March 15, 1977.

Suppose J is a Jacobi matrix (doubly infinite) which is periodic of period n, i.e. $TJT^{-1} = J$ where T is the shift by n-steps. Let M be the vector space over $\mathbb{C} = k$ consisting of doubly-infinite row-vectors with finite support. Then M is a module over the ring $A = k[J, T]$ of doubly-infinite matrices. I can think of a linear map $u : M \longrightarrow k$ as a doubly-infinite column vector. To say that u is a common eigenvector for J, T:

$$Ju = \lambda u \quad , \quad Tu = z u \qquad (\lambda, z) \in \mathbb{C}$$

means that u is a module homomorphism when k is identified w- $A/(J-\lambda, T-z)$. Now I know already that $M/M(J-\lambda)$ is exactly 2-dimensional, and that $M/M(T-z)$ is n-dimensional. In fact it is clear that M is a free module of rank 2 over $k[J]$ and a free module of rank n over $k[T, T^{-1}]$. I know

$$T^2 - \mathrm{tr}\,\Phi(J)\, T + 1 = 0 \qquad (\text{say } c_1 \ldots c_n = a_1 \ldots a_n)$$

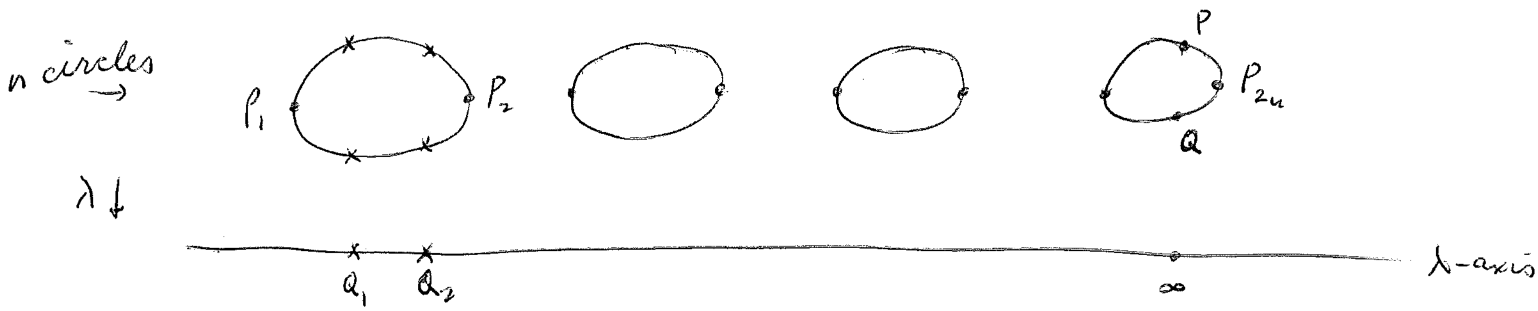
If I suppose this equation is non-singular i.e. putting $\mathrm{tr}\,\Phi(J) = 2\varphi(J)$, that

$$\varphi(J)^2 - 1 = 0$$

has $2n$ simple roots, then M will be a line bundle over the Dedekind domain A.

Next recall that there are two points $Q = (\infty, 0)$, $P = (\infty, \infty)$

missing from the variety of A to make it a complete non-singular curve C. J has simple poles at P, Q and is regular elsewhere. T has a n-th order 0 at Q and an n-th order pole at Q. Picture of real spectrum



From the picture one has

$$0 \longrightarrow \lambda^* K_{\mathbb{P}_1} \longrightarrow K_C \longrightarrow \bigoplus_{i=1}^{2n} k(P_i) \longrightarrow 0$$

so as $K_{\mathbb{P}_1} = \mathcal{O}(-Q_1 - Q_2)$ one has

$$2g - 2 = \deg(K_C) = -4 + 2n \qquad \text{or}$$

$$\boxed{g = n - 1.} \qquad n = g + 1.$$

Next observe that

$$\begin{array}{ccccccc} H^0(\mathcal{O}) & < & H^0(\mathcal{O}(P+Q)) & < & \quad & < & H^0(\mathcal{O}(kP+kQ)) \\ \cup & & \cup & & & & \cup \\ 1 & & 1, J & & & & 1, J, \dots, J^k \end{array}$$

so that $\dim H^0(\mathcal{O}(kP+kQ)) \geq k+1$. Moreover R-R implies

$$H^0(\mathcal{O}(gP+gQ)) = 2g+1-g = g+1$$

hence we conclude (since $H^0(\mathcal{O}(D)) \overset{J}{\hookrightarrow} H^0(\mathcal{O}(D+P+Q))$)

$$H^0(\mathcal{O}(\ell P+\ell Q)) = k1 \oplus \cdots \oplus kJ^{\ell}$$

for $0 \le \ell \le g$. Now for $\ell = g+1 = n$

$$H^0(\mathcal{O}((g+1)P+(g+1)Q)) = k1 \oplus \cdots \oplus kJ^{n} \oplus kT \qquad \dim g+3.$$

Next point will be to extend M to a line bundle over M. Recall as $\lambda \to \infty$

$$\frac{\Phi(\lambda)}{\lambda^n} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \qquad c \text{ non-zero constant}$$

hence the eigenvector for J with eigenvalue λ and T with eigenvalue $z \sim \lambda^{-n}$ ~~[illegible]~~ ought to converge in ~~[illegible]~~ non-negative degrees to a vector with $y_0 = 1$,

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March 16, 1977.

Let's recall the recursion formulas for $(J-\lambda)y = 0$:

$$(y_n \; y_{n+1}) = (y_0 \; y_1)\begin{pmatrix} 0 & -\frac{c_0}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}$$

Now we want to let λ approach infinity and consider the eigenvector with z value asymptotic to const λ^{r}. Here r denotes the period (denoted n above). Now

$$\Phi(\lambda) = \begin{pmatrix} 0 & -\frac{c_0}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_{r-1}}{a_r} \\ 1 & \frac{\lambda - b_r}{a_r} \end{pmatrix} \sim \frac{1}{a_1 \cdots a_r} \lambda^r \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Recursion relations

$$\frac{c_{n-1}}{\lambda^2}\frac{y_{n-1}}{\lambda^{n-1}} + \frac{b_n}{\lambda}\frac{y_n}{\lambda^n} + a_n\frac{y_{n+1}}{\lambda^{n+1}} = \lambda\frac{y_n}{\lambda^n}$$

$$\left(\frac{y_n}{\lambda^n}, \frac{y_{n+1}}{\lambda^{n+1}}\right) = \left(y_0, \frac{y_1}{\lambda}\right)\begin{pmatrix} 0 & -\frac{c_0}{\lambda^2 a_1} \\ 1 & \frac{1}{a_1} - \frac{b_1}{\lambda} \end{pmatrix}\cdots\begin{pmatrix} 0 & -\frac{c_{n-1}}{\lambda^2 a_n} \\ 1 & \frac{1}{a_n} - \frac{b_n}{\lambda} \end{pmatrix}$$

If y^λ denote an eigenvector with the ~~small~~ large z value $z \sim \frac{1}{a_1\cdots a_n}\lambda^{\pm n}$, then we have

$$\left(\frac{y^\lambda_n}{\lambda^n}\ \frac{y^\lambda_{n+1}}{\lambda^{n+1}}\right) = \frac{z}{\lambda^n}\left(y^\lambda_0, \frac{y^\lambda_1}{\lambda}\right).$$

so if we normalize y^λ by requiring $y^\lambda_0 = 1$, we find

$$\frac{y^\lambda_n}{\lambda^n} \longrightarrow \frac{1}{a_1\cdots a_n} \qquad \text{as } \lambda \to \infty \quad \text{for all } n \geq 0.$$

A similar formula should hold for $n < 0$ by periodicity. So therefore I see that if I want the section of M given by $y \mapsto y_0$ to remain regular at the point $\lambda = \infty$, $z = \infty$, then I want also the sections $y \mapsto y_n$ for $n < 0$ to vanish at this point, in fact the section $y \mapsto y_n$ vanishes to the $(-n)$-th order at this point, similarly $y \mapsto y_n$ vanishes to the n-th order at the point $\lambda = \infty$, $z = 0$.

So what you get is a line bundle L on C such that the two filtrations on $M = \Gamma(C - \{P, Q\}, L)$ one obtains from the order of vanishing at P and the order of vanishing at Q are opposite. This means that if we specify $H^0(C, L) \cong k$, then

$$(*) \qquad H^0(C, L \otimes \mathcal{O}(nP - nQ)) \cong k$$

for all n. Conversely given a line bundle L on C satisfying $(*)$ one gets a vector space $M = \Gamma(C - \{P, Q\}, L)$ with two opposite flags, so $M \cong \bigoplus_n H^0(C, L \otimes \mathcal{O}(nP - nQ))$. The shift operator T is given by multiplying by the function T having a pole of order r at one point and a zero of order r at another. The Jacobi matrix comes from multiplying by the function J.

Suppose $k[T, J]$ defined by

$$T^2 - 2\varphi(J)T + 1 = 0$$

$$2\varphi(J) = J^r + \text{lower terms}$$

i.e $a_1 \cdots a_r = c_1 \cdots c_r = 1$. Then we can normalize Jacobi matrices belonging to this equation by requiring $a_1 = a_2 = \cdots = a_r = 1$. It seems then that I have $(r-1)$ possibilities for $c_1, \ldots, c_r$ and r possibilities for $b_1, \ldots, b_r$. It would seem that I am missing a relation for the totality of line bundles is of dimension $2g = 2(r-1) = 2r - 2$. This occurs somewhere in the formula for $\varphi(J)$. In effect going from $3r$ parameters to describe the periodic J-matrices to the equation

$$T^2 - \underbrace{\operatorname{tr}\Phi(\lambda)}_{\substack{\text{poly of deg } r \\ \text{with } r+1 \text{ coeff}}} T + \underbrace{\det\Phi(\lambda)}_{\substack{\text{const.} \\ 1 \text{ coeff.}}}$$

we fix $r+2$ parameters, leaving $3r-(r+2)=2r-2$ parameters.

Probably $(*)$ above isn't strong enough. One wants

$$H^0(C, L(nP+mQ)) = \bigoplus_{-m\le i\le n} H^0(C, L(iP-iQ))$$

which forces (by RR)

$$1-g+\deg(L)+n+m = n+m+1 \qquad n, m \text{ large}$$

$$\Rightarrow \deg(L) = g.$$

On the other hand if $\deg(L)=g$, then

$$h^0(L(iP-iQ)) \ge g+1-g = 1$$

with equality if $h^1(L(iP-iQ)) = 0$. So $(*)$ should be replaced by

$(*)'$ $\quad \deg(L)=g \quad$ and $\quad h^1(L(iP-iQ))=0 \quad$ all $i\in\mathbb{Z}$.

or simpler just add to $(*)$ the condition $\deg(L)=g$.

March 19, 1977:

P.D. Lax: Almost periodic behavior of non-linear waves.
Advances in Math 16(1975), 368-379.

If $\quad U(t)^{-1}L(t)U(t) = L(0)$, then differentiating wrt. t.

$$-U^{-1}U_tU^{-1}LU + U^{-1}L_tU + U^{-1}LU_t = 0$$

so if we set

$$B = U_tU^{-1} \qquad \text{or} \quad U_t = BU,$$

then

$$L_t = BL - LB.$$

Conversely given B, L satisfying this equation if we can solve $U_t = BU$, then we see $U^{-1}LU$ must be constant in t. $\quad U$ unitary $\Longleftrightarrow$ B skew-adjoint

Toda lattice: $\quad u = (u_1, \ldots, u_N)$, T cyclic translation

$$(Tu)_j = u_{j-1}, \qquad u_0 = u_N$$

Thus T translates to the right one step. Now let a denote a diagonal operator and put

$$a_+ = TaT^{-1}$$
$$a_- = T^{-1}aT$$

Let c be another diagonal matrix and put

$$L = T^{-1}a + c + aT$$

This is a periodic Jacobi matrix: $\quad L = \begin{pmatrix} c_1 & a_2 & & & a_1 \\ a_2 & c_2 & & & \\ & & \ddots & & \\ & & & & a_n \\ a_1 & \cdots & & a_n & c_n \end{pmatrix}$

Put $\qquad B = aT - T^{-1}a \quad$; skew-adjoint.

$$BL - LB = [aT - T^{-1}a,\ aT + c + T^{-1}a]$$

$$= aTc - caT + a^2 - T^{-1}a^2T - T^{-1}a^2T + a^2$$
$$- T^{-1}ac + cT^{-1}a$$

$$= a(c_+ - c)T + 2(a^2 - a_-^2) + a_-(c - c_-)T^{-1}$$
$$T^{-1}[a(c_+ - c)]$$

Now if a, c vary wrt t, then

$$L_t = a_t T + c_t + T^{-1}a_t$$

so that if we want $L_t = BL - LB$ we must have

$$\begin{cases} c_t = 2(a^2 - a_-^2) \\ a_t = a(c_+ - c). \end{cases}$$

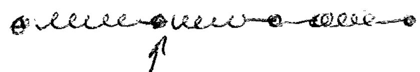
If we have a solution of this non-linear system, then L has spectrum independent of t, in particular the eigenvalue sums

$$\sum \lambda_j = \sum c_j$$

$$\sum \lambda_j^2 = \operatorname{tr} L^2 = \operatorname{tr}(T^{-1}a + c + aT)^2$$
$$= \sum c_j^2 + \operatorname{tr}(T^{-1}aaT + a^2)$$
$$= \sum c_j^2 + 2\sum a_j^2$$

etc. are constant in t.

Now consider a collection of particles with



springs in between. Let q_j be the displacement of the jth particle from equilibrium: $m_j = 1$

$$\frac{d^2 q_j}{dt^2} = f(q_{j+1} - q_j) - f(q_j - q_{j-1})$$

which comes from a Hamiltonian

$$H = \frac{1}{2}\sum p_j^2 + \sum F(q_{j+1} - q_j) \qquad p_j = \frac{dq_j}{dt}$$

$$\frac{dF}{ds} = f$$

Toda considers $f(s) = -e^{-s}$ whence one gets the DE's

$$\frac{dq_j}{dt} = p_j$$

$$\frac{dp_j}{dt} = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}$$

So now put

$$c_j = \frac{1}{2} p_j$$

$$a_j = \frac{1}{2} e^{\frac{1}{2}(q_{j-1} - q_j)}$$

so that $\displaystyle \frac{d}{dt} c_j = 2(a_j^2 - a_{j+1}^2) \qquad \frac{d}{dt} a_j = a_j (c_{j-1} - c_j)$

These are the sames as the D.E. on page 8, so we conclude that the eigenvalue sums $\sum \lambda_i^p$ are integrals of the motion.

<u>March 20, 1977.</u>

Let T be the shift operator on k^n

$$(Tu)_j = u_{j-1}. \qquad T(e_{j-1}) = e_j$$

and suppose we are given an infinite Jacobi matrix

$$L = aT + b + T^{-1}c = aT + b + {}^{T^{-1}}c\,T^{-1}$$

We want to determine all matrices B giving rise to isospectral deformations. We require B to be supported in a band around the diagonal, whence we have a unique representation.

$$B = \sum \beta_j T^j \qquad \text{finite sum}$$

with β_j diagonal. Next the condition on B is that

$$[L, B]$$

should be tridiagonal, i.e. a Jacobi matrix. So

$$LB = \sum (aT + b + {}^{T^{-1}}c\,T^{-1})\,\beta_j T^j$$
$$= \sum \left(a\,{}^{T}\beta_j T^{j+1} + b\beta_j T^j + {}^{T^{-1}}c\,{}^{T^{-1}}\beta_j T^{j-1} \right)$$

$$BL = \sum \left(\beta_j \, {}^{T^j}\!a \, T^{j+1} + \beta_j \, {}^{T^j}\!b \, T^j + \beta_j \, {}^{T^{j-1}}\!c \, T^{j-1} \right)$$

Thus
$$BL - LB = \sum_j \left(\begin{array}{l} \beta_{j-1} \, {}^{T^{j-1}}\!a + \beta_j \, {}^{T^j}\!b + \beta_{j+1} \, {}^{T^j}\!c \\ - a \, {}^{T}\!\beta_{j-1} - b\beta_j - c \, {}^{T^{-1}}\!\beta_{j+1} \end{array} \right) T^j$$

Now suppose that we consider the smallest j such that $\beta_j \neq 0$. Then the coefficient of T^{j-1} in $BL - LB$ is

$$\beta_j \, {}^{T^{j-1}}\!c - {}^{T^{-1}}\!(c\beta_j)$$

This has to be zero if $j \leq -1$ and $BL - LB$ is a J-matrix. So since c is assumed non-zero everywhere one gets a unique possibility up to a scalar for β_j. Namely if we take $B = L^{-j}$ then the degree j term is

$$\left({}^{T^{-1}}\!c \, T^{-1} \right)^{-j} = {}^{T^{-1}}\!c \, {}^{T^{-2}}\!c \cdots {}^{T^{j}}\!c \, T^j$$

Check: if
$$\beta_j = {}^{T^{-1}}\!c \, {}^{T^{-2}}\!c \cdots {}^{T^{j}}\!c$$

then
$$\beta_j \, {}^{T^{j-1}}\!c = {}^{T^{-1}}\!c \cdots {}^{T^{j-1}}\!c = T^{-1}\left(c \, T^{-1}(c) \cdots T^{j}(c)\right).$$

So what is clear is that by adding to B multiples of L we can arrange all negative degree terms to be zero without affecting the bracket $[B, L]$.

Let's review the real symmetric one-sided J-matrix situation. Here one has an equivalence between the following notions

1) triples (H, A, v_0) consisting of a Hilbert space with a bdd self-adjoint operator A and a cyclic vector v_0. $\|v_0\| = 1$

2) measures $d\mu$ on R with bounded support. $\int d\mu = 1$

3) real symmetric J-matrices

$$\begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & b_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad a_i > 0$$

either finite or infinite with bounded entries.

Given $d\mu$, the associated H is $L^2(R, d\mu)$, $A =$ mult. by x and $v_0 =$ the function 1. To get J one constructs by Gram-Schmidt an orthonormal sequence of polys $\phi_0, \phi_1, \cdots$ using the sequence $1, x, x^2, \cdots$. Then J is the matrix of multiplication by x wrt this orthonormal basis. Thus

$$x\,\phi_n(x) = a_n \phi_{n+1}(x) + b_n \phi_n(x) + a_{n-1}\phi_{n-1}(x)$$

from which it is clear that

$$\phi_n(x) = \frac{x^n}{a_1 \cdots a_{n-1}} + \text{lower terms}$$

Symmetry. Suppose $d\mu(-x) = d\mu(x)$ or equivalently (since $d\mu$ is determined by its moments) that $\int x^n d\mu = 0$ for n odd, i.e. $(A^n v_0, v_0) = 0$ for n odd. Then H splits $H = H^{ev} \oplus H^{odd}$ where H^{ev} is spanned by $A^n v_0$ for n even, etc. It's now clear that $\phi_n \in H^{ev}$ for n even $\phi_n \in H^{odd}$ for n odd, hence the b_n are all zero. The converse is also clear. So

$$d\mu \text{ even} \iff b_n = 0 \text{ for all } n.$$

Since $b = 0$ we don't get an interesting B in the form $aT - T^{-1}a$. However

$$L^2 = (aT + T^{-1}a)^2 = (aT(a))T^2 + (a^2 + T^{-1}(a)^2) + T^{-2}(aT(a))$$

so L^2 is a J matrix with shift T^2, so we should try $B = aT(a)T^2 - T^{-2}aT(a)$.

$$[B, L] = [aT(a)T^2 - T^{-2}aT(a),\ aT + T^{-1}a]$$

$$= aT(a)Ta - T^{-1}a^2T(a)T^2 - T^{-2}aT(a)aT + aT^{-1}aT(a)$$

$$= aT(a)^2T - T^{-1}(a)^2aT - T^{-1}aT^{-1}(a)^2 + T^{-1}aT(a)^2$$

$$= a\left(T(a)^2 - T^{-1}(a)^2\right)T + T^{-1}\left(T(a)^2 - T^{-1}(a)^2\right)a$$

Thus we get the D.E.'s

$$(*) \qquad a_t = a\left(T(a)^2 - T^{-1}(a)^2\right)$$

$$\tfrac{1}{2}(a^2)_t = a a_t = a^2\left(T(a^2) - T^{-1}(a^2)\right)$$

$$(2a^2)_t = (2a^2)\left(T(2a^2) - T^{-1}(2a^2)\right)$$

So let us put

$$2a^2 = e^{-R}$$

whence the D.E. becomes

$$-R_t = e^{-T(R)} - e^{-T^{-1}(R)}$$

or

$$\frac{dR_j}{dt} = e^{-R_{j-1}} - e^{-R_{j+1}}$$

say $T(R)_j = R_{j+1}$ to get Kac-vMoerbeke equations.

So we are now at the following point. We have defined, at least formally, an isospectral flow on the set of J-matrices $L = aT + T^{-1}a$ ~~Set~~ ~~$B = aT(a)T^2 - T^{-2}T(a)a$ and solve the system~~ by the DE $(*)$ above. If we restrict to a finite-dimensional setup, say by requiring $a_0 = a_N = 0$, then $(*)$ determines a well-defined vector field on $\{(a_1, \ldots, a_{N-1}) \mid a_i > 0\} \sim \mathbb{R}^{N-1}$ which leaves the spectrum invariant. But we have described these J-matrices in terms of even measures on $\mathbb{R}$ with $\int d\mu = 1$, so the problem now becomes to describe the flow on the set of these measures.

Related problem: Consider all symmetric J-matrices $L = aT + b + T^{-1}a$ supported in $[1,n]$. Let's agree that T is backwards shift $(Tu)_j = u_{j+1}$. Then we want

$$a_i = 0 \qquad i \leq 0 \quad \text{or} \quad i \geq n$$
$$b_i = 0 \qquad i \leq 0 \quad \text{or} \quad i > n.$$

Thus

$$L = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix}$$

This is the special case of an n-periodic J-matrix with $a_n = 0$. Set $B = aT - T^{-1}a$ as before and consider the flow on the set of these J-matrices given by

$$L_t = [B, L]$$

i.e.

$$\begin{cases} a_t = a(T(b) - b) \\ b_t = 2\left(a^2 - T^{-1}(a)^2\right) \end{cases}$$

We know this flow is isospectral, so the problem is now to describe this flow on the space of measures.

The answer turns out to be very simple provided one normalizes in a tricky way. The idea is that because the deformation is iso spectral one has $d\mu(t) = \gamma(t)^2 d\mu(0)$, hence

$$\int x^m \gamma^2 d\mu(0) = (L^m e_1, e_1) \qquad \forall m$$

$$\int x^m 2\gamma\gamma_t \, d\mu(0) = ((L^m)_t e_1, e_1)$$
$$= ([B, L^m] e_1, e_1)$$

because $\varphi(L) \mapsto \varphi(L)_t$ and $\mapsto [B, \varphi(L)]$ are two derivations on polys in L which coincide on L.

$$([B, L^m]e_1, e_1) = -(L^m B e_1, e_1) - (L^m e_1, B e_1) = -2(L^m e_1, B e_1)$$

now
$$Be_1 = (aT - T^{-1}a)e_1 = 0 - a_1 e_2 = -a_1 e_2$$
$$Le_1 = (aT + b + T^{-1}a)\, e_1 = b_1 e_1 + a_1 e_2$$

$$\therefore \quad -2(L^m e_1, B e_1) = 2(L^m e_1, a_1 e_2)$$
$$= 2(L^m e_1, Le_1 - b_1 e_1)$$

Thus
$$\int x^m \gamma\gamma_t \, d\mu(0) = (L^{m+1} e_1, e_1) - b_1 (L^m e_1, e_1)$$
$$= \int (x^{m+1} - b_1 x^m) \gamma^2 d\mu(0) \qquad \forall m$$

so
$$\gamma\gamma_t = (x - b_1)\gamma^2 \qquad \text{or}$$
$$\gamma_t = (x - b_1)\gamma \qquad \text{where}$$
$$b_1 = (Le_1, e_1) = \int x \gamma^2 d\mu(0).$$

But the way to interpret this is to recall that γ is restrained by the condition $\int \gamma^2 d\mu(0) = 1$. Thus

$$\frac{d}{dt}\log\gamma = \frac{\gamma_t}{\gamma} = x - b_1 \quad \text{or} \quad \log\gamma = xt - \int_0^t b_1\,dt + \log\gamma(0)$$

$$\gamma(t) = e^{xt}\, e^{-\int_0^t b_1}\, \gamma(0)$$

$$\gamma(t) = \frac{e^{xt}\gamma(0)}{e^{\int_0^t b_1\,dt}}$$

$$1 = \int \gamma^2 d\mu_0 = e^{-2\int_0^t b_1 dt} \int e^{2xt}\gamma(0)^2 d\mu(0)$$

But $\gamma(0) = 1$, so one get

$$\boxed{\gamma(t,x) = \frac{e^{xt}}{\left(\int e^{2xt}\, d\mu(0)\right)^{1/2}}}$$

which shows that up to the normalization constant this flow is

$$d\mu(t) = e^{2xt}\, d\mu(0)/\text{norm}.$$

March 21, 1977.

We've been considering the space of J-matrices:

$$L = aT + b + T^{-1}a$$

On this space we have a flow given by

$$L_t = [B, L] = [aT - T^{-1}a, L]$$
$$= a(T(b) - b)T + 2(a^2 - T^{-1}(a^2)) + T^{-1}(T(b) - b)a$$

i.e.
$$\begin{cases} \dot{a} = a(T(b) - b) \\ \dot{b} = 2(a^2 - T^{-1}(a^2)) \end{cases}$$

which can be written

$$\begin{cases} (4a^2)^\circ = 4a^2(T(2b) - 2b) \\ (2b)^\circ = (4a^2) - T^{-1}(4a^2). \end{cases}$$

Put

$$\beta = 2b$$
$$4a^2 = e^v$$

whence the flow becomes

$$(*) \quad \begin{cases} \dot{v} = T(\beta) - \beta \\ \dot{\beta} = e^v - e^{T^{-1}(v)} \end{cases} \qquad \begin{cases} \dot{v}_j = \beta_{j+1} - \beta_j \\ \dot{\beta}_j = e^{v_j} - e^{v_{j-1}} \end{cases}$$

Now I know that

$$tr(L^2) = \sum b_i^2 + 2a_i^2 = \frac{1}{2}\left(\sum_i \frac{1}{2}\beta_i^2 + e^{v_i}\right)$$

is invariant under the flow, so the problem is to construct a symplectic structure on the space of L so that the flow belongs to the Hamiltonian

$$H = \frac{1}{2}\sum \beta_i^2 + e^{v_i}$$

Thus I seek a 2-form (non-degenerate and closed), say

$$\omega = \sum \gamma_{ij}\, d\beta_i\, dv_j .$$

The vector field X_f corresponding to f wrt ω is calculated as follows:

$$X_f = \sum s_i \frac{\partial}{\partial \beta_i} + t_i \frac{\partial}{\partial v_i}$$

$$\iota(X_f)\omega = \sum s_i \gamma_{ik}\, dv_k - t_i \gamma_{ki}\, d\beta_k = df$$

$$\therefore \sum_i s_i \gamma_{ik} = \frac{\partial f}{\partial v_k} \qquad \sum_i \gamma_{ki} t_i = -\frac{\partial f}{\partial \beta_k}$$

If $\tau^{-1} = \gamma$

$$s_i = \sum_k \frac{\partial f}{\partial v_k} \tau_{ki} \qquad t_i = -\sum \tau_{ik} \frac{\partial f}{\partial \beta_k}$$

$$X_f = \sum \frac{\partial f}{\partial v_k} \tau_{ki} \frac{\partial}{\partial \beta_i} - \frac{\partial f}{\partial \beta_k} \tau_{ik} \frac{\partial}{\partial v_i}$$

So next ~~the~~ Hamilton's equations for the flow X_H are

$$\dot{\beta}_i = \{H, \beta_i\} = X_H \beta_i = \sum_k \frac{\partial H}{\partial v_k} \tau_{ki} = \sum_k e^{v_k} \tau_{ki} = e^{v_i} - e^{v_{i-1}}$$

$$\dot{v}_i = \{H, v_i\} = X_H v_i = -\sum_k \frac{\partial H}{\partial \beta_k} \tau_{ik} = -\sum \beta_k \tau_{ik} = \beta_{i+1} - \beta_i$$

So it's clear we have to have

$$\begin{cases} \tau_{ii} = 1 \\ \tau_{i,i+1} = -1 \end{cases} \qquad \text{rest } 0.$$

Unfortunately τ is singular unless we restrict to submanifolds with $\sum_i v_i$, $\sum b_i$ constant. But we only need τ to have the Poisson bracket, hence flows.

See if we can find a B_k such that $\dot{L} = [B_k, L]$ corresponds to the flow

$$d\mu(t) = e^{tx^k} d\nu \Big/ \int e^{tx^k} d\nu. \qquad d\nu = d\mu(0)$$

Then

$$\frac{\int x^m e^{tx^k} d\nu}{\int e^{tx^k} d\nu} = (L^m e_1, e_1)$$

$$([B_k, L^m] e_1, e_1) = \frac{\int x^{m+k} e^{tx^k} d\nu}{\int e^{tx^k} d\nu} - \frac{\int x^m e^{tx^k} d\nu \cdot \int x^k e^{tx^k} d\nu}{\left(\int e^{tx^k} d\nu\right)^2}$$

or

$$([B_k, L^m] e_1, e_1) = (L^{m+k} e_1, e_1) - (L^m e_1, e_1)(L^k e_1, e_1)$$

$$\|$$

$B_k^* = -B_k$ $\qquad -2(L^m e_1, Be_1)$ $\qquad$ this is to hold for all m.

$$\boxed{-2B_k e_1 = L^k e_1 - (L^k e_1, e_1) e_1}$$

Now it's clear how to get B_k. Write

$$L^k = T^{-k}a_k + \cdots + T^{-1}a_1 + a_0 + a_1T + \cdots + a_kT^k$$

$$L^k : \begin{pmatrix} a_{0,1} & a_{1,1} & a_{2,1} & \ddots \\ a_{1,1} & \ddots & \ddots & \ddots \\ a_{2,1} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \end{pmatrix}$$

Now put $\quad B_k = \frac{1}{2}\left(-T^{-k}a_k - \cdots - T^{-1}a_1 + a_1T + \cdots + a_kT^k\right)$

whence

$$L^k e_1 = a_{k1}e_{k+1} + \cdots + a_{01}e_1$$

$$-2B_k e_1 = +a_{k1}e_{k+1} + \cdots + a_{11}e_2$$

$$\therefore \quad -2B_k e_1 = L^k e_1 - (L^k e_1, e_1)$$

[redacted] Moreover

$$[B_k, L] = [B_k + \tfrac{1}{2}L^k, L] = [\tfrac{1}{2}a_0 + a_1T + \cdots + a_kT^k,\ T^{-1}a + b + aT]$$

is a symmetric operator with terms c_jT^j for $j \geq -1$, hence it is a J-matrix, so the flow defined by $\dot{L} = [B_k, L]$ is indeed the flow defined by $d\mu(t) = e^{tx^k}d\nu/\text{norm}$.

Note: The above derivation hold for [redacted] one-sided J-matrices which can be understood in terms of measures. However one can use the same B_k in the periodic and two-sided cases, [redacted] even when a has vanishing components.

[redacted]

So from now on we work with $B = \frac{1}{2}(-T^{-1}a + aT)$ instead of the old B. The DE $\dot{L} = [B, L]$ becomes

$$\dot{a} = \frac{1}{2} a (T(b) - b)$$
$$\dot{b} = a^2 - T^{-1}(a^2)$$

or

$$\begin{cases} (a^2)^{\bullet} = (a^2)(T(b) - b) \\ b^{\bullet} = a^2 - T^{-1}(a^2) \end{cases}$$

Potential Scattering: Consider the operator

$$L = -\Delta + q$$

on $\mathbb{R}$ where q has compact support. One wants to understand L by comparison with

$$L_0 = -\Delta = D^2 \qquad D = \frac{1}{i}\frac{d}{dx}$$

The spectrum of L_0 consists of $\{\lambda = \xi^2 \mid \xi \in \mathbb{R}\}$ and there are two "eigenfunctions"

$$e^{i\xi x}, \; e^{-i\xi x}$$

for each λ except for $\lambda = 0$. Fix $\lambda \in \mathbb{C}$ and consider eigenfunctions:

$$Lu = \lambda u$$

Because q has compact support we have that u

is of the form

$$u(x) = Ae^{i\xi x} + Be^{-i\xi x}$$

for $x \gg 0$ and also for $x \ll 0$ with different constants A, B. Now λ will be in the spectrum of L when u is bounded. This happens if λ if > 0. If $\lambda < 0$, say $\lambda = -a^2$, then it might happen that there is a solution u with $u = Ae^{ax}$ $x \ll 0$ and $u = Be^{-ax}$ for $x \gg 0$. This gives us bound states for L. If $\lambda = 0$, then by a limiting process from $\lambda > 0$, we get an eigenfunction which is constant and $\neq 0$ for $x \ll 0$ and $x \gg 0$.

If we stay perpendicular to the bound states then the operator L has eigenvalues ≥ 0, hence it has at least one self-adjoint square root. Perhaps this square root exists as a pseudo-differential operator of order 1. In any case the question arises as to whether the eigenfunction of L with eigenvalue ξ^2, which for $x \ll 0$ agrees with $e^{i\xi x}$, has the form (Const)$e^{i\xi x}$ for $x \gg 0$. Also whether this constant is of absolute value 1.

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$L = -\Delta + q$ on $\mathbb{R}$ with q compact support. If $\sqrt{\lambda} \in \mathbb{R}$, then $\lambda \geq 0$ and conversely. Suppose $\lambda \notin \mathbb{R}_{\geq 0}$. Then $(L-\lambda)u = 0$ has a unique solution up to a scalar which decays exponentially as $x \to +\infty$. Indeed, once x is beyond the support of q, then any u such that $(L-\lambda)u=0$ has the form $ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x}$, and exactly one of the roots $\pm\sqrt{\lambda}$ has a positive imaginary part (since $\sqrt{\lambda} \notin \mathbb{R}$), say $\mathrm{Im}(\sqrt{\lambda}) > 0$, whence $u = ae^{i\sqrt{\lambda}x}$ decays exponentially as $x \to +\infty$. Similarly there is a unique solution of $(L-\lambda)u = 0$ which decays exponentially as $x \to -\infty$.

So suppose we label these u^+ and u^-:

$$\begin{cases} (L-\lambda)u^+ = (L-\lambda)^-u = 0 \\ u^+ = e^{i\xi x} & x \gg 0 \\ u^- = e^{-i\xi x} & x \ll 0 \\ \xi^2 = \lambda \qquad \mathrm{Im}(\xi) > 0. \end{cases}$$

Now we can construct the Green's function $G(x,y,\lambda)$ which satisfies

$$(L_x - \lambda)\, G(x,y,\lambda) = \delta(x-y)$$

and

$$G(x,y,\lambda) \text{ decays exponentially as } x \to \pm\infty.$$

namely
$$G(x,y,\lambda) = \begin{cases} a(y)\, u^+(x) & x > y \\ b(y)\, u^-(x) & x < y \end{cases}$$

for suitable a, b. First G is to be continuous so

$$a(x)\, u^+(x) = b(x)\, u^-(x)$$

Next

$$1 = \lim_{\varepsilon \searrow 0} \int_{y-\varepsilon}^{y+\varepsilon} (-\Delta + q - \lambda) G\, dx = \lim_{\varepsilon \searrow 0} \int_{y-\varepsilon}^{y+\varepsilon} (-\Delta G)\, dx$$

$$= \lim \left[-\frac{dG}{dx} \right]_{y-\varepsilon}^{y+\varepsilon} = \left.\frac{dG}{dx}\right|_{y-} - \left.\frac{dG}{dx}\right|_{y+}$$

~~Thus~~ Thus $\frac{dG}{dx}$ jumps -1 in crossing y:

$$b(x) \frac{du^-}{dx}(x) - a(x) \frac{du^+}{dx}(x) = 1$$

So we get the equations

$$a(x)\, u^+(x) - b(x)\, u^-(x) = 0$$

$$a(x)\, \dot{u}^+(x) - b(x)\, \dot{u}^-(x) = -1$$

so

$$a(x) = \frac{\begin{vmatrix} 0 & -u^-(x) \\ -1 & -\dot{u}^-(x) \end{vmatrix}}{-W} = + \frac{u^-(x)}{W} \qquad W = \begin{vmatrix} u^+ & u^- \\ \dot{u}^+ & \dot{u}^- \end{vmatrix}$$

$$b(x) = \frac{\begin{vmatrix} u^+ & 0 \\ \dot{u}^+ & -1 \end{vmatrix}}{-W} = \frac{u^+(x)}{W}$$

Note

$$\frac{dW}{dx} = \underset{\overset{\shortparallel}{0}}{\begin{vmatrix} \dot{u}^+ & \dot{u}^- \\ \dot{u}^+ & \dot{u}^- \end{vmatrix}} + \begin{vmatrix} u^+ & u^- \\ \ddot{u}^+ & \ddot{u}^- \end{vmatrix} = \begin{vmatrix} u^+ & u^- \\ (q-\lambda)u^+ & (q-\lambda)u^- \end{vmatrix} = 0$$

so W is constant. Thus we get

$$G(x,y,\lambda) = \begin{cases} +\dfrac{u^-(y)\,u^+(x)}{W} & x > y \\[2ex] \dfrac{u^+(y)\,u^-(x)}{W} & x < y \end{cases}$$

Example: $q=0$. $\quad u^+(x) = e^{i\xi x} \qquad u^-(x) = e^{-i\xi x}$

$$W = \begin{vmatrix} e^{i\xi x} & e^{-i\xi x} \\ i\xi e^{i\xi x} & -i\xi e^{-i\xi x} \end{vmatrix} = -2i\xi$$

So

$$G(x,y,\lambda) = \frac{e^{i\xi x_>}\, e^{-i\xi x_<}}{-2i\xi}$$

where $x_> = \max(x,y)$, $x_< = \min(x,y)$.

Notice that by definition as operators one has

$$G = (L-\lambda)^{-1}$$

so G is the resolvent of the operator L.

In the above calculation it is essential that not only $\lambda \notin \mathbb{R}_{\geq 0}$ but that also u^+ and u^- are independent. They will become dependent at certain negative values of λ where L has a point spectrum.

So what happens to u^+, u^- as $\mathrm{Im}(\xi) \searrow 0$

March 24, 1977.

Recall that there is a 1-1 correspondence between (bounded entire) one-sided real symmetric J-matrices $\begin{pmatrix} b_1 & a_1 & \\ a_2 & b_2 & \ddots \\ & a_2 & \ddots \end{pmatrix}$ with $a_i > 0$ and measures $d\mu(x)$ on $\mathbb{R}$ with bounded infinite support and $\int d\mu(x) = 1$. To obtain the J matrix corresponding to $d\mu(x)$ one constructs an orthonormal sequence of polynomials $\phi_0(x), \phi_1(x), \ldots$ (orthonormal wrt $\|f\|^2 = \int |f|^2 d\mu$) by applying Gram-Schmidt ~~to~~ the sequence $1, x, x^2, \ldots$. Then J is the matrix of mult. by x relative to this orth basis ϕ_n:

$$x\phi_{n-1} = a_n\phi_n + b_n\phi_{n-1} + a_{n-1}\phi_{n-2}$$

($\phi_n = (n+1)$'th basis element).

To go from a J-matrix to a measure one needs a version of the spectral theorem. For each $\lambda \in \mathbb{C}$, let $\psi(\lambda)$ denote the unique solution of

$$J\psi(\lambda) = \lambda\psi(\lambda)$$

$$\psi(\lambda)_1 = 1$$

Let J_n be the $n \times n$ truncation of J. The eigenvectors for J_n are those $\psi(\lambda)_{\leq n}$ such that $\psi(\lambda)_{n+1} = 0$. We know the eigenvalues of J_n are simple ($\exists$ cyclic vector) hence if f has support in $[1, n]$ we have an eigenfunction expansion

$$f = \sum_{\lambda \in \mathrm{Spec}(J_n)} (f, \psi(\lambda))\psi(\lambda)\, r_\lambda$$

where $r_\lambda = \left(\sum_{i=1}^{n} |\psi(\lambda)_i|^2 \right)^{-1}$. Taking $f = e_1$ and using that

$$(e_1, \psi(\lambda)) = \psi(\lambda)_1 = 1$$

we get

$$1 = (e_1, e_1) = \left(\sum_{\lambda \in Sp(J_n)} (e_1, \psi(\lambda))(\psi(\lambda), e_1) r_\lambda \right)$$

$$1 = \sum_{\lambda \in Sp(J_n)} r_\lambda$$

which gives us a measure of mass 1 supported on the spectrum of J_n. [redacted] Call this measure $d\mu^n$ so that we have

$$f = \int (f, \psi(\lambda)) \psi(\lambda) \, d\mu^n(\lambda)$$

whenever f has support in $[1, n]$. Now let $n \to \infty$. By weak compactness the sequence $d\mu^n$ has a limit point (maybe here we use J is bounded) $d\mu$ and we have

$$f = \int (f, \psi(\lambda)) \, \psi(\lambda) \, d\mu(\lambda)$$

for all f with bounded support. [redacted]
The measure $d\mu(\lambda)$ is uniquely determined because:

$$J^k e_1 = \int \lambda^k \psi(\lambda) \, d\mu(\lambda)$$

$$(J^k e_1, e_1) = \int \lambda^k d\mu(\lambda).$$

The next problem to understand is scattering. Somehow we want to look a perturbation of a given J-matrix. The new measure will be a multiple of the old together with some point masses.

So we first work out the example

$$J = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

which leads to recursion equations with constant coefficients

$$(Ju)_n = \frac{1}{2} u_{n-1} + \frac{1}{2} u_{n+1} = \lambda u_n$$

Try $\quad u_n = \omega^n$

$$\frac{1}{2}\omega^{n-1} + \frac{1}{2}\omega^{n+1} = \lambda \omega^n$$

$$\frac{1}{2}\left(\omega^{-1} + \omega - 2\lambda\right) = 0$$

$$\omega^2 - 2\lambda\omega + 1 = 0$$

$$\omega = \lambda \pm \sqrt{\lambda^2 - 1}$$

So

$$\psi(\lambda)_n = \frac{\left(\lambda + \sqrt{\lambda^2-1}\right)^n - \left(\lambda - \sqrt{\lambda^2-1}\right)^n}{2\sqrt{\lambda^2-1}}$$

Suppose $\quad \psi(\lambda)_n = 0$

Then $\lambda + \sqrt{\lambda^2-1}$ and $\lambda - \sqrt{\lambda^2-1}$ have the same absolute value $\Rightarrow$ have abs. value 1 and they are conjugate points on the unitt circle. So $-1 \le \lambda \le 1$. Put

$\lambda = \cos\theta$. Actually $\psi(\lambda)$ is bounded exactly when $\lambda \pm \sqrt{\lambda^2-1}$ are on S^1, hence exactly when $\lambda \in [-1,1]$.
If $\lambda = \cos\theta$, then

$$\psi(\lambda)_n = \frac{e^{in\theta} - e^{-in\theta}}{2i\sin\theta} = \frac{\sin n\theta}{\sin\theta}$$

This is zero when $\theta = \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(n-1)\pi}{n}$, so in the limit as $n \to \infty$ it is clear that we find the spectrum of J is $[-1,1]$. Let's first find the measure of $0 \le \theta \le \pi$ which gives the expansion formula:

$$f_n = \int_0^\pi (f, \psi(\lambda))\, \psi(\lambda)_n \, d\nu(\theta)$$

$f = e_k$

$$\delta_{kn} = \int_0^\pi \frac{\sin k\theta}{\sin\theta} \frac{\sin n\theta}{\sin\theta}\, d\nu(\theta)$$

Thus

$$d\nu(\theta) = \frac{2\sin^2\theta\, d\theta}{\pi}$$

If $\quad \lambda = \cos\theta, \quad d\lambda = -\sin\theta\, d\theta, \quad d\theta = -\frac{d\lambda}{\sin\theta}$

$$d\nu = \frac{2\sin^2\theta\, d\theta}{\pi} = -\frac{2}{\pi}\sin\theta\, d\lambda = -\frac{2}{\pi}\sqrt{1-\lambda^2}\, d\lambda$$

Therefore

$$f_n = -\int_1^0 (f, \psi(\lambda))\, \psi(\lambda)\, \frac{2}{\pi}\sqrt{1-\lambda^2}\, d\lambda$$

$$\boxed{d\mu(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}\, d\lambda}$$