April 11, 1977.

The spectral function of a SL problem.
Start with the problem on a finite interval.

$$(1) \qquad \begin{cases} Lu = \lambda u \\ u(0) = u(b) = 0 \end{cases}$$

Let $u_\lambda(x)$ be the solution of

$$\begin{cases} L u_\lambda = \lambda u_\lambda \\ u_\lambda(0) = 0 \\ u_\lambda'(0) = 1 \end{cases}$$

so that the eigenvalues are the roots of $u_\lambda(b) = 0$. Let $\lambda_1 < \lambda_2 < \ldots\ldots$ be the eigenvalues.

If $\quad f = \sum a_j u_{\lambda_j} \quad$ then

$$(f, u_{\lambda_j}) = a_j \|u_{\lambda_j}\|^2 \qquad\qquad a_j = \frac{(f, u_{\lambda_j})}{\|u_{\lambda_j}\|^2}$$

$$\|f\|^2 = \sum |a_j|^2 \|u_{\lambda_j}\|^2 = \sum \frac{|(f, u_{\lambda_j})|^2}{\|u_{\lambda_j}\|^2}$$

These formulas can be written

$$f = \int u_\lambda (f, u_\lambda)\, d\rho(\lambda) \qquad\qquad \|f\|^2 = \int |(f, u_\lambda)|^2\, d\rho(\lambda)$$

where $d\rho(\lambda)$ is the measure with mass $\|u_{\lambda_j}\|^{-2}$ at λ_j.

According to the spectral theorem

$$L = \int \lambda\, dE_\lambda$$

where E_λ is the orthogonal projection on the subspace spanned by the u_{λ_j} with $\lambda_j \le \lambda$. Thus

$$E_\mu f = \int_{-\infty}^{\mu} u_\lambda (f, u_\lambda)\, d\rho(\lambda)$$

so E_μ is represented by the kernel

$$e(x,y,\mu) = \int_{-\infty}^{\mu} u_\lambda(x)\, \overline{u_\lambda(y)}\, d\rho(\lambda).$$

Hörmander calls $e(x,y,\mu)$ the spectral function of the self-adjoint operator defined by (1). He computes an asymptotic expansion of $e(x,y,\mu)$ as $\mu \to \infty$. Since

$$N(\lambda) = \text{number of } \lambda_j \le \lambda = tr(E_\lambda) = \int_0^b e(x,x,\lambda)\,dx$$

he obtains from his asymptotic estimate for $e(x,y,\lambda)$ an asymptotic estimate for $N(\lambda)$.

Note that $\int_{-\infty}^{\infty} dE_\lambda = I$, so that one should have

$$\lim_{\lambda\to+\infty} e(x,y,\lambda) = \delta(x-y)$$

April 13, 1977

Consider the Schroedinger equation

(1) $$\frac{d^2\psi}{dx^2} + (\lambda - q)\psi = 0$$

and let us try to construct a solution of the form

$$\psi(x,\lambda) = e^{iS(x,\lambda)} u(x,\lambda).$$

$$\frac{d^2\psi}{dx^2} + (\lambda-q)\psi = \frac{d}{dx}\left(e^{iS}(iS_x u + u_x)\right) + (\lambda-q)\psi$$

$$= e^{iS}\left(-S_x^2 u + 2iS_x u_x + iS_{xx} u + u_{xx} + (\lambda-q)u\right)$$

Let us choose S so that it satisfies the eikonal equation

$$S_x^2 = \lambda - q$$

Specifically put $S_x = \sqrt{\lambda - q}$ and $S(x,\lambda) = \int_0^x \sqrt{\lambda - q(t)}\,dt$

so that $S(0,\lambda) = 0$ and

$$S(x,\lambda) \sim \sqrt{\lambda}\,x \qquad \lambda \to \infty \qquad x \text{ bdd.}$$

To have ψ a solution we need in addition

$$2i S_x u_x + i S_{xx} u + u_{xx} = 0$$

$$2i\sqrt{\lambda - q}\, u_x + \frac{i}{2}\frac{-q'(x)}{\sqrt{\lambda - q(x)}}\, u + u_{xx} = 0$$

or
$$u_x = \frac{i}{2}(\lambda - q)^{-1/2} u_{xx} + \frac{q'(x)}{4}(\lambda - q)^{-1} u$$

$$= \lambda^{-\frac{1}{2}}\left(1 + \frac{q}{2\lambda} + \cdots\right)\frac{i}{2} u_{xx} + \lambda^{-1}\left(1 + \frac{q}{\lambda} + \frac{q^2}{\lambda^2} + \cdots\right)\frac{q'}{4} u$$

From this equation we can construct by interation a formal solution

$$u(x,\lambda) = u_0(x) + u_1(x)\lambda^{-1/2} + u_2(x)\lambda^{-1} + u_3(x)\lambda^{-3/2} + \cdots\cdots$$

with $u(0,\lambda) = 1$. For example from

$$u_0'(x) + u_1'(x)\lambda^{-1/2} + u_2'(x)\lambda^{-1} + \cdots = \lambda^{-\frac{1}{2}}\left(1 + \frac{q}{2\lambda} + \frac{3}{8}\frac{q^2}{\lambda^2} + \cdots\right)\frac{i}{2}\left(u_0'' + u_1''\lambda^{-\frac{1}{2}} + \cdots\right)$$
$$+ \lambda^{-1}\left(1 + \frac{q}{\lambda} + \frac{q^2}{\lambda^2} + \cdots\right)\frac{q'}{4}\left(u_0 + u_1\lambda^{-\frac{1}{2}} + \cdots\right)$$

we get
$$u_0'(x) = 0 \quad \text{hence} \quad u_0(x) = 1.$$
$$u_1'(x) = \frac{i}{2}u_0'' = 0 \quad \text{hence} \quad u_1(x) = 0$$
$$u_2'(x) = \frac{q'}{4}u_0 = \frac{q'}{4} \qquad \text{hence} \quad u_2(x) = \frac{q(x) - q(0)}{4}$$

Note on the other hand that if $q' = 0$, then $u(x,\lambda)$ is to satisfy

$$2i\sqrt{\lambda - q}\, u_x + u_{xx} = 0$$

$$2i\sqrt{\lambda - q}\, x + \ln(u_x) = \text{Const}$$

$$u_x = u_x(0,\lambda)\, e^{-2i\sqrt{\lambda - q}\, x}$$

$$\cancel{u(x,\lambda) = \frac{du}{dx}(0,\lambda)\, e^{-2i\sqrt{\lambda - q}\, x} + \text{const}}$$

$$u(x,\lambda) - \underset{\substack{\shortparallel \\ 1}}{u(0,\lambda)} = u_x(0,\lambda)\,\frac{e^{-2i\sqrt{\lambda - q}\,x} - 1}{-2\sqrt{\lambda-q}\,x}$$

But it's clear that $u(x,\lambda)$ won't have an asymptotic expansion at $\lambda \to \infty$ unless $u(x,\lambda) = 1$.

So I see that the Schrödinger equation (1) has a unique formal solution of the form

$$\psi(x,\lambda) = e^{iS(x,\lambda)}\,u(x,\lambda)$$

with $S(x,\lambda) = \int_0^x \sqrt{\lambda - q(t)}\,dt$ and where u is a formal series

$$u(x,\lambda) = u_0(x) + u_1(x)\lambda^{-1/2} + u_2(x)\lambda^{-1} + \cdots$$

where $u(0,\lambda) = 1$.

However it should be possible to construct a solution $\psi^+(x,\lambda)$ of the Schroedinger equation with

$$\psi^+(0,\lambda) = 1$$
$$\psi^+(x,\lambda) \sim e^{i\sqrt{\lambda}\,x} \quad \text{as } \lambda \to \infty$$

(Now observe that the virtue of the phase function $S(x,\lambda)$ seems to be that in the asymptotic expansion $u(x,\lambda) = 1 + O(\lambda^{-1})$ uniformly for x bdd. Thus

$$e^{-i\sqrt{\lambda}x}\, e^{i\int_0^x \sqrt{\lambda - q(t)}\,dt} = e^{i\sqrt{\lambda}\int_0^x \left((1-\frac{q}{\lambda})^{1/2} - 1\right)dt} = e^{i\sqrt{\lambda}\int_0^x \left(-\frac{1}{2}\frac{q}{\lambda} + \cdots\right)dt}$$

$$= 1 - \frac{i}{2}\lambda^{-1/2}\int_0^x q(t)\,dt + O(\lambda^{-1})$$

so one could instead find a solution of the form

$$\psi(x,\lambda) = e^{i\sqrt{\lambda}\,x}\left(v_0(x) + v_1(x)\lambda^{-1/2} + \cdots\right) \qquad \psi(0,\lambda) = 1$$

except that one only has $v_0(x) = 1$ and $v_1 \neq 0$ in fact $v_1 = -\frac{i}{2}\int_0^x q(t)\,dt$.)

April 15, 1977

$$u_2'\lambda^{-1} + u_3'\lambda^{-3/2} + \cdots = \lambda^{-1/2}\left(1+\frac{q}{2\lambda}+\frac{3}{8}\frac{q^2}{\lambda^2}+\cdots\right)\left(\frac{i}{2}\right)\left(u_2''\lambda^{-1}+u_3''\lambda^{-3/2}+\cdots\right)$$
$$+\lambda^{-1}\left(1+\frac{q}{\lambda}+\frac{q^2}{\lambda^2}+\cdots\right)\left(\frac{q'}{4}\right)\left(1+u_2\lambda^{-1}+u_3\lambda^{-2}+\cdots\right)$$

$u_2' = \frac{q'}{4}$ coeff of λ^{-1}

$u_3' = \frac{i}{2}u_2''$ " $\lambda^{-3/2}$ $\qquad u_3' = \frac{i}{2}\frac{q''}{4} = \frac{iq''}{8}$

$u_4' = \frac{q'}{4}(q+u_2)$ " λ^{-2}

so $\quad u_2 = \dfrac{q(x)-q(0)}{4}$

$$u_3 = i\,\frac{q'(x)-q'(0)}{8}$$

so
$$\varphi(x,\lambda) = e^{i\int_0^x(\lambda-q)^{1/2}dt}\left[1+\frac{1}{4}(q(x)-q(0))\lambda^{-1}+\frac{i}{8}(q'(x)-q'(0))\lambda^{-3/2}+\cdots\right]$$

is the asymptotic formula. This shows that we probably need to assume q is C^∞ in order to get the full asymptotic expansion.

Question: Let $f(z)$ be an entire function of z. When is it of the form $e^{a_1 z}u_1(\frac{1}{z}) + e^{a_2 z}u_2(\frac{1}{z})$ with u_1 and u_2 holomorphic at 0, and how unique is this representation?

For example take a component $\varphi(x,\lambda)$ in the solution matrix for $\dfrac{d^2\psi}{dx^2} + (\lambda - q(x))\psi = 0$ with q analytic near zero. Then does one have a representation

$$\varphi(x,\lambda) = e^{i\lambda^{1/2}x}u_1(x,\lambda) + e^{-i\lambda^{1/2}x}u_2(x,\lambda)$$

where u_i are analytic in $x, \lambda^{1/2}$ near $\lambda^{1/2} = \infty$?

Example: Take $P = \frac{1}{i}\frac{d}{dx}$ on $[0, 2\pi]$ with the periodic boundary conditions $u(x+2\pi) = u(x)$. Better to replace the line by S^1. Then

$$\left(e^{itP} f\right)(x) = \left(e^{t\frac{d}{dx}} f\right)(x) = f(x+t)$$

$$= \int_{S^1} \delta(x+t-y)\, f(y)\, dy$$

so $\delta(x+t-y)$ is the kernel representing e^{itP}. On the other hand the spectral function $e(x, y, \lambda)$ is defined by

$$P = \int \lambda\, dE_\lambda \qquad (E_\lambda f)(x) = \int e(x,y,\lambda) f(y)\, dy$$

so

$$e^{itP} = \int e^{it\lambda}\, dE_\lambda$$

or taking kernels

$$\delta(x+t-y) = \int e^{it\lambda}\, de(x,y,\lambda) = \int e^{it\lambda} \frac{de(x,y,\lambda)}{d\lambda}\, d\lambda$$

Fourier inversion gives

$$\frac{de_\lambda(x,y)}{d\lambda} = \frac{1}{2\pi} \int e^{-it\lambda}\, \delta(x+t-y)\, dt$$

contributes where the image in S^1 of $t = y - x$.

$$= \frac{1}{2\pi} {\sum_{n \in \mathbb{Z}}}' e^{-i(y-x+n)\lambda}$$

$$= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\lambda}\, e^{i\lambda x}\, \overline{e^{i\lambda y}}$$

$$= \frac{1}{2\pi}\, \delta_{\mathbb{Z}}(\lambda)\, e^{i\lambda x}\, \overline{e^{i\lambda y}}$$

Thus

$$e_\lambda(x,y) = {\sum_{n \leq \lambda}}' \frac{e^{inx} e^{-iny}}{2\pi}.$$

Perhaps a better formula, since λ can go to $-\infty$ is

$$\boxed{e_\lambda(x,y) - e_\mu(x,y) = {\sum_{\mu < n \leq \lambda}}' \frac{e^{inx} e^{-iny}}{2\pi}}$$

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Formula for systems:

$$\frac{d}{dx}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}i\lambda & \bar{p}\\ p & -i\lambda\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$\frac{1}{i}\begin{pmatrix}\frac{d}{dx} & -\bar{p}\\ p & -\frac{d}{dx}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \lambda\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

When p is constant u_1, u_2 are both killed by

$$\begin{vmatrix} i\lambda - \frac{d}{dx} & \bar{p}\\ p & -i\lambda - \frac{d}{dx}\end{vmatrix} = \frac{d^2}{dx^2} + (\lambda^2 - p\bar{p}).$$

I want to understand Hörmander's method for obtaining the eigenvalue distribution.

Example: Consider $P = \frac{1}{i}\frac{d}{dx}$

operating on functions $\psi(x)$ $0 \leq x \leq 2\pi$. We require periodic boundary conditions.

$$\psi(x+2\pi) = e^{i\theta}\psi(x).$$

Now Hörmander considers the operator e^{itP}. First one notes that P has the eigenfunctions

$$e^{i\lambda x} \qquad \text{which has } \int_0^{2\pi} |e^{i\lambda x}|^2 dx = 2\pi$$

where $e^{2\pi i\lambda} = e^{i\theta}$ i.e. $2\pi\lambda = \theta + 2n\pi$

or $\lambda = n + \frac{\theta}{2\pi}$ $n \in \mathbb{Z}$.

Hence e^{itP} is represented by the kernel

$$\sum_n e^{it(n+\frac{\theta}{2\pi})} \frac{e^{i(n+\frac{\theta}{2\pi})x}\, e^{-i(n+\frac{\theta}{2\pi})y}}{2\pi}$$

$$= e^{i(t+x-y)\frac{\theta}{2\pi}}\, \delta_{2\pi\mathbb{Z}}(t+x-y)$$

In the above $0 \le x, y \le 2\pi$ but $t \in \mathbb{R}$.

Here's how to get the eigenvalue distribution from e^{itP}. Since

$$P = \int \lambda\, dE_\lambda \quad , \qquad e^{itP} = \int e^{it\lambda}\, dE_\lambda$$

is represented by the kernel $\int e^{it\lambda}\, de_\lambda(x,y)$

where $e_\lambda(x,y)$ is the kernel representing E_λ. Thus we can get $\frac{\partial e_\lambda}{\partial \lambda}(x,y)$ by Fourier inversion:

$$\frac{\partial e_\lambda}{\partial \lambda}(x,y) = \frac{1}{2\pi}\int e^{-it\lambda}\, e^{i(t+x-y)\frac{\theta}{2\pi}}\, \delta_{2\pi\mathbb{Z}}(t+x-y)\, dt$$

$$= \sum_n \frac{1}{2\pi} e^{-i(-x+y+2\pi n)\lambda}\, e^{in\theta}$$

$$= e^{i\lambda x - i\lambda y}\, \delta_{\mathbb{Z}}\left(\lambda - \frac{\theta}{2\pi}\right)$$

For some reason which is not yet clear the asymptotics of the eigenvalue distribution depend only on knowing the kernel of e^{itP} for small t.

Example: $$\frac{d^2\psi}{dx^2} + (\lambda - q)\psi = 0 \qquad q \text{ constant}$$

$$\psi(0) = \psi(\pi) = 0.$$

The eigenfunctions are

$$\begin{cases} \psi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) & n = 1, 2, \ldots \\ \lambda_n = q + n^2 \end{cases}$$

Now consider the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} - q\psi$$

$$\psi(0,t) = \psi(\pi,t) = 0$$

This has the general solution

$$\psi(x,t) = \sum_{n=1}^{\infty} \left(a_n e^{i\omega_n t} + b_n e^{-i\omega_n t} \right) \psi_n(x)$$

where $\omega_n^2 = \lambda_n$ i.e. $\omega_n = \sqrt{n^2 + q}$ $n = 1, 2, \ldots$

Suppose q positive so there is no ambiguity in ω_n.

Introduce $P^{1/2}$. If

P is rep. by $\frac{2}{\pi} \sum_{n=1}^{\infty} (n^2 + q) \sin nx \sin ny$

then

$P^{1/2}$ is rep. by $\frac{2}{\pi} \sum_{n=1}^{\infty} (n^2 + q)^{1/2} \sin nx \sin ny$

and $e^{-itP^{1/2}}$ is rep. by $\frac{2}{\pi} \sum_{n=1}^{\infty} e^{-it(n^2+q)^{1/2}} \sin nx \sin ny$

On the other hand if $u(x) = \frac{1}{2\pi} \int e^{+ix\xi} \tilde{u}(\xi)\, d\xi$ has compact support, then

$$(Pu)(x) = \frac{1}{2\pi} \int e^{+ix\xi} (\xi^2 + q)\, \tilde{u}(\xi)\, d\xi$$

so that the obvious definition of $P^{1/2}$ is

$$(P^{1/2}u)(x) = \frac{1}{2\pi}\int e^{-ix\xi}(\xi^2+q)^{1/2}\tilde{u}(\xi)\,d\xi$$

except that we still have to make $(\xi^2+q)^{1/2}$ precise.

Follow Hormander + take the positive square root. Then you have

$$e^{-itP^{1/2}} \quad \text{approximately represented by} \quad \frac{1}{2\pi}\int e^{ix\xi}e^{-it|(\xi^2+q)^{1/2}|}e^{-iy\xi}\,d\xi$$

in some sense. The question is how can one use this approximate representation to get at the actual eigenvalue distribution. Note that the above gadget does not depend on the boundary conditions or even on the size of the interval.

Recall

$$e^{-itP^{1/2}} \leftrightarrow \sum e^{-it\omega_n}\psi_n(x)\overline{\psi_n(y)}$$

so

$$tr(e^{-itP^{1/2}}) = \sum_n e^{-it\omega_n} = \int e^{-it\omega}\,dN(\omega)$$

where $N(\omega) = \text{card}\{\omega_n \mid \omega_n \leq \omega\}$. Thus if I replace the actual kernel of $e^{-itP^{1/2}}$ by the approximate one I find the approximation

$$tr(e^{-itP^{1/2}}) \doteq \frac{1}{2\pi}\int e^{-it|(\xi^2+q)^{1/2}|}\,d\xi \underbrace{\int_0^{\pi} 1\,dx}_{\text{volume}}$$

$$\|$$

$$\frac{1}{2}\int e^{-it|(\xi^2+q)^{1/2}|}\,d\xi$$

Suppose $q=0$. This is $\int_0^\infty e^{-it\xi}\,d\xi$ so we get the approximation $N(\omega) = \omega$.

Suppose q not zero. To evaluate:

$$\int_0^\infty e^{-it(\xi^2+q)^{1/2}}\, d\xi$$

$$(\xi^2+q)^{1/2} - \xi = \xi\left(\left(1+\frac{q}{\xi}\right)^{1/2}-1\right) \doteq \frac{q}{2} - \frac{q^2}{8\xi}$$

$$\doteq \int_0^\infty e^{-it(\xi+\frac{q}{2})}\, d\xi = \int_{q/2}^\infty e^{-it\xi}\, d\xi$$

$$\text{so} \quad N(\omega) \doteq \int_{q/2}^{\omega} d\xi = \omega - q/2$$

which is consistent with $\omega_n = (n^2+q)^{1/2} \sim n + \frac{q}{2}$.

So the result to be understood is why is it possible to replace the operator $e^{-itP^{1/2}}$ which is defined using the boundary conditions by the Fourier integral operator.

So what we maybe should begin with is the case of the resolvent for the operator $P = -\frac{d^2}{dx^2} + q$ with same boundary conditions. Here we calculate $(\lambda - P)^{-1}$ as a pseudo-differential operator.

Suppose $(Gf)(x) = \frac{1}{2\pi}\int e^{ix\xi} g(x,\xi)\tilde{f}(\xi)\, d\xi$ satisfies $(P-\lambda)Gf = f$ for $f \in C_0^\infty(0,\pi)$. Here $P = -\frac{d^2}{dx^2} + q$.

$$\cancel{\left((P-\lambda)Gf\right)(x) = \frac{1}{2\pi}\int -\frac{d}{dx}\left(e^{ix\xi}\left[(-i\xi)g + g_x\right]\right)\tilde{f}(\xi)\, d\xi}$$

$$\cancel{= \frac{1}{2\pi}\int e^{ix\xi}\Big[}$$

$$\big((P-\lambda)Gf\big)(x) = \frac{1}{2\pi}\int\Big[-\frac{d}{dx}\, e^{ix\xi}(i\xi g + g_x) + e^{ix\xi}(q-\lambda)g\Big]\tilde{f}(\xi)\,d\xi$$

$$= \frac{1}{2\pi}\int e^{ix\xi}\Big[-(i\xi)(i\xi g + g_x) - (i\xi g_x + g_{xx}) + (q-\lambda)g\Big]\tilde{f}(\xi)\,d\xi$$

$$= \frac{1}{2\pi}\int e^{ix\xi}\Big[\,\xi^2 g - 2i\xi g_x - g_{xx} + (q-\lambda)g\Big]\tilde{f}(\xi)\,d\xi$$

We will have $(P-\lambda)Gf = f$ if $g(x,\xi)$ satisfies

$$(*)\qquad \xi^2 g - 2i\xi g_x - g_{xx} + (q-\lambda)g = 1.$$

I claim one can always find a unique formal solution of $(*)$ of the form

$$(2)\qquad \hat{g}(x,\xi) = \sum_{n\geq n_0} a_n(x)\,\xi^{-n}$$

In effect we get the recurrence relations

$$\sum_n a_n(x)\xi^{2-n} - 2i a_n'(x)\xi^{1-n} - a_n''\xi^{-n} + (q-\lambda)a_n\xi^{-n} = 1$$

or

$$a_n(x) - 2i a_{n-1}'(x) - a_{n-2}''(x) + (q-\lambda)a_{n-2}(x) = \begin{cases} 1 & n=2 \\ 0 & n\neq 2. \end{cases}$$

Thus starting with $a_0(x) = 1$ we can grind out the rest of the coefficients.

The next point is that having constructed the formal solution (2) to (1) we can then find a C^∞ function $g(x,\xi)$ which has $\hat{g}$ as asymptotic expansion as $\xi\to\infty$. If G is then defined using $g(x,\xi)$, we have

$$\big[(P-\lambda)Gf\big)(x) = \frac{1}{2\pi}\int e^{ix\xi}\, h(x,\xi)\tilde{f}(\xi)\,d\xi$$

where h is $1+$ a C^∞ function with 0 asymptotic expansion.

i.e. $h(x,\xi)$ is rapidly decreasing as $\xi \to \infty$. It follows that the kernel representing $(P-\lambda)G - I$

$$\frac{1}{2\pi}\int e^{ix\xi}\, h(x,\xi)\, e^{-iy\xi}\, d\xi$$

is a C^∞ function of x, y.

But suppose now that boundary conditions are given, so that the operator $R_\lambda = (P-\lambda)^{-1}$ exists for λ not an eigenvalue (e.g. λ not real). By Schwarz kernel thm. R_λ is given by a kernel and from

$$(P-\lambda)G = I + K$$

where K has a C^∞ kernel we get

$$(P-\lambda)(G - R_\lambda) = K$$

hence by regularity $G - R_\lambda$ has a C^∞-kernel. Thus modifying the function g without changing its asymptotic expansion one arranges that $R_\lambda = G$ in particular that Gf satisfies the boundary conditions.

But there should be a better reason that once one has exhibited the symbols for e^{-tP}, P^{-s}, $e^{-itP^{1/2}}$ that the Fourier integral operators associated to these symbols agree with these operators up to C^∞ kernels.

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April 18, 1977

Here's what I understand of the Hormander eigenvalue distribution theory so far.

Let P be an elliptic operator, say $\frac{1}{i}\frac{d}{dx}$ on $(0,2\pi)$ to fix the ideas. Let $\hat{P}$ be a self-adjoint extension, e.g. the one defined by the boundary conditions

$$\psi(2\pi) = e^{i\theta}\psi(0).$$

By the spectral theorem we can define the operator $e^{-it\hat{P}}$. There we find the eigenfunctions

$$\psi_n(x) = \frac{e^{i\lambda_n x}}{\sqrt{2\pi}} \qquad \lambda_n = n + \frac{\theta}{2\pi} \qquad n \in \mathbb{Z}$$

whence

$$e^{-it\hat{P}} \longleftrightarrow \sum_n e^{-it(n+\frac{\theta}{2\pi})} \frac{e^{i(n+\frac{\theta}{2\pi})x}\, e^{-i(n+\frac{\theta}{2\pi})y}}{2\pi}$$

$$= e^{i(x-y-t)\theta/2\pi}\, \delta_{2\pi\mathbb{Z}}(x-y-t)$$

On the other hand using the symbol of P one can write down a Fourier integral operator candidate for $e^{-it\hat{P}}$:

$$(Pu)(x) = \frac{1}{2\pi}\int e^{ix\xi}\,\xi\,\hat{u}(\xi)\,d\xi \qquad u \in C_0^\infty(0,2\pi)$$

so the candidate for e^{-itP} has the kernel

$$\frac{1}{2\pi}\int e^{ix\xi}e^{-it\xi}e^{-iy\xi}\,d\xi = \delta(x-y-t)$$

What I want to understand. Given a Schroedinger equation

$$\frac{d^2\psi}{dx^2} + (\lambda - q)\psi = 0$$

on $0 \le x \le \infty$, let $S(x,\lambda)$ be the solution matrix

$$S(x,\lambda) = \begin{pmatrix} \psi & \varphi \\ \psi' & \varphi' \end{pmatrix}$$

$$\frac{d}{dx} S(x,\lambda) = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix} S(x,\lambda) \qquad S(0,\lambda) = I.$$

Put $\lambda = \mu^2$ and write $S(x,\mu)$ instead of $S(x,\mu^2)$.

Assume now that $q(x) \nearrow +\infty$ as $x \longrightarrow +\infty$, whence the spectrum is discrete and one is in the limit point case at $x = \infty$. Then for each complex number λ (not any eigenvalue λ) there is a unique number $m(\lambda)$ such that

$$\varphi(x,\lambda) + m(\lambda)\psi(x,\lambda)$$

is square integrable.

Maybe we would do better to introduce the solution $\chi(x,\lambda)$ which vanishes at ∞. It should be normalized somehow. Note that $\tilde{\chi}(x,\lambda) = f(\lambda)\chi(x,\lambda)$ still vanishes at $x = +\infty$, and that $f(\lambda) = 0 \Rightarrow \tilde{\chi}(x,\lambda) \equiv 0$. Thus perhaps $\chi(x,\lambda)$ is uniquely defined if we require it has some sort of growth at $\lambda = \infty$.

What we ultimately want is to compute the Fourier transform:

$$\chi(x,\lambda) = \frac{1}{2\pi}\int e^{i\lambda\tau}\, \tilde{\chi}(x,\tau)\, d\tau$$

Let's begin again. Suppose $\psi(x,\lambda)$ is a solution of the Schroedinger equation

$$-\frac{d^2\psi}{dx^2} + q(x)\psi = \lambda\psi$$

whose initial values $\psi(x_0,\lambda)$, $\frac{\partial\psi}{\partial x}(x_0,\lambda)$ are independent of λ. Does it follow that $\psi(x,\lambda)$ has an asymptotic expansion

$$\psi(x,\lambda) = e^{i\sqrt{\lambda}(x-x_0)} a_+(x,\lambda) + e^{-i\sqrt{\lambda}(x-x_0)} a_-(x,\lambda)$$

where a_+, a_- are holom. in $\lambda^{1/2}$ at ∞?

Example of the Bessel D.E.

$$\left(-\frac{d^2}{dx^2} + e^{2x}\right)\psi = \mu^2\psi$$

with solution

$$\psi(x,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{e^x}{2}(e^{\alpha}+e^{-\alpha})} e^{i\mu\alpha}\, d\alpha$$

$$= \frac{1}{2\pi}\int_0^{\infty} e^{-\frac{e^x}{2}(t+t^{-1})} t^{i\mu}\,\frac{dt}{t}$$

$$= \frac{1}{2\pi} K_{i\mu}(e^x)$$

Suppose we take the Fourier transform of ψ

$$u(x,\alpha) = \tilde{\psi}(x,\alpha) = \int_{-\infty}^{\infty} e^{-i\mu\alpha}\psi(x,\mu)\, d\mu$$

$$= e^{-\frac{e^x}{2}(e^{\alpha}+e^{-\alpha})}$$

In general this will satisfy the DE.

$$-\frac{\partial^2 u}{\partial x^2} + e^{2x}u = \int \mu^2 e^{-i\mu\alpha}\psi(x,\mu)\, d\mu = -\frac{\partial^2 u}{\partial\alpha^2}$$

Thus $u(x,\alpha)$ satisfies the wave equation:

$$\frac{\partial^2 u}{\partial \alpha^2} = \frac{\partial^2 u}{\partial x^2} - e^{2x} u \, .$$

So now our problem appears to be to locate a potential $q(x)$ such that the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - q(x) u$$

has a solution $u(x,t)$ rapidly decreasing as $x \mapsto +\infty$ and $t \to \pm\infty$ with $u(0,t)$ prescribed.