April 2, 1977. In

eigenvalue dist. for $K_S(2\pi)$ using Bohr-Summerfeld rules.

(1) $$\frac{d}{dx}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

let us change variables

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2i}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \frac{1}{\sqrt{2i}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \in SU_2$$

$$\frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \zeta - qi & -i\zeta + q \\ \zeta + ri & i\zeta + r \end{pmatrix}$$

$$= \frac{1}{2i}\begin{pmatrix} -qi - ri & -2i\zeta + q - r \\ 2i\zeta + q - r & qi + ri \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{q+r}{2} & -\zeta + \frac{q-r}{2i} \\ \zeta + \frac{q-r}{2i} & \frac{q+r}{2} \end{pmatrix}$$

Instead suppose we try

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{-2i}}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$+\frac{1}{2i}\begin{pmatrix} -1 & +1 \\ +i & +i \end{pmatrix}\begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -\zeta + qi & -i\zeta + q \\ -\zeta - ri & i\zeta + r \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -q-r & 2\zeta + (q-r)i \\ -2\zeta + (q-r)i & q+r \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{q+r}{2} & \zeta + \frac{(q-r)i}{2} \\ -\zeta + \frac{(q-r)i}{2} & \frac{q+r}{2} \end{pmatrix}$$

Thus we get a system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{q+r}{2} & -\zeta + \frac{q-r}{2i} \\ \zeta + \frac{q-r}{2i} & \frac{q+r}{2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

which is of the form ② $\frac{dX}{dx} = \left(A_0 + \lambda\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)X$ with

$A_0$ real & symmetric and $\lambda = -\zeta$, i.e. provided $r = \bar{q}$. Thus we have a 1-1 correspondence between equations of the form ② with $A_0$ real & symmetric and equations ① with $r = \bar{q}$.

Now ① can be written

$$\zeta v_1 = -\frac{1}{i}\frac{d}{dx}v_1 + \frac{q}{i}v_2$$

$$\zeta v_2 = -\frac{r}{i}v_1 + \frac{1}{i}\frac{dv_2}{dx}$$

or

$$\zeta\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\frac{d}{dx} & -iq \\ ir & -i\frac{d}{dx} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and the matrix ↑ is formally self-adjoint when $r = \bar{q}$.

Thus if I put

$$L = i\begin{pmatrix} \frac{d}{dx} & -\bar{r} \\ r & -\frac{d}{dx} \end{pmatrix}$$

Then $L \cdot V = \zeta \cdot V \qquad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

with boundary conditions

$$\begin{cases} v_1(0) = e^{i\theta_0} v_2(0) \\ v_1(1) = e^{i\theta_1} v_2(1) \end{cases}$$

is a self-adjoint eigenvalue problem. One has Green's formula

$$\begin{pmatrix}w_1\\ w_2\end{pmatrix}^{\dagger} L\begin{pmatrix}v_1\\ v_2\end{pmatrix} - \overline{\begin{pmatrix}v_1\\ v_2\end{pmatrix}^{\dagger} L\begin{pmatrix}w_1\\ w_2\end{pmatrix}} = i\frac{d}{dx}\left(v_1\bar{w}_1 - v_2\bar{w}_2\right)$$

as can easily be checked.

I have to work out carefully the relation between

i) The solution matrix $S(\lambda)$

ii) The spectral measure

iii) The Green's function

Let's work on a finite interval $[0,b]$ and let

$$S(x,\lambda) = \begin{pmatrix}\psi_1(x,\lambda) & \varphi_1(x,\lambda)\\ \psi_2(x,\lambda) & \varphi_2(x,\lambda)\end{pmatrix}$$

be the solution matrix so that

$$LS = \lambda S \qquad L = \begin{pmatrix} i\partial & -i\bar{r}\\ ir & -i\partial\end{pmatrix}.$$
$$S(0,\lambda) = I$$

Next I have to specify boundary conditions at both endpoints. Thus I have the problem:

$$\begin{cases} L\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \lambda\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\ v_1(0) = e^{i\theta_0}v_2(0)\\ v_1(b) = e^{i\theta_b}v_2(b)\end{cases}$$

$$\left(1,\ e^{i\theta_0}\right)\cdot\begin{pmatrix}v_1(0)\\ v_2(0)\end{pmatrix} = 0$$

Now from $S$ I should be able to construct the Green's function which is a matrix $G(x,y,\lambda)$ satisfying

$$(L_x - \lambda I)\, G(x,y,\lambda) = -I \cdot \delta(x-y)$$
$$\& \quad (1, -e^{i\theta_j}) \cdot G(j,y,\lambda) = 0 \qquad j = 0, b.$$

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How to determine eigenvalues in terms of $S$. Consider the unique solution (up to a scalar multiple) satisfying the boundary condition at $0$.

$$S\begin{pmatrix} e^{i\theta_0} \\ 1 \end{pmatrix} = e^{i\theta_0}\psi + \varphi$$

Then $\lambda$ will be an eigenvalues if this satisfies the bdry. condition at $b$; i.e.

$$0 = (1, e^{+i\theta_b}) \cdot S\begin{pmatrix} e^{i\theta_0} \\ 1 \end{pmatrix} = (1, -e^{+i\theta_b})\begin{pmatrix} \psi_1 e^{i\theta_0} + \varphi_1 \\ \psi_2 e^{i\theta_0} + \varphi_2 \end{pmatrix}$$

$$= \psi_1 e^{i\theta_0} + \varphi_1 - e^{i\theta_b}\psi_2 e^{i\theta_0} - e^{i\theta_b}\varphi_2$$

$$= \operatorname{tr}\begin{pmatrix} \psi_1 & \varphi_1 \\ \psi_2 & \varphi_2 \end{pmatrix}\begin{pmatrix} e^{i\theta_0} & -e^{i\theta_b}e^{i\theta_0} \\ 1 & -e^{i\theta_b} \end{pmatrix}$$

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Goal: I want to find an eigenvalue problem of the above sort whose eigenvalues are (up to a transf. $\frac{1}{2} + i\mathbb{R} \simeq \mathbb{R}$) are the zeroes of $\zeta$. In fact I want $\zeta(\lambda)$ to be the function obtained from $S(\lambda)$ in the above $\lambda$ whose zeroes give the eigenvalues. Thus I want

$$\zeta(\lambda) = \operatorname{tr}(S(\lambda)T)$$

for some matrix $T$ which comes from self-adjoint boundary conditions.

There are two (maybe 3) types of $T$ that arise: In the case of periodic boundary conditions $T$ is non-singular and it is real (in the case where we use the upper half plane as the region contracted by $S(\lambda)$ for $\operatorname{Im}\lambda > 0$). In this case we can arrange $\det(T) = \pm 1$.
~~$\det(T) = 1$ doesn't occur for then $\operatorname{tr}(S(\lambda)T) = 0$~~
$\det(T) = -1$ maybe doesn't occur for then $\det(S(\lambda)T) = -1$ $\operatorname{tr}(S(\lambda)T) = 0$ $\Rightarrow$ $S(\lambda)T$ has eigenvalues $\pm 1$.

Suppose $\det(T) = +1$. Then we've seen that $\frac{1}{2}\operatorname{tr}(S(\lambda)T) \in [-1, 1]$ forces $\lambda$ to be real. In other words it is impossible for $S(\lambda)$ to die down as $\lambda \to \infty$ and oscillate.

It is necessary to understand what kind of boundary conditions lead to a self-adjoint problem. By Green's formula (page 62) one must have that $v_1\bar{w}_1 - v_2\bar{w}_2$ has the same value at both 0 and 1, whence $v, w$ satisfy the boundary conditions.

Suppose $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ non-singular preserves $v_1\bar{w}_1 - v_2\bar{w}_2$ i.e.

$$(av_1 + bv_2)\overline{(aw_1 + bw_2)} - (cv_1 + dv_2)\overline{(cw_1 + dw_2)} = v_1\bar{w}_1 - v_2\bar{w}_2$$

i.e.

$$a\bar{a} - c\bar{c} = 1$$
$$a\bar{b} - c\bar{d} = 0$$
$$b\bar{a} - d\bar{c} = 0 \qquad \Leftrightarrow \qquad b = \lambda\bar{c} \quad d = \lambda\bar{a} \quad \text{same } \lambda$$
$$b\bar{b} - d\bar{d} = -1 \qquad \Rightarrow \qquad |\lambda|^2|a|^2 - |\lambda|^2|c|^2 = 1$$
$$\Rightarrow \quad |\lambda|^2 = 1. \qquad \bar{c} = \lambda^{-1}b = \bar{\lambda}b \Rightarrow c = \lambda\bar{b}$$

Thus $\quad T = \begin{pmatrix} a & \lambda\bar{c} \\ c & \lambda\bar{a} \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda\bar{b} & \lambda\bar{a} \end{pmatrix}$

where $\quad |\lambda| = 1, \quad |a|^2 - |b|^2 = 1.$

Suppose $T$ preserves $|z| = 1$ i.e.

$$z\bar{z} = 1 \implies \overline{\frac{az+b}{cz+d}} = \frac{cz+d}{az+b}$$

$$\bar{a}a + \bar{b}az + b\bar{a}\bar{z} + b\bar{b} = c\bar{c}z\bar{z} + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d}$$

$$\implies \quad \bar{a}a + b\bar{b} = c\bar{c} + d\bar{d}$$

$$\left.\begin{aligned} a\bar{b} &= c\bar{d} \\ \bar{a}b &= d\bar{c} \end{aligned}\right\} \qquad b = \lambda\bar{c}, \quad d = \lambda\bar{a}$$

$$|a|^2 + |\lambda|^2|c|^2 = |c|^2 + |\lambda|^2|a|^2$$

$$(1-|\lambda|^2)|a|^2 = (1-|\lambda|^2)|c|^2$$

Now $\quad |T| = \begin{vmatrix} a & \lambda\bar{c} \\ c & \lambda\bar{a} \end{vmatrix} = \lambda(|a|^2 - |c|^2) \neq 0 \implies |\lambda|^2 = 1.$

Therefore those $T$ preserving $|z| = 1$ are of the form $\begin{pmatrix} a & b \\ \lambda\bar{b} & \lambda\bar{a} \end{pmatrix}$ with $|\lambda| = 1$.

<u>Prop.</u> The set of non-singular $T$ which give rise to self-adjoint boundary problems of the "periodic type" are those matrices of the form

$$\begin{pmatrix} a & b \\ \lambda\bar{b} & \lambda\bar{a} \end{pmatrix} \quad \text{with} \quad \begin{aligned} &|\lambda| = 1 \\ &|a|^2 - |b|^2 = 1. \end{aligned}$$

These $T$ preserves $|z| = 1$.

Now if I am only interested in the zeroes of the entire function

$\mathrm{tr}(S(\lambda)T)$ I can multiply $T$ by a scalar and so arrange for $\det(T) = 1$.

But then we've seen that $\frac{1}{2}\mathrm{tr}(S(\lambda)T) \in [-1, 1]$ implies $\lambda$ is real which means we don't get a decaying function of $\lambda$. However $\mathrm{tr}(S(\lambda)T) = 0$ doesn't give the eigenvalues, see page 67

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Note that the possible boundary values of a function $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ form a 4-dim'l space $\mathbb{C}^2 \times \mathbb{C}^2$ and that those boundary values of a solution of $(L-\lambda)(u) = 0$ form a 2-dim'l space $=$ graph $S(\lambda)$. Boundary conditions amount to giving a 2-dimensional subspace $B$ of $\mathbb{C}^2 \times \mathbb{C}^2$. $\lambda$ is an eigenvalue when $B \cap \Gamma_{S(\lambda)} \neq 0$. What has this to do with the trace?

For periodic boundary conditions one is given an isomorphism $T : V_0 \xrightarrow{\sim} V_b$ where $V_0 =$ space of $u(0)$ and $V_b$ the space of $u(b)$. Then $B = \{ (v, w) \in V_0 \times V_b \mid Tv = w \}$ $=$ graph$(T)$ and $\lambda$ is an eigenvalue when

$$B \cap \Gamma_{S(\lambda)} \neq 0$$

i.e. $\exists\ v \in V_0$ with $S(\lambda) v = Tv$ or finally that $T^{-1}S(\lambda)$ has an eigenvector with eigenvalue 1.

Now on $V_0$ and $V_b$ we have the sesquilinear form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \longmapsto v_1 \bar{w}_1 - v_2 \bar{w}_2$$

which has to be preserved by $T$ if $T$ gives rise to a self-adjoint problem. Moreover this means $T$ has the form $T = \begin{pmatrix} a & b \\ \varepsilon\bar{b} & \varepsilon\bar{a} \end{pmatrix}$ $|a|^2 - |b|^2 = 1$, $|\varepsilon| = 1$

so $\det(T) = \varepsilon$ has absolute value 1. Thus

$$\det\left(T^{-1}S(\lambda)\right) = \varepsilon^{-1}$$

so if one of the eigenvalues is 1 the other must be $\varepsilon^{-1}$. In other words

$$\begin{array}{l}\lambda \text{ is an eigenvalue} \\ \text{for } L, T \text{ iff}\end{array} \iff \operatorname{tr}\left(T^{-1}S(\lambda)\right) = 1 + \varepsilon^{-1}$$

$$\varepsilon = \det(T)$$

Take $T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -\bar{i} \end{pmatrix}$, $\quad \varepsilon = \det(T) = -1$

$$\lambda \text{ eigenvalue} \iff \operatorname{tr}\left(T^{-1}S(\lambda)\right) = 0$$

$$\begin{array}{c} \shortparallel \\ \frac{1}{i}\operatorname{tr}\left(S(\lambda)\right) \end{array}$$

More generally suppose $T$ given with $\det(T) = \varepsilon$, one has $\det\left(\varepsilon^{1/2}T^{-1}\right) = 1$. Then

$$\operatorname{tr}\left(T^{-1}S(\lambda)\right) = 1 + \varepsilon^{-1} \iff \operatorname{tr}\left(\left(\varepsilon^{1/2}T^{-1}\right)S(\lambda)\right) = \varepsilon^{1/2} + \varepsilon^{-1/2}$$

Therefore we see that the entire function $\operatorname{tr}(T^{-1}S(\lambda)) - 1 - \varepsilon^{-1/2}$ whose zeroes are the eigenvalues can always be put in the form

$$\operatorname{tr}\left(S(\lambda)T_0\right) - \cos\theta$$

for some $T_0 = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ of determinant 1 and angle $\theta$.

Therefore from the periodic boundary problem we cannot get an entire function with real zeroes which dies as $\lambda \to \pm\infty$

So the next thing to understand is what limits we can take. Suppose

$$T_0 = \begin{pmatrix} \frac{1}{\sqrt{1-a^2}} & \frac{a}{\sqrt{1-a^2}} \\ \frac{a}{\sqrt{1-a^2}} & \frac{1}{\sqrt{1-a^2}} \end{pmatrix}$$

and we let $a \nearrow 1$. Look at

$$\left[ tr\left( S(\lambda) T_0 \right) - \cos\theta \right] \sqrt{1-a^2}$$

$$= tr\left( S(\lambda) \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \right) - \sqrt{1-a^2}\cos\theta$$

as $a \to 1$. One gets $f(\lambda) = tr\left( S(\lambda) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)$ except this time there might not be an interval $[-\varepsilon, \varepsilon]$ such that $f(\lambda) \in [-\varepsilon, \varepsilon] \Rightarrow \lambda \in \mathbb{R}$. We ask then about limits

$$tr\left( S(\lambda) \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \right) - \sqrt{|a|^2 - |b|^2}\, \cos\theta$$

as the determinant goes to zero. Might as well assume $a \to e^{i\varphi_1}$, $b \to e^{i\varphi_2}$. The limit is

$$tr\left( S(\lambda) \begin{pmatrix} e^{i\varphi_1} & e^{i\varphi_2} \\ e^{-\varphi_2} & e^{-i\varphi_1} \end{pmatrix} \right)$$

which is essentially the same as the one we got on page 63.

Calculate $S(\lambda)$ in the constant coefficient case

$$L\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\partial & -i\bar{r} \\ ir & -i\partial \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\partial v_1 = -i\lambda v_1 + \bar{r} v_2$$
$$\partial v_2 = r v_1 + i\lambda v_2$$

$$\partial \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i\lambda & \bar{r} \\ r & i\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$S(t,\lambda) = e^{t\begin{pmatrix} -i\lambda & \bar{r} \\ r & i\lambda \end{pmatrix}}$$

$$\det \begin{pmatrix} -i\lambda & \bar{r} \\ r & i\lambda \end{pmatrix} = \lambda^2 - r\bar{r}$$

so the eigenvalues are $\omega^2 = -\lambda^2 + r\bar{r}$

$\pm\omega$ where $\omega = i\sqrt{\lambda^2 - r\bar{r}}$.

$$\begin{pmatrix} -i\lambda & \bar{r} \\ r & i\lambda \end{pmatrix} \begin{pmatrix} \bar{r} & \bar{r} \\ i\lambda+\omega & i\lambda-\omega \end{pmatrix} = \begin{pmatrix} \bar{r} & \bar{r} \\ i\lambda+\omega & i\lambda-\omega \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

$$S(t,\lambda) \begin{pmatrix} \bar{r} & \bar{r} \\ i\lambda+\omega & i\lambda-\omega \end{pmatrix} = \begin{pmatrix} \bar{r} & \bar{r} \\ i\lambda+\omega & i\lambda-\omega \end{pmatrix} \begin{pmatrix} e^{t\omega} & 0 \\ 0 & e^{-t\omega} \end{pmatrix}$$

$$S(t,\lambda) \begin{pmatrix} \bar{r} & \bar{r} \\ i\lambda+\omega & i\lambda-\omega \end{pmatrix} \begin{pmatrix} i\lambda-\omega & -\bar{r} \\ -i\lambda-\omega & \bar{r} \end{pmatrix} = \begin{pmatrix} \bar{r} e^{t\omega} & \bar{r} e^{-t\omega} \\ (i\lambda+\omega)e^{t\omega} & (i\lambda-\omega)e^{-t\omega} \end{pmatrix} \begin{pmatrix} i\lambda-\omega & -\bar{r} \\ -i\lambda-\omega & \bar{r} \end{pmatrix}$$

$$S(t,\lambda) \begin{pmatrix} -2\omega\bar{r} & 0 \\ 0 & -2\omega\bar{r} \end{pmatrix} = \begin{pmatrix} \bar{r}e^{t\omega}(i\lambda-\omega) + \bar{r}e^{-t\omega}(-i\lambda-\omega) & -\bar{r}^2 e^{t\omega} + \bar{r}^2 e^{-t\omega} \\ -r\bar{r}e^{t\omega} + r\bar{r}e^{-t\omega} & -(i\lambda+\omega)e^{t\omega}\bar{r} + (i\lambda-\omega)e^{-t\omega}\bar{r} \end{pmatrix}$$

$$\boxed{S(t,\lambda) = \frac{1}{2\omega} \begin{pmatrix} e^{t\omega}(-i\lambda+\omega) + e^{-t\omega}(i\lambda+\omega) & \bar{r}e^{t\omega} - \bar{r}e^{-t\omega} \\ +re^{t\omega} - re^{-t\omega} & e^{t\omega}(i\lambda+\omega) + (-i\lambda+\omega)e^{-t\omega} \end{pmatrix}}$$

$$\operatorname{tr} S(t,\lambda) = \frac{1}{2\omega}\left(e^{t\omega}2\omega + e^{-t\omega}2\omega\right) = \frac{e^{t\omega}+e^{-t\omega}}{2} \cdot 2$$
$$= 2\cos\left(\sqrt{\lambda^2 - r\bar{r}}\, t\right)$$

Take $T=\begin{pmatrix} i & -i \\ i & -i \end{pmatrix}$ and suppose $r=\bar{r}$

$$tr\left(S(t,\lambda)\,T\right) = \frac{i}{2\omega}\left((-2\lambda)e^{t\omega} + (2\lambda)e^{-t\omega}\right) = 2\,\frac{\lambda}{\omega}\,\frac{e^{t\omega}-e^{-t\omega}}{2i}$$

$$= 2\,\frac{\lambda}{i\sqrt{\lambda^2-r\bar{r}}}\,\sin\left(\sqrt{\lambda^2-r\bar{r}}\right)t.$$

Unfortunately it does not seem to be possible to get a $tr(S(t,\lambda)T)$ which dies as $\lambda\to\pm\infty$. In effect $\omega\sim i\lambda$ as $|\lambda|\to\infty$, hence as $\lambda\to\pm\infty$

$$S(t,\lambda) \sim \begin{pmatrix} e^{-t\omega} & 0 \\ 0 & e^{+t\omega} \end{pmatrix}$$

which is what one gets by putting $r=0$.

April 4, 1977

Bessel function

$$K_s(c) = \int_0^\infty e^{-\frac{c}{2}(t+t^{-1})}\, t^s \frac{dt}{t} \qquad K_s(c) = K_{-s}(c)$$

$$sK_s(c) = \int_0^\infty e^{-\frac{c}{2}(t+t^{-1})}\, s t^{s-1} dt = -\int_0^\infty e^{-\frac{c}{2}(t+t^{-1})}\left(-\frac{c}{2}(1-t^{-2})\right) t^s dt$$

$$= \int_0^\infty e^{-(c/2)(t+t^{-1})}\, c\,\frac{1}{2}(t-t^{-1})\, t^{s-1} dt$$

$$s^2 K_s(c) = -\int_0^\infty \left[e^{-\frac{c}{2}(t+t^{-1})}\frac{c}{2}(t-t^{-1})\right]' t^s dt$$

$$= \int_0^\infty e^{-\frac{c}{2}(t+t^{-1})}\left[\frac{c^2}{4}(1-t^{-2})(t-t^{-1}) - \frac{c}{2}(1+t^{-2})\right] t\, t^s \frac{dt}{t}$$

$$= \int_0^\infty e^{-c/2(t+t^{-1})}\left[\frac{c^2}{4}(t+t^{-1})^2 - c^2 - \frac{c}{2}(t+t^{-1})\right] t^s \frac{dt}{t}$$

$$= c^2\frac{d^2}{dc^2}K_s(c) - c^2 K_s(c) + c\frac{d}{ds}K_s(c)$$

Thus $K_s(c)$ satisfies the Bessel D.E. (imaginary argument)

$$\left(c\frac{d}{dc}\right)^2 K_s - (c^2+s^2)K_s = 0$$

So now put $c = e^x$ $\quad dc = e^x dx \quad \frac{dc}{c} = dx$

$$\left(\frac{d}{dx}\right)^2 K_s - (e^{2x}+s^2)K_s = 0$$

Finally [redacted] put [redacted] $s = i\lambda$ so this becomes

(*) $$\left(-\left(\frac{d}{dx}\right)^2 + e^{2x}\right)K = \lambda^2 K$$

The SL DE ⊛ is of the limit point type at $x = +\infty$, and the eigenvalues are discrete corresponding to those $\lambda$ such that $K_{i\lambda}(a) = 0$.
fixed endpt

So the function to look at here is the function
$$f(\lambda) = K_{i\lambda}(c) = \int_0^\infty e^{-\frac{c}{2}(t+t^{-1})} t^{i\lambda} \frac{dt}{t}$$
where $c$ is fixed. The Riemann-Lebesgue Lemma says that $f(\lambda)$ is a rapidly decreasing function of $\lambda$. Now I'd like to get an idea of the distribution of zeroes of $f(\lambda)$.

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Another example: Schroedinger equation for a simple harmonic oscillator:

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right)\psi = E\psi$$

The eigenfunctions are

$$E = \left(n + \frac{1}{2}\right) \qquad \psi_n = \frac{e^{-x^2/2} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}$$
$$= \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \left(x - \frac{d}{dx}\right)^n e^{-x^2/2}$$

---

Laguerre polys. $L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x} x^n) = \left(\frac{d}{dx} - 1\right)^n (x^n)$
satisfy
$$xy'' + (1-x)y' + ny = 0$$
$$xe^{-x}y'' + (1-x)e^{-x}y' + ne^{-x}y = 0$$

Put $u = e^{-x/2} y$ or $y = e^{x/2} u$

$$y' = e^{x/2}\left(u' + \tfrac{1}{2}u\right)$$

$$y'' = e^{x/2}\left(u'' + \tfrac{1}{2}u' + \tfrac{1}{2}u' + \tfrac{1}{4}u\right)$$

$$x\left(u'' + u' + \tfrac{1}{4}u\right) + (1-x)\left(u' + \tfrac{1}{2}u\right) + n u = 0$$

$$xu'' + \cancel{xu'} + \tfrac{1}{4}xu + u' + \tfrac{1}{2}u - \cancel{xu'} - \tfrac{1}{2}xu + nu$$

$$(xu')' + \left(n + \tfrac{1}{2} - \tfrac{1}{4}x\right)u = 0$$

Now put $x = e^z$ $\quad x\frac{d}{dx} = \frac{d}{dz}$ so

$$\frac{d^2u}{dz^2} + \left(n + \tfrac{1}{2} - \tfrac{1}{4}e^z\right)u = 0$$

$$-\frac{d^2u}{dz^2} + \left(\tfrac{1}{4}e^z\right)u = \left(n + \tfrac{1}{2}\right)u$$

Put $2w = z$

$$\left(-\frac{d^2u}{dw^2} + e^{w/2}\right)u = (4n+2)u$$

How to convert a Sturm-Liouville system to standard form. Start with

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + (\lambda r - q)y = 0 \qquad p, r > 0$$

First get symbol correct: $\quad p^{1/2}(x)\frac{d}{dx} = \frac{d}{dz}$ or $dz = p^{-1/2}dx$

$$z = \int_{x_0}^{x} p^{-1/2}\,dx$$

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) = p^{-1/2}\frac{d}{dz}\left(p^{1/2}\frac{dy}{dz}\right) = p^{-1/2}\left[p^{1/2}\frac{d^2y}{dz^2} + \frac{1}{2}\overbrace{p^{-1/2}\frac{dp}{dz}}^{\frac{dp}{dx}}\frac{dy}{dz}\right]$$

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$$\frac{d^2y}{dz^2} + \Big(\overbrace{\tfrac{1}{2}p^{-1/2}\frac{dp}{dx}}^{\frac{1}{2}\frac{d}{dz}(\ln p)}\Big)\frac{dy}{dz} + (\lambda r - q)y = 0$$

Now put $y = \alpha u$. Better suppose $r=1$.

$$\alpha u'' + 2\alpha' u' + \alpha'' u + \frac{1}{2}\frac{d}{dz}(\ln p)(\alpha u' + \alpha' u) + (\lambda r - q)\alpha u = 0$$

$$u'' + \left[2\frac{\alpha'}{\alpha} + \frac{1}{2}\frac{d}{dz}(\ln p)\right]u' + \left[\lambda + \frac{\alpha''}{\alpha} + \frac{\alpha'}{\alpha}\frac{1}{2}\frac{d}{dz}(\ln p) - q\right]u = 0$$

$$\therefore \qquad 2\frac{\alpha'}{\alpha} + \frac{1}{2}\frac{d}{dx}(\ln p) = 2\frac{d}{dz}\left(\ln(\alpha\cdot p^{1/4})\right) = 0$$

$$\alpha = p^{-1/4}$$

So the substitution is

$$z = \int^x p^{-1/2}dx \qquad y = p^{-1/4}u$$

which means that $y$ should have the approximate form

$$y = \frac{1}{p^{1/4}(x)}\left(A e^{i\sqrt{\lambda}\int^x p^{-1/2}dx} + B e^{-i\sqrt{\lambda}\int^x p^{-1/2}dx}\right)$$

constants

If $r$ is not 1 one first replaces $y$ by $\tilde{y} = r^{1/2}y$ whence

$$r^{-1/2}\frac{d}{dx}\left(p\frac{d}{dx}(r^{-1/2}\tilde{y})\right) = \frac{d}{dx}\left(r^{-1}p\frac{d\tilde{y}}{dx}\right) + q\tilde{y}$$

Then the above substitutions becomes

$$z = \int^x \left(\frac{r}{p}\right)^{1/2}dx \qquad y = r^{-1/2}\left(\frac{p}{r}\right)^{-1/4}u = (pr)^{-1/4}u$$

So try this in the case of

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(n+\frac{1}{2}-\frac{1}{4}x\right)y = 0$$

$$z = \int^x x^{-1/2}dx = 2x^{1/2} \qquad dz = x^{-1/2}dx$$

$$x = \frac{z^2}{4} \qquad y = x^{-1/4}u = \left(\frac{z}{2}\right)^{-1/2}u = \sqrt{2}\,z^{-1/2}u$$

$$dx = \frac{z\,dz}{2} \qquad \frac{d}{dx} = \frac{2}{z}\frac{d}{dz}$$

$$\frac{2}{z}\frac{d}{dz}\left(\frac{z^2}{\cancel{4}}\frac{\cancel{2}}{z}\frac{d}{dz}\right)\left(z^{-1/2}u\right) + \left(n+\frac{1}{2}-\frac{z^2}{16}\right)z^{-1/2}u = 0$$

$$\frac{1}{z^{1/2}}\frac{d}{dz}\left(z\left(-\frac{1}{2}z^{-3/2}u+z^{-1/2}u'\right)\right) + \left(n+\frac{1}{2}-\frac{z^2}{16}\right)u = 0$$

$$z^{-1/2}\frac{d}{dz}\left(-\frac{1}{2}z^{-1/2}u + z^{1/2}u'\right)$$

$$z^{-1/2}\left(\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)z^{-3/2}u + \cancel{\left(-\frac{1}{2}z^{-1/2}\right)u'} + \cancel{\left(\frac{1}{2}z^{-1/2}u'\right)} + z^{1/2}u''\right)$$

$$\frac{1}{4}z^{-2}u + u'' + \left(n+\frac{1}{2}-\frac{z^2}{16}\right)u = 0$$

$$\frac{d^2u}{dz^2} + \left(n+\frac{1}{2}+\frac{1}{4}z^{-2}-\frac{1}{16}z^2\right)u = 0$$

$$\boxed{\left[-\frac{d^2}{dz^2} + \left(-\frac{1}{2}-\frac{z^{-2}}{4}+\frac{z^2}{16}\right)\right]u = nu}$$

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Example: Take a self-adjoint first order operator on functions

$$L = \frac{1}{i}\frac{du}{dx} + pu = \lambda q u \qquad p, q \text{ real}$$

The solutions are

$$u = C e^{i\int(\lambda q - p)}$$

Since

$$(Lu, v) - (u, Lv) = \int_0^b \left(\frac{1}{i} u'\bar{v} - u\,\overline{\frac{1}{i}v'}\right) dx = \int_0^b \frac{1}{i}[u'\bar{v} + u\bar{v}']\,dx$$

$$= \frac{1}{i}\left[u(b)\bar{v}(b) - u(0)\bar{v}(0)\right]$$

the only finite interval self-adjoint boundary conditions are

$$u(b) = \zeta\, u(0) \text{ with } |\zeta| = 1$$

The eigenvalues are given by

$$e^{i\int_0^b(\lambda q - p)\,dx} = \zeta = e^{i\theta}$$

$$\lambda \int_0^b q\,dx = \theta + \int_0^b p\,dx + 2\pi n \qquad n \in \mathbb{Z}$$

This holds even if $q$ becomes negative. Notice that if $q > 0$, then we can change $x$ to make $q = 1$, whence $\int_0^b q\,dx \nearrow \infty$ as $b \to \infty$, so the spectrum becomes continuous. Actually the operator $L = \frac{1}{i}\frac{d}{dx}$ is not self-adjoint on $[0, \infty)$; better to say, it doesn't have a self-adjoint extension to $L^2(0, \infty)$.

Next I want to understand the eigenvalue distribution for

$$\begin{cases} -\dfrac{d^2u}{dx^2} + pu = \lambda u \\ u(0) = u(\pi) = 0 \end{cases}$$

If $p = 0$, then the eigenvalues and eigenfunctions are:

$$u_n = \sin(nx) \qquad \lambda = n^2 \qquad n = 1, 2, \ldots\ldots$$

---

Consider $\quad -\dfrac{d^2u}{dx^2} + qu = \lambda u \quad$ as a Schroedinger equation on $[0, \infty)$ with boundary condition $u(0) = 0$, $u \in L^2$. Assume $q(x) \nearrow +\infty$ as $x \to +\infty$. The classical motion corresponding to the Schroedinger equation is

$$\underset{\text{K.E.}}{\underset{\uparrow}{\dot{x}^2}} + \underset{\text{P.E.}}{\underset{\uparrow}{q}} = \underset{\text{Energy}}{\underset{\uparrow}{\lambda}} \qquad\qquad \dot{x} = \sqrt{\lambda - q}$$

with $x = 0$ as a reflecting wall. The Bohr condition is

$$\text{action} = \oint \underset{\text{mom.}}{\underset{\uparrow}{p}}\, d\underset{\text{pos.}}{\underset{\uparrow}{q}} = n \cdot h \qquad\qquad \begin{matrix} h \text{ constant} \\ n \text{ an integer} \end{matrix}$$

which in this notation should be

$$\int \sqrt{\lambda - q}\, dx = nh$$

where the integral is taken over one period of motion at energy $\lambda$. This equation in $\lambda$ has a solution $\tilde{\lambda}_n$ for each

$n$ and the conjecture is that one gets in this way the asymptotic distribution of eigenvalues.

Ex: $$-\frac{d^2u}{dx^2} + x^2 u = \lambda u \qquad \text{on } (-\infty, \infty)$$

$$\dot{x}^2 + x^2 = \lambda$$

$$x = \sqrt{\lambda}\sin(t - t_0)$$

$$\dot{x} = \sqrt{\lambda}\cos(t - t_0)$$

$$\oint \dot{x}\,dx = \int_0^{2\pi} \lambda \cos^2(t - t_0)\,dt = \pi\lambda = nh$$

Hence we get the asymptotic distribution $\lambda_n \sim \text{const.}\, n$ for the eigenvalues. Now one knows in this case that

$$\lambda_n = 2\left(n + \frac{1}{2}\right)$$

so we see the constant $h$ should be $2\pi$.

Ex. $\rho = e^{2x}$  $x = \ln u,\ u = e^x$  $u = \lambda^{1/2}\cos\theta$

$$2\int_0^{\log\lambda^{1/2}} \sqrt{\lambda - e^{2x}}\,dx = 2\int_1^{\lambda^{1/2}} \sqrt{\lambda - u^2}\,\frac{du}{u} = 2\int_0^{\cos^{-1}(\lambda^{-1/2})} \lambda^{1/2}\sin\theta\,\frac{\lambda^{1/2}\sin\theta\,d\theta}{\lambda^{1/2}\cos\theta}$$

$$= 2\lambda^{1/2}\int_0^{\cos^{-1}(\lambda^{-1/2})} \left(\frac{1}{\cos\theta} - \cos\theta\right) d\theta$$

$$= 2\lambda^{1/2}\left[ \left[\log\frac{\lambda^{1/2} + \lambda^{1/2}\sin\theta}{\lambda^{1/2}\cos\theta}\right]_0^{\cos^{-1}(\lambda^{-1/2})} - \sin\theta\Big]_0^{\cos^{-1}(\lambda^{-1/2})} \right]$$

$$= 2\lambda^{1/2}\left[ \left[\log\frac{\lambda^{1/2} + \sqrt{\lambda - u^2}}{u}\right]_{\lambda^{1/2}}^{1} - \sqrt{1 - \lambda^{-1}} \right]$$

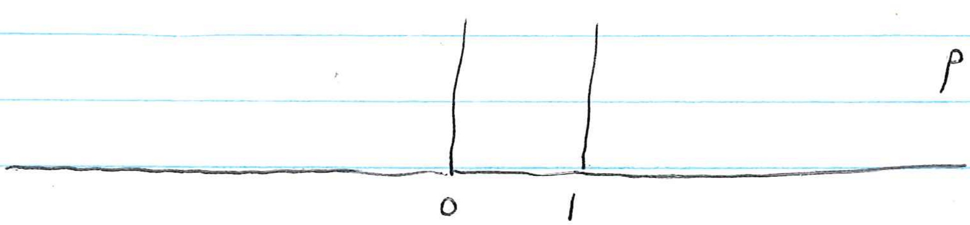
$$= 2\lambda^{1/2}\left[ \log\left(\lambda^{1/2} + \sqrt{\lambda - 1}\right) - \sqrt{1 - \lambda^{-1}} \right]$$

$$= 2\lambda^{1/2}\log\left(\lambda^{1/2} + \sqrt{\lambda - 1}\right) - 2\sqrt{\lambda - 1}$$

So the asymptotic equation for $\lambda_n$ appears to be

$$\boxed{2\lambda_n^{1/2}\log\left(\lambda_n^{1/2} + \sqrt{\lambda_n - 1}\right) - 2\sqrt{\lambda_n - 1} = 2\pi n}$$

Example: Consider a particle in a potential well




$$p(x) = \begin{cases} +\infty & x < 0 \\ 0 & 0 \le x \le 1 \\ +\infty & x > 1. \end{cases}$$

$$\left.\begin{array}{l} -\dfrac{d^2u}{dx^2} = \lambda u \\ u(0) = u(1) = 0 \end{array}\right\} \text{ has eigenfunctions } u = \sin n\pi x$$

eigenvalues $\lambda_n = n^2\pi^2$  $n = 1, 2, \cdots$

classical motion is $\dot{x}^2 = \lambda$ i.e. $\dot{x} = \pm\sqrt{\lambda}$ with reflection at $x=0$ and $x=1$. So the action of a period is

$$\oint \dot{x}\,dx = 2\int_0^1 \sqrt{\lambda}\,dx = 2\sqrt{\lambda}$$

gives $2\sqrt{\lambda_n} = 2\pi n$ or $\lambda_n = \pi^2 n^2$.

---

Remark: Frequently results on eigenvalues are phrased in terms of

$$N(T) = \text{number of eigenvalues} \leq T$$

Thus if the eigenvalues are listed $\lambda_1 \leq \lambda_2 \leq \cdots$ then

$$N(T) = n \iff \lambda_n \leq T < \lambda_{n+1}$$

Therefore to say that $\oint \sqrt{\lambda_n - p}\,dx \doteq 2\pi n$ approximately is the same thing as saying

$$\boxed{\begin{matrix}\text{number of}\\ \lambda_j \leq \lambda\end{matrix} = N(\lambda) \doteq \frac{1}{2\pi}\oint \sqrt{\lambda - p}\,dx.}$$

Now to make the connection with $\zeta(s)$ we want to think of $s$ as $\frac{1}{2} + iu$, where $u = \sqrt{\lambda}$ if $\lambda$ occurs for a 2nd order SL operator.

$$8(1-s)\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \int_0^\infty \left\{ \sum_n e^{-\pi n^2 t^2}\left((2\pi n^2 t^2)^2 - 3(2\pi n^2 t^2)\right) \right\} t^s \frac{dt}{t}$$

I've been trying to relate this to $2\int_0^\infty e^{-\pi(t^2+t^{-2})}\, t^s \frac{dt}{t} =$

$$\int_0^\infty e^{-\pi(t+t^{-1})}\, t^{s/2} \frac{dt}{t} = K_{s/2}(2\pi).$$ But our above calculations

indicate that the number of zeroes of $K_{iu}(2\pi)$

with $0 \leq u \leq T$ should be estimated by

$$\frac{1}{2}\,\frac{1}{2\pi}\,2\int_{\log(2\pi)}^{\log(T)} \sqrt{T^2 - e^{2x}}\,dx \sim \frac{1}{2\pi}\,T\log T$$

because roots $\pm u$ belong to same $\lambda = u^2$ value

so we get $\oint$

[redacted] Thus we get the correct leading term.

still off by a factor of 2 (see p. 82)

$$(*)\quad \begin{pmatrix} \frac{1}{i}\partial & \frac{1}{i}p \\ i\bar{p} & i\partial \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{i}\partial - \lambda & \frac{1}{i}p \\ i\bar{p} & i\partial - \lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

If [redacted] $p$ is constant then $u_1, u_2$ are killed by

$$\begin{vmatrix} \frac{1}{i}\partial - \lambda & \frac{1}{i}p \\ i\bar{p} & i\partial - \lambda \end{vmatrix} = \partial^2 + \lambda^2 - p\bar{p}$$

i.e. $u_1, u_2$ are solutions of

$$\left(-\partial^2 + p\bar{p}\right) u = \lambda^2 u$$

Notice also that if $\binom{u_1}{u_2}$ is an eigenfunction with eigenvalue $\lambda$, then if $p$ is real, one sees by conjugating $(*)$ that $\binom{\bar{u}_1}{\bar{u}_2}$ is an eigenfunction with eigenvalue $-\lambda$.

It seems interesting to consider the case $p = e^x$ in view of the equation

$$\left(-\frac{d^2}{dx^2} + e^{2x}\right)K = (-s^2)K$$

satisfied by $K = K_s(e^x)$.

---

Here's the correct version of the estimate of zeroes for

$$\int_0^\infty e^{-\pi(t^2+t^{-2})}t^s \frac{dt}{t} = \frac{1}{2}\int_0^\infty e^{-\pi(t+t^{-1})}t^{s/2}\frac{dt}{t} = \frac{1}{2}K_{s/2}(2\pi).$$

First since $K_{iu}(c)$ satisfies $\left(-\frac{d^2}{dx^2}+e^{2x}\right)K = \underset{\lambda}{\underset{\parallel}{u^2}}K$
we have

$$\text{card}\left\{\lambda_j \le \lambda \;\middle|\; K_{i\sqrt{\lambda_j}}(c)=0\right\} \sim \frac{1}{2\pi}\oint \sqrt{\lambda - e^{2x}}\,dx$$

$$\parallel \qquad\qquad = \frac{1}{\pi}\int_c^{\log\lambda^{1/2}} \sqrt{\lambda - e^{2x}}\,dx$$

$$\text{card}\left\{\sqrt{\lambda_j} \le \sqrt{\lambda} \;\middle|\; K_{i\sqrt{\lambda_j}}(c)=0\right\}$$

$$\sim \frac{1}{\pi}\lambda^{1/2}\log\lambda^{1/2}$$

~~But~~ Thus

$$\text{card}\left\{u_j \le T \;\middle|\; K_{iu_j}(c)=0\right\} \sim \frac{1}{\pi}T\log T$$

$$\text{card}\left\{u_j \le 2T \;\middle|\; K_{iu_j/2}(c)=0\right\} \sim \frac{1}{\pi}T\log T$$

replace $2T$ by $T$

$$\text{card}\left\{u_j \le T \;\middle|\; K_{iu_j/2}(c)=0\right\} \sim \frac{1}{\pi}\frac{T}{2}\log\frac{T}{2}$$

$$\sim \frac{1}{2\pi}T\log T$$

April 10, 1977:

Suppose one has a SL operator $L$ with eigenvalues $\lambda_j$ and eigenfunctions $u_j$. In order to understand the distribution of the eigenvalues one introduces various functions of $L$. For example we have the resolvent

$$(\lambda - L)^{-1}$$

or the heat operator $e^{-tL}$ or the $\zeta$-operator $L^{-s}$, or Hörmanders operator $e^{itL^{1/2}}$. One then takes these operators and represents them by kernels. The restriction of the kernel to the diagonal then gives information on the eigenvalues.

$$\operatorname{tr}(e^{-tL}) = \sum_j{}' e^{-t\lambda_j}$$

$$\operatorname{tr}(L^{-s}) = \sum_j{}' \lambda_j^{-s}$$

$$\operatorname{tr}(e^{itL^{1/2}}) = \sum{}' e^{it\lambda_j^{1/2}}$$ this exists as a distribution.

For some reason Hörmander's work with $e^{itL^{1/2}}$ gives much better results. One point is that $e^{itL^{1/2}}$ is a Fourier integral operator with phase function related to the classical geodesics.

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According to a theorem of Hartman (reference can be found in Vol. 2 of Dunford-Schwartz) one has

$$N(\lambda) = \frac{1}{\pi}\int_0^{g^{-1}(\lambda)} \sqrt{\lambda - g(x)}\,dx + O(1)$$ if $g$ continuous increasing convex and $g(+\infty) = \infty$

Thus for $K_S(2\pi)$ we compute:-

$$\frac{1}{\pi}\int_{\log 2\pi}^{\log \lambda^{1/2}} \sqrt{\lambda - e^{2x}}\, dx = \frac{1}{\pi}\int_{2\pi}^{\lambda^{1/2}} \sqrt{\lambda - u^2}\,\frac{du}{u}$$

$$= \frac{1}{\pi}\left[-\lambda^{1/2}\log\left(\frac{\lambda^{1/2}+\sqrt{\lambda-u^2}}{u}\right) + \sqrt{\lambda - u^2}\right]_{2\pi}^{\lambda^{1/2}}$$

$$= \frac{1}{\pi}\lambda^{1/2}\log\left(\frac{\lambda^{1/2}+\sqrt{\lambda-4\pi^2}}{2\pi}\right) - \frac{1}{\pi}\sqrt{\lambda - 4\pi^2}$$

Note that $\sqrt{\lambda - a} - \lambda^{1/2} = \lambda^{1/2}\left(\left(1-\frac{a}{\lambda}\right)^{1/2} - 1\right) = -\frac{a}{2}\lambda^{-1/2} + \ldots$

$$\log\left(\lambda^{1/2} + \sqrt{\lambda - a}\right) = \log\left(2\lambda^{1/2} + \lambda^{1/2}\left(-\frac{a}{2\lambda}+\ldots\right)\right)$$

$$= \log\left(2\lambda^{1/2}\right) + \underset{O\left(\frac{1}{\lambda}\right)}{\underset{\shortparallel}{\log\left(1 + \frac{1}{2}\left(-\frac{a}{2\lambda}+\ldots\right)\right)}}$$

Thus

$$N(\lambda) = \frac{\lambda^{1/2}}{\pi}\log\left(\frac{\lambda^{1/2}}{\pi}\right) - \frac{\lambda^{1/2}}{\pi} + O(1)$$

Now put $T = 2\lambda^{1/2}$ to get comparison with $\int$. Edwards book gives

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

# Nonlinear-Evolution Equations of Physical Significance*

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We present the inverse scattering method which provides a means of solution of the initial-value problem for a broad class of nonlinear evolution equations. Special cases include the sine-Gordon equation, the sinh-Gordon equation, the Benney-Newell equation, the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, and generalizations.

One of the most exciting recent advances in applied mathematics and theoretical physics has been a method of solution for the initial value problem for certain nonlinear partial differential equations which arise naturally in many scientific areas.[1-4] There are four key steps in the method of solution. (1) First, set up an appropriate, linear scattering (eigenvalue) problem in the "space" variable where the solution of the nonlinear evolution equation plays the role of the potential. (2) Choose the "time" dependence of the eigenfunctions in such a way that the eigenvalues remain time invariant as the potential evolves according to the evolution equation. Although it is yet to be proved, the ability to achieve this step appears to depend on the existence of an infinite number of independent conservation laws for the evolution equation. (3) Solve the direct scattering problem at the initial "time" and determine the "time" dependence of the scattering data. (4) Do the inverse scattering problem at later "times"; namely, knowing the (discrete) eigenvalues corresponding to the bound states and knowing the time dependence of the other scattering data, reconstruct the potential. The final step can be written in terms of a linear integral equation (or a coupled set of linear integral equations) from which one can compute the solution to the evolution equation for all time.

We have found that many nonlinear evolution equations can be solved by the following scattering problem:

$$\begin{aligned} \partial v_1/\partial x + i\zeta v_1 &= q(x,t)v_2, \\ \partial v_2/\partial x - i\zeta v_2 &= r(x,t)v_1. \end{aligned} \tag{1}$$

Choose the time dependence of the eigenfunctions $v_1$ and $v_2$ to be

$$\begin{aligned} \partial v_1/\partial t &= A(x,t,\zeta)v_1 + B(x,t,\zeta)v_2, \\ \partial v_2/\partial t &= C(x,t,\zeta)v_1 - A(x,t,\zeta)v_2. \end{aligned} \tag{2}$$

The eigenvalues $\zeta$ are time invariant when

$$\begin{aligned} \partial A/\partial x &= qC - rB, \\ \partial B/\partial x + 2i\zeta B &= 2\partial q/\partial t - 2Aq, \\ \partial C/\partial x - 2i\zeta C &= \partial r/\partial t + 2Ar. \end{aligned} \tag{3}$$

Equations (3) are obtained by cross differentiation of the systems (1) and (2). Finite expansions of $A$, $B$, and $C$ in terms of the parameter $2i\zeta$ allow us to determine the class of evolution equations which can be solved by the inverse scattering method. It can be verified that the following evolution equations belong to this class.

*Class I.*—Take $A = -4i\zeta^3 - 2iqr\zeta + r\partial q/\partial x - q\partial r/\partial x$ and find

$$\begin{aligned} \partial q/\partial t - 6rq\partial q/\partial x + \partial^3 q/\partial x^3 &= 0, \\ \partial r/\partial t - 6rq\partial r/\partial x + \partial^3 r/\partial x^3 &= 0. \end{aligned} \tag{4}$$

When $r = -1$, (4) reduces to the Korteweg-de Vries (KdV) equation, and the system of equations (1) reduces to the Schrödinger equation[1]

$$\partial^2 v_2/\partial x^2 + [\zeta^2 + q(x,t)]v_2 = 0. \tag{5}$$

When $r = \pm q$, (4) reduces to the modified KdV equation[5]

$$\partial q/\partial t \mp 6q^2 \partial q/\partial x + \partial^3 q/\partial x^3 = 0. \tag{6}$$

When $r = +q$, $q$ real, the eigenvalue problem posed by (1) is self-adjoint, and hence all eigenvalues are real. In this case no solitons arise, and the final state can be shown to decay algebraically in time.[6] When $r = -q$ a class of paired permanent waves (with complex eigenvalues) arise in addition to the individual solitons.

*Class II.*—Take $A = a/\zeta$, and find

$$\begin{aligned} &\partial a/\partial x = \tfrac{1}{2}i\partial(qr)/\partial t, \quad \partial^2 q/\partial x\partial t = -4iaq, \\ &\partial^2 r/\partial x\partial t = -4iar. \end{aligned} \tag{7}$$

When $a = \frac{1}{4}i\cos u$, $r = -q = \frac{1}{2}\partial u/\partial x$, the sine-Gordon equation

$$\partial^2 u/\partial x\partial t = \sin u \tag{8}$$

is obtained. Equations (6) and (8) are solved by the same scattering problem.[3] In addition to the traveling kink and antikink solutions, the only other localized stable solutions are soliton states (breathers) which oscillate in time and which correspond to paired complex eigenvalues. The eigenvalues corresponding to the modes of a given breather in its own rest frame all lie on the circle $\zeta\zeta^* = \frac{1}{4}$. In its rest frame, a particular breather solution is

$$u(x,t) = 4\tan^{-1}\{[(1-\omega^2)/\omega^2]^{1/2}\cos\omega(T-T_0) \times \operatorname{sech}[1-\omega^2)^{1/2}(X-X_0)]\}, \tag{9}$$

where $\omega = -2\,\mathrm{Re}\zeta$ and $x = \frac{1}{2}(X+T)$, $t=\frac{1}{2}(X-T)$. These solutions have been obtained by Lamb[7] and Seeger, Donth, and Kochendorfer[8] from the Bäcklund transformation and have been observed numerically.[9,10]

If $a = \frac{1}{4}i\cosh u$, $r = q = \frac{1}{2}\partial u/\partial x$, the sinh-Gordon equation

$$\partial^2 u/\partial x\partial t = \sinh u \tag{10}$$

is obtained. Since the eigenvalue equation is self-adjoint, no solitons arise and the final state will decay in time.

*Class III.*—Take $A = -2i\zeta^2 - irq$ and find

$$i\partial q/\partial t + \partial^2 q/\partial x^2 - 2q^2 r = 0,$$
$$i\partial r/\partial t - \partial^2 r/\partial x^2 + 2qr^2 = 0. \tag{11}$$

In the special case $r = -q^*$ $(+q^*)$, this corresponds to the equation describing the evolution of the envelope of an almost monochromatic wave[11,12] when the Benjamin-Feir instability[13,14] is operative (inoperative). The case $r=-q^*$ has been solved by Zakharov and Shabat.[2] When $r = q^*$, the eigenvalue problem (1) is again self-adjoint and thus no solitons arise. The solution decays algebraically in time and has a self-similar structure.[6]

These equations (4), (7), and (11) are special cases of the general class of evolution equations which can be derived and solved by our procedure. The generalized Korteweg–de Vries equation[15] can also be obtained by this procedure when $r = 1$.

The direct and inverse scattering analysis will be published elsewhere. Here we merely quote the result.

Given $q(x,0)$, $r(x,0)$ sufficiently smooth and decaying sufficiently fast as $|x| \to \infty$, the solutions $q(x,t)$, $r(x,t)$ for all time are given by

$$q(x,t) = -2K_1(x,x),$$
$$r(x,t) = -2\bar{K}_2(x,x), \tag{12}$$

where

$$K(x,y) - \begin{bmatrix}1\\0\end{bmatrix}\bar{F}(x+y) - \int_x^\infty \bar{K}(x,s)\bar{F}(s+y)\,ds = 0,$$
$$\bar{K}(x,y) + \begin{bmatrix}0\\1\end{bmatrix}F(x+y) + \int_x^\infty K(x,s)F(s+y)\,ds = 0, \tag{13}$$

and

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\xi,t)}{a(\xi)}e^{i\xi x}d\xi - i\sum_k C_k\exp(i\zeta_k x),$$
$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{b}(\xi,t)}{\bar{a}(\xi)}e^{-i\xi x}d\xi + i\sum_k \bar{C}_k\exp(-i\bar{\zeta}_k x), \tag{14}$$

and

$$K(x,y) = \begin{bmatrix}K_1(x,y)\\K_2(x,y)\end{bmatrix}, \quad \bar{K} = \begin{bmatrix}\bar{K}_1(x,y)\\\bar{K}_2(x,y)\end{bmatrix}.$$

The $\zeta_k$ $(\bar{\zeta}_k)$ are the eigenvalues of (1) which lie in the upper (lower) half plane; the $a, b, C_k, \bar{a}, \bar{b}, \bar{C}_k$ are the scattering data and have the time dependences

$$a(\zeta) = a_0(\zeta), \quad \bar{a}(\zeta) = \bar{a}_0(\zeta),$$
$$b(\xi,t) = b_0(\xi)\exp[-2A_0(\xi)t],$$
$$\bar{b}(\xi,t) = \bar{b}_0(\xi)\exp[2A_0(\xi)t], \tag{15}$$
$$C_k = C_{k0}\exp[-2A_0(\zeta_k)t],$$
$$\bar{C}_k = \bar{C}_{k0}\exp[2A_0(\bar{\zeta}_k)t], \text{ and } A_0(\zeta) = \lim_{|x|\to\infty} A(x,\zeta;t).$$

The eigenvalues and the various constants are determined by solving the eigenvalue problem (1) at the initial time. Following Zakharov and Shabat,[2] an infinite set of conservation laws can be found for the above systems and correspond in the case of Eq. (6) (also for KdV) to the polynomial conserved densities of integer rank.[6]

We note also that a large class of linear problems can be solved by this method. For such cases take $r \equiv 0$, and the procedure reduces to the Fourier-transform approach. The conservation laws for these cases are trivial.

The solution of the system of linear integral equations can readily be found in closed form when the reflection coefficients $b_0(\xi)$ and $\bar{b}_0(\xi)$ are identically zero. This requires a very special class of initial conditions. For arbitrary initial conditions, (13) can be solved asymptotically (large $t$) following Ref. 6. The general asymptotic solution is essentially a sequence of kinks

Newell, and Segur, [1]. AKNS consider the first order matrix operator

$$L = \begin{pmatrix} -\partial & q \\ r & \partial \end{pmatrix}. \tag{5.1}$$

The analysis of AKNS suggests to seek $B$ of the form

$$B = RL^{-1}, \tag{5.2}$$

where

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.3}$$

Setting $B$ as given by (5.2) into (2.4) gives

$$L_t = R - LRL^{-1}.$$

Multiplying by $L$ on the right we get

$$L_tL = RL - LR. \tag{5.4}$$

We proceed now to solve this equation for $R$ of form (5.3) when $L$ is of form (5.1).

A straightforward calculation gives

$$RL = \begin{pmatrix} -a\partial + br & aq + b\partial \\ -c\partial + dr & cq + d\partial \end{pmatrix},$$

$$LR = \begin{pmatrix} -\partial a + qc & -\partial b + qd \\ ra + \partial c & rb + \partial d \end{pmatrix}.$$

So

$$RL - LR = \begin{pmatrix} a_x - qc + br & 2b\partial + b_x + aq - qd \\ -2c\partial - c_x + dr - ra & -d_x + cq - rb \end{pmatrix}. \tag{5.5}$$

Differentiating (5.1) we get

$$L_t = \begin{pmatrix} 0 & q_t \\ r_t & 0 \end{pmatrix}.$$

A straightforward calculation gives

$$L_tL = \begin{pmatrix} q_tr & q_t\partial \\ -r_t\partial & r_tq \end{pmatrix}. \tag{5.6}$$



Substituting (5.5) and (5.6) into (5.4) we get 4 sets of relations from the 4 components:

$$\begin{aligned} &\text{(i)} \quad q_tr = a_x - qc + br, \\ &\text{(ii)} \quad q_t = 2b, \qquad b_x + aq - qd = 0, \\ &\text{(iii)} \quad r_t = 2c, \qquad -c_x + dr - ra = 0, \\ &\text{(iv)} \quad r_tq = -d_x + cq - rb. \end{aligned} \tag{5.7}$$

Substituting the first relation in (5.7ii) into (5.7i) and the first relation in (5.7iii) into (5.7iv) we get

$$br + qc = a_x \tag{5.8$_1$}$$

and

$$qc + rb = -d_x\,. \tag{5.8$_2$}$$

Subtracting these two we get

$$a_x + d_x = 0,$$

which we satisfy by setting $d = -a$. Substituting this into the second relation in (5.7ii) and the second relation in (5.7iii) gives

$$b_x = (d - a)q = -2aq, \tag{5.9$_1$}$$

$$c_x = (d - a)r = -2ar. \tag{5.9$_2$}$$

Multiply (5.9$_1$) by $c$, (5.9$_2$) by $b$, and (5.8$_1$) by $2a$, and add; we get

$$cb_x + bc_x + 2aa_x = 0;$$

from this we conclude that

$$cb + a^2 = \text{const.}$$

We take that constant to be 1; so

$$a = (1 - bc)^{1/2}. \tag{5.10}$$

Relations (5.9) and (5.10) constitute a system of differential equations for $b$ and $c$; if initial values are specified, $b$ and $c$ are uniquely determined in terms of $q$ and $r$. The first relations in (5.7ii) and (5.7iii):

$$q_t = 2b, \qquad r_t = 2c \tag{5.11}$$

is a system of evolution equations for $q$ and $r$; the right side is a nonlocal function of $q$ and $r$.

Equation (5.11) is particularly simple when $q = r$; in this case we choose $b = c$; the resulting system occurs in the theory of self-induced transparency, see [8]. Relation (5.10) suggests the parametrization

$$b = \sin u, \qquad a = \cos u. \tag{5.12}$$

Substituting this into (5.9) gives

$$\cos u u_x = -2 \cos u q$$

from which we deduce

$$q = -\tfrac{1}{2} u_x .$$

Substituting this into (5.11) and using (5.12) we get

$$u_{xt} + 4 \sin u = 0, \tag{5.13}$$

the so-called sine-Gordon equation. For application of these ideas to solutions of the sine-Gordon equation we refer the reader to [1].

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