has a unique representation

$$m(\lambda) = a\lambda + b + \int_{-\infty}^{\infty} \frac{d\sigma(x)}{x-\lambda}$$

where a, b real and $a \geq 0$. This ~~function~~ measure $d\sigma(x)$ is the so-called spectral measure.

May 26, 1977

Strings: Consider

$$\frac{d^2u}{dx^2} + \lambda^2 \rho(x) u = 0$$

where $\rho \geq 0$ is a positive density function on an interval $0 \leq x \leq b$. One has

$$0 = \int_0^b u(u'' + \lambda^2 \rho u)\, dx = [uu']_0^b - \int_0^b (u')^2 dx + \lambda^2 \int_0^b \rho u^2 dx$$

Hence if any of the boundary conditions $u = 0$, or, $u' = 0$, or periodic conditions, are used one has

$$\lambda^2 = \frac{\int_0^b (u')^2 dx}{\int_0^b u^2 \rho dx} \geq 0$$

so λ is real. Thus the spectrum is real and symmetric.