March 25, 1977

Approximating Sturm-Liouville DE with a Jacobi matrix. Start with

$$(1) \qquad -\frac{1}{2}\frac{d^2\psi}{dx^2} + q\psi = E\psi$$

and note that

$$\frac{1}{2}\Big[\psi((n+1)h) - 2\psi(nh) + \psi((n-1)h)\Big]\frac{1}{h^2} \longrightarrow \frac{1}{2}\frac{d^2\psi}{dx^2}(x)$$

as $h \to 0$ and $nh \to x$. Thus

$$\frac{1}{2}\Big[\psi((n+1)h) + \psi((n-1)h)\Big] = \Big(1 - Eh^2 + \overset{q(nh)}{q}\,h^2\Big)\psi(nh)$$

will yield (1) as $h \to 0$ and $nh \to x$. An even simpler candidate is

$$(2) \qquad \frac{1}{2}\Big[\psi((n+1)h) + \psi((n-1)h)\Big] = e^{-h^2E + h^2 q(nh)}\,\psi(nh).$$

Put $\sigma(n) = h^2 q(nh)$, $\lambda = e^{-h^2E}$. Then (2) can be rewritten

$$\frac{1}{2}\Big[e^{-\frac{1}{2}(\sigma(n)+\sigma(n-1))}\,e^{\frac{1}{2}\sigma(n-1)}\psi((n-1)h) + e^{-\frac{1}{2}(\sigma(n)+\sigma(n+1))}\,e^{\frac{1}{2}\sigma(n+1)}\psi((n+1)h)\Big]$$

$$= \lambda\, e^{\frac{1}{2}\sigma(n)}\psi(nh).$$

Hence if we put $y_n = e^{\frac{1}{2}\sigma(n)}\psi(nh)$ and $a_n = \frac{1}{2}e^{-\frac{1}{2}(\sigma(n)+\sigma(n+1))}$ we get

$$\boxed{a_{n-1}\,y_{n-1} + a_n\,y_{n+1} = \lambda\,y_n}$$

which is the eigenvalue problem for $L = aT + T^{-1}a$

Note that there is an equivalence between the systems

$$\lambda y_k = a_{k-1} y_{k-1} + a_{k} y_{k+1}$$

$$\lambda m_k \psi_k = \frac{1}{2}\left(\psi_{k-1} + \psi_{k+1}\right)$$

given by $y_k = m_k^{1/2}\psi_k$, $a_k = (m_k m_{k+1})^{-1/2}/2$.

Also the usual way of taking the limit of the discrete string

$$m_k \ddot{y}_k = a_{k-1}(y_{k-1} - y_k) + a_k(y_{k+1} - y_k)$$

is to let $y_k = u(kh)$, $m_k = h\rho(kh)$, $a_k = T(kh)/h$ and let $h \to 0$. The limiting equation is

$$\rho u_{tt} = (T u_x)_x$$

<u>March 29, 1977</u>

Inverse scattering à la Kac:

Recall we have an equivalence between J-matrices $L = \begin{pmatrix} 0 & a_1 & & \\ a_1 & 0 & a_2 & \\ & a_2 & 0 & \ddots \\ & & \ddots & \end{pmatrix}$ $a_i > 0$ bounded and bounded symmetric measures $d\mu$ on $\mathbb{R}$ of mass 1. In the case $a_i = \frac{1}{2}$ for all i we found the measure: $d\mu(x) = \frac{2}{\pi}\sqrt{1-x^2}\,dx$.

To go from $d\mu$ to L one constructs the orthonormal sequence of polynomials belonging to $d\mu$. To go from L to $d\mu$ one constructs eigenfunctions $\psi(\lambda)$:

$$\begin{cases} L\psi(\lambda) = \lambda\psi(\lambda) \\ \psi(\lambda)_0 = 0 \\ \psi(\lambda)_1 = 1. \end{cases}$$

Then $d\mu$ is the measure such that

$$e_1 = \int \psi(\lambda)\, d\mu(\lambda)$$

and to find it one can truncate L to an $(n\times n)$-matrix L_n whose eigenfunctions will be those $\psi(\lambda) \ni \psi(\lambda)_{n+1} = 0$. One then gets a measure $d\mu_n(\lambda)$ supported on n points such that

$$e_1 = \int \psi(\lambda)\, d\mu_n(\lambda)$$

in degrees $\leq n$. Now let $n \to \infty$.

In the example with all $a_i = \frac{1}{2}$ one finds

$$\psi(\lambda)_n = \frac{\sin n\theta}{\sin\theta} \qquad \text{where} \quad \begin{array}{l} \lambda = \cos\theta \\ 0 < \theta < \pi \end{array}$$

for the eigenfunctions:

$$\int \psi(\lambda)\, d\mu(\lambda) = \int_0^\pi \frac{\sin n\theta}{\sin\theta}\left(\frac{2}{\pi}\sin^2\theta\, d\theta\right) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$$

So now we want to consider the case where L is a perturbation of the case $L^{(0)}$ with all $a_n = \frac{1}{2}$. Thus L has $a_n = \frac{1}{2}$ for $n \geq n_0$. In this case we know that

$$\psi(\lambda)_n = \left(c_1(\theta) e^{in\theta} + \overline{c_1(\theta)} e^{-in\theta}\right)/2 \qquad \begin{matrix} \lambda = \cos\theta \\ 0 < \theta < \pi \end{matrix}$$

for n large, and also that there are a finite number of $\lambda_n \in \mathbb{R}$ with $|\lambda| > 1$ such that $\psi(\lambda) \in \ell^2$. In effect if we start with the eigenfunction

$$n \longmapsto \begin{cases} \left(\lambda - \sqrt{\lambda^2-1}\right)^n & \lambda > 1 \\ \left(\lambda + \sqrt{\lambda^2-1}\right)^n & \lambda < -1 \end{cases}$$

for n large and calculate its value at 0 which get an algebraic equation for λ which has only finitely many roots. What I have to understand now is how to relate $d\mu(\lambda)$ to the scattering data.

Let λ be an eigenvalue such that $\psi(\lambda) \in \ell^2$. Then

$$d\mu(x) = \frac{1}{\|\psi(\lambda)\|^2}\,\delta_\lambda(x)$$

for x near λ.

Let E_n denote the projection on the first n coordinates. If $p(X)$ is a poly of degree $\leq n$, one has

$$p(L)e_1 = \int p(x)\,\psi(x)\,d\mu(x)$$

$$\big(p(L)e_1, E_n\psi(y)\big) = \int p(x)\big(\psi(x), E_n\psi(y)\big)\,d\mu(x)$$

$$\|$$

$$\big(p(L)e_1, \psi(y)\big) = \big(e_1, \bar{p}(y)\psi(y)\big) = p(y)$$

Thus we have

$$\lim_{n\to\infty} \int p(x)\big(\psi(x), E_n\psi(y)\big)\,d\mu(x) = p(y)$$

for all polynomials $p(X)$, which means as measures

$$\lim_{n\to\infty} \big(\psi(x), E_n\psi(y)\big)\,d\mu(x) = \delta(x-y)\,dx$$

Now if $y \in$ point spectrum this gives

$$\|\psi(y)\|^2\,d\mu(x) = \delta(x-y)\,dx$$

for x near y. On the other hand suppose $-1 < y < 1$, whence

$$\psi(x)_n \sim \operatorname{Re}\, c(\theta)e^{in\theta} \qquad x = \cos\theta \qquad 0<\theta<\pi$$

$$\psi(y)_n \sim \operatorname{Re}\, c(\theta')e^{in\theta'} \qquad y = \cos\theta'$$

We can simplify this by writing

$$\psi(x)_n \sim A(\theta)\sin\big(n\theta - \delta(\theta)\big)$$

where $\delta(\theta)$ is a phase shift.

$$(\psi(x), E_n \psi(y)) = \sum_{k=1}^{n} \psi(x)_k \psi(y)_k$$

$$= \sum_{k=1}^{n_0} \psi(x)_k \psi(y)_k + \sum_{k=n_0+1}^{n} A(\theta)A(\theta')\sin\big(k\theta-\delta(\theta)\big)\sin\big(k\theta'-\delta(\theta')\big)$$

Since I expect $d\mu(x) = \rho(\theta)d\theta$ for $0<\theta<\pi$, the Riemann-Lebesgue lemma should tell me what

$$A(\theta)A(\theta')\sum_{k=n_0+1}^{n} \sin(k\theta-\delta)\sin(k\theta'-\delta')\rho(\theta)\,d\theta$$

converges to as $n \to \infty$.

$$\sin\alpha\,\sin\beta = \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta)$$

$$\sum_{k=n_0+1}^{n} \cos(kx+\varepsilon) = \mathrm{Re}\sum_{n_0<k\le n} e^{i\varepsilon}e^{ikx} = \mathrm{Re}\Big(e^{i\varepsilon}\frac{e^{i(n+1)x}-e^{i(n_0+1)x}}{e^{ix}-1}\Big)$$

$$= \mathrm{Re}\Big(e^{i\varepsilon}\frac{e^{inx}-e^{in_0x}}{e^{+ix}-1}\Big)$$

$$\sum_{k=n_0+1}^{n} \cos\big(k(\theta+\theta')-\delta-\delta'\big)\,d\theta = \mathrm{Re}\; e^{-i(\delta+\delta')}\frac{e^{in(\theta+\theta')}-e^{in_0(\theta+\theta')}}{e^{-i(\theta+\theta')}-1}\,d\theta$$

$$\longrightarrow \mathrm{Re}\Big(e^{-i(\delta+\delta')}\frac{e^{in_0(\theta+\theta')}}{1-e^{-i(\theta+\theta')}}\Big)d\theta$$

doesn't blow up for $0<\theta,\theta'<\pi$

$$\sum_{k=n_0+1}^{n} \cos\big(k(\theta-\theta')+\delta-\delta'\big)\,d\theta = \mathrm{Re}\Big(e^{i(\delta-\delta')}\frac{e^{in(\theta-\theta')}-e^{in_0(\theta-\theta')}}{e^{-i(\theta-\theta')}-1}\Big)d\theta$$

$$\longrightarrow \mathrm{Re}\Big(e^{i(\delta-\delta')}\frac{e^{in_0(\theta-\theta')}}{1-e^{-i(\theta-\theta')}}\Big)d\theta$$

when $\theta\neq\theta'$. ?

Problem: What is $\sum_{n=1}^{\infty} e^{in\theta}$?

First interpretation. Put $z = e^{i\theta}$

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \qquad \text{if} \quad |z|<1.$$

Hence Abel summation gives the value $\frac{e^{i\theta}}{1-e^{i\theta}}$ for $\sum_{n=1}^{\infty} e^{in\theta}$

$$\frac{e^{i\theta}}{1-e^{i\theta}} = \frac{-e^{i\frac{\theta}{2}}}{e^{i\theta/2}-e^{-i\theta/2}} = \frac{i}{2}\,\frac{e^{\frac{i\theta}{2}}}{\sin\frac{\theta}{2}}$$

$$= -\frac{1}{2} + i\,\frac{1}{2}\cot\left(\frac{\theta}{2}\right).$$

This holds for $\theta \neq 0$. In general $\sum_{n=1}^{\infty} e^{-in\theta}$ is a distribution on S^1 which is given by the formula

$$\sum_{n=1}^{\infty} e^{-in\theta} = \pi\delta(0) - \frac{1}{2} + i\,\frac{1}{2}\cot\left(\frac{\theta}{2}\right).$$

Here integration with $\cot\left(\frac{\theta}{2}\right)$ is defined in the principal value sense — for test functions vanishing at zero, integration is well-defined & one extends this to be zero on constant functions.

March 30, 1977.

Scattering: Suppose one has a two-sided J-matrix L (symmetric + pos. off-diagonal entries) which agrees with $L_0 = \frac{1}{2}T + \frac{1}{2}T^{-1}$ ouside of a finite region. For each λ we get an automorphism $S(\lambda)$ of the 2-diml. space $\mathrm{Ker}(L_0-\lambda)$ as follows. Given $y \in \mathrm{Ker}(L_0 - \lambda)$, let $A_-y \in \mathrm{Ker}(L-\lambda)$ the unique element agreeing with y in large negative degrees, and $A_+y \in \mathrm{Ker}(L-\lambda)$ the element agreeing with y is large positive degrees. Then $S(\lambda)$ is defined by

$$A_+(S(\lambda)y) = A_-y.$$

Thus the perturbation L of L_0 gives rise to an automorphism S of the two-plane bundle over $\mathbb{C}$ determined by the eigensolutions of L_0.

How can one relate the spectrum to the scattering operator $S(\lambda)$? First you have to fix the boundary conditions to get a self-adjoint problem. In the one-sided problem you require $y_0 = 0$ and y should be bounded at ∞. Here the multiplicity is one for each λ. In the two-sided problem one requires boundedness in each direction, and the multiplicity is 2 for each λ. So now we have to parametrize our eigenfunctions. This we do by the $-\infty$ boundary condition. So introduce for each θ, $-\pi < \theta \leq \pi$ the eigenfunction $\phi(\theta)$:

$$\begin{cases} L\phi(\theta) = \cos\theta \, \phi(\theta) \\ \phi(\theta)_n = e^{in\theta} \qquad n \ll 0 \end{cases}$$

This parameterizes the continuous spectrum eigenfunctions. In addition we have certain bound states

$$L\phi_+(\tau) = \cosh(\tau)\,\phi_+(\tau)$$

$$\phi_+(\tau)_n = \begin{cases} e^{+n\tau} & n \ll 0 \\ \text{const}\cdot e^{-n\tau} & n \gg 0 \end{cases}$$

$$L\phi_-(\tau) = -\cosh(\tau)\,\phi_+(\tau)$$

$$\phi_-(\tau)_n = \begin{cases} e^{n\tau} & n \ll 0 \\ \text{const}\cdot e^{-n\tau} & n \gg 0. \end{cases}$$

What remains is to find the spectral (Plancherel) measure which gives an expansion thus:

$$f = \int_{-\pi}^{\pi} (f, \phi(\theta))\,\phi(\theta)\,d\nu(\theta) \quad + \quad \text{discrete part.}$$

For example if $L = \frac{1}{2}T + \frac{1}{2}T^{-1}$ then $\phi(\theta)_n = e^{in\theta}$ for all n and we have the Fourier expansion

$$f_n = \int_{-\pi}^{\pi} g(\theta)e^{in\theta}\,\frac{d\theta}{2\pi} \qquad g(\theta) = \sum_n{}' f_n e^{-in\theta}$$

so that $d\nu(\theta) = \frac{1}{2\pi}\,d\theta.$

It seems to be desirable to introduce the Green's function $G(m,n,\lambda)$ which is the resolvent of L. Thus $G(\ ,n,\lambda)$ is the solution of

$$(\lambda - L)\, G(\ ,n,\lambda) = \delta_n$$

which dies exponentially when $\lambda \notin$ spectrum of L. Since

$$G_\lambda = \frac{1}{\lambda - L} = \frac{1}{\lambda}\,\frac{1}{1-\frac{L}{\lambda}} \sim \frac{1}{\lambda} \qquad \text{as } |\lambda| \to \infty$$

one sees that

$$\frac{1}{2\pi i}\oint G_\lambda \, d\lambda = I$$

i.e.

$$\frac{1}{2\pi i}\oint G(m,n,\lambda)\, d\lambda = \delta_{mn}$$

where the circle of integration is large. Now deform this circle down to around the singularities of G_λ and you should get the desired expansion theorem.

Describe the scattering as follows. Denote by $\psi(\lambda)$ the eigenfunction with

$$L\psi(\lambda) = \lambda\,\psi(\lambda)$$

$$\psi(\lambda)_n = \left(\lambda + \sqrt{\lambda^2-1}\right)^n \qquad n \ll 0$$

Here $\sqrt{\lambda^2-1}$ is the branch off $-1 \le \lambda \le 1$ which is asymptotic to λ as $|\lambda| \to \infty$. Thus

$$\psi(\lambda) = A_-\left(n \mapsto \left(\lambda + \sqrt{\lambda^2-1}\right)^n\right)$$

For n large and positive one has

$$\psi(\lambda)_n = c(\lambda)\left(\lambda + \sqrt{\lambda^2-1}\right)^n + b(\lambda)\left(\lambda - \sqrt{\lambda^2-1}\right)^n$$

where $c(\lambda)$ = transmission coefficient and $b(\lambda)$ = reflection coefficient.

Let $\phi(\lambda)_n = (\lambda$ [illegible] $\sqrt{\lambda^2-1})^n$ for all n and $\tilde{\phi}(\lambda)_n = (\lambda$ [illegible] $\sqrt{\lambda^2-1})^n$ for all n. Then the above says

$$\psi(\lambda) = A_-(\lambda)\,\phi(\lambda)$$

$$\psi(\lambda) = A_+(\lambda)\left[c(\lambda)\phi(\lambda) + b(\lambda)\tilde{\phi}(\lambda)\right]$$

Note $\phi(\lambda)$, $\tilde{\phi}(\lambda)$ form a natural basis for $\operatorname{Ker}(L_0-\lambda)$. Let $\tilde{\psi}(\lambda)$ be the eigenfunction of L with

$$L\tilde{\psi}(\lambda) = \lambda\tilde{\psi}(\lambda)$$

$$\tilde{\psi}(\lambda)_n = (\lambda \text{ [illegible] } \sqrt{\lambda^2-1})^n$$

i.e. $$\tilde{\psi}(\lambda) = A_-(\lambda)\,\tilde{\phi}(\lambda)$$

or equivalently $\tilde{\psi}(\lambda)$ = analytic continuation of $\psi(\lambda)$ across the cut. Define $\tilde{c}(\lambda)$, $\tilde{b}(\lambda)$ by

$$\tilde{\psi}(\lambda) = A_+(\lambda)\left[\tilde{c}(\lambda)\tilde{\phi}(\lambda) + \tilde{b}(\lambda)\phi(\lambda)\right]$$

Then

$$S(\lambda)\phi(\lambda) = c(\lambda)\phi(\lambda) + b(\lambda)\tilde{\phi}(\lambda)$$

$$S(\lambda)\tilde{\phi}(\lambda) = \tilde{b}(\lambda)\phi(\lambda) + \tilde{c}(\lambda)\tilde{\phi}(\lambda)$$

or in the basis $\phi(\lambda)$, $\tilde{\phi}(\lambda)$ one has

$$S(\lambda) = \begin{pmatrix} c(\lambda) & \tilde{b}(\lambda) \\ b(\lambda) & \tilde{c}(\lambda) \end{pmatrix}$$

It seems clear that $A_{\pm}(\lambda)$, $\delta(\lambda)$ are entire functions of λ. But $\phi(\lambda)$, $\tilde{\phi}(\lambda)$ forms a basis for $\mathrm{Ker}(L_0-\lambda)$ only for $\lambda \neq \pm 1$.

Note that $\lambda - \sqrt{\lambda^2-1} \sim \lambda\left(1-\left(1-\frac{1}{\lambda^2}\right)^{1/2}\right) = \lambda\left(1-\left(1-\frac{1}{2\lambda^2}\right)\right) = \frac{1}{2\lambda}$, $\lambda + \sqrt{\lambda^2-1} \sim 2\lambda$ for λ large, so that

$$\phi(\lambda)_n = \left(\lambda+\sqrt{\lambda^2-1}\right)^n \quad \text{and } \psi(\lambda)$$

dies exponentially for $n \to -\infty$, whereas

$$\tilde{\phi}(\lambda)_n = \left(\lambda-\sqrt{\lambda^2-1}\right)^n \quad \text{and } \tilde{\psi}(\lambda)$$

NO

dies exponentially for $n \to +\infty$.

So when we construct G_λ we shall use

$$G(m,n,\lambda) = \begin{cases} \alpha(\lambda)\,\psi(\lambda)_m & m \le n \\ \beta(\lambda)\,\tilde{\psi}(\lambda)_m & m \ge n \end{cases}$$

Our first equation is

$$\alpha(\lambda)\psi(\lambda)_n = \beta(\lambda)\tilde{\psi}(\lambda)_n$$

$$\Big(\big(\lambda - L\big)G(\ ,n,\lambda)\Big)_n = 1$$

$$\|$$

$$\underbrace{a_{n-1}G(n-1,n,\lambda) + (b_n-\lambda)G(n,n,\lambda)}_{-a_n\alpha(\lambda)\psi(\lambda)_{n+1}} + \underbrace{a_n G(n+1,n,\lambda)}_{a_n\beta(\lambda)\tilde{\psi}(\lambda)_{n+1}} = -1$$

$$-a_n\alpha(\lambda)\psi(\lambda)_{n+1} \quad a_n\beta(\lambda)\tilde{\psi}(\lambda)_{n+1} = -1$$

Two equations because

$$\alpha(\lambda)\psi(\lambda)_n - \beta(\lambda)\tilde{\psi}(\lambda)_n = 0$$

$$+\alpha(\lambda)\psi(\lambda)_{n+1} - \beta(\lambda)\tilde{\psi}(\lambda)_{n+1} = +\frac{1}{a_n}$$

Note that

$$a_n \begin{vmatrix} \psi(\lambda)_n & \tilde{\psi}(\lambda)_n \\ \psi(\lambda)_{n+1} & \tilde{\psi}(\lambda)_{n+1} \end{vmatrix} = \begin{vmatrix} \psi(\lambda)_n & \tilde{\psi}(\lambda)_n \\ -a_{n-1}\psi(\lambda)_{n-1} & -a_{n-1}\tilde{\psi}(\lambda)_{n-1} \end{vmatrix} = a_{n-1} \begin{vmatrix} \psi(\lambda)_{n-1} & \tilde{\psi}(\lambda)_{n-1} \\ \psi(\lambda)_n & \tilde{\psi}(\lambda)_n \end{vmatrix}$$

hence this Wronskian is constant, and we can evaluate it for n large and ~~negative~~ positive

I see now that I have the wrong definition of $\tilde{\psi}(\lambda)$. What I want here is for

$$\tilde{\psi}(\lambda) \sim \left(\lambda - \sqrt{\lambda^2-1}\right)^n \qquad \text{as} \quad n \longrightarrow +\infty$$

ie

$$\tilde{\psi}(\lambda) = A_+(\lambda)\,\tilde{\phi}(\lambda)$$

Now I recall

$$\psi(\lambda) = c(\lambda)A_+(\lambda)\phi(\lambda) + b(\lambda)A_+(\lambda)\tilde{\phi}(\lambda)$$

$$= c(\lambda)\left(\lambda + \sqrt{\lambda^2-1}\right)^n + b(\lambda)\left(\lambda - \sqrt{\lambda^2-1}\right)^n \qquad n \gg 0$$

so that for $n \gg 0$

$$a_n \begin{vmatrix} \psi(\lambda)_n & -\tilde{\psi}(\lambda)_n \\ \psi(\lambda)_{n+1} & -\tilde{\psi}(\lambda)_{n+1} \end{vmatrix} = a_n \begin{vmatrix} c(\lambda)(\lambda+\sqrt{\lambda^2-1})^n + b(\lambda)(\lambda-\sqrt{\lambda^2-1})^n & -(\lambda-\sqrt{\lambda^2-1})^n \\ c(\lambda)(\lambda+\sqrt{\lambda^2-1})^{n+1} + b(\lambda)(\lambda-\sqrt{\lambda^2-1})^{n+1} & -(\lambda-\sqrt{\lambda^2-1})^{n+1} \end{vmatrix}$$

$$= \frac{1}{2}c(\lambda)\left[\left(\lambda-\sqrt{\lambda^2-1}\right) - \left(\lambda+\sqrt{\lambda^2-1}\right)\right]$$

$$= +c(\lambda)\sqrt{\lambda^2-1}$$

This holds for all n.

Now return to the equations

$$\begin{cases} \psi(\lambda)_n \, \alpha(\lambda) - \tilde{\psi}(\lambda)_n \, \beta(\lambda) = 0 \\ a_n \psi(\lambda)_{n+1} \, \alpha(\lambda) - a_n \tilde{\psi}(\lambda)_{n+1} \, \beta(\lambda) = 1 \end{cases}$$

$$\alpha(\lambda) = \frac{\begin{vmatrix} 0 & -\tilde{\psi}(\lambda)_n \\ 1 & -a_n\tilde{\psi}(\lambda)_{n+1} \end{vmatrix}}{+c(\lambda)\sqrt{\lambda^2-1}} = \frac{\tilde{\psi}(\lambda)_n}{c(\lambda)\sqrt{\lambda^2-1}}$$

$$\beta(\lambda) = \frac{\begin{vmatrix} \psi(\lambda)_n & 0 \\ a_n\psi(\lambda)_{n+1} & 1 \end{vmatrix}}{+c(\lambda)\sqrt{\lambda^2-1}} = \frac{\psi(\lambda)_n}{c(\lambda)\sqrt{\lambda^2-1}}$$

Hence

$$G(m,n,\lambda) = \begin{cases} \dfrac{\psi(\lambda)_m \, \tilde{\psi}(\lambda)_n}{c(\lambda)\sqrt{\lambda^2-1}} & m \le n \\[2ex] \dfrac{\psi(\lambda)_n \, \tilde{\psi}(\lambda)_m}{c(\lambda)\sqrt{\lambda^2-1}} & m \ge n \end{cases}$$

$$\boxed{G(m,n,\lambda) = \frac{\psi(\lambda)_{m_<} \, \tilde{\psi}(\lambda)_{m_>}}{c(\lambda)\sqrt{\lambda^2-1}}}$$

$m_< = \min(m,n)$
$m_> = \max(m,n)$

For example if $L = L_0$, then

$$\psi(\lambda)_n = \left(\lambda + \sqrt{\lambda^2-1}\right)^n, \qquad \tilde{\psi}(\lambda)_n = \left(\lambda - \sqrt{\lambda^2-1}\right)^n, \qquad c(\lambda) = 1$$

so

$$G(m,n,\lambda) = \frac{\left(\lambda+\sqrt{\lambda^2-1}\right)^{m_<}\left(\lambda-\sqrt{\lambda^2-1}\right)^{m_>}}{\sqrt{\lambda^2-1}}$$

$$= \frac{\left(\lambda-\sqrt{\lambda^2-1}\right)^{|m-n|}}{\sqrt{\lambda^2-1}}$$

as a check:

$$\tfrac{1}{2}G(n-1,n,\lambda) - \lambda\, G(n,n,\lambda) + \tfrac{1}{2}G(n+1,n,\lambda)$$

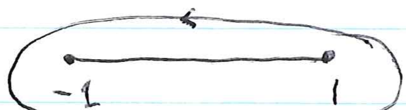
$$= \frac{\lambda - \sqrt{\lambda^2-1} - 2\lambda + \lambda - \sqrt{\lambda^2-1}}{2\sqrt{\lambda^2-1}} = -1 \quad \checkmark .$$

Now compute

$$\frac{1}{2\pi i}\oint G(m,n,\lambda)\, d\lambda$$

Take first the case $L = L_0$, and put $d = |m-n|$. Put $\lambda = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right)$, $z = re^{i\theta}$ where $r > 1$, $0 \leq \theta \leq 2\pi$. This gives a contour

$$\lambda \sim \tfrac{1}{2} r e^{i\theta} \qquad r \gg 1.$$



$$\lambda = \cos\theta \qquad r = 1.$$

$$d\lambda = \left(\frac{1}{2} - \frac{1}{2z^2}\right) dz \qquad \sqrt{\lambda^2 - 1} = \sqrt{\tfrac{1}{4}\left(z^2 - 2 + \tfrac{1}{z^2}\right)} = \tfrac{1}{2}\left(z - \tfrac{1}{z}\right)$$

$$\frac{d\lambda}{\sqrt{\lambda^2-1}} = \frac{dz}{z} = \frac{ire^{i\theta}\, d\theta}{re^{i\theta}} = i\, d\theta$$

$$\lambda - \sqrt{\lambda^2 - 1} = \frac{1}{z}. \qquad \text{Thus}$$

$$\frac{1}{2\pi i}\oint G(m,n,\lambda)\, d\lambda = \frac{1}{2\pi}\int_0^{2\pi} z^{-|m-n|}\, d\theta = \delta_{m,n}$$

In the general case we get what? Note that we we put $\lambda = \cos\theta$ we sort of specify what branch of $\sqrt{\lambda^2-1}$ to take, namely $\sqrt{\lambda^2-1} = i\sin\theta$. So

$$\psi(\cos\theta)_n = e^{in\theta} \qquad n \ll 0$$

$$\tilde{\psi}(\cos\theta)_n = e^{-in\theta} \qquad n \gg 0$$

(Think of θ as having come from $\mathrm{Im}(\theta) < 0$ i.e $|e^{i\theta}| > 1$).

So we get

$$\frac{1}{2\pi i}\oint G(m,n,\lambda)\,d\lambda = \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi(\cos\theta)_{m_<}\,\tilde{\psi}(\cos\theta)_{m_>}}{c(\cos\theta)}\,d\theta \quad + \text{ discrete part}$$

March 31, 1977.

Review. We consider a J-matrix $L = aT + b + T^{-1}a$ where $a_n = \frac{1}{2}$ $b_n = 0$ for $|n|$ large. For each $\lambda \in \mathbb{C}$ we have a unique eigenfunction $\psi(\lambda)$ with

$$(1) \qquad \begin{aligned} L\psi(\lambda) &= \lambda\psi(\lambda) \\ \psi(\lambda)_n &= \left(\lambda + \sqrt{\lambda^2 - 1}\right)^n \qquad n \ll 0. \end{aligned}$$

To be more accurate we cut the complex plane along the segment $[-1, 1]$. Off this segment we take the branch of $\sqrt{\lambda^2-1}$ which is asymptotic to λ for large $|\lambda|$. On the cut we use the parameterization $\lambda = \cos\theta$ to distinguish the upper and lower parts of the cut.

Thus $\psi(\lambda)$ is defined for $\lambda \notin [-1, 1]$ by (1). It decays exponentially as $n \to -\infty$. On the cut we define $\psi(\theta)$ by

$$\begin{aligned} L\psi(\theta) &= \cos\theta\,\psi(\theta) \\ \psi(\theta) &= e^{in\theta} \qquad n \ll 0. \end{aligned}$$

Similarly we can define $\tilde{\psi}(\lambda)$ to be the eigenfunction with $\quad \tilde{\varphi}(\lambda) = \left(\lambda - \sqrt{\lambda^2-1}\right)^n \qquad n \gg 0.$ This is for λ off the cut, and specializing to the cut we get

$$\begin{cases} L\tilde{\varphi}(\theta) = \cos\theta\, \tilde{\varphi}(\theta) \\ \tilde{\varphi}(\theta) = e^{-in\theta} \qquad n \gg 0 \end{cases}$$

Now for $n \gg 0$ one has

$$\psi(\lambda)_n = c(\lambda)\left(\lambda + \sqrt{\lambda^2-1}\right)^n + b(\lambda)\left(\lambda - \sqrt{\lambda^2-1}\right)^n$$

hence
$$\psi(\lambda) = c(\lambda)\,\gamma(\lambda) + b(\lambda)\,\tilde{\varphi}(\lambda)$$

where $\gamma(\lambda)$ is the eigenfunction asymptotic to $\left(\lambda + \sqrt{\lambda^2-1}\right)^n$ (= analytic continuation of $\tilde{\varphi}(\lambda)$ across the cut). This looks better on the cut:

$$\begin{cases} \psi(\theta) = c(\theta)e^{in\theta} + b(\theta)e^{-in\theta} & n \gg 0 \\ \psi(\theta) = e^{in\theta} & n \ll 0 \end{cases}$$

$c(\lambda)$ = transmission coeff., $b(\lambda)$ = reflection coeff.

Notice that if $c(\lambda) = 0$, then $\psi(\lambda) = b(\lambda)\tilde{\varphi}(\lambda)$ dies exponentially in both directions, hence we get a bound state or point eigenvalue, and conversely.

Compute Wronskian

$$W = a_n \begin{vmatrix} \psi(\lambda)_n & \tilde{\varphi}(\lambda)_n \\ \psi(\lambda)_{n+1} & \tilde{\varphi}(\lambda)_{n+1} \end{vmatrix} = \begin{vmatrix} \psi(\lambda)_n & \tilde{\varphi}(\lambda)_n \\ -b_n\psi(\lambda)_n + -a_{n-1}\psi(\lambda)_{n-1} & -b_n\tilde{\varphi}(\lambda)_n + -a_{n-1}\tilde{\varphi}(\lambda)_{n-1} \end{vmatrix} = a_{n-1}\begin{vmatrix} \psi(\lambda)_{n-1} & \tilde{\varphi}(\lambda)_{n-1} \\ \psi(\lambda)_n & \tilde{\varphi}(\lambda)_n \end{vmatrix}$$

is constant. Evaluate for n large

$$W = \frac{1}{2}\begin{vmatrix} c(\lambda)(\lambda+\sqrt{\lambda^2-1})^n + b(\lambda)\ldots & (\lambda-\sqrt{\lambda^2-1})^n \\ c(\lambda)(\lambda+\sqrt{\lambda^2-1})^{n+1} + b(\lambda)\ldots & (\lambda-\sqrt{\lambda^2-1})^{n+1} \end{vmatrix} = \frac{1}{2}\left[c(\lambda)\left(\lambda-\sqrt{\lambda^2-1}-\lambda-\sqrt{\lambda^2-1}\right)\right]$$

$$= -c(\lambda)\sqrt{\lambda^2-1}.$$

Now put $\boxed{G(m,n,\lambda) = \frac{\psi(\lambda)_{m<}\tilde{\psi}(\lambda)_{m>}}{c(\lambda)\sqrt{\lambda^2-1}}}$

and then $(L-\lambda)G(\cdot,n,\lambda)_m = 0 \qquad m \neq n$ whereas

$$(L-\lambda)G(\cdot,n,\lambda)_n = \frac{a_{n-1}\psi(\lambda)_{n-1}\tilde{\psi}(\lambda)_n + (b_n-\lambda)\psi(\lambda)_n\tilde{\psi}(\lambda)_n + a_n\psi(\lambda)_n\tilde{\psi}(\lambda)_{n+1}}{c(\lambda)\sqrt{\lambda^2-1}}$$

$$= \frac{-a_n\psi(\lambda)_{n+1}\tilde{\psi}(\lambda_n) + a_n\psi(\lambda)_n\tilde{\psi}(\lambda)_{n+1}}{c(\lambda)\sqrt{\lambda^2-1}} = -1$$

Thus G is the Green's function. Now take the contour integral

$$\delta_{mn} = \frac{1}{2\pi i}\oint G(m,n,\lambda)\,d\lambda \qquad \frac{d\lambda}{\sqrt{\lambda^2-1}} = \frac{-\cos\theta\,d\theta}{-i\sin\theta} = i\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{\psi(\theta)_{m<}\tilde{\psi}(\theta)_{m>}}{c(\theta)}\,d\theta + \text{sum over the zeroes of } c(\lambda)$$

Next let's consider the effects of conjugating:

~~[illegible]~~

$$\left.\begin{aligned} L\psi(\theta) &= \cos\theta\,\psi(\theta) \\ \psi(\theta) &= e^{-in\theta} \quad n \ll 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} L\overline{\psi(\theta)} &= \cos\theta\,\overline{\psi(\theta)} \\ \overline{\psi(\theta)} &= e^{-in\theta} \quad n \ll 0 \end{aligned}$$

$$\therefore \boxed{\overline{\psi(\theta)} = \psi(-\theta)}$$

similarly $\boxed{\overline{\tilde{\psi}(\theta)} = \tilde{\psi}(-\theta)}$

$$\psi(\theta) = c(\theta) e^{in\theta} + b(\theta) e^{-in\theta} \qquad n >> 0$$

$$\Rightarrow \quad \psi(\theta) = c(\theta)\tilde{\psi}(-\theta) + b(\theta)\tilde{\psi}(\theta)$$

$$\Rightarrow \quad \psi(-\theta) = \overline{c(\theta)}\,\tilde{\psi}(\theta) + \overline{b(\theta)}\,\tilde{\psi}(-\theta)$$

$$= c(-\theta)\tilde{\psi}(-\theta) + b(-\theta)\tilde{\psi}(\theta)$$

$$\therefore \quad \overline{c(\theta)} = b(-\theta)$$

$$b(\theta) = \overline{c(-\theta)}.$$

Suppose we scatter the other way:

$$\tilde{\psi}(\lambda)_n = f(\lambda)\left(\lambda - \sqrt{\lambda^2-1}\right)^n + g(\lambda)\left(\lambda+\sqrt{\lambda^2-1}\right)^n \qquad n << 0$$

$$W = \frac{1}{2}\begin{vmatrix} \left(\lambda+\sqrt{\lambda^2-1}\right)^n & f(\lambda)\left(\lambda-\sqrt{\lambda^2-1}\right)^n + \\ \left(\lambda+\sqrt{\lambda^2-1}\right)^{n+1} & f(\lambda)\left(\lambda-\sqrt{\lambda^2-1}\right)^{n+1} + \end{vmatrix}$$

$$= \frac{1}{2} f(\lambda)\left[\lambda - \sqrt{\lambda^2-1} - \lambda - \sqrt{\lambda^2-1}\right] = -f(\lambda)\sqrt{\lambda^2-1}$$

Conclude $f(\lambda) = c(\lambda)$.

Better

$$\psi(\lambda)_n = \left(\lambda+\sqrt{\lambda^2-1}\right)^n \qquad n << 0$$

$$\psi_1(\lambda)_n = \left(\lambda-\sqrt{\lambda^2-1}\right)^n \qquad "$$

$$\tilde{\psi}_1(\lambda)_n = \left(\lambda+\sqrt{\lambda^2-1}\right)^n \qquad n >> 0$$

$$\tilde{\psi}(\lambda)_n = \left(\lambda-\sqrt{\lambda^2-1}\right)^n \qquad n >> 0$$

$$\psi(\lambda) = c\,\tilde{\psi}_1(\lambda) + b\,\tilde{\psi}(\lambda)$$

$$\psi_1(\lambda) = c_1\tilde{\psi}_1(\lambda) + b_1\tilde{\psi}(\lambda) \qquad S(\lambda) = \begin{pmatrix} c & c_1 \\ b & b_1 \end{pmatrix}$$

Because of the Wronskian

$$\det S(\lambda) = 1$$

Hence $$S(\lambda)^{-1} = \begin{pmatrix} b_1 & -c_1 \\ -b & c \end{pmatrix}$$

so

$$(\tilde\psi_1 \;\; \tilde\psi) = (\psi \;\; \psi_1)\begin{pmatrix} b_1 & -c_1 \\ -b & c \end{pmatrix}$$

$$= (b_1\psi - b\psi_1 \quad -c_1\psi + c\psi_1)$$

$$\therefore\ \tilde\psi = -c_1\psi + c\psi_1$$

Now when $\lambda = \cos\theta$, $\tilde\psi_1(\lambda) = \tilde\psi(-\theta)$ so our equations become

$$(\psi(\theta) \;\; \psi(-\theta)) = (\tilde\psi(-\theta) \;\; \tilde\psi(\theta))\begin{pmatrix} c(\theta) & c(-\theta) \\ b(\theta) & b(-\theta) \end{pmatrix}$$

$$\begin{matrix} \parallel \\ \begin{pmatrix} c & \bar b \\ b & \bar c \end{pmatrix} \end{matrix}$$

so we get $S(\theta) = \begin{pmatrix} c(\theta) & \overline{b(\theta)} \\ b(\theta) & \overline{c(\theta)} \end{pmatrix}$ has det=1:

$$|c|^2 - |b|^2 = 1$$

I seems to be unable to derive the Plancherel measure. For example suppose λ is a root of $c(\lambda)$, whence the contribution to the contour integral coming from λ is

$m \le n$
$$\frac{\psi(\lambda)_m\,\tilde\psi(\lambda)_n}{c'(\lambda)\sqrt{\lambda^2-1}} = \frac{\psi(\lambda)_m\,\psi(\lambda)_n}{c'(\lambda)\,b(\lambda)\sqrt{\lambda^2-1}}$$

I still want to show $c'(\lambda)\, b(\lambda) \overset{\sqrt{\lambda^2-1}}{\wedge}= \|\psi(\lambda)\|^2$ where $c(\lambda)=0$. The problem with this is that the Green's function has not been closely related to the l^2 structure and self-adjoint nature of L yet.

Work out G-function for one-sided case:

$$L\phi(\lambda) = \lambda\, \phi(\lambda) \qquad \phi(\lambda)_0 = 0 \quad \phi(\lambda)_1 = 1.$$

$$\tilde{\psi}(\lambda) \text{ as before } = \left(\lambda - \sqrt{\lambda^2-1}\right)^n \qquad n \gg 0$$

$$\phi(\lambda) = a(\lambda)\left(\lambda + \sqrt{\lambda^2-1}\right)^n + b(\lambda)\left(\lambda - \sqrt{\lambda^2-1}\right)^n \qquad n \gg 0$$

$$G(m,n,\lambda) = \begin{cases} \alpha(\lambda)\, \phi(\lambda)_m & m \leq n \\ \beta(\lambda)\, \tilde{\psi}(\lambda)_m & m \geq n \end{cases}$$

$$\alpha(\lambda)\, \phi(\lambda)_n - \beta(\lambda)\, \tilde{\psi}(\lambda)_n = 0$$

$$\underbrace{a_{n-1}\, \alpha(\lambda)\, \phi(\lambda)_{n-1} + (b_n - \lambda)\, \alpha(\lambda)\, \phi(\lambda)_n}_{-a_n\, \alpha(\lambda)\, \phi(\lambda)_{n+1}} + a_n\, \beta(\lambda)\, \tilde{\psi}(\lambda)_{n+1} = -1$$

$$\alpha(\lambda)\, a_n \phi(\lambda)_{n+1} - \beta(\lambda)\, a_n\, \tilde{\psi}(\lambda)_{n+1} = 1.$$

$$W = a_n \begin{vmatrix} \phi(\lambda)_n & \tilde{\psi}(\lambda)_n \\ \phi(\lambda)_{n+1} & \tilde{\psi}(\lambda)_{n+1} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} a(\lambda)\left(\lambda+\sqrt{\lambda^2-1}\right)^n + b(\lambda)\ldots & \left(\lambda-\sqrt{\lambda^2-1}\right)^n \\ a(\lambda)\left(\lambda+\sqrt{\lambda^2-1}\right)^n + b(\lambda)\ldots & \left(\lambda-\sqrt{\lambda^2-1}\right)^{n+1} \end{vmatrix}$$

$$= -a(\lambda)\sqrt{\lambda^2-1}$$

$$\alpha(\lambda) = \frac{\begin{vmatrix} 0 & \tilde{\psi}(\lambda)_n \\ 1 & \end{vmatrix}}{-a(\lambda)\sqrt{\lambda^2-1}} = \frac{\tilde{\psi}(\lambda)_n}{a(\lambda)\sqrt{\lambda^2-1}} \qquad +\beta(\lambda) = \frac{\begin{vmatrix} \phi(\lambda)_n & 0 \\ & 1 \end{vmatrix}}{+a(\lambda)\sqrt{\lambda^2-1}}$$

$$= \frac{\phi(\lambda)_n}{a(\lambda)\sqrt{\lambda^2-1}}$$

$$G(m,n,\lambda) = \frac{\phi(\lambda)_{m<}\,\tilde{\psi}(\lambda)_{m>}}{a(\lambda)\sqrt{\lambda^2-1}}$$

$m \le n$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\phi(\theta)_m\,\tilde{\psi}(\theta)_n}{a(\theta)}\,d\theta = \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(\theta)_m\,\tilde{\psi}(\theta)_n}{a(\theta)}d\theta + \int_{-\pi}^{0}\frac{\phi(-\theta)_m\,\tilde{\psi}(-\theta)_n}{a(-\theta)}(-d\theta)$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\frac{\phi(\theta)_m\,\tilde{\psi}(\theta)_n}{a(\theta)} + \frac{\phi(-\theta)_m\,\tilde{\psi}(-\theta)_n}{a(-\theta)}\right)d\theta$$

Now

$$\phi(\theta)_n = a(\theta)e^{in\theta} + b(\theta)e^{-in\theta} \qquad n>>0$$

$$= a(\theta)\,\tilde{\psi}(-\theta)_n + b(\theta)\,\tilde{\psi}(\theta)_n \qquad \forall n$$

But $\phi(\theta) = \phi(-\theta)$ so $b(\theta) = a(-\theta)$

Thus the above integral is

$$\frac{1}{2\pi}\int_0^{\pi}\phi(\theta)_m\,\frac{a(-\theta)\tilde{\psi}(\theta)_n + a(\theta)\tilde{\psi}(-\theta)_n}{a(\theta)a(-\theta)}\,d\theta$$

$\phi(\theta) = \overline{\phi(\theta)}$
$\Rightarrow \overline{a(\theta)} = b(\theta)$
$= a(-\theta)$.

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(\theta)_m\,\phi(\theta)_n}{|a(\theta)|^2}\,d\theta \quad .$$

If λ is a point eigenvalue, i.e $a(\lambda)=0$ the contribution of λ to the contour integral of G is

$$\frac{\phi(\lambda)_m\,\phi(\lambda)_n}{a'(\lambda)\,b(\lambda)\sqrt{\lambda^2-1}}$$

April 1, 1977

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. To calculate the image of $P_1(\mathbb{R})$ under S^{-1} i.e. the set

$$\Delta = \left\{ m \in P_1\mathbb{C} \;\middle|\; \frac{am+b}{cm+d} \in P_1\mathbb{R} \right\}$$

This will be a circle in $\mathbb{C}$ provided $\infty \notin \Delta$, i.e. $\frac{a}{c} \notin P_1\mathbb{R}$, i.e. $\begin{vmatrix} a & \bar{a} \\ c & \bar{c} \end{vmatrix} = a\bar{c} - \bar{a}c \neq 0$.

The center of Δ reflected through Δ is $m = \infty$ which goes to $\frac{a}{c}$ which reflects thru $\mathbb{R}$ to $\frac{\bar{a}}{\bar{c}}$ which comes from

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \left(\frac{\bar{a}}{\bar{c}} \right) = \frac{d\bar{a} - b\bar{c}}{a\bar{c} - c\bar{a}} = \frac{\begin{vmatrix} \bar{a} & b \\ \bar{c} & d \end{vmatrix}}{\begin{vmatrix} a & \bar{a} \\ c & \bar{c} \end{vmatrix}}$$

This is the center of Δ. Any straight line joining $m =$ center Δ, $m = \infty$ corresponds to a circle containing $z = \frac{\bar{a}}{\bar{c}}$ and $z = \frac{a}{c}$. Take the circle $|z| = \frac{|a|}{|c|}$. The two m-points corresponding to $\pm \frac{|a|}{|c|}$ lie on opposite sides of Δ hence

$$\text{diam}(\Delta) = \left| \frac{d\left|\frac{a}{c}\right| - b}{-c\left|\frac{a}{c}\right| + a} + \frac{+d\left|\frac{a}{c}\right| + b}{+c\left|\frac{a}{c}\right| + a} \right|$$

Better

$$\text{rad}(\Delta) = \left| \frac{\bar{a}d - \bar{c}b}{a\bar{c} - c\bar{a}} + \frac{d}{c} \right| = \left| \frac{\cancel{\bar{a}dc} - c\bar{c}b + ad\bar{c} - \cancel{\bar{a}cd}}{(a\bar{c} - c\bar{a})c} \right|$$

$$= \left| \frac{\bar{c}(ad - bc)}{(a\bar{c} - c\bar{a})c} \right| = \left| \frac{1}{a\bar{c} - c\bar{a}} \right|$$

More clearly

$$\mathrm{rad}(\Delta) = \left| \overset{\text{center}}{\overset{\downarrow}{\frac{\bar a d - \bar c b}{a\bar c - \bar a c}}} - \overset{\text{point corres. to } m=\infty}{\overset{\downarrow}{\frac{d}{-c}}} \right| = \left| \frac{\cancel{\bar a c d} - \bar c bc + \bar c ad - \cancel{a \bar c d}}{(a\bar c - \bar a c)\, c} \right|$$

$$= \left| \frac{1}{\begin{vmatrix} a & \bar a \\ c & \bar c \end{vmatrix}} \right|$$

One applies this to a S-L system.

$$\frac{d}{dt}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

Let the solution matrix starting at $t=0$ be

$$S(t,\lambda) = \begin{pmatrix} \psi(t,\lambda) & \varphi(t,\lambda) \\ \psi'(t,\lambda) & \varphi'(t,\lambda) \end{pmatrix} \qquad S(0,\lambda) = I$$

and let $S = S(b,\lambda)$. Then we can parameterize the solutions by $m \in \mathbb{P}^1\mathbb{C}$, using $m \mapsto m\psi + \varphi$. The radius of the circle Δ_b in the m-plane corresponding to real values at $t=b$ is given by

$$\mathrm{rad}(\Delta_b) = \left| \frac{1}{\begin{vmatrix} \psi(b) & \bar\psi(b) \\ \psi'(b) & \bar\psi'(b) \end{vmatrix}} \right|$$

But

$$\frac{d}{dt}\begin{vmatrix} \psi & \bar\psi \\ \psi' & \bar\psi' \end{vmatrix} = \begin{vmatrix} \psi & \bar\psi \\ \psi'' & \bar\psi'' \end{vmatrix} = \begin{vmatrix} \psi & \bar\psi \\ (q-\lambda)\psi & (q-\bar\lambda)\bar\psi \end{vmatrix} = -\bar\lambda\psi\bar\psi + \lambda\psi\bar\psi$$

$$= 2i\,\mathrm{Im}(\lambda)\,\psi\bar\psi$$

So integrating from 0 to b we get

$$\begin{vmatrix} \psi & \bar{\psi}^* \\ \psi' & \bar{\psi}' \end{vmatrix}(b) = 2i\,\mathrm{Im}(\lambda) \int_0^b |\psi|^2 dt$$

so
$$\frac{1}{\mathrm{rad}(\Delta(b))^*} = 2|\mathrm{Im}(\lambda)| \int_0^b |\psi|^2 dt$$

Recall ~~that~~ from page 98 (March 4, 1977) that if S is the solution matrix for a system

$$\frac{dX}{dt} = AX \qquad \mathrm{tr}(A) = 0$$

with $\mathrm{Im}(A) = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ such that $q \geq 0$ $r \geq 0$, $\det = -p^2 + qr \geq 0$

then $S(t)$ carries H into H. H = upper half-plane.

Recall
$$\frac{d}{dt} \det(S) = \mathrm{tr}(A) \cdot \det(S) = 0$$

so $\det(S) \equiv 1$.

Next let's see what happens if we change variables:
$$X = UY$$

Then
$$\dot{U}Y + U\dot{Y} = \dot{X} = AX = AUY$$
$$U\dot{Y} = [AU - \dot{U}]Y$$
$$\dot{Y} = [U^{-1}AU - U^{-1}\dot{U}]Y$$

If $X = SX_0$ is the solution with initial value X_0 then
$$Y = U^{-1}X = U^{-1}SU_0Y_0$$
is the solution with initial value Y_0 so the new solution

matrix is

$$\tilde{S} = U^{-1} S U_0$$

Of course we want $\det(U) = 1$ so that

$$0 = \frac{d}{dt}\det(U) = \operatorname{tr}(\dot{U}U^{-1})\det(U) = \operatorname{tr}(\dot{U}U^{-1}) = \operatorname{tr}(U^{-1}\dot{U})$$

and so $\tilde{A} = U^{-1}AU - U^{-1}\dot{U}$ will still have trace zero.

So next consider

$$A = A_0 + \lambda A_1$$

where $A_1 = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ is in the good class. We suppose $U \in SL_2(\mathbb{R})$, then

$$\tilde{A} = \underbrace{\left(U^{-1}A_0U - U^{-1}\dot{U}\right)}_{\tilde{A}_0} + \lambda\underbrace{\left(U^{-1}A_1U\right)}_{\tilde{A}_1}$$

Suppose that $\det(A_1) > 0$ – this is the interesting case that comes from Ising models. Now if we reparametrize t, i.e. change from t to x so that

$$\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx}$$

then we replace our system by

$$\frac{dX}{dx} = \left(\frac{A_0 + \lambda A_1}{\frac{dx}{dt}}\right)X.$$

So choosing $\frac{dx}{dt} = \left(\det(A_1)\right)^{-1/2}$ we can arrange that A_1 have determinant 1.

Now we should be able to choose U so that

$$A_1 = U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^{-1}$$

with U varying smoothly. Note that the centralizer of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $SL_2(\mathbb{R})$ is the rotation group $K = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\}$ and that

$$SL_2(\mathbb{R})/K \xrightarrow{\sim} H$$

Therefore has to be a way of identifying $\left\{\begin{pmatrix} p & q \\ -r & -p \end{pmatrix} \quad \begin{matrix} q, r > 0 \\ -p^2+qr=1 \end{matrix}\right\}$ with H.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -(ac+bd) & a^2+b^2 \\ -c^2-d^2 & ac+bd \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(i) = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{(ac+bd)+i}{c^2+d^2}$$

Thus the 1-1 correspondence is

$$\begin{pmatrix} p & q \\ -r & -p \end{pmatrix} \longleftrightarrow \frac{-p+i}{r}$$

So it's now more or less clear that by these changes of variable we can manipulate our equation into the form

$$\frac{dX}{dt} = (A_0 + \lambda A_1)X$$

where $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We are still free to choose U in the centralizer of A_1, which is the rotation group K, in order to simplify the DE. This changes A_0 to $U^{-1}A_0U - \dot{U}^{-1}U$. Note that under $Ad(K)$ we have

$$sl_2(\mathbb{R}) = \underset{\mathbb{R}\begin{pmatrix}0&1\\-1&0\end{pmatrix}}{\underset{\parallel}{k}} \oplus \underset{\mathbb{R}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\mathbb{R}\begin{pmatrix}0&1\\1&0\end{pmatrix}}{\underset{\parallel}{p}}$$

with $p \simeq \mathbb{R}^2$ with K-action given by doubling θ:

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ -\sin 2\theta & -\cos 2\theta\end{pmatrix}$$

$$= \cos 2\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \sin 2\theta\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}c & -s\\ s & c\end{pmatrix} = \begin{pmatrix}\sin 2\theta & \cos 2\theta\\ \cos 2\theta & -\sin 2\theta\end{pmatrix}$$

$$= \sin 2\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \cos 2\theta\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Note that if $U = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, $\theta = \theta(t)$

$$U^{-1}\dot{U} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}\dot{\theta} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\dot{\theta}$$

Therefore we should be able to do one of two things by choosing U suitably:

i) Make $A_0 \in p$ = symmetric matrices

ii) Make $A_0 \in \mathbb{R}\begin{pmatrix}0&1\\-1&0\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

.

i) $$\frac{dX}{dt} = \left(A_0 + \lambda\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)X$$ A_0 real and symmetric trace zero

or

ii) $$\frac{dX}{dt} = \begin{pmatrix}a & b+\lambda\\ -b-\lambda & -a\end{pmatrix}X$$ with $a \geq 0$

Perhaps a third possibility is

iii) $A_0 \in \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

whence the equations are

iii) $$\frac{dX}{dt} = \begin{pmatrix} 0 & b+\lambda \\ c-\lambda & 0 \end{pmatrix} X \qquad b+c \geq 0$$

So let us now consider the solution matrix

$$S(t,\lambda) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

so that $$\dot{S} = \begin{pmatrix} \dot{\alpha} & \dot{\beta} \\ \dot{\gamma} & \dot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & b+\lambda \\ c-\lambda & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

So we consider $S = S(t,\lambda)$ as mapping $P_1\mathbb{C}$ to $P_1\mathbb{C}$. We want the inverse image Δ of $P_1\mathbb{R}$ under S. According to earlier calculations this has radius

$$\frac{1}{\mathrm{rad}(\Delta)} = \left\| \begin{matrix} \alpha & \bar{\alpha} \\ \gamma & \bar{\gamma} \end{matrix} \right\|$$

$$\frac{d}{dt}\begin{vmatrix} \alpha & \bar{\alpha} \\ \gamma & \bar{\gamma} \end{vmatrix} = \begin{vmatrix} \dot{\alpha} & \bar{\alpha} \\ \dot{\gamma} & \bar{\gamma} \end{vmatrix} + \begin{vmatrix} \alpha & \dot{\bar{\alpha}} \\ \gamma & \dot{\bar{\gamma}} \end{vmatrix} = \begin{vmatrix} (b+\lambda)\gamma & \bar{\alpha} \\ (c-\lambda)\alpha & \bar{\gamma} \end{vmatrix} + \begin{vmatrix} \alpha & (b+\bar{\lambda})\bar{\gamma} \\ \gamma & (c-\bar{\lambda})\bar{\alpha} \end{vmatrix}$$

$$= (b+\lambda)|\gamma|^2 - (c-\lambda)|\alpha|^2 + (c-\bar{\lambda})|\alpha|^2 - (b+\bar{\lambda})|\gamma|^2$$

$$= 2(\mathrm{Im}\,\lambda)\left(|\alpha|^2 + |\gamma|^2\right)$$

Thus

$$\frac{1}{\mathrm{rad}(\Delta)} = 2\,|\mathrm{Im}\,\lambda| \cdot \int_0^t \left(|\alpha|^2 + |\gamma|^2\right) dt$$