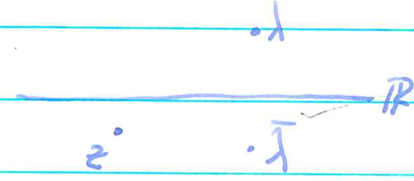
de Branges spaces:

Def: A de Branges fn. E is an entire function such that $\operatorname{Im}\lambda > 0 \Rightarrow |E(\lambda)| > |E(\bar{\lambda})|$.

Examples: 1) $\lambda - z$ where $\operatorname{Im} z < 0$



2) A poly [redacted] having roots in $\operatorname{Im}\lambda \le 0$ and at least one root in $\operatorname{Im}\lambda < 0$. (Note that any de Branges fn is non-zero for $\operatorname{Im}\lambda > 0$, so that all de Branges polys. are of this form)

3) $E(\lambda) = e^{-ia\lambda}$, $a > 0$, since $|E(\lambda)| = e^{a\operatorname{Im}(\lambda)}$.

Def: $B(E)$ is the vector space of entire f satisfying the following growth conditions:

1) $$\int_{\mathbb{R}} \left|\frac{f(\lambda)}{E(\lambda)}\right|^2 d\lambda < \infty$$

2) $\exists\, C, R_0 > 0$ such that for $|\lambda| \ge R_0$

$$\left|\frac{f(\lambda)}{E(\lambda)}\right| < \frac{C}{(\operatorname{Im}\lambda)^{1/2}} \quad \text{for } \operatorname{Im}\lambda > 0$$

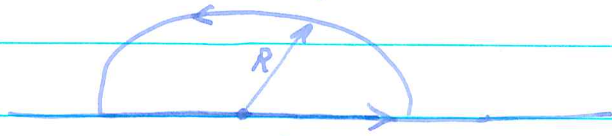
$$\left|\frac{f(\lambda)}{E^{\#}(\lambda)}\right| < \frac{C}{|\operatorname{Im}\lambda|^{1/2}} \quad \text{for } \operatorname{Im}\lambda < 0$$

Here $E^{\#}(\lambda) = \overline{E(\bar{\lambda})}$ is the conjugate analytic fn. to E.

$B(E)$ is equipped with the inner product:

3) $$(f, g) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\lambda)\overline{g(\lambda)}}{|E(\lambda)|^2}\, d\lambda$$

1) $\Rightarrow$ the merom. fn. $\frac{f(\lambda)}{E(\lambda)}$ has no poles on $\mathbb{R}$, hence $\frac{f(\lambda)}{E(\lambda)}$ is analytic for $\operatorname{Im}(\lambda) \geq 0$. Take the semi-circular contour of radius R



and use Cauchy's thm.

$$\frac{1}{2\pi i} \int_{\frown} \frac{f(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} \frac{f(z)}{E(z)} & z \text{ inside } \frown \\ 0 & z \text{ outside} \end{cases}$$

Using 2) we have

$$\left| \frac{1}{2\pi i} \int_{\curvearrowleft} \frac{f(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} \right| \leq \frac{1}{2\pi} \int_0^\pi \left| \frac{f(Re^{i\theta})}{E(Re^{i\theta})} \right| \cdot \left| \frac{Re^{i\theta} i \, d\theta}{Re^{i\theta} - z} \right|$$

$$\leq \int_0^\pi \frac{\text{Const}}{\operatorname{Im}(Re^{i\theta})} d\theta = \int_0^\pi \frac{\text{Const}}{R^{1/2} (\sin\theta)^{1/2}} d\theta$$

$$= \frac{\text{Const}}{R^{1/2}} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty$$

Hence

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\lambda)}{E(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} \frac{f(z)}{E(z)} & \operatorname{Im} z > 0 \\ 0 & \operatorname{Im} z < 0 \end{cases}$$

By a similar argument

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\lambda)}{E^\#(\lambda)} \frac{d\lambda}{\lambda - z} = \begin{cases} 0 & \operatorname{Im} z > 0 \\ -\frac{f(z)}{E^\#(z)} & \operatorname{Im} z < 0 \end{cases}$$

Multiplying former by $E(z)$ and the latter by $E^\#(z)$ and subtracting we get

$$(*) \qquad \frac{1}{2\pi i}\int_{\mathbb{R}} \left\{ \frac{f(\lambda)}{E(\lambda)} E(z) - \frac{f(\lambda)}{E^{\#}(\lambda)} E^{\#}(z) \right\} \frac{d\lambda}{\lambda - z} = f(z)$$

for $\operatorname{Im} z \neq 0$. ~~However the integrand is clearly an analytic function of~~ Note that the integrand has a removable singularity at $\lambda = z$, hence is analytic on $\operatorname{Im}\lambda = 0$ even when z is real. Thus when z is real the integral is the same as if we integrated over the contour:

Inspection shows that the above derivation ~~[illegible]~~ can be modified to yield that the integral for this modified contour is $f(z)$, hence $(*)$ holds for all z.

We can rewrite $(*)$ ~~[illegible]~~ in terms of the inner product:

$$f(z) = (f, J_z)$$

where

$$J_z(\lambda) = \overline{\frac{1}{2\pi i}\left(\frac{E(z)}{E(\lambda)} - \frac{E^{\#}(z)}{E^{\#}(\lambda)} \right) \frac{1}{\lambda - z}}\; |E(\lambda)|^2 \pi$$

$$= \frac{i}{2(\lambda - \bar{z})} \left(E(\lambda)\overline{E(z)} - \overline{E(\lambda)}\, E(\bar{z}) \right)$$

for λ real. ~~[illegible]~~ Let us define J_z for all λ by

$$J_z(\lambda) = \frac{i}{2(\lambda - \bar{z})} \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(\bar{z}) \end{vmatrix} = \frac{i}{2(\lambda - \bar{z})} \left(E(\lambda)\overline{E(z)} - E^{\#}(\lambda) E(\bar{z}) \right) =$$

Then $J_z(\lambda)$ is an entire function of λ, (the singularity at $\lambda = \bar{z}$ is removable). Moreover

~~$\frac{J_z(\lambda)}{E(\lambda)} = \frac{i}{2(\lambda-\bar{z})}\left(E(z) - \frac{E^\#(\lambda)}{E(\lambda)} E(\bar{z})\right)$~~

$$\frac{J_z(\lambda)}{E(\lambda)} = \frac{i}{2(\lambda-\bar{z})}\left(\overline{E(z)} - E(\bar{z})\frac{E^\#(\lambda)}{E(\lambda)}\right)$$

Because E is a deBranges fn. $\left|\frac{E^\#(\lambda)}{E(\lambda)}\right| < 1$ for $\operatorname{Im}\lambda > 0$, hence ≤ 1 for $\operatorname{Im}\lambda \geq 0$. Hence we get for $\operatorname{Im}\lambda \geq 0$, that

$$\left|\frac{J_z(\lambda)}{E(\lambda)}\right| \leq \frac{C}{|\lambda|}$$

for $|\lambda|$ sufficiently big. A similar argument shows

$$\left|\frac{J_z(\lambda)}{E^\#(\lambda)}\right| \leq \frac{C}{|\lambda|}$$

for $\operatorname{Im}\lambda \leq 0$ and $|\lambda| \geq R_0$. Thus it's clear that $J_z \in B(E)$, hence we have proved.

<u>Prop. 1:</u> $J_z = \frac{i}{2(\lambda-\bar{z})}\begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^\#(\lambda) & E^\#(\bar{z}) \end{vmatrix} \in B(E)$ is the point evaluator at z:

$$f(z) = (f, J_z) \qquad \text{all } f \in B(E)$$

Cauchy-Schwarz yields

$$|f(z)| \leq \|f\| \cdot \|J_z\|$$

and $\|J_z\|^2 = J_z(z) = \frac{1}{4\operatorname{Im} z}\left(|E(z)|^2 - |E(\bar{z})|^2\right)$

for $\operatorname{Im} z \neq 0$. To get answers for z real use Hôpital's rule:

$$\lim_{\lambda \to z} \frac{i}{2(\lambda - z)} \begin{vmatrix} E(\lambda) & E(z) \\ \overline{E(\lambda)} & \overline{E(z)} \end{vmatrix} = \frac{i}{2} \begin{vmatrix} E'(z) & E(z) \\ \overline{E'(z)} & \overline{E(z)} \end{vmatrix}$$

$$= \frac{i}{2} \left(\frac{E'(z)}{E(z)} - \overline{\frac{E'(z)}{E(z)}} \right) |E(z)|^2$$

$$= - \frac{d}{dz} \operatorname{Im} \left(\log E(z) \right) \cdot |E(z)|^2$$

$$\therefore \quad J_z(z) = \begin{cases} -|E(z)|^2 \frac{d}{dz} \left(\arg E(z) \right) & z \text{ real}, E(z) \neq 0 \\ 0 & z \text{ real}, E(z) = 0. \end{cases}$$

Prop 2: $B(E)$ is a Hilbert space.

Proof: Let f_n be a Cauchy sequence in $B(E)$. Then the estimate $|f(z)| \leq \|f\| \cdot \|J_z\|$ shows that ~~$f_n(z)$ converges for each z, so we get an~~ $f(z) = \lim f_n(z)$ exists for each $z \in \mathbb{C}$. Moreover as $\|J_z\|$ is a continuous function of z, it is bounded on compacts, hence f_n converges to f uniformly on compact sets, so f is entire. From

$$|f(z)| \leq \lim \|f_n\| \cdot \|J_z\|$$

and the formula for $\|J_z\|$ which gives

$$\frac{\|J_z\|}{|E(z)|} \leq \frac{1}{2 |\operatorname{Im} z|^{1/2}}$$

in the upper half-plane, one sees f satisfies the estimate 2).

From $\int_{-R}^{R} \left|\frac{f(\lambda)}{E(\lambda)}\right|^2 \frac{d\lambda}{\pi} = \lim_{n} \int_{-R}^{R} \left|\frac{f_n(\lambda)}{E(\lambda)}\right| \frac{d\lambda}{\pi} \leq \lim_{n} \|f_n\|$

with a bound independent of R we see that f satisfies the estimate 1), so $f \in B(E)$. Finally given $\varepsilon > 0$ choose n_0 so that $\|f_m - f_n\|^2 < \varepsilon$ for $m, n \geq n_0$. Then

$$\int_{-R}^{R} \left|\frac{f - f_n}{E}\right|^2 \frac{d\lambda}{\pi} = \lim_{m} \int_{-R}^{R} \left|\frac{f_m - f_n}{E}\right|^2 \frac{d\lambda}{\pi} \leq \varepsilon$$

ind. of R, so $\|f - f_n\|^2 < \varepsilon$ for $n \geq n_0$. Thus $f_n \to f$ in $B(E)$.

Example: Suppose E is a poly of degree n having all roots in $\operatorname{Im}\lambda < 0$. If f is a poly of degree $< n$, one has

$$\frac{f(\lambda)}{E(\lambda)}, \quad \frac{f(\lambda)}{E^{\#}(\lambda)} = O\left(\frac{1}{|\lambda|}\right) \quad \text{as } |\lambda| \to \infty$$

so it is clear that $f \in B(E)$. On the other hand it is clear from the formula for $J_z(\lambda)$ that

$$J_z(\lambda) = a_0(\lambda) + a_1(\lambda) z + \dots + a_{n-1}(\lambda) z^{n-1}$$

where the $a_i(\lambda)$ are polys of degree $< n$ in λ, hence $a_i \in B(E)$. Thus

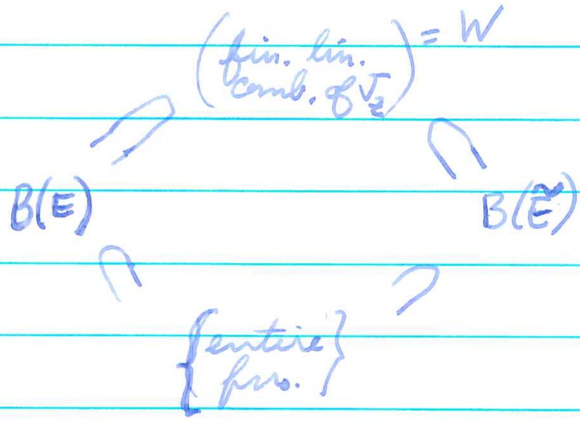
$$f(z) = (f, J_z) = \sum_i (f, a_i) z^i$$

so $B(E)$ consists all the polys. of degree $< n$.

[redacted] Cor. of prop. 2: Finite linear combinations of the J_z, $z \in \mathbb{C}$ are dense in $B(E)$.

Proof: Let W be the closed subspace of $B(E)$ [redacted] spanned by the J_z. If $W < B(E)$, then because $B(E)$ is a Hilbert space, $\exists$ $f \perp W$ with $f \neq 0$. But this is impossible as $f(z) = (f, J_z)$, hence no element of $B(E)$ is orthogonal to all [redacted] J_z.

If $E, \tilde{E}$ are two de Branges functions giving rise to the same $J_z(\lambda)$ for all z, then $B(E) = B(\tilde{E})$ (they have the same elements and the same inner product)



The inner products on $B(E)$, $B(\tilde{E})$ coincide on W because

$$(J_z, J_w) = J_z(w)$$

Rest clear by [redacted] density. [redacted]

$SU(1,1) =$ [redacted] subgroup of $SL_2(\mathbb{C})$ consisting of $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, $|a|^2 - |b|^2 = 1$. Then if we put

$$\tilde{E} = aE + bE^{\#}$$

we have $\quad \tilde{E}^{\#} = \bar{b}E + \bar{a}E^{\#} \quad$ hence

$$\begin{vmatrix} \tilde{E}(\lambda) & \tilde{E}(\bar{z}) \\ \tilde{E}^{\#}(\lambda) & \tilde{E}^{\#}(\bar{z}) \end{vmatrix} = \left| \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(\bar{z}) \end{pmatrix} \right| = \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(\bar{z}) \end{vmatrix}$$

Putting $\lambda = z$ one gets

$$|\tilde{E}(z)|^2 - |\tilde{E}(\bar{z})|^2 = |E(z)|^2 - |E(\bar{z})|^2$$

hence E is a de Branges fn. $\Leftrightarrow$ $\tilde{E}$ is. So we have half of

<u>Prop.</u> If E is a de Branges fn. and $|a|^2 - |b|^2 = 1$, then $\tilde{E} = aE + bE^{\#}$ is a de Branges function such that $B(\tilde{E}) = B(E)$. Conversely every de Branges function $\tilde{E}$ with $B(\tilde{E}) = B(E)$ is uniquely represented in the form $\tilde{E} = aE + bE^{\#}$, i.e. the group $SU(1,1)$ acts simply-transitively on the de Branges functions giving rise to the same de Branges space.

Note that $E, E^{\#}$ are linearly independent functions since if $E = cE^{\#}$, then taking a real point where $E(\lambda) \neq 0$ we find $|c| = 1$, so $|E| = |E^{\#}|$.

Next:

$$-2i(\lambda - \bar{z})\, J_z(\lambda) = E(\lambda)\overline{E(z)} - E^{\#}(\lambda)\, E(\bar{z})$$

$$+2i(\lambda - z)\, J_z(\lambda) = -E(\lambda)\overline{E(\bar{z})} + E^{\#}(\lambda)\, E(z)$$

$\Rightarrow J_z^{\#} = J_{\bar{z}}$

$$\begin{pmatrix} -2i(\lambda - \bar{z})\, J_z(\lambda) \\ +2i(\lambda - z)\, J_z(\lambda) \end{pmatrix} = \begin{pmatrix} \overline{E(z)} & -E(\bar{z}) \\ -\overline{E(\bar{z})} & E(z) \end{pmatrix} \begin{pmatrix} E(\lambda) \\ E^{\#}(\lambda) \end{pmatrix}$$

$$\begin{pmatrix} E(\lambda) \\ E^{\#}(\lambda) \end{pmatrix} = \underbrace{\frac{1}{|E(z)|^2 - |E(\bar z)|^2}}_{4(\mathrm{Im}\, z)\|J_z\|^2} \begin{pmatrix} E(z) & E(\bar z) \\ \overline{E(\bar z)} & \overline{E(z)} \end{pmatrix} \begin{pmatrix} -2i(\lambda - \bar z) J_z \\ +2i(\lambda - z) J_{\bar z} \end{pmatrix}$$

Suppose $\mathrm{Im}\, z > 0$

$$\boxed{\begin{pmatrix} E(\lambda) \\ E^{\#}(\lambda) \end{pmatrix} = \frac{1}{\sqrt{|E(z)|^2 - |E(\bar z)|^2}} \begin{pmatrix} E(z) & E(\bar z) \\ \overline{E(\bar z)} & \overline{E(z)} \end{pmatrix} \begin{pmatrix} -\dfrac{i(\lambda - \bar z) J_z}{(\mathrm{Im}\, z)^{1/2}\|J_z\|} \\ +\dfrac{i(\lambda - z) J_{\bar z}}{(\mathrm{Im}\, z)^{1/2}\|J_z\|} \end{pmatrix}}$$

This formula shows that $SU(1,1)$ acts transitively on the set of de Branges fns giving rise to $B(E)$, and that

$$-\frac{i(\lambda - \bar z) J_z}{(\mathrm{Im}\, z)^{1/2}\|J_z\|}$$

is a de Branges function giving rise to $B(E)$.

It vanishes at $\lambda = \bar z$ and is unique up to a scalar of modulus 1.

We want now to give an intrinsic description of $B(E)$. Notice that $B(E)$ is closed under $f \mapsto f^{\#}$ and that $\|f^{\#}\| = \|f\|$; hence $f \mapsto f^{\#}$ is an anti-linear isometry. Also if $f \in B(E)$ and $f(z) = 0$

with z non-real, then $\frac{\lambda-\bar{z}}{\lambda-z}f(\lambda)$ is asymptotic to f for large λ and it has the same absolute value as f does on the real axis. Hence it also belongs to $B(E)$ and

$$\left\| \frac{\lambda-\bar{z}}{\lambda-z} f(\lambda) \right\| = \|f\|.$$

Prop. 3: Let B be a non-trivial Hilbert space consisting of entire functions. Assume 1) $f \in B \Rightarrow f^{\#} \in B$ and $\|f^{\#}\| = \|f\|$ 2) $f \in B$, $f(z)=0$, $\operatorname{Im} z \neq 0 \Rightarrow \frac{\lambda-\bar{z}}{\lambda-\bar{z}} f \in B$ and has same norm as f, 3) for z non-real $f \mapsto f(z)$ is a bdd linear functional on B (hence representable $f(z) = (f, J_z)$ with $J_z \in B$). Then

$$E(\lambda) = (1-i\lambda)J_i / \|J_i\|$$

is a de Branges function with $B = B(E)$.

Proof: If z non-real, then $\exists f \in B \ni f(z) \neq 0$, for otherwise we would have $\left(\frac{\lambda-\bar{z}}{\lambda-z}\right)^n f \in B$ all n. In particular $J_i \neq 0$. ∎

$$i^n \|f + i^n g\|^2 = i^n\left(\|f\|^2 - i^n (f,g) + i^n (g,f) + \|g\|^2\right)$$

$$\frac{1}{4}\sum_{n=0}^{3} i^n \|f + i^n g\|^2 = (f,g)$$

$$(f^{\#}, g) = \frac{1}{4}\sum i^n \|f^{\#} + i^n g\|^2 = \overline{(f, g^{\#})}$$

$$(f, J_i^{\#}) = \overline{(f^{\#}, J_i)} = \overline{f^{\#}(i)} = f(-i) = (f, J_{-i}) \quad \therefore J_i^{\#} = J_{-i}$$

If $f \in B$, $f(z)=0$, $\operatorname{Im} z \neq 0 \Rightarrow f, \frac{\lambda-\bar z}{\lambda-z} f \in B$

$$\frac{\lambda-\bar z}{\lambda-z} f = \frac{\lambda-z+z-\bar z}{\lambda-z} f = f + (z-\bar z)\frac{1}{\lambda-z} f$$

$$\therefore \quad \frac{z}{\lambda-\bar z} f, \quad \frac{1}{\lambda-\bar z} f \in B$$

Now if $g \in B$, $g(\bar z)=0$, then also $\frac{z}{\lambda-z} g, \frac{1}{\lambda-z} g \in B$

$$\left(\frac{\lambda-\bar z}{\lambda-z} f, g\right) = \frac{1}{4}\sum_0^3 \left\| \frac{\lambda-\bar z}{\lambda-z} f + i^n g \right\|^2$$

$$\| \qquad = \frac{1}{4}\sum_0^3 \left\| \frac{\lambda-\bar z}{\lambda-z}\left(f + i^n \frac{\lambda-z}{\lambda-\bar z} g\right)\right\|^2$$

$$(f,g)+(z-\bar z)\left(\frac{f}{\lambda-z}, g\right)$$

$$= \left(f, \frac{\lambda-z}{\lambda-\bar z} g\right) = (f,g) + (z-\bar z)\left(f, \frac{g}{\lambda-\bar z}\right)$$

$$\left(\frac{f}{\lambda-z}, g\right) = \left(f, \frac{g}{\lambda-\bar z}\right) \quad \text{also} \quad \left(\frac{\lambda f}{\lambda-z}, g\right) = \left(f, \frac{\lambda}{\lambda-\bar z} g\right)$$

Now construct what should turn out to be the point evaluator J_z if the proposition is true

$$K_z(\lambda) = \frac{i}{2(\lambda-\bar z)}\begin{vmatrix} E(\lambda) & E(\bar z) \\ E^{\#}(\lambda) & E^{\#}(\bar z)\end{vmatrix} = \frac{i}{2(\lambda-\bar z)\|J_i\|^2}\begin{vmatrix}(1-i\lambda)J_i(\lambda) & (1-i\bar z)J_i(\bar z) \\ (1+i\lambda)J_i^{\#}(\lambda) & (1+i\bar z)J_i^{\#}(\bar z)\end{vmatrix}$$

(As a check put $z=i$:

$$K_i(\lambda) = \frac{i}{2(\lambda+i)\,J_i(i)}\begin{vmatrix}(1-i\lambda)J_i(\lambda) & 0 \\ * & 2J_{+i}(+i)\end{vmatrix} = J_i(\lambda) \quad)$$

We have

$$2\|J_i\|^2 K_z(\lambda) = \frac{i}{\lambda-\bar z}\Big\{(1+\lambda\bar z + i\bar z - i\lambda)J_i(\lambda)\overline{J_i(z)}$$
$$-(1+\lambda\bar z + i\lambda - i\bar z)\,J_i^{\#}(\lambda)\,J_i(\bar z)\Big\}$$

$$= i\,\frac{1+\lambda\bar z}{\lambda-\bar z}\left(J_i(\lambda)\overline{J_i(z)} - J_i^{\#}(\lambda)J_i(\bar z)\right) + J_i(\lambda)\overline{J_i(z)} + J_i^{\#}(\lambda)J_i(\bar z)$$

Since $J_i, J_i^{\#} = J_{-i} \in B$, it is clear from the above calculations that $K_z \in B$. Furthermore if $h \in B$, $h(z)=0$, then

$$2\|J_i\|^2(h, K_z) = \left(h,\ i\,\frac{1+\lambda z}{\lambda-\bar z}\left(J_i(\lambda)\overline{J_i(z)} - J_i(\lambda)J_i(\bar z)\right)\right)$$

$$+ \left(h,\ J_i(\lambda)\overline{J_i(z)} + J_{-i}(\lambda)J_i(\bar z)\right)$$

$$= -i\left(\frac{1+\lambda\bar z}{\lambda-z}h,\ J_i(\lambda)\overline{J_i(z)} - J_{-i}(\lambda)J_i(\bar z)\right)$$

$$+ h(i)\overline{J_i(z)} + h(-i)J_i(\bar z)$$

$$= \frac{-i+\bar z}{i-z}h(i)\overline{J_i(z)} - \frac{(-i)(1-iz)}{(-i)-z}h(-i)J_i(\bar z) = 0$$

(In class I stated the following version of the calculation on p. 11)

Lemma: If $f \in B$, $f(z)=0$, then $\frac{a\lambda+b}{\lambda-z}f \in B$ for any a,b. Moreover if $g \in B$, $g(\bar z)$ then

$$\left(\frac{a\lambda+b}{\lambda-z}f,\ g\right) = \left(f,\ \frac{\bar a\lambda+\bar b}{\lambda-\bar z}g\right).$$

If $f,g \in B$ we can apply the above to $h(\lambda) = f(\lambda)g(z) - f(z)g(\lambda)$ to get

$$0 = (h, K_z) = g(z)(f, K_z) - f(z)(g, K_z).$$

~~Hence if $K_z \neq 0$, choose g so that $(g, K_z) \neq 0$ and you get~~

~~$$f(z) = \left(f,\ \frac{\overline{g(z)}}{(g,K_z)}K_z\right)$$~~

Notice that

$$\overline{K_z(\lambda)} = \frac{-i}{2(\bar\lambda - \bar z)} \begin{vmatrix} \overline{E(\lambda)} & \overline{E(\bar z)} \\ \overline{E(\bar\lambda)} & \overline{E(z)} \end{vmatrix} = \frac{i}{2(z-\bar\lambda)} \begin{vmatrix} E(z) & E(\bar\lambda) \\ E^{\#}(z) & E^{\#}(\bar\lambda) \end{vmatrix} = K_\lambda(z)$$

Hence taking $g = J_i$ we get

$$(J_i, K_z) = \overline{K_z(i)} = K_i(z) = J_i(z)$$

so

$$J_i(z)(f, K_z) = f(z) J_i(z)$$

hence for all z such that $J_i(z) \neq 0$ we have

$$f(z) = (f, K_z)$$

hence K_z is a point-evaluator for this open dense set of z. I want to prove this for any non-real z.

~~Fix z non-real. Because B consists of [illegible] entire functions it follows that the J_w are dense if w ranges over a countable set with finite accumulation point~~

Suppose $K_z \neq 0$. Then choose w so that $(K_z \cdot J_i)(w) \neq 0$. Then we have $K_w = J_w$, so taking $g = K_w$ we find

$$f(z)\underbrace{(K_w, K_z)} = K_w(z)(f, K_z)$$

$$\overline{(K_z, J_w)} = \overline{K_z(w)} = K_w(z)$$

and since $K_z(w) \neq 0$ we can cancel getting $f(z) = (f, K_z)$ for this z. Thus we see that $K_z \neq 0$ $\Rightarrow$ J_z exists and $J_z = K_z$. The set of these z is open and dense.

Now

$$K_z(z) = \frac{1}{4\,\mathrm{Im}\,z}\left(|E(z)|^2 - |E^{\#}(z)|^2\right)$$

$$= \frac{1}{4\,\mathrm{Im}\,z}\left(|1-iz|^2|J_i(z)|^2 - |1-i\bar{z}|^2|J_i(\bar{z})|^2\right)\Big/\|J_i\|^2$$

For those z with $K_z \neq 0$ we have $K_z(z) = J_z(z) =$ [illegible] $\|J_z\|^2 = \|K_z\|^2 > 0$. So in the upper half plane we have always $|E(z)| \geq |E^{\#}(z)|$. ~~Somehow I have to rule out equality. But if one has equality then because $|1-iz| = |z+i| > |z-i| = |1+iz| = |1-i\bar{z}|$ in UHP we would have to have $|J_i(z)| <$ [illegible]~~ Somehow I have to rule out equality. If $\mathrm{Im}(z) > 0 \Rightarrow E(z) \neq 0$ then I can apply maximum modulus to

$$\frac{E^{\#}(z)}{E(z)}$$

to conclude it is constant if it assumes modulus 1 in $\mathrm{Im}\,z > 0$, which is impossible (for then $K_z(z) \equiv 0$). So now I have to rule out the case $E(z) = 0$ in the UHP, i.e. I have to show $J_i(z) = 0 \Rightarrow$ [illegible] $\mathrm{Im}\,z \leq 0$. ?

Other method is to assume that for any non-real point z the point evaluator J_z exists, i.e. that evaluation $f \mapsto f(z)$ is continuous. If this is assumed, then it is impossible for $J_\lambda(z) = 0$ for all λ ~~with~~ in a ~~dense~~ countable subset with accumulation point, for then $J_z(\lambda) = \overline{J_\lambda(z)} = 0$ so J_z [illegible] $= 0$ and hence all $f \in B$ would vanish at z. Thus there is a λ with $J_\lambda(z) \neq 0$ and so we have $K_z = J_z$ as on page 13.

Example from differential equations:

Consider a first order eigenvalue system

$$Lu = A\frac{du}{dx} + Bu = \lambda C u$$

where A, B, C are matrix functions of x and A is non-singular. The adjoint of L is

$$L^*v = -\frac{d}{dx}(A^*v) + B^*v = -A^*\frac{dv}{dx} + \left(-\frac{dA^*}{dx} + B^*\right)v$$

and Green's formula is

$$v^*(Lu) - (L^*v)^*u = v^*\left(A\frac{du}{dx} + Bu\right) - \left(-A^*\frac{dv}{dx} + \left(-\frac{dA^*}{dx} + B^*\right)v\right)^*u$$

$$= v^*A\frac{du}{dx} + v^*Bu + \frac{dv^*}{dx}Au + v^*\frac{dA}{dx}u - v^*Bu$$

$$= \frac{d}{dx}(v^*Au)$$

~~[illegible]~~

(This is a bad approach. Instead of trying to give the story about how the general system reduces to the particular case to be studied, you should just give the example as directly as possible.)

Consider the system of first order DE's

1) $$A \frac{du}{dx} = \lambda C u$$

where
$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad A = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \qquad C = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix}$$

where α, β are functions of x with α real. Suppose for the moment that α, β are smooth functions on $0 \le x \le b$ and let $\phi(x, \lambda)$ denote the solution of the DE satisfying the initial condition

2) $$\phi(0, \lambda) = \begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}$$

where θ is a fixed real number. It is know that $\phi(x, \lambda)$ is an entire function of λ for each x. If we write the DE

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \frac{du}{dx} = \lambda \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix} u$$

it is clear this DE has the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$, $\lambda \mapsto \bar{\lambda}$ hence

$$\begin{pmatrix} \overline{\phi_2(x, \bar{\lambda})} \\ \overline{\phi_1(x, \bar{\lambda})} \end{pmatrix} = \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix}$$

because both sides are solutions with the same initial values. Thus we have

3) $$\phi_1(x, \lambda) = \phi_2^{\#}(x, \lambda)$$

Green's formula for $L = A\frac{d}{dx} + B$

$$\frac{d}{dx} v^*Au = v^*Lu - (L^*v)^*u$$

in this case is

$$\frac{d}{dx}\frac{1}{i}(\bar{v}_1 u_1 - \bar{v}_2 u_2) = \frac{d}{dx} v^*Au = v^*A\frac{du}{dx} - \left(A^*\frac{dv}{dx}\right)^* u$$

and gives us

$$\frac{1}{i}\frac{d}{dx}\left(|\phi_1|^2 - |\phi_2|^2\right) = \phi^*\lambda C\phi - (\lambda C\phi)^*\phi$$
$$= (\lambda - \bar\lambda)\,\phi^*C\phi$$

or

4) $$|\phi_1(b,\lambda)|^2 - |\phi_2(b,\lambda)|^2 = -2\,\mathrm{Im}\,\lambda\int_0^b \phi^*C\phi\,dx$$

If we suppose $C > 0$ i.e. $\alpha > 0$ and $\alpha^2 - |\beta|^2 > 0$, then the integral is > 0 so we have for $b > 0$

$$\mathrm{Im}\,\lambda > 0 \implies |\phi_2(b,\lambda)| > |\phi_1(b,\lambda)| = |\phi_2^\#(b,\lambda)|$$

and hence $\phi_2(b,\lambda)$ is a de Branges function of λ.

How the associated de Branges space arises:

Consider the Hilbert space $\mathcal{H} = L^2([0,b], C\,dx)$ consisting of measurable u on $[0,b]$ such that

$$\|u\|^2 = \int_0^b u^*Cu\,dx < \infty$$

Inside $\mathcal{H}$ we have the element $\phi_{\bar\lambda}$ for each $\lambda \in \mathbb{C}$. Moreover

$$(\lambda - \bar z)(\phi_{\bar z}, \phi_{\bar\lambda}) = \int_0^b (\bar\lambda C\phi_{\bar\lambda})^*\phi_{\bar z} - \phi_{\bar\lambda}^*(\bar z C\phi_{\bar z})\,dx$$

$$= -\int_0^b \frac{d}{dx}\left(\phi_{\bar\lambda}^* A \phi_{\bar z}\right) dx = -\phi_{\bar\lambda}^* A \phi_{\bar z}\,(b)$$

$$= i\left\{ \overline{\phi_1(b,\bar\lambda)}\,\phi_1(b,\bar z) - \overline{\phi_2(b,\bar\lambda)}\,\phi_2(b,\bar z)\right\}$$

$$= i \begin{vmatrix} \phi_2(b,\lambda) & \phi_2(b,\bar z) \\ \phi_2^{\#}(b,\lambda) & \phi_2^{\#}(b,\bar z) \end{vmatrix}$$

Therefore if $J_z(\lambda)$ is the point evaluator in $B(E)$ where $E(\lambda) = \sqrt{2}\,\phi_2(b,\lambda)$, we have

$$(J_z, J_\lambda) = J_z(\lambda) = (\phi_{\bar z}, \phi_{\bar\lambda}) .$$

Now $B(E)$ is the Hilbert space generated by $\{J_z \mid z \in \mathbb{C}\}$ with the given inner products (J_z, J_λ), so there is a unique isometry

$$B(E) \longrightarrow L^2([0,b], \mathbb{C}\,dx)$$

sending J_z into $\phi_{\bar z}$. Since

$$f(z) = (f, J_z) = \int f(\lambda)\,\overline{J_z(\lambda)}\,\frac{d\lambda}{|E(\lambda)|^2\pi}$$

or

$$f = \int_{\mathbb{R}} J_\lambda \frac{f(\lambda)\,d\lambda}{|E(\lambda)|^2\pi}$$

we want to send f to

$$f \longmapsto \int_{\mathbb{R}} \phi_{\bar\lambda} \frac{f(\lambda)\,d\lambda}{|E(\lambda)|^2\pi} \qquad \text{(not too clear)}$$

So I have the embedding (isometric)

$$B(E) \hookrightarrow L^2([0,b], C\,dx)$$
$$J_\lambda \longmapsto \phi_{\bar\lambda}$$

whose adjoint is

$$u \longmapsto \hat u(\lambda) = (u, \phi_{\bar\lambda})$$

Suppose next that a boundary condition at $x=b$ of the form

$$u_1(b) = e^{-i\theta} u_2(b)$$

is given. Then we get a set of eigenvalues Λ consisting of λ such that $\phi(x,\lambda)$ satisfies the boundary conditions, i.e.

$$\overline{\phi_2(b,\bar\lambda)} = e^{-i\theta} \phi_2(b,\lambda).$$

Because ϕ_2 is a de Branges fn. the eigenvalues have to be real and we have

$$\Lambda = \{\lambda \in \mathbb{R} \mid 2\arg \phi_2(b,\lambda) = \theta \mod 2\pi\}$$

Assume known that the eigenfunctions $\{\phi(\cdot,\lambda),\ \lambda\in\Lambda\}$ form an orthogonal basis for $L^2([0,b], C\,dx)$. Then we see the above isometry is an isom:

$$B(E) \simeq L^2([0,b], C\,dx).$$

and also that $\{J_\lambda,\ \lambda\in\Lambda\}$ is an orthogonal basis for $B(E)$. We are going to prove this in general, but we need some tools first.

Recall that any $f(\lambda)$ holom. in $\operatorname{Im}\lambda > 0$ with $\operatorname{Im} f(\lambda) \geq 0$ can be uniquely represented

(1) $$f(\lambda) = c + p\lambda + \int_{\mathbb{R}} \left(\frac{1}{x-\lambda} - \frac{x}{x^2+1} \right) d\mu(x)$$

where $c \in \mathbb{R}$, $p \geq 0$, and $d\mu$ is a measure on $\mathbb{R}$ with $\int \frac{d\mu}{1+x^2} < \infty$. We have

$$\operatorname{Im} f(\lambda) = p \operatorname{Im}\lambda + \int \frac{\operatorname{Im}\lambda}{|x-\lambda|^2} d\mu(x)$$

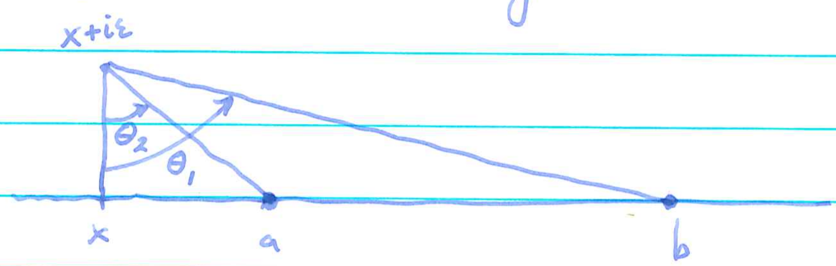
Put $\lambda = t + i\varepsilon$ and integrate over $a \leq t \leq b$.

$$\int_a^b \operatorname{Im} f(t+i\varepsilon)\,dt = p\varepsilon(b-a) + \int_{\mathbb{R}} d\mu(x) \int_a^b \frac{\varepsilon\,dt}{(t-x)^2+\varepsilon^2}$$

Now

(*) $$\varphi_\varepsilon(x) = \int_a^b \frac{\varepsilon\,dt}{(t-x)^2+\varepsilon^2} = \int_{a-x}^{b-x} \frac{\varepsilon\,dt}{t^2+\varepsilon^2} = \left[\arctan \frac{t}{\varepsilon} \right]_{a-x}^{b-x} = \theta_1 - \theta_2$$

where θ_1, θ_2 are the angles indicated:



Thus the integral $\varphi_\varepsilon(x)$ is the angle subtended by $[a,b]$ as viewed from $x+i\varepsilon$, so the limit of $\varphi_\varepsilon(x)$ as $\varepsilon \downarrow 0$ is the function

$$\varphi(x) = \begin{cases} 0 & x \notin [a,b] \\ \pi & x \in (a,b) \\ \frac{\pi}{2} & x = a \text{ or } b. \end{cases}$$

Now one has the bounds

$$0 < \psi_\varepsilon(x) \leq \pi$$

and $\psi_\varepsilon(x)$ is decreasing as $\varepsilon \downarrow 0$ for $x \notin [a,b]$, hence one can bound ψ_ε by ψ_1 independently of ε, and so conclude by dominated convergence that

$$\int d\mu(x)\, \psi_\varepsilon(x) \longrightarrow \int d\mu(x)\, \psi(x)$$

$$= \tfrac{1}{2}\mu\{a\} + \mu((a,b)) + \tfrac{1}{2}\mu(\{b\}).$$

Hence we get the Stieltjes inversion formula: If f is in the form (1), then

$$\frac{1}{2}\mu(\{a\}) + \mu((a,b)) + \frac{1}{2}\mu(\{b\}) = \lim_{\varepsilon \downarrow 0} \int_a^b \operatorname{Im} f(t+i\varepsilon)\, dt$$

As an application suppose $f(\lambda)$ is given by (1) in both $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ (hence $f^{\#}(\lambda) = f(\lambda)$) and that it extends continuously across an interval $a \leq \lambda \leq b$ of the real axis. Then on the interval f is real so we see the measure $d\mu$ has its support outside $[a,b]$.

Suppose now E is a deB fn. and put

$$A = \frac{E + E^{\#}}{2} \qquad B = \frac{E^{\#} - E}{2i}$$

so that $A = A^{\#}$, $B = B^{\#}$ and

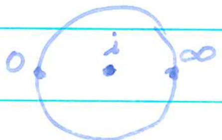
$$\begin{pmatrix} E \\ E^\# \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Then

$$J_z(\lambda) = \frac{i}{2(\lambda - \bar{z})} \left| \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{pmatrix} \right|$$

$$= \frac{-1}{\lambda - \bar{z}} \begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix}$$

Since E is a de B fn. $\frac{E^\#}{E}$ maps $\operatorname{Im} \lambda > 0$ into $|z| < 1$. Now $z \longmapsto w$ where



$$w = \frac{1}{i} \frac{z+1}{z-1}$$

maps $|z| < 1$ onto $\operatorname{Re}(w) > 0$. So

$$f(\lambda) = \frac{1}{i} \frac{\left(\frac{E^\#}{E}\right) + 1}{\left(\frac{E^\#}{E}\right) - 1} = \frac{A}{-B}(\lambda)$$

maps $\operatorname{Im} \lambda > 0$ into $\operatorname{Im} f(\lambda) > 0$. Notice that $f = f^\#$ and that f is a meromorphic fn. of λ whose poles occur at the zeroes of B, which are those λ with $E(\lambda) = E^\#(\lambda)$. Such a λ has to be real. Put $\Lambda_0 = \{\lambda \mid E(\lambda) = E^\#(\lambda)\}$.

Now consider the representation 1) for f and use the remarks about f being analytic on intervals of $\mathbb{R}$ not containing elts. of Λ_0. Then the measure $d\mu$ must be supported in Λ_0:

$$f(\lambda) = -\frac{A}{B}(\lambda) = c + p\lambda + \sum \frac{p_n}{\lambda_n - \lambda}$$

(should have $\frac{1}{\lambda_n - \lambda} + \frac{1}{\lambda_n^2 + 1}$ for convergence)

where λ_n runs over the elements of Λ_0 and

$$p_n = \lim_{\lambda \to \lambda_n} (\lambda_n - \lambda) f(\lambda) = \frac{A(\lambda_n)}{B'(\lambda_n)}$$

This assumes that B has a simple zero at λ_n. Since f has a simple pole at λ_n, if B has a non-simple zero, then A must vanish also at λ_n, so $E(\lambda_n) = 0$. So if we assume E has no real zeroes (a relatively harmless assumption, see below), then $B'(\lambda_n) \neq 0$.

Now

$$J_z(\lambda) = \frac{-1}{\lambda - \bar{z}} \begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix}$$

$$\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n - \bar{z}} = \frac{\lambda - \bar{z}}{(\lambda - \lambda_n)(\bar{z} - \lambda_n)}$$

$$= \frac{1}{\lambda - \bar{z}} \left\{ f(\lambda) - f(\bar{z}) \right\} B(\lambda) B(\bar{z})$$

$$= p\, B(\lambda) B(\bar{z}) + \sum_n \frac{p_n B(\lambda) B(\bar{z})}{(\lambda - \lambda_n)(\bar{z} - \lambda_n)}$$

and

$$J_{\lambda_n}(\lambda) = \frac{-1}{\lambda - \lambda_n} \begin{vmatrix} A(\lambda) & A(\lambda_n) \\ B(\lambda) & 0 \end{vmatrix} = A(\lambda_n) \frac{B(\lambda)}{\lambda - \lambda_n}$$

From this last formula one sees that the J_{λ_n} as λ_n runs over Λ_0 are orthogonal, hence for any $f \in B(E)$ one has from Bessel's inequality

$$\|f\|^2 \geq \sum_n \left| \left(f, \frac{J_{\lambda_n}}{\|J_{\lambda_n}\|} \right) \right|^2 = \sum \frac{|f(\lambda_n)|^2}{\|J_{\lambda_n}\|^2}$$

with equality for all $f \iff \{J_{\lambda_n}\}$ is an orthog. basis. Taking $f = J_z$ we know from Bessel that the following series in $B(E)$ converges

$$\sum_n \left(J_z, \frac{J_{\lambda_n}}{\|J_{\lambda_n}\|} \right) \frac{J_{\lambda_n}(\lambda)}{\|J_{\lambda_n}\|} = \sum_n \frac{J_z(\lambda_n)\, J_{\lambda_n}(\lambda)}{\|J_{\lambda_n}\|^2}$$

$$= \sum \frac{\overline{J_{\lambda_n}(z)}\; J_{\lambda_n}(\lambda)}{A(\lambda_n)\, B'(\lambda_n)}$$

$$= \sum_n{}' \; A(\lambda_n) \frac{\overline{B(z)}}{(\bar z - \lambda_n)} \; A(\lambda_n) \frac{B(\lambda)}{(\lambda - \lambda_n)} \; \frac{1}{A(\lambda_n) B'(\lambda_n)}$$

$$= \sum_n{}' \frac{p_n\, B(\lambda)\, B(\bar z)}{(\lambda - \lambda_n)(\bar z - \lambda_n)}$$

Since convergence in $B(E) \Rightarrow$ unif. convergence on compacts, we get

$$J_z(\lambda) = p\, B(\lambda)\, \overline{B}(z) + \sum_{\lambda_n \in \Lambda_0}{}' \frac{J_z(\lambda_n)\, J_{\lambda_n}(\lambda)}{\|J_{\lambda_n}\|^2}$$

showing that $pB(\lambda) = \frac{p}{2i}(E^\# - E) \in B(E)$. Clearly $E - E^\#$, if it belongs to $B(E)$, is orthog. to all J_{λ_n}. So we obtain the following with $\theta = 0$:

<u>Thm:</u> Assume $E(\lambda)$ is a de B. function with no real zeroes, and put

$$\Lambda_\theta = \{ \lambda \in \mathbb{R} \mid 2 \arg E(\lambda) = \theta \mod 2\pi \}$$
$$= \{ \lambda \in \mathbb{C} \mid e^{-i\theta/2} E(\lambda) - e^{i\theta/2} E^\#(\lambda) = 0 \}.$$

<u>Case 1:</u> If $F(\lambda) = e^{-i\theta/2} E(\lambda) - e^{i\theta/2} E^\#(\lambda) \notin B(E)$, then the elements

$$\frac{J_{\lambda_n}}{\|J_{\lambda_n}\|} \qquad \lambda_n \in \Lambda_\theta$$

form an orthonormal basis for $B(E)$.

Case 2: If $F(\lambda) = e^{-i\theta/2}E(\lambda) - e^{i\theta/2}E^{\#}(\lambda) \in \mathcal{B}(E)$, then these elements ~~form an orthonormal basis for the orthogonal complement of~~ together with $\frac{F(\lambda)}{\|F\|}$ form an orthonormal basis for $\mathcal{B}(E)$.

The general case where $\theta \neq 0$ is reduced to the special case $\theta = 0$ by replacing $E(\lambda)$ by $e^{-i\theta/2}E(\lambda)$.

Above gives in case 2 the formula

$$\|f\|^2 = \sum_n |f(\lambda_n)|^2 \frac{1}{A(\lambda_n)\,B'(\lambda_n)}$$

Understanding de Branges fns. which are polys. and the associated de B. spaces:

Getting rid of real zeroes. Weierstrass thm. says that if λ_n is a sequence of non-zero numbers $\to \infty$, then for suitable $m_n \geq 0$ the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) e^{\frac{\lambda}{\lambda_n} + \frac{1}{2}\left(\frac{\lambda}{\lambda_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{\lambda}{\lambda_n}\right)^{m_n}}$$

converges on compact sets to give an entire fn. with the zeroes λ_i. Consequently if λ_n are the real zeroes of $E(\lambda)$ counted with multiplicity, then $\exists$ entire fn. $F(\lambda)$ with zeroes λ_n such that $F^\# = F$. Then $E_1 = \frac{E}{F}$ is entire, it is a de B. fn. without real zeroes. Moreover it is clear that one gets an isom. of Hilb. spaces

$$B(E_1) \xrightarrow{\sim} B(E)$$

$$f \longmapsto f \cdot F$$

In effect $B(E)$ consists of entire g such that g/E is square-integrable over $\mathbb{R}$ and satisfies the estimates

$$\left|\frac{g}{E}\right| \leq \frac{C}{(\operatorname{Im}\lambda)^{1/2}} \quad \text{for} \quad \operatorname{Im}\lambda > 0, \ |\lambda| > R$$

and a similar estimate in the lower half plane. The square int $\Rightarrow$ multiplicity of g at a real λ is $\geq$ mult. of E $\Rightarrow f = \frac{g}{F}$ is entire. Then $\frac{f}{E_1} = \frac{g}{E_1 F} = \frac{g}{E}$ so $\frac{f}{E_1} \in B(E_1)$, etc.

This arguments shows that one can essentially assume E has no real zeroes in trying to understand these spaces.

Suppose E is a poly of degree $n \geq 1$ with roots in $\mathrm{Im}(\lambda) < 0$. We have seen $\mathcal{B}(E)$ consists of all polys of degree $< n$. Let $\phi_1, \dots, \phi_n$ be the orthonormal basis obtained by applying Gram-Schmidt to $1, \lambda, \dots, \lambda^{n-1}$. Recall the inner product in $\mathcal{B}(E)$ is

$$(f,g) = \int f(\lambda)\overline{g(\lambda)} \, \frac{d\lambda}{|E(\lambda)|^2 \pi}$$

hence provided f, g are of degree $< n-1$ we have

$$(\lambda f, g) = (f, \lambda g).$$

It follows easily that the ϕ_i satisfy the recursion relation

(*) $$\lambda \phi_i = a_i \phi_{i+1} + b_i \phi_i + a_{i-1} \phi_{i-1} \qquad i = 1, \dots, n-1$$

where

$$a_i = (\lambda \phi_i, \phi_{i+1})$$

$$b_i = (\lambda \phi_i, \phi_i) \qquad \text{and } \phi_0 = 0.$$

Note that $a_i > 0$ since the leading coeff. of all the ϕ_i are > 0. From (*) and the initial values

(**) $$\phi_0 = 0, \qquad \phi_1 = \frac{1}{\|1\|}$$

we see that the constants $c_0 = \|1\|$, $a_1, \dots, a_{n-1} > 0$ and the $b_1, \dots, b_{n-1} \in \mathbb{R}$ determine the orth. basis $\phi_1, \dots, \phi_n$ for $\mathcal{B}(E)$ and hence they determine the inner product on $\mathcal{B}(E)$.

Conversely given $c_0, a_1, \dots, a_{n-1} > 0$ and $b_1, \dots, b_{n-1} \in \mathbb{R}$ choose a $a_n > 0$ and define $\phi_0, \dots, \phi_{n+1}$ via (*), (**). We can then put an inner product on $F_{n-1}\mathbb{C}[\lambda]$ = the vector space

of polys. of degree $\leq n-1$, by decreeing that $\phi_1, \dots, \phi_n$ is an orth. basis. I claim this gives a deBranges space $\mathcal{B}$ 1°.
To see this we note that the point evaluator at z in $\mathcal{B}$ is given by

$$J_z(\lambda) = \sum_{i=1}^{n} \phi_i(\lambda)\overline{\phi_i(z)} = \sum_{i=1}^{n} \phi_i(\lambda)\,\phi_i(\bar z)$$

and that by Darboux-Christoffel this is

$$J_z(\lambda) = \frac{-1}{\lambda - \bar z}\begin{vmatrix} \phi_n(\lambda) & \phi_n(\bar z) \\ a_n\phi_{n+1}(\lambda) & a_n\phi_{n+1}(\bar z)\end{vmatrix}.$$

Consequently if we define A, B by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \Theta \begin{pmatrix} \phi_n \\ a_n\phi_{n+1}\end{pmatrix}$$

for some $\Theta \in SL_2(\mathbb{R})$ and put $E = A - iB$, then we have

$$J_z(\lambda) = \frac{-1}{\lambda - \bar z}\begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z)\end{vmatrix} = \frac{i}{2(\lambda - \bar z)}\begin{vmatrix} E(\lambda) & E(\bar z) \\ E^\#(\lambda) & E^\#(\bar z)\end{vmatrix},$$

It follows that for z non-real

$$0 < \|J_z\|^2 = J_z(z) = \frac{1}{4\,\mathrm{Im}(z)}\left(|E(z)|^2 - |E(\bar z)|^2\right)$$

so E is a de B. fn. Also we see that $J_z(\lambda)$ coincides with the pt evaluator at z in $\mathcal{B}(E)$, hence $\mathcal{B} = \mathcal{B}(E)$. Thus we have established an equivalence between the following gadgets:

1) de Branges spaces $\mathcal{B}$ of the form $\mathcal{B}(E)$ where E is a poly of degree n with roots in $\mathrm{Im}\,\lambda < 0$.

2) de Branges spaces $\mathcal{B}$ whose underlying vector space is $F_{n-1}\,\mathbb{C}[\lambda]$.

3) $SU(1,1)$ orbits on the sets of polys. E of degree n with roots in $\mathrm{Im}\,\lambda < 0$.

4) Sequences of polynomials defined by a recursion relation

$$\lambda \phi_i = a_i \phi_{i+1} + b_i \phi_i + a_{i-1} \phi_{i-1} \qquad i = 1, \dots, n-1$$

$$\phi_1 = \frac{1}{c_0} \qquad \phi_0 = 0$$

where $c_0, a_1, \dots a_{n-1} > 0$, and $b_1, \dots, b_{n-1} \in \mathbb{R}$.

5) Inner products on $F_{n-1}\,\mathbb{C}[\lambda]$ satisfying

$$(\lambda f, g) = (f, \lambda g)$$

for $\deg f, \deg g < n-1$, or equivalently positive-definite $n \times n$ Hankel matrices

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{pmatrix}$$

Next let's consider an infinite sequence of orth. polys. $\phi_1, \phi_2, \dots$ defined by a recursion relation of the above sort. Then we get a sequence of de B. spaces

$$\mathcal{B}_1 < \mathcal{B}_2 < \cdots < \mathcal{B}_n < \cdots$$

where $\phi_1, \dots, \phi_n$ spans $\mathcal{B}_n$. Suppose we choose de B. fns. E_n, E_{n-1} for $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$, supposing that $n \geq 2$. Then we have a unique $\Theta_n \in SL_2(\mathbb{R})$ with

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \Theta_n \begin{pmatrix} \phi_n \\ a_n \phi_{n+1} \end{pmatrix}$$

and a similar Θ_{n-1}; here $E_n = A_n - iB_n$ as usual. Hence

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \Theta_n \begin{pmatrix} \phi_n \\ (\lambda - b_n)\phi_n - a_{n-1}\phi_{n-1} \end{pmatrix} = \Theta_n \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ -b_n\phi_n - a_{n-1}\phi_{n-1} \end{pmatrix}$$

$$= \Theta_n \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & \frac{-b_n}{a_{n-1}} \end{pmatrix} \begin{pmatrix} \phi_{n-1} \\ a_{n-1}\phi_n \end{pmatrix}$$

$$= \underbrace{\Theta_n \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & -\frac{b_n}{a_{n-1}} \end{pmatrix} \Theta_{n-1}^{-1}}_{M_n(\lambda)} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

From this formula it is clear that if E_n is given, i.e. Θ_n is given, then there is a unique choice of Θ_{n-1}, i.e. for E_{n-1}, such that $M_n(0) = I$. So we have

Lemma: Given a de B fn. E_n for $\mathcal{B}_n$ $(n \geq 2)$ there is a unique choice of de B. fn. E_{n-1} for $\mathcal{B}_{n-1}$ such that one has

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = M_n(\lambda) \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

where M_n is a linear fn. of λ with $M_n(0) = I$. Conversely any choice for E_{n-1} determines a choice for E_n in the same way.

Notice that if M_n is determined in this way, then

$$M_n(\lambda) = \Theta_n \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \Theta_n^{-1} = I + \lambda\, \Theta_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Theta_n^{-1}$$

If $\Theta_n^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ [redacted], then

$$\Theta_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Theta_n^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} -b & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}$$

so we see M_n has the form

$$M_n(\lambda) = I + \lambda \begin{pmatrix} -\beta_n & -\gamma_n \\ \alpha_n & \beta_n \end{pmatrix}$$

where $\alpha_n, \beta_n, \gamma_n \in \mathbb{R}$, $\alpha_n, \gamma_n \geq 0$, $\beta_n^2 = \alpha_n \gamma_n$ and not all $\alpha_n, \beta_n, \gamma_n$ are 0. [redacted]

~~Prop: Any poly $E_n = A_n - iB_n$, $A_n^{\#} = A_n$, $B_n^{\#} = B_n$ of deg n with roots in $\mathrm{Im}\,\lambda < 0$ determines a sequence [redacted] of non-zero real matrices $\begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & \gamma_i \end{pmatrix}$ $i = 1, \ldots, n$ with $\alpha_i, \gamma_i \geq 0$, $\alpha_i \gamma_i = \beta_i^2$, such that~~

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \left(I + \lambda \begin{pmatrix} -\beta_n & -\gamma_n \\ \alpha_n & \beta_n \end{pmatrix} \right) \cdots \left(I + \lambda \begin{pmatrix} -\beta_1 & -\gamma_1 \\ \alpha_1 & \beta_1 \end{pmatrix} \right) \begin{pmatrix} A_n(0) \\ B_n(0) \end{pmatrix}$$