April 27, 1977

On April 1, I considered the general D.E.

$$\frac{dX}{dt} = AX$$

$$A = A_0(t) + A_1(t)\lambda \qquad A_i \text{ real \& trace } 0.$$

where $A_1(t)$ is of the form $\begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ $\quad -p^2+qr \geq 0$, $q, r \geq 0$.
I considered changing variables: $X = UY$ where $U = U(t)$ is in $SL_2(\mathbb{R})$. Under this change the matrix A is replaced by

$$U^{-1}AU - U^{-1}\dot{U}.$$

Assuming $\det A_1(t) = -p^2+qr > 0$ we can rescale, i.e. change t so as to make this determinant 1. Then we can choose U so as to make $A_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then I can further alter U to make A_0 symmetric, whence I saw that the DE was equivalent (by the change from UHP to unit disk) to the system:

$$\frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ +p & -\frac{d}{dx} \end{pmatrix}\psi = \lambda\psi$$

with p complex.

I want to note now that the solution matrices obtained do not seem to exhaust the class of linear Ising model limits. I recall that a linear Ising model gives a matrix function of the form

$$A_1\begin{pmatrix} \cosh h_1\lambda & \sinh h_1\lambda \\ -\sinh h_1\lambda & \cosh h_1\lambda \end{pmatrix} A_2\begin{pmatrix} & \\ & \end{pmatrix} \cdots\cdots A_n\begin{pmatrix} \cosh h_n\lambda & \sinh h_n\lambda \\ -\sinh h_n\lambda & \cosh h_n\lambda \end{pmatrix}$$

where the $A_i \in SL_2(\mathbb{R})$ and the $h_i \geq 0$.

April 28, 1977

Consider the classical motion associated to the potential e^{2x}:

$$\left(\frac{dx}{dt}\right)^2 + e^{2x} = \lambda^2$$

Solution is

$$x = \log\left(\frac{\lambda}{\cosh \lambda t}\right) \qquad r = e^x = \frac{\lambda}{\cosh(\lambda t)}$$

where one can replace t by $t - t_0$. Check

$$\frac{dx}{dt} = -\frac{d}{dt}\log(\cosh \lambda t) = -\frac{\sinh \lambda t}{\cosh \lambda t}\lambda$$

$$\left(\frac{dx}{dt}\right)^2 + e^{2x} = \lambda^2 \frac{\sinh^2 \lambda t}{\cosh^2 \lambda t} + \frac{\lambda^2}{\cosh^2 \lambda t} = \lambda^2$$

An interesting question is how to relate the motion $r = \frac{\lambda}{\cosh(\lambda t)}$ for all different λ with the basic wave motion $u(x,t) = e^{-r\cosh(t)}$. What one would like is some approximate representation of u as a superposition of classically moving wave packets.

April 29, 1977

$E = h\nu$ $\quad$ E measured in ergs $= gr\frac{cm^2}{sec^2} \Rightarrow h$ in $gr\frac{cm^2}{sec}$

$H = \frac{p^2}{2m} + V.$ $\quad$ As an operator $\quad p = \frac{\hbar}{i}\frac{d}{dx}$

(momentum $= gr\frac{cm}{sec} = gr\frac{cm^2}{sec}\frac{1}{cm}$). Thus Schroedinger's eqn. is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

The time dependent value function is $u(x,t) = \psi(x)e^{-\frac{i}{\hbar}Et}$. (Note $E = h\nu$ means we should have $\frac{E}{h}$ cycles per second, i.e. an angle of $2\pi\frac{E}{h} = \frac{E}{\hbar}$ radians in one second). Time dep. Schroedinger equation is

$$\frac{\hbar}{i}\frac{\partial u}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial x^2} - V(x)u$$

To understand the classical approximation put

$$u(x,t) = e^{\frac{i}{\hbar}S(x,t)}$$

Then

$$-\frac{\hbar}{i}\frac{\partial u}{\partial t} = -\frac{\partial S}{\partial t}\cdot u$$

$$\left(+\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 u = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)(S_x u) = \left(S_x^2 + \frac{\hbar}{i}S_{xx}\right)u$$

i.e.

$$\boxed{\frac{\partial S}{\partial t} + \frac{1}{2m}(S_x)^2 + V(x) + \frac{\hbar}{i}S_{xx} = 0}$$

If we let $\hbar \to 0$ then we get the Hamilton-Jacobi equation for the classical motion.

I notice now that my use of u, ψ is opposite to the physicists' convention. So we change: From now on $\psi = \psi(x,t)$ and $u = u(x, E)$ and we use the expansion formula

$$\psi(x,t) = \int e^{-iEt} u(x,E)\, dE$$

when $\hbar = 1$ and

$$\psi(x,t) = \int e^{-\frac{i}{\hbar}Et} u(x,E)\, dE$$

in general. $\psi(x,t)$ is the state of the system at time t and evolves according to

$$\psi = e^{-\frac{i}{\hbar}Ht}\psi_0 \qquad \psi_0 = \psi(\ ,0)$$

The <u>average value</u> of an operator A when the system is in the state ψ is

$$\langle A\psi, \psi \rangle = \int \psi^* A \psi \, dx.$$

$$\frac{d}{dt}\langle A\psi, \psi \rangle = \frac{d}{dt}\langle e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht}\psi_0, \psi_0 \rangle$$

$$= \left\langle \frac{i}{\hbar}[H, A]\psi, \psi \right\rangle$$

Applying this to position & momentum:

$$\cancel{\frac{i}{\hbar}[H, x] = -\frac{i}{\hbar}\frac{\hbar^2}{2m}\left[\frac{d^2}{dx^2}, x\right] = +\frac{1}{m}\frac{\hbar}{\hbar}\frac{d}{dx} = -\frac{p}{m}}$$

$$\cancel{\frac{i}{\hbar}[H, p] = \frac{i}{\hbar}\left[V, \frac{\hbar}{i}\frac{d}{dx}\right] =}$$

$$\frac{i}{\hbar}[H,x] = -\frac{i}{\hbar}\frac{\hbar^2}{2m}\left[\frac{d^2}{dx^2}, x\right] = \frac{1}{m}\frac{\hbar}{i}\frac{d}{dx} = \frac{p}{m}$$

$$\frac{i}{\hbar}[H,p] = \left[V, \frac{d}{dx}\right] = -\frac{dV}{dx}$$

we get

$$m\frac{d^2}{dt^2}\langle x\psi,\psi\rangle = m\frac{d}{dt}\langle \tfrac{p}{m}\psi,\psi\rangle = -\langle \frac{dV}{dx}\psi,\psi\rangle$$

Thus if ψ is supported in a small neighborhood around $\bar{x} = \langle x\psi,\psi\rangle$ one gets the classical motion:

$$m\frac{d^2\bar{x}}{dt^2} = -V(\bar{x}).$$

<u>April 30, 1977:</u>

Conservation of energy: From

$$\frac{d}{dt}\langle A\psi,\psi\rangle = \langle \frac{i}{\hbar}[H,A]\psi,\psi\rangle$$

one sees

$$\begin{aligned}\langle H\psi,\psi\rangle &= \langle -\frac{\hbar^2}{2m}\psi_{xx} + V\psi,\psi\rangle \\ &= \frac{\hbar^2}{2m}\|\psi_x\|^2 + \langle V\psi,\psi\rangle \\ &= \int\left(\frac{\hbar^2}{2m}\left|\frac{\partial\psi}{\partial x}\right|^2 + V|\psi|^2\right)dx\end{aligned}$$

is constant in time.

April 30, 1977

Find fund. solution of heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(x,t) = \frac{1}{2\pi}\int e^{+ix\xi}\,\hat{u}(\xi,t)\,dt$$

$$u(x,0) = \delta(x) \implies \hat{u}(\xi,0) = 1$$

$$\frac{\partial \hat{u}}{\partial t} = -\xi^2 \hat{u} \implies \hat{u} = e^{-t\xi^2}$$

$$u(x,t) = \frac{1}{2\pi}\int e^{-t\xi^2 + ix\xi}\,d\xi \quad e^{-i^2\frac{x^2}{4t} - \frac{x^2}{4t}}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int e^{-t\xi^2}\,d\xi\,\frac{\sqrt{t}}{\sqrt{t}} = \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \frac{\sqrt{\pi}}{\sqrt{t}}$$

$$\boxed{u = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}}$$

Consider now heat condition on Fourier's ring of period 2π. The eigenfunctions are $e^{-n^2 t} e^{inx}$ so the solution with initial value $\delta_{2\pi\mathbb{Z}}(x)$ is

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx} = \sum_{n\in\mathbb{Z}} \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{(x-2\pi n)^2}{4t}}$$

Put $x \mapsto 2\pi x$, $t \mapsto \pi t^2$

$$\cancel{\frac{1}{2\pi}} \sum e^{-n^2\pi t^2} e^{in2\pi x} = \sum \frac{1}{\cancel{2\pi} t}\, e^{-\frac{(2\pi)^2(x-n)^2}{4\pi t^2}}$$

$$\sum e^{-\pi n^2 t^2} e^{2\pi i n x} = \frac{1}{t}\sum e^{-\pi\left[\frac{x^2}{t^2} - \frac{2xn}{t^2} + \frac{n^2}{t^2}\right]}$$

Put $x \mapsto xt$

$$\sum e^{-\pi n^2 t^2} e^{2\pi i n t x} = \frac{1}{t}\sum e^{-\frac{\pi n^2}{t^2}}\, e^{2\pi n\frac{x}{t}}\, e^{-\pi x^2}$$

Therefore if we put

$$u(x,t) = t^{1/2}\sum_n e^{-\pi n^2 t + 2\pi n t x - \frac{\pi}{2}x^2}$$

$$= \left(e^{\pi x^2} t\right)^{1/2} \sum_n e^{-\pi(nt-x)^2}$$

we have the relation $\quad u(t, ix) = u(\frac{1}{t}, x)\quad$ hence

$$u(t,x) = u\left(\frac{1}{t}, \frac{x}{i}\right) = u(t, -x)$$

May 1, 1977. Start with the basic identity:

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} e^{2\pi i n x} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{t}(x-n)^2}$$

$$= \frac{e^{-\frac{\pi}{t}x^2}}{\sqrt{t}} \sum_n e^{-\pi n^2 \frac{1}{t} + 2\pi n \frac{x}{t}}$$

i.e.

$$\theta(t,x) = \frac{e^{-\frac{\pi}{t}x^2}}{\sqrt{t}}\, \theta\left(\frac{1}{t}, \frac{x}{it}\right)$$

[redacted] $\theta(t,x)$ is periodic in x of period 1 $\Rightarrow \theta\left(\frac{1}{t}, \frac{x}{it}\right)$ is periodic in x of period it. Thus if I let $x = a+ib$ [redacted], a fixed, $b \to \pm\infty$, then $\theta\left(\frac{1}{t}, \frac{x}{it}\right)$ should be bounded, so $\theta(t,x)$ grows like

$$e^{-\frac{\pi}{t}(a+ib)^2} \sim e^{+\frac{\pi}{t}b^2}$$

May 1, 1977

Asymptotic expansions.

$$(1)\qquad -\frac{d^2u}{dx^2} + qu = \lambda^2 u$$

Put $\quad u = e^{iS(x,\lambda)}$.

$$-\frac{d}{dx}\left(e^{iS}\, iS_x\right) = -e^{iS}\left(-S_x^2 + iS_{xx}\right) = (\lambda^2 - q)e^{iS}$$

$$(2)\qquad S_x^2 - (\lambda^2 - q) - iS_{xx} = 0$$

I claim there is a unique asymptotic expansion

$$S = \lambda x + u_0(x) + u_1(x)\lambda^{-1} + \cdots$$

such that $S(0,\lambda) = 0$. To see this write

$$S = \lambda x + T$$

whence (2) becomes

$$(\lambda + T_x)^2 - \lambda^2 + q - iT_{xx} = 0$$

$$2\lambda T_x + T_x^2 + q - iT_{xx} = 0$$

and it's clear one can successively solve for the coefficients of

$$T_x = u_0' + u_1'\lambda^{-1} + u_2'\lambda^{-2} + \cdots$$

Note $u_0' = 0$.

Through terms up to λ^{-1} one gets

$$S_x = \lambda - \frac{q}{2}\lambda^{-1} - i\frac{q'}{4}\lambda^{-2}$$

which agrees up to factors independent of x with

$$(\lambda^2-g)^{-1/4}\, e^{i\int_0^x (\lambda^2-g)^{1/2}}$$

One obtains another asymptotic solution by replacing λ by $-\lambda$.

<u>Question</u>: What have these asymptotic solutions to do with real solutions?

Rewrite (1) in the form

$$\frac{1}{\lambda^2}\frac{d^2u}{dx^2} + \left(1-\frac{g(x)}{\lambda^2}\right)u = 0$$

and put $y=\lambda x$ whence we get

$$(1)' \qquad \frac{d^2u}{dy^2} + \left(1-\frac{1}{\lambda^2}g\left(\frac{y}{\lambda}\right)\right)u = 0$$

Assuming g analytic near 0, then $\frac{1}{\lambda^2}g\left(\frac{y}{\lambda}\right)$ is analytic around $\frac{1}{\lambda}=0$, so any solution of $(1)'$ with initial values independent of λ should be holomorphic in $\frac{1}{\lambda}$. The problem is that if $\Phi(y,\frac{1}{\lambda})$ is the solution matrix for $(1)'$ starting at $y=0$ is $\Phi(\lambda x, \frac{1}{\lambda})$ holomorphic in $\frac{1}{\lambda}$, aside from the $e^{\pm i\lambda x}$ parts.

May 2, 1977

Consider

$$\frac{d^2u}{dy^2} + \left[1 - \left(q\left(\frac{y}{\lambda}\right)\frac{1}{\lambda^2}\right)\right]u = 0$$

Put $\varepsilon = \frac{1}{\lambda}$ and $u = e^{iy}v$.

$$\left(\frac{d^2u}{dy^2} + u\right) = e^{iy}\frac{d^2v}{dy^2} + 2ie^{iy}\frac{dv}{dy} + \cancel{(-e^{iy})}v + \cancel{e^{iy}v}$$

$$= \left(+ \varepsilon^2 q(\varepsilon y)\right)e^{iy}v$$

or

$$\frac{d^2v}{dy^2} + 2i\frac{dv}{dy} = \varepsilon^2 q(\varepsilon y)v$$

Look for a series solution $v = a_0(y) + a_1(y)\varepsilon + \cdots$. If $q(\varepsilon y) = q_0 + q_1\varepsilon y + q_2\varepsilon^2 y^2 + \cdots$ then comparing coeffecients of powers of ε we get

$$\frac{d^2a_i}{dy^2} + \underset{\sqrt{-1}}{2i}\frac{da_i}{dy} = q_{i-2}y^{i-2}a_0 + \cdots + q_1 y\, a_{i-3} + q_0 a_{i-2}$$

Assuming inductively that a_j is a poly in y of degree $\leq j-1$, and using that $\frac{d}{dy} + 2i$ acts as an isomorphism on the polys of degree $\leq n$ for any n, one sees there is a unique choice for a_i as a poly. in y and its degree is $\leq i-1$, up to an additive constant. So we can make a_i unique by requiring it to vanish at 0.

It remains to ~~show~~ decide whether the resulting series for $v(y,\varepsilon)$ is convergent. Can you find a formula for $\left(\frac{d}{dy} + 2i\right)^{-1}$ on polynomials?

Sobering example:

$$\left(1-\frac{d}{dz}\right)^{-1}\left(\frac{1}{z}\right) = \left(1+\frac{d}{dz}+\frac{d^2}{dz^2}+\cdots\right)\left(z^{-1}\right)$$

$$= z^{-1} + (-1)z^{-2} + (-1)(-2)z^{-3} + \cdots + (-1)^{n-1}(n-1)!\,z^{-n} + \cdots$$

is a divergent series. Obvious method of trying to invert $1-\frac{d}{dz}$ is by

$$(*) \qquad \left(1-\frac{d}{dz}\right)^{-1} f \;\text{``}=\text{''}\; -e^{z}\int^{z} e^{-t} f(t)\,dt$$

where the initial point for the integration has to be specified. Note that the above has to be related to the well-known asymptotic expansion for the exponential integral

$$\int_z^\infty e^{-t}\,\frac{dt}{t} = e^{-z}\left(z^{-1} - z^{-2} + 2z^{-3} - \cdots + (-1)^{n-1}(n-1)!\,z^{-n} + \cdots\right)$$

Next let us compare both sides of $(*)$ when f is a polynomial and the initial point is $z=a$. One has

$$-e^{z}\int_a^z e^{-t} f(t)\,dt = \left[\left(1+\frac{d}{dz}+\frac{d^2}{dz^2}+\cdots\right)f(z)\right]_a^z$$

which is not what I want e.g. $f(z)=z$ gives

$$\left[z+1\right]_a^z = z-a$$

which is correct only for $a=-1$, whereas $f(z)=z^2$ gives

$$\left[z^2+2z+2\right]_a^z$$

which is correct $\Leftrightarrow$ $a^2+2a+2=0$.

However it is clear that what I want is

the formula

$$\boxed{\left(1-\frac{d}{dz}\right)^{-1} f(z) = e^{z}\int_{z}^{\infty} e^{-t} f(t)\,dt}$$

To prove this works note

$$\int_z^\infty e^{-t}t^n\,dt = \left[-e^{-t}t^n\right]_z^\infty + n\int_z^\infty e^{-t}t^{n-1}\,dt$$

$$= e^{-z}z^n + n\int_z^\infty e^{-t}t^{n-1}\,dt$$

so

$$e^z\int_z^\infty e^{-t}t^n\,dt = z^n + n\left(z^{n-1} + (n-1)(z^{n-2} \cdots\right.$$

$$= z^n + nz^{n-1} + n(n-1)z^{n-2} + \cdots$$

$$= \left(1 + \frac{d}{dz} + \frac{d^2}{dz^2} + \cdots\right)z^n$$

Observe that the constant term is $n!$ hence this operator on polynomial will not extend to series to give convergence

March 4, 1977.

We consider

$$(1) \qquad -\frac{d^2u}{dx^2} + q(x)u = \lambda^2 u$$

around $x=0$, $q(x)$ being supposed analytic if we want. We have seen that we can find unique formal solutions of the form

$$(2) \qquad e^{ix\lambda}\left(a_0(x) + a_1(x)\lambda^{-1} + \cdots\cdots\right)$$

$$e^{-ix\lambda}\left(b_0(x) + b_1(x)\lambda^{-1} + \cdots\right)$$

where $a_i(0) = b_i(0) = 0 \quad i>0$ and $a_0(x) = b_0(x) = 1$.

Moreover one has

$$e^{ix\lambda}\left(a_0(x) + a_1(x)\lambda^{-1} + \cdots\right) = \left(1 - \frac{q - q(0)}{\lambda^2}\right)^{-1/4} e^{i\int_0^x (\lambda^2 - q)^{1/2} dx} + O(\lambda^{-3}).$$

If $u = ae^{i\lambda x}$, then (1) becomes

$$-\left(a'' e^{i\lambda x} + 2a' i\lambda e^{i\lambda x} + a(i\lambda)^2 e^{i\lambda x}\right) + qae^{i\lambda x} = \lambda^2 a e^{i\lambda x}$$

or

$$\boxed{a'' + 2i\lambda a' = qa}$$

so (2) is constructed by a process which probably will lead to a divergent series.

Fröman approach (N. and P. Fröman, JWKB approximation, Contributions to the theory, North-Holland Amsterdam 1965). One seeks a solution of (1) in the form

$$u = a(x)e^{i\lambda x} + b(x)e^{-i\lambda x}$$

subject to the variation of constants condition

$$a' e^{i\lambda x} + b' e^{-i\lambda x} = 0.$$

Then
$$u = a e^{i\lambda x} + b e^{-i\lambda x}$$
$$u' = a(i\lambda e^{i\lambda x}) + b(-i\lambda e^{-i\lambda x})$$
$$u'' = a'(i\lambda e^{i\lambda x}) + b'(-i\lambda e^{-i\lambda x}) + (-\lambda^2)u$$

so
$$a' e^{i\lambda x} + b' e^{-i\lambda x} = 0$$
$$a'(i\lambda e^{i\lambda x}) + b'(-i\lambda e^{-i\lambda x}) = q a e^{i\lambda x} + q b e^{-i\lambda x}$$

$$a'(2i\lambda e^{i\lambda x}) = q a e^{i\lambda x} + q b e^{-i\lambda x}$$
$$b'(2i\lambda e^{-i\lambda x}) = -q a e^{i\lambda x} - q b e^{-i\lambda x}$$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{q}{2i\lambda}\begin{pmatrix} 1 & e^{-2i\lambda x} \\ -e^{2i\lambda x} & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

They put
$$M = \frac{q(x)}{2i\lambda}\begin{pmatrix} 1 & e^{-2i\lambda x} \\ -e^{2i\lambda x} & -1 \end{pmatrix}$$
and denote by $F(x, x_0, \lambda)$ the solution matrix for the above DE starting at x_0:
$$F(x, x_0) = I + \int_{x_0}^{x} M(y)\, F(y, x_0)\, dy$$

Consider finding asymptotic solutions for
$$A\frac{du}{dx} + Bu = i\lambda u$$
of the form $u = e^{iS} v$, where $v = v_0 + v_1 \frac{1}{\lambda} + \cdots$

$$A\left[iS_x v + v_x\right] + Bv = i\lambda v$$

So we want $\frac{\lambda}{S_x}$ to be an eigenvalue for A. Supposing the eigenvalues of $A(x)$ distinct $\neq 0$ at each x they should depend smoothly on x, hence we get eigenvalues $\lambda_1(x), \lambda_2(x)$ for $A(x)$. This gives the equation

$$S_x = \frac{\lambda}{\lambda_i(x)} \qquad \text{for } S. \qquad i = 1,2$$

It follows that v_0 must be a multiple, depending on λ, x times the eigenvector corresponding to the choice of eigenvalue. Suppose we fix one of the eigenvalues $\lambda(x)$ and let $e(x)$ be a smooth choice of eigenvector:

$$A(x)e(x) = \lambda(x)e(x) \qquad e(x) \neq 0.$$

$$S_x = \frac{\lambda}{\lambda(x)}.$$

$$\left[Ai\frac{\lambda}{\lambda(x)} - i\lambda\right]v + Av_x + Bv = 0$$

It is clear $v_0 = f(x)e(x)$ some f. The next equation is

$$i\left(A\frac{1}{\lambda(x)} - 1\right)v_1 + A(v_0)_x + Bv_0 = 0$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$i\left(A\frac{1}{\lambda(x)} - 1\right)v_1 + f_x\lambda(x)e(x) + f\left(A(e)_x + Be\right) = 0$$

By proper choice of f_x one can get an arbitrary multiple of $e(x)$, and by choosing v_1 on the complement of e the first term can be made an arbitrary vector in the complement. Thus it's clear how to grind out the asymptotic series: one determines the e-part of v_{n-1} and the complement-to-e-part of v_n at the n_n^{th} stage.

So apply this to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{du}{dx} + \begin{pmatrix} 0 & -\bar{p} \\ p & 0 \end{pmatrix} u = i\lambda u$$

Here the eigenvalues of A are ± 1, hence $S_x = \pm\lambda$ so $e^{iS} = e^{\pm i\lambda x}$.

$$\begin{pmatrix} \frac{d}{dx} - i\lambda & -\bar{p} \\ p & -\frac{d}{dx} - i\lambda \end{pmatrix} u = 0 \qquad u = e^{i\lambda x} v$$

$$\left(\frac{d}{dx} + i\lambda\right) u = e^{i\lambda x}\left(\frac{dv}{dx} + 2i\lambda v\right)$$

$$\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} - 2i\lambda \end{pmatrix} v = 0$$

Suppose $v = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \lambda^{-1} + \cdots$. Then comparing coeffs. of λ^{-1} one gets $b_0 = 0$.

$$(a_0' + a_1'\lambda^{-1} + \cdots) - \bar{p}(b_0 + b_1\lambda^{-1} + \cdots) = 0$$

$$p(a_0 + a_1\lambda^{-1} + \cdots) - (b_0' + b_1'\lambda^{-1} + \cdots) - 2i\lambda(b_0 + b_1\lambda^{-1} + \cdots) = 0$$

$$2i\lambda b_0 = 0 \qquad a_0' = \bar{p} b_0$$

$$\Downarrow$$

$$b_0 = 0 \implies a_0' = 0 \quad . \quad \text{So can suppose } a_0 = 1.$$

$$p a_0 - \cancel{b_0'} - 2ib_1 = 0 \implies b_1 = \frac{p}{2i}$$

$$a_1' = \bar{p} b_1 = \frac{|p|^2}{2i} \qquad a_1 = \int_0^x \frac{|p|^2}{2i}\, dx$$

$$2ib_2 = pa_1 - b_1' = p\int_0^x \frac{|p|^2}{2i}dx - \frac{p'}{2i}$$

$$b_2 = \frac{p}{2i}\int_0^x \frac{|p|^2}{2i}dx + \frac{p'}{4}$$

$$a_2' = \bar{p}b_2 = \frac{|p|^2}{2i}\int_0^x \frac{|p|^2}{2i}dx + \frac{p'\bar{p}}{4}$$

$$a_2 = \frac{1}{2}\left(\int_0^x \frac{|p|^2}{2i}dx\right)^2 + \int_0^x \frac{p'\bar{p}}{4}dx$$

Thus we get

$$a = e^{-i\lambda^{-1}\int_0^x \frac{|p|^2}{2}dx}\left(1 + \lambda^{-2}\int_0^x \frac{\bar{p}p'}{4}dx + \cdots\right)$$

$$b = e^{-i\lambda^{-1}\int_0^x \frac{|p|^2}{2}dx}\left(\lambda^{-1}\frac{p}{2i} + \lambda^{-2}\frac{p'}{4} + \cdots\right)$$

Recall that

$$\sqrt{\lambda^2 - |p|^2} = \lambda - \lambda^{-1}\frac{|p|^2}{2} + O(\lambda^{-3})$$

$$\int_0^x \sqrt{\lambda^2 - |p|^2}\,dx = \lambda x - \lambda^{-1}\int_0^x \frac{|p|^2}{2}dx + O(\lambda^{-3})$$

hence our asymptotic solution can be written

$$\boxed{u = e^{i\int_0^x \sqrt{\lambda^2 - |p|^2}\,dx}\begin{pmatrix} 1 + \lambda^{-2}\int_0^x \frac{\bar{p}p'}{4}dx + \cdots \\ \lambda^{-1}\frac{p}{2i} + \lambda^{-2}\frac{p'}{4} + \cdots \end{pmatrix}}$$

What is the Froman approach here? If $\frac{dX}{dt} = AX$ has the solution matrix S, then to solve

$$\frac{dX}{dt} = AX + B \qquad \qquad \frac{dS}{dt} = AS, \quad S(0) = I$$

put $X = SY$ with $Y = Y(t)$. Then

$$\frac{d(SY)}{dt} = ASY + S\frac{dY}{dt} = ASY + B$$

$$\frac{dY}{dt} = S^{-1}B$$

or

$$Y = \int_0^t S^{-1}B\,dt + Y(0)$$ [redacted]

Apply this to our [redacted] system

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u = \underbrace{\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}}_{A} u + \underbrace{\begin{pmatrix} 0 & \bar{p} \\ p & 0 \end{pmatrix}}_{B} u$$

$$S = \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix}$$

$$S^{-1}u = S^{-1}u(0) + $$ [redacted] $$\int_0^x S^{-1}\begin{pmatrix} 0 & \bar{p} \\ p & 0 \end{pmatrix} u$$

$$e^{-i\lambda x}u_1(x) = e^{-i\lambda x}u_1(0) + \int_0^x e^{-2i\lambda y}\bar{p}(y)\left(e^{i\lambda y}u_2(y)\right)dy$$

$$e^{+i\lambda x}u_2(x) = e^{+i\lambda x}u_2(0) + \int_0^x e^{+2i\lambda y}p(y)\left(e^{-i\lambda y}u_1(y)\right)dy$$

Thus if I put $\boxed{u = Sv}$ one has

$$\boxed{v(x) = v(0) + \int_0^x \begin{pmatrix} 0 & e^{-2i\lambda y}\bar{p}(y) \\ e^{2i\lambda y}p(y) & 0 \end{pmatrix} v(y)\,dy}$$

May 11, 1977

Add to April 1 the following:

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

$$S = \begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}$$ solution matrix starting at $x = 0$

$\Delta_b = S(b,\lambda)^{-1}\, P_1(\mathbb{R})$ is a circle of radius:

$$\frac{1}{r(\Delta_b)} = \left| \begin{vmatrix} \psi(b) & \bar{\psi}(b) \\ \psi'(b) & \bar{\psi}'(b) \end{vmatrix} \right| \qquad \lambda \notin \mathbb{R}$$

and center

$$c(\Delta_b) = \frac{\begin{vmatrix} \bar{\psi}(b) & \varphi(b) \\ \bar{\psi}'(b) & \varphi'(b) \end{vmatrix}}{\begin{vmatrix} \psi(b) & \bar{\psi}(b) \\ \psi'(b) & \bar{\psi}'(b) \end{vmatrix}}$$

But

$$\frac{d}{dx}\begin{vmatrix} \bar{\psi} & \varphi \\ \bar{\psi}' & \varphi' \end{vmatrix} = \begin{vmatrix} \bar{\psi} & \varphi \\ (q-\bar{\lambda})\bar{\psi} & (q-\lambda)\varphi \end{vmatrix} = (\bar{\lambda}-\lambda)\varphi\bar{\psi} = -2i\,\mathrm{Im}(\lambda)\varphi\bar{\psi}$$

$$\begin{vmatrix} \bar{\psi} & \varphi \\ \bar{\psi}' & \varphi' \end{vmatrix}(b) = 1 + \int_0^b (\bar{\lambda}-\lambda)\varphi\bar{\psi} = 1 - 2i\,\mathrm{Im}\lambda \int \varphi\bar{\psi}$$

$$\begin{vmatrix} \psi & \bar{\psi} \\ \psi' & \bar{\psi}' \end{vmatrix}(b) = \int_0^b (\lambda-\bar{\lambda})\psi\bar{\psi} = 2i\,\mathrm{Im}\,\lambda \int \psi\bar{\psi}$$

So if $\mathrm{Im}\,\lambda > 0$ one has

$$\frac{1}{r(\Delta_b)} = 2\,\mathrm{Im}(\lambda)\int_0^b |\psi|^2 dx \qquad r(\Delta_b) = \frac{1}{2\,\mathrm{Im}(\lambda)\int_0^b |\psi|^2 dx}$$

$$c(\Delta_b) = \frac{1 - i2\,\mathrm{Im}(\lambda)\int_0^b \varphi\bar{\psi}\,dx}{i2\,\mathrm{Im}(\lambda)\int_0^b |\psi|^2 dx}$$

But note that if we minimize

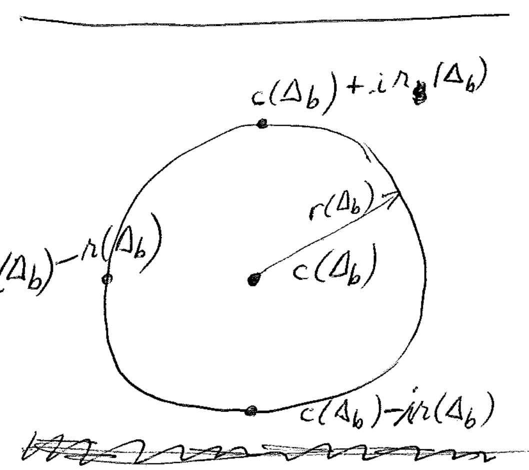
$$\|m\psi+\varphi\|^2 = \int_0^b (m\psi+\varphi)^2 dx$$

then we have

$$(m\psi+\varphi,\psi) = m\|\psi\|^2 + (\varphi,\psi)$$

or

$$m = -\frac{(\varphi,\psi)}{\|\varphi\|^2}$$



Notice also that

$$c(\Delta_b) - \frac{r(\Delta_b)}{i} = -\frac{\int_0^b \varphi\bar{\psi}\,dx}{\int_0^b |\psi|^2 dx}$$

Therefore we see that

$$m_b(\lambda) = -\frac{\int_0^b \varphi\bar{\psi}\,dx}{\int_0^b |\psi|^2 dx}$$

is on Δ_b. In fact it seems that it is the point on Δ_b closed to the real axis, (Because $S(x,\lambda)$ shrinks the UHP $S^{-1}(b,\lambda)$ carries $P_1(\mathbb{R})$ into the lower half plane.)

So we get the formula

$$m_\infty(\lambda) = \lim_{b\to\infty} -\frac{\int_0^b \varphi\bar{\psi}\,dx}{\int_0^b |\psi|^2 dx}$$

which might also be valid for λ real.

Simpler formula for m_∞:

$$m_\infty = \lim_{b\to\infty} S(b,\lambda)^{-1}(r) = \lim_{b\to\infty} \frac{r\varphi'(b,\lambda) - \varphi(b,\lambda)}{-r\psi'(b,\lambda) + \psi(b,\lambda)}$$

for any real number r including ∞. Thus if $r=0$ we get

$$m_\infty(\lambda) = \lim_{b\to\infty} -\frac{\varphi(b,\lambda)}{\psi(b,\lambda)} .$$

Argument: For each λ there should exist a unique up-to-scalar $u(x,\lambda)$ which dies at $x=+\infty$. Thus we get a line bundle over $\mathbb{C}$ whose fibre at λ is the line of solutions u dying at ∞. This line bundle has to be trivial hence we can trivialize it and obtain $u(x,\lambda) = a(\lambda)\psi(x,\lambda) + b(\lambda)\varphi(x,\lambda)$ unique up to an invertible function $e^{g(\lambda)}$ of λ. Now what I want to arrange is for $u(x,\lambda)$ to be of exponential type and rapidly decreasing as $\lambda \to \pm\infty$.