April 20, 1977:

Philosophy: Up to now you have been thinking in terms of solutions of

(1) $$-\frac{d^2\psi}{dx^2} + q\psi = \lambda^2\psi$$

with λ constant, and x varying globally. But what you want to do is to think globally in λ and locally in x so that you can take the Fourier transform with respect to λ and get the wave equation.

So suppose we work around $x=0$. Let $c(x,\lambda)$, $s(x,\lambda)$ denote the solutions of (1) with

$$\begin{pmatrix} c & s \\ c' & s' \end{pmatrix}(0,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It should be true that c and s have asymptotic expansions in λ

(2) $$c(x,\lambda) = e^{i\lambda x}\Big(\underbrace{a_0(x)}_{=1/2} + a_1(x)\lambda^{-1} + \cdots\Big) + e^{-i\lambda x}\Big(\bar{a}_0(x) + \bar{a}_1(x)\lambda^{-1} + \cdots$$

which can be found formally. Note that better asymptotic formulas to the first (2nd) order can be found if one uses

$$e^{i\int_0^x \sqrt{\lambda^2 - q}} \sim e^{i\left(\lambda x - \frac{\int_0^x q}{2}\lambda^{-1}\right)}$$

I think that (2) implies the Fourier transform of $c(x,\lambda)$ with respect to λ has support in $[-|x|, |x|]$ with singularities at the ends.

Suppose $c(x,\lambda)$ as above. Then put

$$\tilde{c}(x,y) = \int e^{-i\lambda y} c(x,\lambda)\, d\lambda .$$

One has

$$\tilde{c}(0,y) = \int e^{-i\lambda y}\, d\lambda = \frac{1}{2\pi}\delta(y)$$

$$\frac{\partial \tilde{c}}{\partial x}(0,y) = 0$$

and $\tilde{c}$ satisfies the wave equation

$$\frac{\partial^2 \tilde{c}}{\partial y^2} = \frac{\partial^2 \tilde{c}}{\partial x^2} - q(x)\tilde{c}$$

Similarly $\tilde{s}(x,\lambda)$ satisfies the wave equation with the initial conditions

$$\tilde{s}(0,y) = 0$$

$$\frac{\partial \tilde{s}}{\partial x}(0,y) = \int e^{-i\lambda y} \frac{\partial s}{\partial x}(x,\lambda)\, d\lambda = \int e^{-i\lambda y}\, d\lambda = \frac{1}{2\pi}\delta(y).$$

Put

$$\psi(x,\lambda) = \int_0^x \frac{\sin \lambda(x-y)}{\lambda} f(y)\, dy$$

Then $\psi(0,\lambda) = 0$

$$\psi_x(x,\lambda) = \int_0^x \cos \lambda(x-y) f(y)\, dy \qquad \psi_x(0,\lambda) = 0$$

$$\psi_{xx}(0,\lambda) = f(x) + \int_0^x -\lambda \sin(x-y) f(y)\, dy.$$

Thus $\quad \dfrac{\partial^2 \psi}{\partial x^2} + \lambda^2 \psi = f(x) \quad$ and $\quad \psi(0) = \psi_x(0) = 0$.

It follows that $c(x,\lambda)$ satisfies the integral equation

$$c(x,\lambda) = \cos(\lambda x) + \int_0^x \frac{\sin \lambda(x-y)}{\lambda}\, q(y)\, c(y,\lambda)\, dy$$

because $\quad \dfrac{d^2 c}{dx^2} + \lambda^2 c = q \cdot c$.

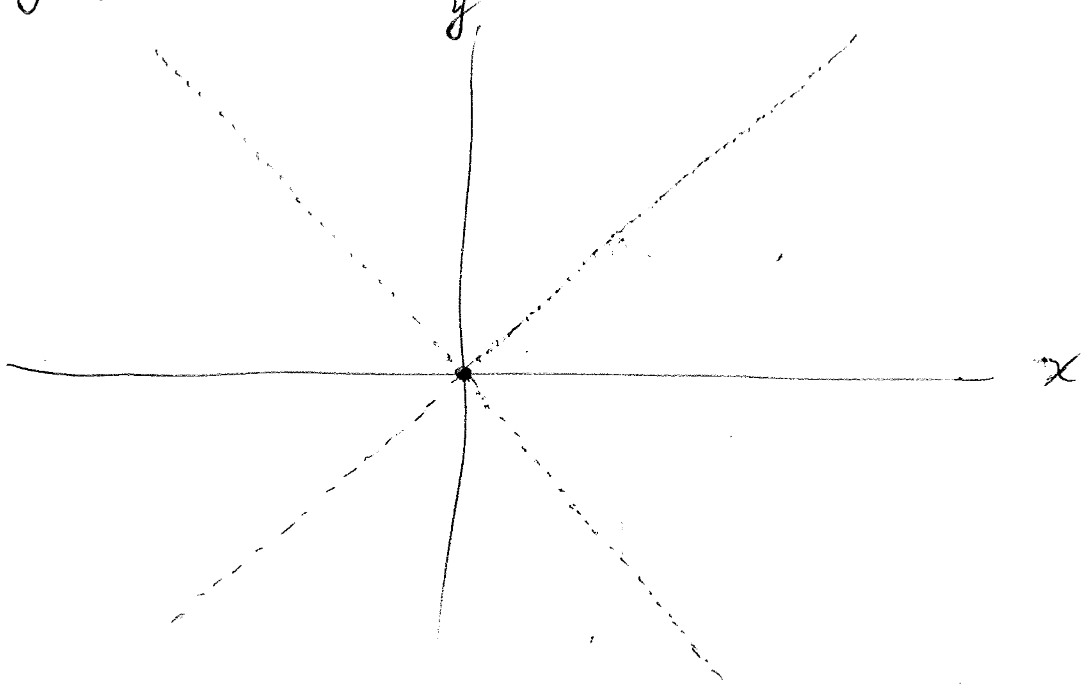
A basic fact about Wave equations is unique solvability of the Cauchy problem across non-characteristic hypersurfaces. In particular singularities propagate [redacted] along characteristics. Let's consider the operator

$$L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} + q$$

and a fundamental solution for it:

$$L v = \delta_{(0,0)}$$

Then the singularities of v can only lie on the characteristics issueing from the origin



It would seem that there exists a Green's function g which is zero for $x<0$. Then g would be supported in $\{(x,y) \mid x \geq 0,\ |y| \leq x\}$.

Example: If $q=0$, then the function

$$g(x,y) = \begin{cases} 1 & 0 \leq x \geq |y| \\ 0 & \text{otherwise} \end{cases}$$

which is 1 in the x-forward cone and 0 outsides satisfies $Lg=0$ away from 0 because locally g is a function of $x-y$ or of $x+y$ away from 0. In fact

$$g(x,y) = H(x-y)\,H(x+y)$$

up to a constant.

$$L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} = -\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

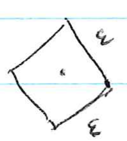
$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\frac{\partial y}{\partial u} \qquad \begin{cases} x = u + v \\ y = -u + v \end{cases}$$

$$L = -\frac{\partial^2}{\partial u \partial v}$$

$$H(x-y)\,H(x+y) = H(2u)\,H(2v)$$

$$L\{H(x-y)\,H(x+y)\} = -\frac{\partial H(2u)}{\partial u} \cdot \frac{\partial H(2v)}{\partial v} = -4\,\delta(u)\,\delta(v)$$

$$= -2\,\delta(x)\,\delta(y)$$



Change L to $\qquad L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - q.$

$$\oint M\,dx + N\,dy = \iint\left(-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right)dx\,dy$$

If $g = H(x+y)H(x-y)$ and $\rho \in C_0^\infty(\mathbb{R}^2)$

$$\iint \rho\, Lg = \iint \left(\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2}\right) dx\, dy = \text{[redacted]}$$

$$= \iint_{u \geq 0\ v \geq 0} \frac{\partial^2 \rho}{\partial u \partial v}\, 2\, du\, dv = 2 \int_{u \geq 0} -\frac{\partial \rho}{\partial u}(u, 0)$$

$$= 2\rho(0,0)$$

Thus

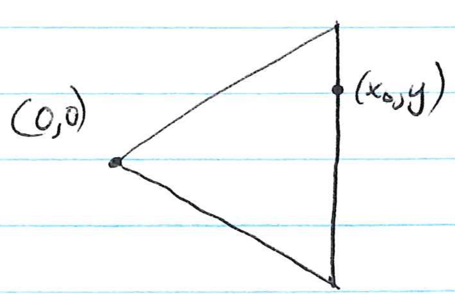
$$\boxed{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\left(\frac{H(x+y)\, H(x-y)}{2}\right) = \delta(x)\, \delta(y).}$$

Use the variables ξ, η for u, v:

$$x = \xi + \eta$$
$$y = -\xi + \eta$$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \xi \partial \eta}$$

The fundamental solution g of $\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} - \rho g = \delta$ can be used to solve the Cauchy problem. Thus given a solution u one can expresses $u(0,0)$ as an integral of the values $u(x_0, y)$, $u_x(x_0, y)$ for $|y| \leq x$.



April 22, 1977

Suppose good boundary conditions for

$$\frac{d^2\psi}{dx^2} + (\lambda^2 - q)\psi = 0$$

are given on $0 \le x \le b$, so that we have eigenvalues $\pm\lambda_n$ and eigenfunctions $\psi_n(x)$ for $n \ge 1$. Suppose 0 not an eigenvalue and $\int_0^b |\psi_n|^2\,dx = 1$. Form the kernel

$$K_t(x,x') = \frac{1}{2}\sum_{n\ge 1} e^{i\lambda_n t}\,\psi_n(x)\overline{\psi_n(x')} + \frac{1}{2}\sum_{n\ge 1} e^{-i\lambda_n t}\,\psi_n(x)\overline{\psi_n(x')}$$

$$= \sum_{n\ge 1} \cos\lambda_n t\;\psi_n(x)\overline{\psi_n(x')}$$

Then if

$$(K_t * f)(x) = \int_0^b K_t(x,x')f(x')\,dx'$$

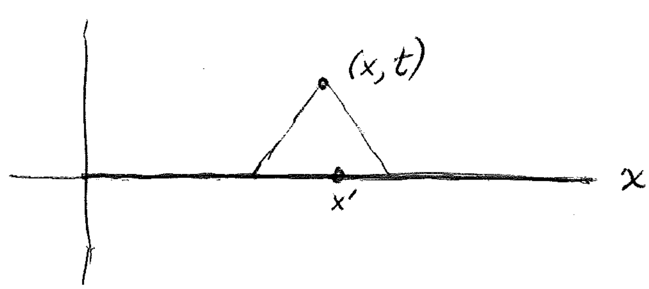
one has

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + q\right)K_t * f = 0$$

$$K_0 * f = f$$

$$\left(\frac{\partial K}{\partial t} * f\right)_{t=0} = 0$$

Thus $(K_t * f)(x)$ is a solution of the Cauchy problem for the wave equation with the initial values $f, 0$ along the line $t=0$



so we maybe know from the theory of these wave equations that $K_t(x,x')$ should have its support in $|x'-x| \leq t$. Assume so.

Next let's shift to systems. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$$\underbrace{\frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix}}_{P} \psi = \lambda \psi$$

P is a first order self-adjoint operator which is elliptic as its symbol is $\xi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (In fact it has the property that the full symbol

$$\begin{vmatrix} \xi & -\frac{\bar{p}}{i} \\ \frac{p}{i} & -\xi \end{vmatrix} = -\xi^2 + \frac{p\bar{p}}{i^2} = -(\xi^2 + p\bar{p})$$

doesn't vanish for ξ real.)

Next consider e^{-itP} which will yield solutions of the Cauchy problem

$$\left(\frac{1}{i}\frac{\partial}{\partial t} + P\right)\left(e^{-itP} f\right) = 0$$

$$\left(e^{-itP} f\right)_{t=0} = f$$

As above if $P\psi_n = \lambda_n \psi_n$ are the normalized eigenfunctions and eigenvalues, P is represented by the kernel

$$\sum_n \lambda_n \psi_n(x)\, \psi_n(x')^* \qquad \text{2x2 matrix}$$

so e^{-itP} is represented by the kernel

$$k_t(x,x') = \sum_n{}' e^{-it\lambda_n} \psi_n(x)\psi_n(x')^*$$

The wave equation under consideration is

$$\left(\frac{\partial}{\partial t} + iP\right) u = 0$$

or

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix}0 & -\bar{p}\\ p & 0\end{pmatrix}\right] u = 0$$

Weyl equation (neutrinos)

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial}{\partial t} + \underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{=\sigma_1}\frac{\partial}{\partial x} + \underbrace{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}_{=\sigma_2}\frac{\partial}{\partial y} + \underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{=\sigma_3}\frac{\partial}{\partial z}\right] u = 0$$

where the σ_i are the Pauli spin matrices (inf. rotations around x, y, z axes).

The example I want to handle is $p = e^x$. If I put $r = e^x$, then

$$\frac{d}{dx} = \frac{d}{dr}\frac{dr}{dx} = \frac{d}{dr} e^x = r\frac{d}{dr}$$

hence we get the equation

$$\frac{\partial}{\partial t} u + r\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\partial}{\partial r} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right] u = 0$$

which we want to view as analogous to

$$\frac{\partial^2 u}{\partial t^2} = \left(r\frac{\partial}{\partial r}\right)^2 u - r^2 u$$

which has the solution
$$u = e^{-r\cosh t}$$

<u>April 23, 1977</u>

$q = q(x)$

Consider again $\quad Lu = \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + q u \quad$ on $\quad 0 \le x < \infty$.

Put
$$H = \frac{1}{2}\int_0^\infty \left(u_t^2 + u_x^2 + q u^2\right) dx$$
assuming this converges. Then $H = H(t)$ satisfies
$$\frac{dH}{dt} = \int_0^\infty \left(u_t u_{tt} + u_x u_{xt} + q u u_t\right) dx$$
$$= \Big[u_x u_t\Big]_0^\infty + \int_0^\infty u_t\left(u_{tt} - u_{xx} + q u\right) dx$$
$$= -u_x(0,t)\, u_t(0,t)$$

So the energy remains constant if $u_x(0,t) \equiv 0$ which means $x = 0$ is a reflecting endpoint, or if $u(0,t) = 0$ which means that the endpoint 0 is held fixed. Another case to consider is
$$u_x(0,t) = c\, u(0,t)$$
for then $\quad -u_x(0,t)\, u_t(0,t) = -c\, u(0,t)\, u_t(0,t) =$
$$= -\frac{c}{2}\frac{d}{dt}\left[u(0,t)^2\right]$$
Here $\quad H(t) = -\dfrac{c}{2}\, u(0,t)^2 + \text{constant}.$

Examples: $u = e^{-r\cosh t}$ which satisfies

$$\frac{\partial^2 u}{\partial t^2} = \left(r\frac{\partial}{\partial r}\right)^2 u - r^2 u$$

Now $$\psi(r,\lambda) = \int e^{i\lambda t} e^{-r\cosh t}\, dt = K_{i\lambda}(r)$$

is never identically zero in r for any λ, and $K_{i\lambda}(r) \longrightarrow 0$ rapidly uniformly in λ as $r \to +\infty$.

In general given $\frac{\partial^2 \psi}{\partial x^2} + (\lambda^2 - q)\psi = 0$ on $0 \leq x < \infty$

where $q(x) \nearrow +\infty$ as $x \to +\infty$, it should be true that the spectrum is discrete for any of the boundary conditions at 0. Moreover there should exist for any complex number λ a solution $\psi(x,\lambda)$ unique up to a scalar [redacted] which [redacted] is square integrable. I conjecture that it should always be possible to normalize $\psi(x,\lambda)$ as a function of λ in the following way:

1) $\psi(x,\lambda)$ should be holomorphic in λ and of exponential type [redacted] rapidly decreasing along [redacted] the real axis. This means that its Fourier transform $u(x,t)$ should be rapidly decreasing.

2) $\psi(x,\lambda)$ not identically zero in x for each λ.

The thing to prove first is that the eigenvalues are discrete. A possible method is to prove, using WKB, the existence of $\psi(x,\lambda)$ of the form

$$\psi(x,\lambda) \doteq (q-\lambda)^{-1/4} e^{-\int^x (q-\lambda)^{1/2}}$$

Once you have this, you have $\psi(x,\lambda)$ defined and it only remains to establish the properties for fixed x.

<u>April 24, 1977</u>

$$\frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -p \\ p & -\frac{d}{dx} \end{pmatrix}\psi = \lambda\psi$$

Take $p = e^x$ and change independent variable to $x = \log r$. $\quad \frac{d}{dx} = r\frac{d}{dr}$, $e^x = r$

$$\begin{pmatrix} \frac{d}{dr} & -1 \\ 1 & -\frac{d}{dr} \end{pmatrix}\psi = \frac{i\lambda}{r}\psi = \frac{k}{r}\psi \qquad \text{if } k = i\lambda$$

$$\boxed{\begin{aligned} \frac{d\psi_1}{dr} - \frac{k}{r}\psi_1 &= \psi_2 \\ \frac{d\psi_2}{dr} + \frac{k}{r}\psi_2 &= \psi_1 \end{aligned}}$$

$$\left(\frac{d}{dr} + \frac{k}{r}\right)\left(\frac{d}{dr} - \frac{k}{r}\right)\psi_1 = \psi_1 \qquad \left(\frac{d}{dr} - \frac{k}{r}\right)\left(\frac{d}{dr} + \frac{k}{r}\right)\psi_2 = \psi_2$$

$$\left(\frac{d^2}{dr^2} + \frac{k}{r^2} - \frac{k^2}{r^2}\right)\psi_1 = \psi_1$$

$$\boxed{\frac{d^2\psi_1}{dr^2} - \frac{k(k-1)}{r^2}\psi_1 = \psi_1} \qquad \boxed{\frac{d^2\psi_2}{dr^2} - \frac{k(k+1)}{r^2}\psi_2 = \psi_2}$$

I'm interested in working on the interval $[a,\infty)$ for some $a>0$. Hence asymptotically as $r \to \infty$ one should have

$$\begin{cases} \psi_1 \sim c\,e^{-r} \\ \psi_2 \sim -c\,e^{-r} \end{cases} \qquad c \text{ constant.}$$

Since I expect the solutions ψ to be something like Bessel functions, let's try power series expansions

$$\psi = \text{[redacted]}\ r^{\mu} \sum_{n \geq 0}' \begin{pmatrix} a_n \\ b_n \end{pmatrix} r^n$$

$$\frac{d\psi}{dr} = \mu r^{\mu-1} \sum \begin{pmatrix} a_n \\ b_n \end{pmatrix} r^n + r^{\mu} \sum \begin{pmatrix} a_n \\ b_n \end{pmatrix} n r^{n-1}$$

$$\frac{k}{r}\begin{pmatrix} -\psi_1 \\ \psi_2 \end{pmatrix} = k r^{\mu-1} \sum \begin{pmatrix} -a_n \\ b_n \end{pmatrix} r^n$$

$$\begin{pmatrix} -\psi_2 \\ -\psi_1 \end{pmatrix} = r^{\mu} \sum \begin{pmatrix} -b_n \\ -a_n \end{pmatrix} r^n$$

$$0 = r^{\mu-1}\ \text{[redacted]} \Big[(\mu\ \text{[redacted]}) \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + k \begin{pmatrix} -a_0 \\ b_0 \end{pmatrix}$$

$$+ \sum_{n \geq 1}^{\infty} \left\{ (\mu+n) \begin{pmatrix} a_n \\ b_n \end{pmatrix} + k \begin{pmatrix} -a_n \\ b_n \end{pmatrix} - \begin{pmatrix} b_{n-1} \\ a_{n-1} \end{pmatrix} \right\} r^{n-1} \Big]$$

The indicial equation is:

$$(\mu - k)\, a_0 = 0$$
$$(\mu + k)\, b_0 = 0$$

Other equations:

$$(\mu + n - k)\, a_n = b_{n-1}$$
$$(\mu + n + k)\, b_n = a_{n-1}$$

so the roots are $\mu = \pm k$. Assume the difference of the indicial roots is not integral i.e. $2k \notin \mathbb{Z}$. Then from each root we get a solution and the two solutions are linearly independent. [redacted]

The root $\mu = k$. $a_0 = 1$, $b_0 = 0$

$$a_n = \frac{b_{n-1}}{\mu+n-k} = \text{[redacted]}\ \frac{a_{n-2}}{(\mu+n-k)(\mu+n+k-1)} = \frac{a_{n-2}}{n(n+2k-1)}$$

$$b_n = \frac{a_{n-1}}{(\mu+n+k)} = \frac{b_{n-2}}{(\mu+n+k)(\mu+n-k-1)} = \frac{b_{n-2}}{(n+2k)(n-1)}$$

$$\begin{array}{cc}
1 & 0 \\
0 & \dfrac{r}{2k+1} \\
\dfrac{r^2}{2(2k+1)} & 0 \\
0 & \dfrac{r^3}{2(2k+1)(2k+3)} \\
\dfrac{r^4}{2\cdot 4\cdot(2k+1)(2k+3)} & 0 \\
0 & \dfrac{r^5}{2\cdot 4\,(2k+1)(2k+3)(2k+5)}
\end{array}$$

It seems that $\dfrac{d}{dr}\left(r^{-k}\psi_1\right) = r^{-k}\psi_2$

$$\frac{d}{dr}\left(r^{k}\psi_2\right) = r^k\psi_1$$

as they should be.

$$\psi_1 = r^k \sum_{m=0}^{\infty} \frac{r^{2m}}{2\cdot 4\cdots 2m \quad (2k+1)(2k+3)\cdots(2k+2m-1)}$$

$$= r^k \sum_{m=0}^{\infty} \left(\frac{r}{2}\right)^{2m} \frac{1}{m!} \frac{1}{(k+\frac{1}{2})(k+\frac{1}{2}+1)\cdots(k+\frac{1}{2}+m-1)} \cdot \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}$$

$$= r^k \sum_{m=0}^{\infty} \left(\frac{r}{2}\right)^{2m} \frac{1}{m!} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2}+m)}$$

$$= r^{1/2}\left(\frac{2}{i}\right)^{k-\frac{1}{2}}\Gamma\left(k+\tfrac{1}{2}\right)\frac{(ir)^{k-\frac{1}{2}}}{2^{k-\frac{1}{2}}}\sum_{m=0}^{\infty}\left(\frac{r}{2}\right)^{2m}\frac{1}{m!}\,\frac{1}{\Gamma\left(m+1+k-\frac{1}{2}\right)}$$

But

$$J_\lambda(iz) = \left(\frac{iz}{2}\right)^{\lambda}\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m}\frac{1}{m!}\,\frac{1}{\Gamma(m+1+\lambda)}$$

so it appears that

$$\psi_1 = \left(\frac{2}{i}\right)^{k-\frac{1}{2}}\Gamma\left(k+\tfrac{1}{2}\right) r^{1/2} J_{k-\frac{1}{2}}(ir)$$

Similarly

$$\psi_2 = r^k\frac{r}{2(k+\frac{1}{2})}\sum_{m=0}^{\infty}\frac{r^{2m}}{2^{2m}}\,\frac{1}{m!}\,\frac{1}{(k+\frac{3}{2})\cdots(k+\frac{1}{2}+m)}\,\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}$$

$$= \frac{r^{1/2}}{2}\left(\frac{2}{i}\right)^{k+\frac{1}{2}}\Gamma\left(k+\tfrac{1}{2}\right)\frac{(ir)^{k+\frac{1}{2}}}{2^{k+\frac{1}{2}}}\sum_{m=0}^{\infty}\left(\frac{r}{2}\right)^{2m}\frac{1}{m!}\,\frac{1}{\Gamma\left(m+1+k+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\left(\frac{2}{i}\right)^{k+\frac{1}{2}}\Gamma\left(k+\tfrac{1}{2}\right) r^{1/2} J_{k+\frac{1}{2}}(ir)$$

so

$$\boxed{\begin{aligned}\psi_1 &= c\, r^{1/2} J_{k-\frac{1}{2}}(ir)\\ \psi_2 &= \frac{1}{i}\, c\, r^{1/2} J_{k+\frac{1}{2}}(ir).\end{aligned}}$$

where c is a constant (k fixed). Can check this using

$$\frac{d}{dz}J_\lambda(z) = \frac{\lambda}{z}J_\lambda(z) - J_{\lambda+1}(z)$$

$$\frac{d}{dr}\left(r^{1/2}J_{k-\frac{1}{2}}(ir)\right) = r^{1/2}\left[\frac{k-\frac{1}{2}}{ir}J_{k-\frac{1}{2}}(ir) - J_{k+\frac{1}{2}}(ir)\right]i + \frac{1}{2}r^{-1/2}J_{k-\frac{1}{2}}(ir)$$

$$= \frac{k}{r}\left(r^{1/2}J_{k-\frac{1}{2}}(ir)\right) + \frac{1}{i}\left(r^{1/2}J_{k+\frac{1}{2}}(ir)\right)$$

so it works. Other solution should be

$$\boxed{\begin{aligned} \bar{\psi}_1 &= r^{1/2}J_{-k+\frac{1}{2}}(ir) \\ \bar{\psi}_2 &= i\,r^{1/2}J_{-k-\frac{1}{2}}(ir) \end{aligned}}$$

Check
$$\frac{d\psi_2}{dr} = ir^{1/2}\left[\frac{-k-\frac{1}{2}}{ir}J_{-k-\frac{1}{2}}(ir) - J_{-k+\frac{1}{2}}(ir)\right]i + i\,\frac{1}{2}r^{-1/2}J_{-k-\frac{1}{2}}(ir)$$

$$= -\frac{k}{r}\left(ir^{1/2}J_{-k-\frac{1}{2}}(ir)\right) + r^{1/2}J_{-k+\frac{1}{2}}(ir)$$

$$K_s(r) = \int_{-\infty}^{\infty} e^{-r\cosh t}\, e^{st}\, dt$$

$$\frac{d}{dr}K_s(r) = \int e^{-r\cosh t}(-\cosh t)e^{st}dt$$

$$= -\frac{1}{2}\int e^{-r\cosh t}\left(e^{t}+e^{-t}\right)e^{st}dt$$

$$\boxed{\frac{dK_s(r)}{dr} = -\frac{1}{2}\left(K_{s+1}(r) + K_{s-1}(r)\right)}$$

$$sK_s(r) = \int_{-\infty}^{\infty} e^{-r\cosh t}\, s e^{st}\, dt = -\int\left(e^{-r\cosh t}\right)' e^{st}\, dt$$

$$sK_s(r) = \int e^{-r\cosh t}\, r\sinh t\, dt = \frac{r}{2}\int e^{-r\cosh t}(e^t - e^{-t})e^{st}dt$$

$$\boxed{\frac{s}{r}K_s(r) = \frac{1}{2}\left(K_{s+1}(r) - K_{s-1}(r)\right)}$$

$$\frac{dK_s}{dr} + \frac{s}{r}K_s = -K_{s-1}(r) \qquad \frac{dK_s}{dr} - \frac{s}{r}K_s = -K_{s+1}$$

$$\boxed{\frac{dK_s}{dr} = -\frac{s}{r}K_s - K_{s-1} = \frac{s}{r}K_s - K_{s+1}}$$

Hence

$$\frac{dK_{s-\frac{1}{2}}}{dr} - \frac{(s-\frac{1}{2})}{r}K_{s-\frac{1}{2}} = -K_{s+\frac{1}{2}}$$

$$\frac{dK_{s+\frac{1}{2}}}{dr} + \frac{(s+\frac{1}{2})}{r}K_{s+\frac{1}{2}} = -K_{s-\frac{1}{2}}$$

Now if we put $\psi = r^{1/2}\varphi$ in the equations

$$\frac{d\psi_1}{dr} - \frac{s}{r}\psi_1 = \psi_2$$

(*)

$$\frac{d\psi_2}{dr} + \frac{s}{r}\psi_2 = \psi_1$$

we get $\quad r^{1/2}\frac{d\varphi_1}{dr} + \frac{1}{2}r^{-1/2}\varphi_1 - \frac{s}{r}r^{1/2}\varphi_1 = r^{1/2}\varphi_2$

$$\frac{d\varphi_1}{dr} - \frac{(s-\frac{1}{2})}{r}\varphi_1 = \varphi_2 \qquad \text{etc.}$$

hence we see that the equations (*) have the solution

$$\psi = \begin{pmatrix} r^{1/2} K_{s-\frac{1}{2}}(r) \\ -r^{1/2} K_{s+\frac{1}{2}}(r) \end{pmatrix}$$

This should be the unique solution of ⊛ which vanishes as $r \longrightarrow +\infty$. Consequently I should know that for any real θ the equation

$$K_{i\lambda-\frac{1}{2}}(r) = e^{i\theta} K_{i\lambda+\frac{1}{2}}(r)$$

has only real solutions in λ for r real >0.

Since

$$\psi = \int_{-\infty}^{\infty} r^{1/2} e^{-r\cosh t} \begin{pmatrix} e^{-\frac{1}{2}t} \\ -e^{\frac{1}{2}t} \end{pmatrix} e^{st}\, dt$$

one has

$$\boxed{\begin{gathered} u(r,t) = r^{1/2} e^{-r\cosh t} \begin{pmatrix} e^{-\frac{1}{2}t} \\ -e^{\frac{1}{2}t} \end{pmatrix} \\ \text{is "the" "privileged" solution of the wave equation} \\ \frac{1}{i}\frac{\partial u}{\partial t} + \frac{1}{i}\begin{pmatrix} r\frac{\partial}{\partial r} & -r \\ r & -r\frac{\partial}{\partial r} \end{pmatrix} u = 0 \end{gathered}}$$

Furtherwork:

Does "privileged" have a sense?

Continued fraction expansion for K_{s-1}/K_s.

April 25, 1977

Consider again

$$(1) \qquad \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & p \\ p & -\frac{d}{dx} \end{pmatrix}\psi = \lambda\psi \qquad\qquad p \text{ real}$$

on $0 \le x < \infty$ and let the solution matrix be $S(x,\lambda)$. The columns of $S(x,\lambda)$ are vector functions $e(x,\lambda)$, $\tilde{e}(x,\lambda)$ satisfying the DE with initial values $\binom{1}{0}$ and $\binom{0}{1}$ at $x=0$. I prefer the notation:

$$S(x,\lambda) = \left(\psi^1(x,\lambda),\ \psi^2(x,\lambda)\right)$$

For example if $p=0$, then

$$S(x,\lambda) = \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix}$$

Now let's suppose that $p \nearrow +\infty$ as $x \nearrow +\infty$. Then there should be a unique solution which dies at ∞, unique up to a scalar multiple, which we can write uniquely as

$$m(\lambda)\psi^1(x,\lambda) + \psi^2(x,\lambda).$$

Here $m(\lambda)$ is a meromorphic function of λ whose poles occur at those λ such that $\psi^1(x,\lambda)$ dies at ∞. So if we factor

$$m(\lambda) = \frac{m_1(\lambda)}{m_2(\lambda)}$$

with $m_i(\lambda)$ entire and relatively prime, then we get a solution

$$\psi(x,\lambda) = m_1(\lambda)\psi^1(x,\lambda) + m_2(\lambda)\psi^2(x,\lambda)$$

entire in λ, not identically zero in x for any λ, which dies at $x = +\infty$. Clearly $\psi(x,\lambda)$ is unique up to multiplying by an invertible entire function ρ of λ. If we can produce a $\psi(x,\lambda)$ which is of exponential type, then the only possible variation of it would be by a ρ of the form $e^{a\lambda+b}$, a, b constants.

Put $u(x,t) = (e^{-itP}f)(x)$, where $P = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -p \\ p & -\frac{d}{dx} \end{pmatrix}$.
Then u satisfies

$$\frac{1}{i}\frac{\partial u}{\partial t} = -Pu \qquad \text{or}$$

$$(2) \qquad \boxed{\frac{\partial u}{\partial t} + \begin{pmatrix} \frac{\partial}{\partial x} & -p \\ p & -\frac{\partial}{\partial x} \end{pmatrix} u = 0}$$

and $\qquad u(x,0) = f(x)$. Thus the operator e^{-itP} solves the Cauchy problem on $t=0$ for the wave equation (2). Notice also that if $\psi(x,\lambda)$ satisfies (1) i.e.

$$(1) \qquad P\psi(\cdot,\lambda) = \lambda\psi(\cdot,\lambda)$$

then assuming we can Fourier transform in λ we get that

$$(3) \qquad u(x,t) = \int e^{-i\lambda t}\psi(x,\lambda)\,d\lambda$$

satisfies $\qquad \frac{1}{i}\frac{\partial u}{\partial t} = -\int e^{-i\lambda t}\lambda\psi(x,\lambda)\,d\lambda = -Pu$

Thus the Fourier transform (3) sets up a correspondence between solutions of (1) and (2).

The problem now is to understand solutions of the wave equation (2). Think globally in t, λ and (more or less) locally in x. Philosophy: The totality of all $u(x,t)$ solving the wave equation vanishing at $x=\infty$ and rapidly decreasing in t can be identified with the totality of functions $a(\lambda)\psi(\lambda,x)$ where $\psi(x,\lambda)$ is the good solution near $x=\infty$ described at the top of page 19.

April 26, 1977 I want to consider the problem of relating solution matrices to fundamental solutions for the wave equation. Let's start with the example

$$P = \frac{1}{i}\frac{d}{dx}$$

Here we want to relate the solution of

$$\begin{cases} \dfrac{1}{i}\dfrac{d\psi}{dx} = \lambda\psi \\ \psi(0,\lambda) = 1 \end{cases}$$

which is $\psi(x,\lambda) = e^{i\lambda x}$ to a fundamental solution E for P which is a solution of

$$(1) \qquad PE = \delta.$$

Better to write $(P-\lambda)E_\lambda = \delta$. Suppose $\lambda = 0$. Then the solutions of $P\psi = 0$ are the constants and a particular fundamental solution for (1) is

$$E = i\theta$$

where θ is the Heaviside fn. $\begin{cases} 0 & x<0 \\ 1/2 & x=0 \\ 1 & x>0 \end{cases}$.

Thus the possible fundamental solutions for P are

$$i\theta + \text{const}.$$

Hence there are unique forward and backward fundamental solutions.

The same holds for

$$(2) \qquad (P-\lambda)E_\lambda = \delta .$$

The solutions are
$$E_\lambda = i\theta e^{i\lambda x} + (\text{const})e^{-i\lambda x}$$
$$= e^{-i\lambda x}(i\theta + \text{const})$$

so again there are unique forward and backward fdl. solutions. Notation

forward $\quad E_\lambda^+ = e^{i\lambda x} i\,\theta(x)$

backward $\quad E_\lambda^- = -e^{i\lambda x} i\,\theta(-x) = e^{i\lambda x} i\left(\theta(x) - 1\right)$

One has

$$E_\lambda^+ - E_\lambda^- = e^{i\lambda(x)} i = i\,\psi(x)$$

(this solution of $P\psi = \lambda\psi$ with initial value i).

Now take Fourier tranform:

$$\tilde{E} = \int e^{-i\lambda t} e^{i\lambda x}(i\theta(x) + c)\,d\lambda \quad = 2\pi\,\delta(x-t)(i\theta(x) + c)$$

$$= 2\pi i\,\delta(x-t)\left[\theta(x) + c_1\right]$$

$$\tilde{E}^+(x,t) = 2\pi i\,\delta(x-t)\,\theta(x)$$

should be solutions of

$$\frac{1}{i}\frac{\partial}{\partial t}\tilde{E} + P\tilde{E} = 2\pi\,\delta(x)\,\delta(t)$$

Check

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\delta(x-t)\,\theta(x)\right) = \left[\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\delta(x-t)\right]\theta(x) + \delta(x-t)\left[\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\theta(x)\right]$$

$$= \delta(x-t)\,\delta(x)$$

$$= \delta(t)\,\delta(x)$$

~~Changing notation slightly~~

Prop. For the wave equation $\left(\frac{1}{i}\frac{\partial}{\partial t}+\frac{1}{i}P\right)(u)=0$ one has the fundamental solutions:

forward: $\quad F^+(x,t) = i\,\delta(x-t)\,\theta(x) = \frac{1}{2\pi}\int e^{-i\lambda t}\left(e^{i\lambda x}\,i\,\theta(x)\right)dx$

backward: $\quad F^-(x,t) = i\,\delta(x-t)\left[\theta(x)-1\right]$

$$F^+ - F^- = i\,\delta(x-t) = \frac{1}{2\pi}\int e^{-i\lambda t}\left(e^{-i\lambda x}\,i\right)dx$$

solution of $\frac{1}{i}\frac{d\psi}{dx}=\lambda\psi$ with $\psi(0,\lambda)=1$.

Next consider the system

$$P\psi = \frac{1}{i}\begin{pmatrix}\frac{d}{dx} & 0\\ 0 & -\frac{d}{dx}\end{pmatrix} = \lambda\psi$$

The solution matrix for initial values at $x=0$ is

$$\psi = \begin{pmatrix} e^{-i\lambda x} & 0\\ 0 & e^{-i\lambda x}\end{pmatrix}$$

~~Forward fundamental solution:~~ $\cancel{E^+(x,\lambda) = \begin{pmatrix} e^{-i\lambda x}\, i\,\theta(x)\\ -e^{-i\lambda x}\, i\,\theta(x)\end{pmatrix}}$

~~Backward:~~ $\cancel{E^-(x,\lambda) = \begin{pmatrix} e^{i\lambda x}\, i\,(\theta(x)-1)\\ -e^{-i\lambda x}\, i\,(\theta(x)-1)\end{pmatrix}}$

Fundamental solns.

$$E^{+}(x,\lambda) = \begin{pmatrix} e^{i\lambda x} i\,\theta(x) & 0 \\ 0 & -e^{-i\lambda x} i\,\theta(x) \end{pmatrix}$$

$$E^{-}(x,\lambda) = \begin{pmatrix} e^{i\lambda x} i\,(\theta(x)-1) & 0 \\ 0 & -e^{-i\lambda x} i\,(\theta(x)-1) \end{pmatrix}$$

Again $\quad E^{+} - E^{-} = i \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix}$

So for the wave equation

$$\frac{1}{i}\frac{\partial u}{\partial t} + Pu = \frac{1}{i}\begin{pmatrix} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \end{pmatrix} u = 0$$

one has the fundamental solutions

$$F^{+}(x,t) = \begin{pmatrix} i\,\delta(x-t)\theta(x) & 0 \\ 0 & -i\,\delta(x+t)\theta(x) \end{pmatrix}$$

F^{-} same with $\theta(x) \longmapsto \theta(x) - 1$

and $\quad F^{+} - F^{-} = \begin{pmatrix} i\,\delta(x-t) & 0 \\ 0 & -i\,\delta(x+t) \end{pmatrix} = \frac{1}{2\pi}\int e^{-i\lambda t}\, i \begin{pmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{pmatrix} d\lambda$

matrix solution of $P\psi = \lambda\psi$
with initial value I
at $x = 0$

Note that $\delta(x-t)\,\theta(x) = \delta(x-t)\,\theta(x+t)$, so we can write

$$F^+(x,t) = \begin{pmatrix} i\,\delta(x-t)\,\theta(x+t) & 0 \\ 0 & -i\,\delta(x+t)\,\theta(x-t) \end{pmatrix}$$

Return to Hormander's analysis in the case of

$$P\psi = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -p \\ p & -\frac{d}{dx} \end{pmatrix}\psi = \lambda\psi$$

and suppose we work on $0 \leq x \leq b$ finite with given self-adjoint boundary conditions. Then we get eigenfunctions + values $\psi_n(x)$ λ_n and can form

$$e^{-itP} = \sum e^{-it\lambda_n}\,\psi_n(x)\,\psi_n(y)^* \qquad \|\psi_n\| = 1.$$

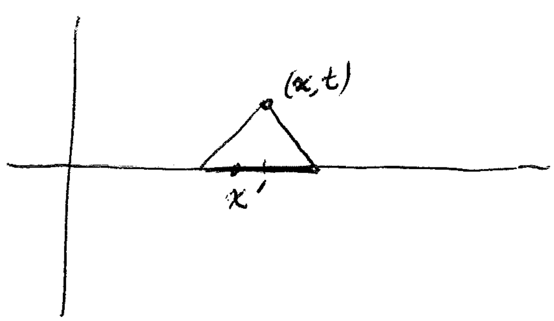
$$= K_t(x,y).$$

This satisfies the Cauchy problem

$$\frac{1}{i}\frac{\partial K}{\partial t} + PK = 0$$

$$K_0(x,y) = \delta(x-y)$$

as well as the boundary conditions at $x=0$, $x=b$. But the point is that the value at (x,t) is determined by $\delta(x',y)$ for $|x'-x| \leq |t|$.



Hence $K_t(x,y)$ is independent of the boundary conditions for t small, i.e. $|t| \leq x$ and $\leq b-x$. Now one chooses a $\rho(\lambda)$ such that $\hat{\rho}(t)$ is supported in $|t| \leq x$. Then $\hat{\rho}(t) K_t(x,y)$ or at least its singularities in x is known.

To be specific suppose $P = \frac{1}{i}\frac{d}{dx}$ whence the wave equation is $\frac{\partial K}{\partial t} + \frac{\partial K}{\partial x} = 0$ so K is a function of $x-t$. Then

$$K_t(x,y) = \delta(x-t-y)$$

near $t=0$. So $\hat{\rho}(t)\,\delta(x-t-y)$ has inverse transform

$$\frac{1}{2\pi}\int e^{it\lambda}\hat{\rho}(t)\,\delta(x-t-y)\,dy = \frac{1}{2\pi}\int e^{it\lambda} K_t(x,y)\hat{\rho}(t)\,dt$$

$$\int \rho(\lambda-\mu)\, e^{i\mu(x-y)}\, d\mu = \sum \rho(\lambda-\lambda_n)\psi_n(x)\psi_n(y)^*$$

so if we take $x=y$ we get

$$\int \rho(\mu)\,d\mu = \sum \rho(\lambda-\lambda_n)\,\psi_n(x)\psi_n(x)^*$$

showing the right side is independent of λ. If one thinks of the RHS as giving an average of $\psi_n(x)\psi_n(x)^*$ for λ_n in some big neighborhood of λ, the above is clearly consistent with the measure

$$de_\lambda(x,x) = \sum \psi_n(x)\psi_n(x)^*\,\delta(\lambda-\lambda_n)$$

being asymptotically equivalent to Lebesgue measure dx.