<u>February 18, 1976.</u> Tate cohomology.

Let G be a finite group. Let $P_\cdot$ be a resolution of $\mathbb{Z}$ by free f.t. $\mathbb{Z}[G]$-modules. Then

$$P^\vee = \operatorname{Hom}_{\mathbb{Z}}(P_\cdot, \mathbb{Z}[G]) = \operatorname{Hom}_{\mathbb{Z}}(P_\cdot, \mathbb{Z})$$

is a resolution of $\mathbb{Z}$ to the right as $H^i(G, \mathbb{Z}[G]) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}$

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow \cdots$$

Splice these together to get an acyclic complex of free f.t. $\mathbb{Z}[G]$ mod

$$\hat{W}_G : \quad \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow \cdots\cdots$$

Then for any complex M of $\mathbb{Z}[G]$-modules one puts

$$\hat{H}^i(G, M) = H^i(\hat{W}_G \otimes_G M)$$

$$H^i(G, M) = H^i(P^\vee \otimes_G M) = H^i(\operatorname{Hom}_G(P, M)) \qquad *$$

$$H_i(G, M) = H_i(P \otimes_G M)$$

whence we get an exact sequence

$$\cdots \longrightarrow H_{-i}(G, M) \longrightarrow H^i(G, M) \longrightarrow \hat{H}^i(G, M) \longrightarrow H_{-i-1}(G, M) \rightarrow$$

~~flat $\mathbb{Z}[G]$-modules. In effect M is the inductive limit of acyclic complexes bdd in the positive direction~~

~~Question: Are these functors defined on the derived category? It suffices to know that M acyclic~~

* This holds only if M is bdd. below (~ cochain complex)

$$(P^\vee \otimes_G M)^k = \coprod_{a+b=k} (P^\vee)^a \otimes_G M^b = \coprod_{a+b=k} \mathrm{Hom}_G(P_{+a}, M^b)$$

$$= \coprod_{-a+b=k} \mathrm{Hom}_G(P^{+a}, M^b)$$

whereas

$$\mathrm{Hom}_G(P, M)^k = \prod_{b-a=k} \mathrm{Hom}_G(P^a, M^b)$$

so I ought to be careful to use Tate cohomology $\hat{H}^i(G, M)$ with complexes bdd. below. (i.e. in D^+)

<u>Note</u>: $M \in D^+$ + M acyclic $\Longrightarrow$ M is a filt. ind. limit of acyclic bdd. complexes:

$$\begin{array}{ccccccc}
\longrightarrow & M^i & \longrightarrow & Z^{i+1} & \longrightarrow & 0 & \\
 & \| & & \cap & & & \\
\longrightarrow & M^i & \longrightarrow & M^{i+1} & \longrightarrow & M^{i+2} &
\end{array}$$

$\therefore$ M acyclic $\Longrightarrow$ $P^\vee \otimes_G M$, $P \otimes_G M$, $\hat{W}_G \otimes M$ also acyclic.

<u>Note:</u> $\hat{W}_G = \varinjlim_p (\hat{W}_G)_{\leq p}$

where $(\hat{W}_G)_{\leq p} \in C^+$ is a complex of f.t. free $\mathbb{Z}[G]$-mods.

and

$$\hat{H}^i(G, M) = \varinjlim_p H^i((\hat{W}_G)_{\leq p} \otimes_G M)$$

$$= \varinjlim_p H^i(\mathrm{Hom}_G(\underbrace{((\hat{W}_G)_{\leq p})^\vee}_{\in D^-}, M))$$

Put $K(p) = ((\hat{W}_G)_{\leq p})^\vee$

Let X be a G-space and let F be a complex of G-sheaves on X bdd below (in C^+). Then we define

$$\hat{H}^i_G(X; F) = \hat{H}^i_G(R\Gamma(X, F))$$

$$= \varinjlim_p H^i(\mathrm{Hom}_G(K(p), R\Gamma(X, F)))$$

$$= \varinjlim_p H^i(G, R\Gamma(X, \mathrm{Hom}(K(p), F)))$$

$$= \varinjlim_p H^i_G(X, \underline{\mathrm{Hom}}(K(p), F))$$

So it's clear from this formula that one is going to have all the good properties. Note that $\{K(p)\}$ is an inverse system of complexes bdd. above.

20 Feb 76 Geometric significance of Tate coh.

$A = \mathbb{Z}/p\mathbb{Z}$. Let X be a smooth manifold on which A acts. X^A is a submanifold and the normal bundle of X^A in X has an A-action, so it breaks up according to the irreducible representations of A over $\mathbb{R}$. If p is odd, this means that the normal bundle can be given a complex structure, hence we will have a Thom isom

$$H_A^*(X, X-X^A) \xleftarrow{\sim} H_A^{*-d}(X^A)$$

where $d =$ codim. of X^A in X. Same is true for $p=2$.

$$\begin{tikzcd}
\arrow[r] & H_A^{*-d}(X^A) \arrow[r] \arrow[d, phantom, "\wr"] & H_A^*(X) \arrow[r] & H_A^*(X-X^A) \arrow[r, "\delta"] \arrow[d, phantom, "\wr"] & {} \\
& {[H_A^* \otimes H^*(X^A)]^{*-d}} & & H^*(X-X^A/A) &
\end{tikzcd}$$

Now the composite $H_A^{*-d}(X^A) \to H_A^*(X) \to H_A^*(X^A)$ is multiplication by the Euler class of the normal bundle of X^A in X, and calculation shows this Euler class is a non-zero divisor. Thus the structure of $H_A^*(X)$ is given by an exact sequence:

$$0 \longrightarrow [H_A \otimes H(X^A)]^{*-d} \longrightarrow H_A^*(X) \longrightarrow H^*(X-X^A/A) \longrightarrow 0$$

This sequence is not a homotopy invariant of X because we could multiply X by a representation of A + change d.

Duality theorem:

$$A = k[T_0, \dots, T_r] \qquad \mathfrak{m} = (T_0, \dots, T_r) \subset A$$

$M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a graded f.t. A-module

$$\begin{tikzcd} Y = \operatorname{Spec}(A) - \mathfrak{m} \arrow[r, hook, "j"] \arrow[d, "p"] & \operatorname{Spec}(A) \\ \mathbb{P}^r & \end{tikzcd}$$

If $\tilde{M} = F$ is the sheaf assoc. to M on $\mathbb{P}^r$ we have

$$p^*F = j^*M$$

$$p_* p^* F = \bigoplus_{n \in \mathbb{Z}} F(n)$$

so

$$H^i(\mathbb{P}^r, \bigoplus F(n)) = H^i(Y, p^*F) \qquad p \text{ affine}$$

$$= H^i(Y, j^*M)$$

But we have

$$0 \to H^0_{\mathfrak{m}}(M) \to M \longrightarrow H^0(Y, j^*M) \to H^1_{\mathfrak{m}}(M) \to 0$$

$$H^i(Y, j^*M) = H^{i+1}_{\mathfrak{m}}(M) \qquad i \geq 1.$$

Therefore one gets

$$\boxed{\begin{array}{c} H^i(\mathbb{P}^r, \bigoplus_{n \in \mathbb{Z}} F(n)) = H^{i+1}_{\mathfrak{m}}(M) \qquad i \geq 1 \\[1em] 0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0(\mathbb{P}^r, \bigoplus_{n \in \mathbb{Z}} F(n)) \to H^1_{\mathfrak{m}}(M) \to 0 \end{array}}$$

Grothendieck duality thm.

$$H^i(\mathbb{P}^r, F(n))^\vee = \operatorname{Ext}^{r-i}_{\mathbb{P}^r}(F(n), \mathcal{O}(-r-1))$$

$$\Uparrow$$

$$E_2^{pq} = H^p(\mathbb{P}^r, \underline{\operatorname{Ext}}^q(F, \mathcal{O}(-r-1-n)))$$

$n \ll 0$ the $H^p = 0$ $p>0$.

Thus for $n \ll 0$ one has

$$H^i(\mathbb{P}^r, F(n))^\vee = H^0(\mathbb{P}^r, \underline{\operatorname{Ext}}^{r-i}(F, \mathcal{O})(-r-1-n)).$$

Consequently $H^{i+1}_{m}(M)$ finite length $\iff$ $\underline{\operatorname{Ext}}^{r-i}(F, \mathcal{O}) = 0$ (for $i \geq 1$).

Observe that I have two procedures for killing off free orbit types

$$H^i_G(X, F) \longrightarrow \hat{H}^i_G(X, F)$$

$$H^*_G(X, F) \longrightarrow H^*_G(X, F)[e^{-1}]$$

where e is the Euler class of some representation. Can I relate these two procedures? Both arise from pro-objects in the derived category of G-modules.

For example, suppose $e \in H^2_G$ is represented by an extension

$$0 \longrightarrow \Lambda \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \Lambda \longrightarrow 0 \qquad \Lambda = \mathbb{F}_p$$

Let $P_\cdot$ be a free $\Lambda[G]$-resolution of Λ. An element of $H_G^{i+2d}(M)$ is rep. by

$$\begin{array}{ccccc} \longrightarrow & P_{i+2d} & \longrightarrow & \cdots & \longrightarrow P_0 \\ & \downarrow & & & \\ & M & & & \end{array}$$

Multiplying by e gives the element rep. by

$$\begin{array}{cccccccccc} & P_{i+2d+2} & & & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \\ & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \longrightarrow & P_{i+2d} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ & \downarrow & & & & & & & & \\ & M & & & & & & & & \end{array}$$

Thus we can arrange an inverse system of complexes

$$K(d): \quad \underset{\deg i}{P_{i+2d}} \longrightarrow \cdots \longrightarrow \underset{\deg -2d}{P_0}$$

and

$$\left(H_G(M)[e^{-1}]\right)^i = \varinjlim_d H^i\{\mathrm{Hom}_G(K(d), M)\}$$

For G-cyclic, it is clear that $\hat{H}_G^i(M) = \left(H_G(M)[e^{-1}]\right)^\wedge$.

<u>Question</u>: What sort of modified cohomologies can be constructed in this manner?

Suppose $\tilde{H}^*_G(X, F)$ is a modified cohomology theory. Say $\tilde{H}^*(G, M)$ is. Then we can consider those subgroups H such that

$$N \longmapsto \tilde{H}^*(G, \operatorname{Ind}_{H \to G}(N)) = 0$$

If $\mathcal{F}$ is the family of these subgroups, then $\mathcal{F}$ satisfies:
$$\begin{cases} H' \subseteq H \in \mathcal{F} \Longrightarrow H' \in \mathcal{F} \\ H \in \mathcal{F} \Longrightarrow gHg^{-1} \in \mathcal{F}. \end{cases}$$

So the question is whether I can construct a modified cohomology associated to such a family $\mathcal{F}$.

Relative homological algebra: Suppose A is a k-algebra, $i: k \to A$ the structural homomorphism. Then for each A-module M, I have a "standard" resolution

$$\Rrightarrow A \otimes_k A \otimes_k M \rightrightarrows A \otimes_k M \longrightarrow M \longrightarrow 0$$

with the following properties.

i) i^* of the resolution splits

ii) the A-modules $A \otimes_k N$ with N a k-module are relatively projective, i.e. lifting for exact sequences split over k.

One knows that such a relative-projective resolution of M is unique up to homotopy, and one can define relative Ext's.

So suppose a subgroup H of G is given. Then we construct a relative injective resolution

$$0 \to \Lambda \to I^0 \to I^1 \to I^2 \to \cdots\cdots$$

and form the following inverse system of complexes

$K(0) = \Lambda$

$K(1)$: $\Lambda \to I^0$

$K(2)$: $\Lambda \to I^0 \to I^1$

and take

$$\varinjlim_{k} \operatorname{Ext}^i_G(K(k), M) = \tilde{H}^i(G, M)$$

Now if $N \to Q^\cdot$ is an injective resolution of an H-module N, then $i_* N \to i_* Q^\cdot$ is an injective resolution of the G-module $i_* N$. So

$$\operatorname{Ext}^i_G(K(k), i_* N) = H^i\{\operatorname{Hom}_G(K(k), i_* Q^\cdot)\}$$
$$= H^i\{\operatorname{Hom}_H(K(k), Q^\cdot)\}$$

$$= \mathrm{Ext}^{i}_{H}(K(k), N) = \mathrm{Ext}^{i+k}_{H}(H^k K(k), N)$$

But as we go from k to $k+1$ the map is zero

$$\underset{-i}{0 \longrightarrow N} \longrightarrow \cdots\cdots \longrightarrow \underset{k}{H^k K(k)} \longrightarrow 0$$

$$0 \longrightarrow H^k K(k) \longrightarrow I^{k+1} \longrightarrow H^{k+1}(K(k+1)) \to 0$$

splits.

Let C be the category of finite G-sets, G finite and let R be a crible in C. One has then Čech cohomology for any G-module F

$$H^p(R, F) = R^p \varprojlim_{C/R} F$$

and one has a spectral sequence

$$E_2^{pq} = H^p(R, \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(G, F)$$

Tate coh is outside of this theory because one wants to ignore the free G-sets, which are in R. Somehow what's going on is that we have some sort of .

cohomological localization process which replaces a sheaf F by an inductive system of complexes:

$$k \longmapsto R\mathrm{Hom}(K(k), F).$$

Brown's theorem: Let G be a finite group and let J be the poset of non-trivial p-subgroups of G. Then for any G-module M

$$\hat{H}^i(G, M) \longrightarrow \hat{H}^i_G(J, M)$$

isomorphism when localized at p (this means on the p-primary components since both sides are torsion).

Proof: Suppose $f: X \to Y$ is a G-space map and we wish to show

$$f^*: H_G(Y, M)_{(p)} \longrightarrow H_G(X, M)_{(p)}$$

is an isomorphism. Then by transfer theory it is enough to do this for a Sylow subgroup of G.

Another point: To calculate $H^*_G(X, M)$ we can consider covering by fixpt. sets.

Idea: G finite group. Let $C(G)$ denote the category of transitive G-sets. In $C(G)$ we have interesting cribles. Take the case where G is cyclic of prime order p. Then $C(G)$ has 2 objects; it is the cone on the category G. If X is a G-space it divides up into 2 strata

$$X = X^G \cup (X - X^G)$$

where X^G is closed in X. Let F be a G sheaf on X; we have a local cohomology sequence

$$\cdots \longrightarrow H^i_G(X, X - X^G; F) \longrightarrow H^i_G(X, F) \longrightarrow H^i_G(X - X^G, F) \xrightarrow{\delta}$$

Go back to Tate cohomology. One replaces $M \in D^+(G\text{-mods})$ by $\{Hom(K(k), M)\}$ where $K(k)$ is the inverse system of truncations of the Tate cx. Then $Hom(K(k), M)$ is in $D^+(G\text{-mods})$ and it is coh. trivial in each dimension.

To construct $K(k)$ one starts with

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}$$

$$\mathbb{Z} \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow \cdots$$

and then $K(k)$ is the complex

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_0^{\vee} \longrightarrow \cdots \longrightarrow P_k^{\vee} \longrightarrow 0 \cdots$$

degree: $\quad -1 \quad 0 \quad k$

This is a complex of free f.t. $\mathbb{Z}[G]$-modules split over $\mathbb{Z}$. Note

$$K(k) \text{ homot. equiv}/\mathbb{Z} \text{ to } H^k K(k)[k]$$

$$\therefore \operatorname{Hom}(K(k), M) \text{ ——— }/\mathbb{Z} \text{ to } H^k(K(k))^{\vee}[-k] \otimes M$$

But $0 \longrightarrow H^k(K(k))^{\vee} \longrightarrow P_k \longrightarrow \cdots$

Thus if we let Z_k be defined by

$$0 \longrightarrow Z_k \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

we have a quis

$$\operatorname{Hom}(K(k), M) \longleftarrow Z_k[-k] \otimes M.$$

So now we can see explicitly what the replacement $M \longmapsto \{\operatorname{Hom}(K(k), M)\}$ consists of. One embeds M into the complex

$$\begin{array}{lcccccc}
 & & & & 0 \longrightarrow & M & \longrightarrow \\
 & & & & & \cap & \\
 & 0 \longrightarrow & P_k \otimes M & \longrightarrow \cdots \longrightarrow & P_0 \otimes M \longrightarrow & M & \longrightarrow 0 \\
\xrightarrow{\text{quis}} & & \cup & & & & \\
 & 0 \longrightarrow & Z_k \otimes M & \longrightarrow 0 & & &
\end{array}$$

Note that the cofibres of the map

$$M[0] \longrightarrow Z_k[k] \otimes M$$

thus constructed [redacted] is built up out of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} N$ modules, that is, modules whose Tate cohomology is trivial. Therefore what it looks like I am after is a "largest" complex with trivial Tate cohomology mapping to M.

"Trivial Tate cohomology" [redacted] means roughly "built up out of the chains complexes of free G-spaces.

Now in general I am trying to construct a cohomology theory which will ignore modules of the form $\mathrm{Ind}_{H \to G}(N)$ for all subgroups in a certain class $\mathcal{F}$ closed under subgroups and conjugates.

Feb. 23, 1976

Let H be a subgroup of G, $i : H \to G$ the inclusion, and let M be a G-module. One has a canonical isom

$$i_! i^* M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \xrightarrow{\sim} \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} M = i_! \mathbb{Z} \otimes M$$

for there is a natural imprimitivity system on the right.

Let us form the complex

$$(*) \qquad \substack{\rightarrow\\\rightarrow\\\rightarrow} \underset{-2}{(i_! i^*)^2 \mathbb{Z}} \rightrightarrows \underset{-1}{(i_! i^*)\mathbb{Z}} \longrightarrow \underset{0}{\mathbb{Z}} \longrightarrow 0$$

and denote by $J(n)$ the truncated subcomplex which is zero in degrees $< -n$ and the same in degrees $\geq -n$.

Suppose M is a bounded complex of G-modules. We form the ind-complex

$$n \longmapsto J(n) \otimes M$$

Note that $J(n)$ is free over $\mathbb{Z}$ in each dimension.

<u>Prop.</u> If M is of the form $i_! N$, then the maps $J(n) \otimes M \longrightarrow J(n+1) \otimes M$ are null-homotopic.

<u>Proof:</u> $J(n) \otimes i_! N = i_! (i^* J(n) \otimes N)$ and one knows that i^* of the standard complex $(*)$ has a

contracting homotopy. This implies trivially that $i^* J(n) \longrightarrow i^* J(n+1)$ is null-homotopic.

Suppose we have an exact sequence of bdd. complexes

$$
(**) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J(n)\otimes M' & \longrightarrow & J(n)\otimes M & \longrightarrow & J(n)\otimes M'' & \longrightarrow & 0
\end{array}
$$

of G-modules.

Supposing that the ind objects $\{J(n)\otimes M'\}$ and $\{J(n)\otimes M''\}$ are zero, in the category of complexes modulo homotopy, I can't conclude the same for $\{J(n)\otimes M\}$ unless I know that the exact sequence (**) is split locally. So we should work in the derived category.

Now suppose M is such that $M \longrightarrow J(n)\otimes M$ is the zero map in the derived category. Then M is a retract of the fibre of this map which is the total complex assoc. to the double complex

$$
\cdots 0 \longrightarrow \underset{k-1}{(i_! i^*)^k M} \longrightarrow \cdots \longrightarrow \underset{0}{i_! i^* M} \cdots
$$

Thus M is a retract of a complex built up out of complexes of the form $i_! N$.

<u>Prop</u>: The following conditions are equivalent for a bdd complex of G-modules:

i) The inductive system $\{J(n) \otimes M\}$ is essentially zero in $D^b(G\text{-mods})$.

ii) $\exists\, n$ such that $M \longrightarrow J(n) \otimes M$ is the zero map in $D^b(G)$

iii) M is a retract of a complex M' having a finite filtration

$$0 \subset F_1 M' \subset \cdots \subset F_p M' = M'$$

such that $F_i M' / F_{i-1} M'$ is of the form $i_! N$.

<u>Proof</u>: [redacted] Clear from the preceding.

<u>Remark</u>: The above ought to hold for $D^+(G)$, and maybe $D(G)$.

Condition iii) above shows the construction given does not depend on the choice of $J(n)$. In fact we get similar results by taking any resolution of $\mathbb{Z}$ by modules of the form $i_! N$ which splits when i^* is applied.

Suppose now we have a complex F in the class described in the proposition, and a map $F \to M$. Then for n large

$$\begin{array}{ccc} F & \longrightarrow & M \\ {\scriptstyle 0}\downarrow & & \downarrow \\ J(n)\otimes F & \longrightarrow & J(n)\otimes M \end{array}$$

so F factors through the fibres of $M \to J(n)\otimes M$, which I should denote $\bar{J}(n)\otimes M$.

$$0 \longrightarrow \varinjlim_n [F, \bar{J}(n)\otimes M] \xrightarrow{\sim} [F, M] \longrightarrow 0$$

Thus we have a universal property for the ind system $\{\bar{J}(n)\otimes M\}$.

So now define

$$\tilde{H}^i(G, M) = \varinjlim_n H^i(G, J(n)\otimes M)$$

This is the "localized" cohomology of M wrt $i: H \to G$.

<u>Variants</u>: Let S be a G-set and denote by S_G the cofibred category over G associated to S. We have a functor

$$S_G \xrightarrow{i} pt_G = \text{"}G\text{"}$$

and adjoint functors

$$\mathrm{Funct}(S_G, ab) \underset{i_*}{\overset{i_!}{\underset{i^*}{\rightleftarrows}}\!\!\rightarrow} \mathrm{Funct}(pt_G, ab)$$

with $i_!$ exact because i is cofibred with discrete fibres:

$$(i_! F)(Y) = \varinjlim_{X \in i/Y} F(X) = \varinjlim_{X \in i^{-1}(Y)} F(X)$$

Preceding construction ought to generalize easily.

Suppose now that X is a G-space, say a G-polyhedron. Let $C(X)$ be the group of chains of X. Then one defines

$$H^i_G(X, M) = H^i_G(\underbrace{\mathrm{Hom}(C(X), M)}_{\substack{\shortparallel \\ C^\cdot(X,M)}})$$

The localized cohomology is defined to be

$$\tilde{H}^i_G(X, M) = \varinjlim_n H^i_G(J(n) \otimes C^\cdot(X, M)).$$

Summary: For each family $\mathcal{F}$ of subgroups of G closed under conjugation and subgroups, I have a universal ind-object in the derived category $\Gamma_{\mathcal{F}}(M) \longrightarrow M$.

It has the universal property that each member of $\Gamma_{\mathcal{F}}(M)$ is built out of modules of the form $\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} N$ with $H \in \mathcal{F}$ and that any such complex mapping to M uniquely factors thru $\Gamma_{\mathcal{F}}(M)$.

We seem to also have a pro-object:

$$\begin{array}{ccccccccc}
 & & 0 & \to & M & \to & \iota_*\iota^*M & \to \cdots \to & (\iota_*\iota^*)^k M \\
 & & & & \downarrow & & & & \\
\cdots & & 0 & \to & M & \to & 0 & \to \cdots & \\
 & & & & \downarrow & & & & \\
 & & & & \iota_*\iota^*M & \to & \cdots & \to & (\iota_*\iota^*)^k M
\end{array}$$

k th-term of the pro-object. Something like the "completion" of M

So because every object M of $D^b(G)$ has the "ind-filtration" $\Gamma_{\mathcal{F}}(M)$ we should get some sort of "localized" cohomology with respect to a subgroup H such that

$$\Gamma_{\leq H, < H}(M)$$

picks up the orbit type G/H of M if say $M = C_\circ(X)$

Let $f: X \to Y$ be a map of G-spaces, and suppose we want to show

$$f^*: H_G(Y) \longrightarrow H_G(X)$$

is an isomorphism. Let us consider the category $\mathcal{C}$ of G-spaces Z such that

$$H_G(Y\times Z) \longrightarrow H_G(X\times Z)$$

is an isomorphism. Assume that $\mathcal{C}$ contains all Z of the form G/A where A is an abelian subgroup of G. By Kunneth, if $Z \in \mathcal{C}$ and if S is a trivial G-space, then $Z\times S \in \mathcal{C}$:

$$\begin{aligned} H_G(Y\times Z\times S) &= H_G\left(PG\times^G(Y\times Z\times S)\right) \\ &= H_G\left([PG\times^G(Y\times Z)]\times S\right) \end{aligned}$$

I now want to show that $\mathcal{C}$ contains all G-spaces with abelian isotropy groups. Try arguing by induction on the number of isotropy groups. Better to look at the map $Z \to Z/G$

$$\begin{array}{ccccc} & & PG\times^G(Y\times Z) & \longrightarrow & Z/G \\ & & \cup & & \cup \\ PG\times^{G_z} Y & = & PG\times^G(Y\times G_z) & \longrightarrow & G/G_z \\ \wr\!\wr & & & & \\ PG_z\times^{G_z} Y & & & & \end{array}$$

so I should get a spectral sequence

$$E_2^{pq} = H^p(Z/G\,,\ Gz \longmapsto H^q_{G_z}(Y)) \Rightarrow H^{p+q}_G(Y \times Z)$$

and so therefore $Z \in C$ if all isotropy groups of C are abelian. Now take a faithful representation V of G, whence the flag manifold of V has only abelian isotropy groups. This shows $\mathbb{P}V \in \mathcal{C}$. But recall that one has an exact sequence

$0 \longrightarrow H^*_G(Y$ [redacted] $) \longrightarrow H^*_G(Y \times \mathbb{P}V) \rightrightarrows H^*_G(Y \times (\mathbb{P}V)^2)$

hence it seems we can prove the following

<u>Theorem:</u> Let $f: X \to$ [redacted] Y be a map of G-spaces such that for all abelian subgroups A of G one has that $H^*_A(Y) \xrightarrow{\sim} H^*_A(X)$. Then [redacted] for all G-spaces Z one has $H^*_G(Y \times Z) \xrightarrow{\sim} H^*_G(X \times Z)$.

(ridiculous: If $f^*: H^*(Y) \xrightarrow{\sim} H^*(X)$, then $H^*_G(Y) \xrightarrow{\sim} H^*_G(X)$ by the spec seq.)

[redacted] It should be possible to get down to an elementary abelian p-group [redacted] if I consider cohomology modulo p.

Conjecture: Let $\mathcal{A}$ be the family of elementary abelian p groups in G and let M be a bounded complex of $\mathbb{Z}/p[G]$-modules. Then $\Gamma_{\mathcal{A}}(M)$ is isomorphic to M.

[redacted] Idea of the proof. Let V be a faithful cx. representation of G and let [redacted] X be the flag manifold of V. Then M should be a retract of $C(X, M)$, and $C(X, M)$ should involve only elem. p-ab. [redacted] subgroups.

Question: Does there exist a contractible G-space X having elementary p-abelian isotropy groups; X should be a polyhedron and contractible should perhaps be replaced by acyclic mod p.

Feb. 25, 1976 Brown's paper on χ for groups.

Suppose Γ is a group having a subgroup Γ' of finite index with $cd(\Gamma') < \infty$. (One says then that $vcd(\Gamma) < \infty$). By Wall $\exists$ finite-dimensional CW cx Y' which is a $B\Gamma'$, so $\tilde{Y}'$ is a finite-dim. contractible CW cx on which Γ' acts freely. Now let X be the multiplicative induction of $\tilde{Y}'$ from Γ' to Γ:

$$X = \text{sections of } \quad \Gamma \times^{\Gamma'} \tilde{Y}' \longrightarrow \Gamma/\Gamma'$$

X is a finite-dimensional CW on which Γ acts; the isotropy groups of Γ on X are finite subgroups; for each finite subgroup H of Γ, X^H is contractible.

Let S be the poset of finite non-trivial subgroups of Γ. We have

$$X = X_{\text{free}} \amalg \bigcup_{H \in S} X^H$$

Since X^H is contractible for each H in S I know that $X' = \bigcup X^H$ has the homotopy type of the simplicial complex $K(S)$ assoc. to S:

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{H_0 \subset H_1} X^{H_0} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{H_0} X^{H_0} \arrow[r] \arrow[d] & \bigcup X^H \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{H_0 \subset H_1} pt \arrow[r, shift left] \arrow[r, shift right] & \coprod_{H_0} pt &
\end{tikzcd}$$

So I get then a long exact sequence

$$
\begin{array}{ccccccc}
\cdots\longrightarrow & H^*_\Gamma(X,X') & \longrightarrow & H^*_\Gamma(X) & \longrightarrow & H^*_\Gamma(X') & \longrightarrow\cdots \\
(*) & \wr\wr & & \wr\wr & & \wr\wr & \\
& H^*(X/\Gamma, X'/\Gamma) & & H^*_\Gamma & & H^*_\Gamma(S) &
\end{array}
$$

and an isomorphism on the Tate cohomology

$$\hat{H}^*_\Gamma \xrightarrow{\sim} \hat{H}^*_\Gamma(S).$$

Moreover Brown's machinery is sufficient to establish under suitable finiteness conditions that the exact sequence (*) leads to a formula for χ:

$$\chi(\Gamma) \equiv \chi_\Gamma(S) \quad (\mathrm{mod}\ \mathbb{Z})$$

In the case where Γ is arithmetic, one knows one should take X to be the symmetric space with its corners added.

~~(p)-version: This time suppose Γ' is normal in Γ and let Γ_p be such that Γ_p/Γ' = Sylow p-subgroup of Γ/Γ'. Then one looks only at the action of Γ_p on X. All the isotropy groups inject into Γ_p/Γ', hence they are~~

<u>Question</u>: Can one always assign an Euler characteristic in $\mathbb{Q}/\mathbb{Z}$ to Tate cohomology?

(p)-version. Let S_p be the poset of non-trivial p-subgroups of Γ, and put

$$X'_{(p)} = \bigcup_{H \in S_p} X^H$$

Then the isotropy groups of points in $X - X'_{(p)}$ are p'-groups, so

$$H^*_\Gamma(X, X'_{(p)})_{(p)} \simeq H^*(X/\Gamma, X'_{(p)}/\Gamma)_{(p)}$$

is bounded. Thus we get

$$(\hat{H}^*_\Gamma)_{(p)} \simeq \hat{H}^*_\Gamma(X'_{(p)})_{(p)}$$

which isn't very useful because $X'_{(p)}$ doesn't have its isotropy groups in S_p.

Suppose $\Gamma' \triangleleft \Gamma$ and let Γ_p/Γ' be a Sylow p-subgroup of Γ/Γ'. Then all isotropy groups of Γ_p on X are p-groups, so Γ_p acts freely on $X - X'_{(p)}$ so

$$\hat{H}^*_{\Gamma_p} \xrightarrow{\sim} \hat{H}^*_{\Gamma_p}(X'_{(p)})$$

?

Theorem: $$\left(\hat{H}^*_\Gamma\right)_{(p)} \xrightarrow{\sim} \hat{H}^*_\Gamma(S_p)_{(p)}.$$

Proof: By transfer theory it suffices to show
$$\hat{H}^*_{\Gamma_p} \xrightarrow{\sim} \hat{H}^*_{\Gamma_p}(S_p)$$
and by the non-p-results it suffices to show that for any finite non-trivial H in Γ_p that $K(S_p)^H$ is contractible, which is true because H is a p-group and if H normalizes a p-group P, then HP is a p-group.

Theorem: $$\chi(\Gamma) - \chi_\Gamma(S_p) \in \mathbb{Z}_{(p)}$$ integers localized at p

Proof: $$\chi(\Gamma) = \frac{1}{(\Gamma:\Gamma_p)}\chi(\Gamma_p)$$
$$\chi_\Gamma(S_p) = \frac{1}{(\Gamma:\Gamma_p)}\chi_{\Gamma_p}(S_p)$$
and since $(\Gamma:\Gamma_p)$ is a p-unit it suffices to show
$$\chi(\Gamma_p) - \chi_{\Gamma_p}(S_p) \in \mathbb{Z}$$
But this follows from the non-p-results.

If Γ finite we get $$\frac{1-\chi(S_p)}{|\Gamma_p|} \in \mathbb{Z}$$

For example, suppose one is in the case of periodic cohomology, i.e. all Sylow p-subgroups cyclic or generalized quaternion. Then each H in S_p contains a unique cyclic subgroup of order p, so $K(S_p)$ is homotopy equivalent to the set of cyclic subgroups of order p. $\chi(S_p) = (G:N)$ where N is the normalizer of some cyclic subgroup A of order p.

Let G_p act on G/N. Let H be the stabilizer in G_p of a cyclic group B of order p. Then BH is a p-group whose unique order p subgroup is B, so $B \subset H \subset G_p$, so B is the unique order p-subgroup in G_p. Thus G_p acts freely on all the other order p subgroups. This proves $|G_p|$ divides $1 - \chi(S_p)$.

Exactly what is happening? Γ finite group acts on S_p, so one has $\chi_\Gamma(S_p) = \dfrac{\chi(S_p)}{|\Gamma|}$

But one knows that for every $e \neq H \subset \Gamma_p$, that $(S_p)^H$ is contractible, so

$$\chi_{\Gamma_p}(S_p) \equiv \chi_{\Gamma_p}\Big(\bigcup_{H \in S_p(\Gamma)} S_p^H\Big) \pmod{\mathbb{Z}}$$

$$= \chi_{\Gamma_p}(pt)$$

So what's happening is this: Look at Γ_p acting on the simplicial complex $K(S_p)$. We know that the non-free part is

$$\bigcup_{H \in S(\Gamma_p)} K(S_p)^H$$

and that $K(S_p)^H \sim pt$, so that the non-free part has the homotopy type of $K(S(\Gamma_p))$ which is contractible. Thus

$$\chi(S_p) = \underset{\overset{\parallel}{1}}{\chi(S_p{}^{\text{non-}}_{\text{free}})} + \chi(S_p, S_p{}^{\text{non-}}_{\text{free}})$$

and the latter is divisible by $|\Gamma_p|$.

<u>Question</u>: What is the smallest class $\mathcal{C}$ of subgroups of G containing the elementary abelian groups such that if A is elementary abelian and A normalizes C in $\mathcal{C}$, then $AC \in \mathcal{C}$?

Question: Let G be a finite group. Can I find a G-polyhedron X such that i) the isotropy groups of X are abelian ii) $\forall$ abelian subgroup H, X^H is acyclic?

[redacted] A bigger question is what sort of G-spaces can be found with some sort of acyclic properties.

[redacted]

Fact. Suppose X is a [redacted] space which is the union of subspaces X_i $i \in I$ where I is a poset. Assume each X_i is contractible and that for each $x \in X$, $\{i \mid x \in X_i\}$ is contractible. Then X is a classifying space for I.

~~Suppose X is a G-space such that for [illegible] abelian subgroups A the components of X^A [illegible] all contractible, and all isotropy groups [illegible]~~

Feb 27, 1976 Relative cohomology.

Let H be a subgroup of G, let M and M' be G-modules. One has the concept of a relative-projective $(\mathbb{Z}[G], \mathbb{Z}[H])$ resolution of M. It is a sequence of G-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

which splits over H and where each P_i is relatively-projective, i.e. a retract of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ for some H-module N (can take $N = P_i$). Such a resolution is unique up to homotopy; the standard example is to take

$$\cdots \longrightarrow \mathbb{Z}[(G/H)^2] \longrightarrow \mathbb{Z}[G/H] \longrightarrow \mathbb{Z} \longrightarrow 0$$

and to tensor with M. Next one defines relative Ext's by

$$\operatorname{Ext}^i_{\mathbb{Z}[G], \mathbb{Z}[H]}(M, M') = H^i\left\{\operatorname{Hom}_{\mathbb{Z}[G]}(P_\bullet, M')\right\}.$$

Taking $M = \mathbb{Z}$, we get relative cohomology groups

$$H^i((G,H); M') = H^i \operatorname{Hom}\left\{\nu \mapsto H^0_G((G/H)^{\nu+1}; M')\right\}$$

Special case: $H \triangleleft G$. Then H acts trivially on G/H so

$$H^0_G((G/H)^{\nu+1}; M') = H^0_{G/H}((G/H)^{\nu+1}; M'^H)$$

and thus

$$H^i((G,H);M') = H^i(G/H, M'^H)$$

More generally

$$\boxed{\mathrm{Ext}^i_{\mathbb{Z}[G],\mathbb{Z}[H]}(M,M') = H^i(G/H, \mathrm{Hom}_{\mathbb{Z}[H]}(M,M'))}$$

This shows that H-split short exact sequences will give rise to long exact sequences of Exts.

Here's another interpretation of relative cohomology: The semi-simplicial G-set

$$\cdots\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; (G/H)^2 \rightrightarrows G/H$$

is the "nerve" of the covering $G/H \to pt$ in the category of G-sets, hence one has a spectral sequence

$$E_2^{pq} = H^p\{\nu \longmapsto H^q_G((G/H)^{\nu+1}; M')\} \Longrightarrow H^{p+q}(G;M')$$

so

$$E_2^{p0} = H^p((G,H);M')$$

i.e. the relative cohomology is the cohomology of the base for this spectral sequence.

So one sees the relative cohomology $H^i((G,H);M')$ is not simply-related to the localized cohomology constructed previously:

$$\begin{tikzcd}
 & & \varinjlim_n H^i(G, J(n)\otimes M) \\
H^i(G/H, M^H) \arrow[r] & H^i(G;M) \arrow[ur] & \\
\varinjlim_n H^i(G, \bar{J}(n)\otimes M) \arrow[u] & &
\end{tikzcd}$$

$$\bar{J}(n) : \cdots 0 \to \mathbb{Z}[(G/H)^n] \to \cdots \to \mathbb{Z}[G/H] \to 0 \cdots\cdots$$

~~It is clear that one has~~

$$\begin{tikzcd}
H^i(G;M) \arrow[r] \arrow[d] & \varinjlim H^i(G, J(n)\otimes M) \arrow[d] \\
H^i(H;M) \arrow[r] & \varinjlim H^i_G(G/H; J(n)\otimes M) = 0
\end{tikzcd}$$

~~which says nothing.~~

It's clear one has when $H \lhd G$

$$\varinjlim_n H^i(G/H, (J(n)\otimes M)^H) = \lim_{n\to\infty} H^i(G/H, J(n)\otimes M^H)$$

$$= \hat{H}^i(G/H, M^H)$$

Thus we get a commutative square:

$$\begin{tikzcd}
H^i(G/H, M^H) \arrow[r] \arrow[d] & H^i(G,M) \arrow[d] \\
\hat{H}^i(G/H, M^H) \arrow[r] & \varinjlim_n H^i(G, J_H(n)\otimes M)
\end{tikzcd}$$

How do we get the spectral sequence for the extension $H \to G \to G/H$? We take the complex

$$P_\cdot : \qquad \mathbb{Z}[(G/H)^2] \rightrightarrows \mathbb{Z}[G/H]$$

and form

$$\mathrm{Hom}(P_\bullet, M) =$$

$$\begin{array}{ccccc} \iota_*\iota^* M & \rightrightarrows & (\iota_*\iota^*)^2 M & \Rrightarrow & \cdots \\ \wr\!\wr & & \wr\!\wr & & \\ \mathbb{Z}[G/H]\otimes M & & \mathbb{Z}[(G/H)^2]\otimes M & & \text{if } (G:H)<\infty . \end{array}$$

Now filter this in a standard way to get the spectral sequence. Better Form the descent ~~spectral~~ spectral sequence using $J(n)\otimes M$.

$$E_2^{pq} = H^p\left(\nu \longmapsto H^q_G\left((G/H)^{\nu+1}; J(n)\otimes M\right)\right) \Rightarrow H^{p+q}_G(J(n)\otimes M)$$

So for a normal subgroup H we get the spectral sequence

$$\boxed{E_2^{pq} = \hat{H}^p(G/H; H^q(H,M)) \Rightarrow \varinjlim H^{p+q}(G, J_H(n)\otimes M)}$$

February 28, 1976

Change the theorem of page 22:

Theorem: Let $M \in D^+(G\text{-mods})$ and suppose [redacted] $\hat{H}^*_A(M) = 0$ for all abelian subgroups $A < G$. Then $\hat{H}^*_G(M) = 0$.

Proof: $E_2^{pq} = H^p(G|F, Gx \mapsto \hat{H}^q_{G_x}(M)) \Rightarrow \hat{H}^{p+q}_G(F, M)$ and $\hat{H}^*_G(M) \hookrightarrow \hat{H}^*_G(F; M)$. Here F[redacted] is the flag manifold of a faithful representation of G.

Example: Let G be a finite simple non-abelian group; [redacted] suppose G minimal simple, i.e. all proper subgroups are solvable. Let T be the poset of proper subgroups of G.

Lemma: If $0 < H < G$, then T^H is contractible.

Proof: If $K \in T^H$, then K is normalized by H, so KH is a subgroup containing K, H; but $KH < G$ otherwise K would be normal in G. Thus $K \leq KH \geq H$ so T^H is contractible by the cone construction.

[redacted] Fix $0 < H < G$. Then

$$T \supset \bigcup_{0 < H' \leq H} T^{H'}$$

and [redacted] I know the latter is contractible. Thus because H acts freely on the complement, we have

$$\chi(T) \equiv 1 \mod (|H|).$$

Since H is arbitrary we could take H to be each of the Sylow subgroups of G, hence we conclude

$$\chi(T) \equiv 1 \mod (|G|).$$

More generally, given any group G I can consider the poset T_G of non-zero solvable subgroups. The same argument shows that for any solvable subgroup $H \in T_G$, T_G^H is contractible, and again

$$\bigcup_{0<H'\leq H} T_G^{H'} \sim \{H' \mid 0<H'\leq H\} \sim pt$$

whence again we have that $\chi(T_G) \equiv 1 \mod |G_p|$ for all p, hence $\chi(T_G) \equiv 1 \pmod{|G|}$ for any finite group.

February 29, 1976 — Burnsides ring.

Let G be a finite group. The Burnside ring $A(G)$ of G is the Grothendieck group of the category of finite G-sets. It is a free $\mathbb{Z}$-module with basis $[G/H]$, where H runs over representatives for the different conjugacy classes of subgroups.

Let J denote the poset of transitive G-sets; say $[G/H] < [G/K]$ if $\exists$ map $G/K \to G/H$, i.e. if K is conjugate to a subgroup of H. Thus when we specialize G/K to G/H the isotropy group increases. The largest member of G is the free orbit class.

For each subgroup H of G we get a homomorphism

$$\lambda_H : A(G) \longrightarrow \mathbb{Z}$$

$$\lambda_H [X] = \operatorname{card}(X^H)$$

$$= \operatorname{card}\{\operatorname{Hom}_G(G/H, X)\}.$$

This last formula shows λ_H depends only on the conjugacy class of H.

<u>Grothendieck idea</u>: Let X be a G-polyhedra, and let $A_G(X)$ be the Grothendieck group of the category of constructible G-sheaves of sets on X

?

If X is a G-polyhedron, then we can associate a class $\gamma_X \in A(G)$ as follows

$$\gamma_X = \sum (-1)^p [X_p]$$

where X_p is the set of p-simplices of X. Note

$$\lambda_H(\gamma_X) = \sum (-1)^p \lambda_H [X_p] = \sum (-1)^p [X_p^H] = \chi(X^H).$$

Thus γ_X ~~keeps~~ track of the Euler characteristics of the different fixpt subspaces.

<u>Prop 1</u>: $A(G) \xrightarrow{\{\lambda_H\}} \prod_{H \in J} \mathbb{Z}$ is injective.

<u>Proof</u>: Observe for $H, K \in J$ (~~say~~ J is a specific family of subgps.)

$$\lambda_H(G/K) = \operatorname{card} \operatorname{Hom}_G(G/H, G/K)$$

$$= \begin{cases} (N(H):H) & K = H \\ 0 & \text{unless } H \to K \end{cases}$$

Choose a linear ordering of J compatible with the natural ordering. Then one sees the matrix

$$H, K \longmapsto \lambda_H(G/K)$$

is upper triangular for this ordering with non-zero diagonal entries.

<u>Prop 2:</u> $\operatorname{Spec}\left(\prod_J \mathbb{Z}\right) \twoheadrightarrow \operatorname{Spec} A(G)$

By C-Seidenberg the image is closed; it is dense because of prop. 1.

Fix a prime p. Then we know that

$$\operatorname{Spec}\left(\prod_J \mathbb{Z}/p\right) \longrightarrow \operatorname{Spec} A(G)\otimes \mathbb{Z}/p$$

is surjective. Observe that $N(K)/K$ acts freely on the right of G/K, hence also freely on the right of $(G/K)^H$. Thus $\lambda_H(G/K) \equiv 0 \mod (N(K)/K)$ for all H, which shows that all λ_H mod p kill G/K when $(N(K):K) \equiv 0 \pmod p$.

<u>Prop. 3:</u> $N(K)/K \equiv 0 \pmod p \iff [G/K]$ nilpotent in $A(G)/p$.

Let $J_p \subset J$ consist of $[G/K] \ni N(K):K \equiv 0 \pmod p$, and let J_p' be the complement. From the formula

$$\lambda_H(G/K) = \begin{cases} (N(H):H) & \text{if } H = K \\ 0 & \text{unless } H \to K \end{cases}$$

one sees that as H, K run over J_p' we get a triangular matrix with invertible diagonal entries mod p. Hence

<u>Prop. 4:</u> $(A(G)/p)_{\mathrm{red}} \xrightarrow{\sim} (\mathbb{Z}/p)^{J_p'}$.

Prop 5: If $H \triangleleft H'$ and H'/H is a p-group, then
$$\lambda_H \equiv \lambda_{H'} \pmod p.$$

Proof:
$$\chi(X^{H'}) = \chi((X^H)^{H/H'})$$
$$\equiv \chi(X^H).$$

So given a subgroup H, let P/H be a Sylow p subgroup of $N(H)/H$, whence $\lambda_p \equiv \lambda_H$ by the above. So it's clear that starting from H we can construct a chain

$$H = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_n$$

where H_i/H_{i-1} is a p-group, and where $N(H_n) : H_n$ is prime to p. Thus we see that $\lambda_H \equiv \lambda_{H'}$ where $H' \in J_p'$.

Notice also that starting from $K \in J_p'$ we have a largest quotient group K/K' which is a p-group. Take K to be H_n as above, and let i be least such that $H_i \supset K'$ but $H_{i-1} \not\supset K'$. Then put $K'' = K' \cap H_{i-1}$

$$\begin{array}{ccc} H_{i-1} & \triangleleft & H_i \\ \cup & & \cup \\ K'' & \triangleleft & K' \end{array}$$

Now K'' might not be normal in K, but if $g \in K$, then $K'' \cap gK''g^{-1}$ is normal in K'; the intersection of normal subgroups of p-power index is again a normal subgroup of p-power index. Thus it is clear that K'' must be K' because of the minimality of K'. $\therefore$ $K' \subset H$.

<u>Prop 6:</u> To each subgp H of G $\exists !$ $K \in J_p'$ such that $K' \subset H \subset K$ where K/K' is the largest p-quotient group of K.

For the uniqueness, see page 43.

When is $\mathrm{Spec}(A(G))$ connected?

<u>Prop 7:</u> If G is simple and $\neq 1$, then $\mathrm{Spec}(A(G))$ is not connected.

<u>Proof:</u> We have a surjection

$$\begin{array}{c} \mathrm{Spec}(\mathbb{Z}^J) \longrightarrow \mathrm{Spec}\, A(G) \\ \| \\ \mathrm{Spec}\,\mathbb{Z} \times J \end{array}$$

which is an isomorphism over all primes not dividing $|G|$. For $p \mid |G|$ we have to pinch together the layers belonging to the different groups in J_p'.

If G is simple, better if $H^1(G, \mathbb{Z}) = 0$, then consider the element $[G/G]$ of J corresponding to the subgroup G. It belongs to J_p' for all p and yet it does not get pinched to any other subgroup. Hence [redacted] the map $\mathrm{Spec}(\mathbb{Z}) \to$ [redacted] $\mathrm{Spec}\, A(G)$ corresponding to this subgroup is a connected components. QED.

[redacted]

[redacted] Observe that every finite group G has a maximal solvable quotient group. $G/D_\infty(G)$ For if G/K_1, G/K_2 are solvable, then

$$G/K_1 \cap K_2 \hookrightarrow G/K_1 \times G/K_2$$

and any subgroup of a solvable group is solvable. Next suppose H is a subgroup of G. If $N(H)/H \neq 1$ then $\exists\ H \triangleleft H_1$ with H_1/H abelian and non-trivial. Thus we get a chain

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n$$

such that $NH_n = H_n$. Put $K = H_n$ and let K/K' be the maximal solvable quotient group of K. [redacted] ~~[illegible]~~ Note that $K' = (K', K')$. If H_i contains K', then

$$K'/K' \cap H_{i-1} \hookrightarrow H_i/H_{i-1} \quad \text{abelian}$$

so $K' \cap H_{i-1} = K'$ i.e. $K' \subset H_{i-1}$. Thus $H \supset K'$.

Prop. 8: Let H be a subgroup of G. There exists a subgroup K such that $K = N(K)$ and $D_\infty(K) \subset H \subset K$.

Revise:

Prop. 9: For any subgroup H of G, there is a unique subgroup K of H such that

i) $K \trianglelefteq H$ and H/K is solvable (resp. a p-grp)

ii) K has no non-trivial solvable quotients (resp. no p-grp quotients)

iii) $H^1(K,\mathbb{Z}) = 0$ (resp: $H^1(K, \mathbb{Z}/p) = 0$).

Namely, you take H/K ~~largest~~ quotient group of H such that H/K is solvable (resp. a p-group), i.e. $K = D_\infty(H)$. Condition i) implies $K = D_\infty(H)$ so K is unique.

Uniqueness in prop 6: Let $K' \subset H \subset K$ where K/K' is the maximal p group quotient of K and where $N(K):K$ is prime to p. $N(K) \subset N(K')$, because K' is char. in K. Can find P/K' Sylow subgroup of $N(K')/K'$ containing K/K'. Then $K = P$ because otherwise $K \triangleleft K'' \subset P$ contradiction. Such a P is unique up to $\neq$ conjugacy.

Dress thm: $\pi_0\{\mathrm{Spec}\, A(G)\} \simeq$ conjugacy classes of subgps K with $H^1(K,\mathbb{Z}) = 0$

Proof: Call the set on the right J^*. Map

$$J \longrightarrow J^*$$

by sending H to $D_\infty(H)$. Thus we get a map

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Z}) \times J \arrow[rr] \arrow[dr] & & \mathrm{Spec}(\mathbb{Z}) \times J^* \\ & \mathrm{Spec}\, A(G) \arrow[ur, dashed] & \end{tikzcd}$$

which I claim factors as indicated, and moreover that the fibres over J^* of $\mathrm{Spec}\, A(G)$ are connected. Suppose then that two elements of J are connected at p. It is enough for the dotted arrow to show that if H/K is a p-group, then $D_\infty(K) = D_\infty(H)$ and this is clear. To see that the fibres are connected, it suffices to show that we can get from the section of $\mathrm{Spec}\, A(G)/\mathrm{Spec}\, \mathbb{Z}$ assoc. to H to the section assoc. to $D_\infty(H)$ by means of a sequence of connections at various primes. Thus I have only to show that $\exists$ a sequence

$$D_\infty(H) = H_n \triangleleft H_{n-1} \triangleleft \cdots\cdots \triangleleft H_0 = H$$

with H_i/H_{i+1} a p-group for different p. This is obvious as $H/D_\infty(H)$ is solvable.

Cor: $\mathrm{Spec}\{A(G)\}$ connected $\Longleftrightarrow$ G solvable

Suppose G - minimal simple. Then $\operatorname{Spec} A(G)$ has two components. The fixpt map

$$\lambda_G : A(G) \longrightarrow \mathbb{Z}$$

is a projection onto a direct factor. This means there is an idempotent e such that

$$\begin{cases} \lambda_G(e) = 1 \\ \lambda_H(e) = 0 & \text{all} \quad 0 \leq H < G . \end{cases}$$

In fact idempotence follows from these formulas and the fact that $A(G) \hookrightarrow \mathbb{Z}^{J}$. Working with $1-e$ then we can find a G-space X such that

$$(1) \qquad \begin{cases} \chi(X^H) = 1 & 0 \leq H < G \\ = 0 & H = G. \end{cases}$$

But in fact I have seen that the poset T of proper subgroups of G is a G-space such that

$$(2) \qquad \begin{array}{ll} T^H \sim pt & 0 < H < G \\ T^G = \phi & \end{array}$$

and I know that $\chi(T) \equiv 1 \pmod{|G|}$. Since

$$\chi\left(\left(G/e\right)^H\right) = \begin{cases} 0 & 0 < H \leq G \\ |G| & H = G \end{cases}$$

we can therefore add free gadgets to T to get a G-space X with (1). [redacted] Maybe Oliver proves we can [redacted] embed T in a contractible G-space X such that T−X is G-free.

<u>Question:</u> Can we generalize this construction to construct the different idempotents in $A(G)$?

[redacted] For example, let's assume G arbitrary and put T_G = non-zero solvable subgroups of G. If H is a subgroup non-solvable, I don't know anything about $(T_G)^H$. [redacted]

Following Oliver we define an ideal $\Delta(G) \subset A(G)$ as follows. It consists of $[X]-1$ where X is <u>contractible.</u> Note that if X is equivariantly contractible, then $\chi(X^H)=1$ for all H so $[X]=1$. Note from

$$X \longrightarrow \mathrm{Cone}(X) \longrightarrow \Sigma(X)$$

that [redacted] $[X]+[\Sigma(X)] =$ [redacted] 2 ?

Better [redacted]

$$[\Sigma X]-1 = [\Sigma X, pt] = [CX, X] = 1-[X]$$

e.g. if $X=\emptyset$ $\Sigma X = S^0$; $X=S^n$ $\Sigma X = S^{n+1}$

$$\therefore \quad [\Sigma X]-1 = -([X]-1).$$

Now $\Delta(G)$ clearly closed under $+$; replacing X by $\Sigma^2 X$ one can suppose $X^G \neq \phi$, whence wedge gives the desired operation.

Better: $\Delta(G)$ consists of all elements of the form $[X]-[X']$, where X, X' are G-spaces such that $\exists$ G-map $f: X \to X'$ which is a homotopy equivalence. Clearly this forms an ideal, and one has

$$[X]-[X'] = [\mathrm{Cone}(f)] - [1]$$

so that this definition is the same as Oliver's.

Problem: Describe $\mathrm{Spec}\{A(G)/\Delta(G)\} \subset \mathrm{Spec}\, A(G)$.

P.A. Smith thm. If X is acyclic mod p, then so is X^P where P is a p-group.

Let $\Delta_p(G)$ consist of $[X]-[X']$ where X, X' are G-spaces such that $\exists$ a G-map $f: X \to X'$ which is a $H^*(\ ,\mathbb{Z}/p)$-isomorphism. Again elements are of the form $[X]-1$ where X is a G-space which is acyclic modulo p. Notice that if X is acyclic mod p, then from the exact sequence

$$\cdots\cdots \xrightarrow{p} \tilde{H}_0(X,\mathbb{Z}) \longrightarrow \tilde{H}_0(X,\mathbb{Z}/p) \longrightarrow 0$$

and the fact that X is a finite complex, we conclude that $\tilde{H}_*(X,\mathbb{Z})$ is finite and of order prime to p. In particular $\tilde{H}_*(X,\mathbb{Q})=0$, so $\chi(X)=1$! So from the Smith theorem if P is a p-subgrp of G, then for X mod p acyclic we have

$$X \text{ acyclic mod } p \Longrightarrow X^P \text{ acyclic mod } p \Longrightarrow \chi(X^P)=1.$$

Therefore

$$\boxed{\lambda_P(\Delta_p(G))=0, \quad \text{for all } p\text{-groups } P}$$

<u>Corollary</u>: $\Delta_p(G)=0$ if G is a p-group, (hence $\Delta(G)=0$ also as $\Delta(G)\subset\Delta_p(G)$).

<u>Oliver theorem</u>: Suppose $\mathcal{F}$ is a family of subgroups containing the p-subgroups of G for all primes p, let X be a finite G-complex, and assume $\exists$ $\xi\in\Delta(G)$ such that

$$\lambda_H([X]-1)=\lambda_H(\xi) \qquad \forall\, H\notin\mathcal{F}$$

Then $\exists$ contractible finite G-complex $Y\supseteq X$ such that $[Y]-1=\xi$ and $\ni$ all isotropy groups of $Y-X$ are in $\mathcal{F}$.

In other words if you take $\mathcal{F}$ to be the all p-subgroups for different p, then you can prescribe the set

$$\tau(Y) = \{ y \in Y \mid G_y \text{ not a } p\text{-group} \}$$

, for a contractible Y, arbitrarily except for Euler characteristic considerations.

<u>Conjecture</u>: $\operatorname{Spec}\{A(G)/\Delta_p(G)\} = \operatorname{Spec}\mathbb{Z} \times \left\{ \begin{matrix}\text{conjugacy classes} \\ \text{of } p\text{-subgps.}\end{matrix} \right\}$ / pinched together over p.

= subset of $\operatorname{Spec} A(G)$ which is the union of the sections corresponding to the conjugacy classes of p-subgroups.

Thus $\sqrt{\Delta_p(G)} = \bigcap_{P \ p\text{-subgp}} \operatorname{Ker}(\lambda_P)$.

Idea: Can you carry out the following program?

a) G minimal simple (non-abelian) $\Longrightarrow \exists$ G-space $X \ni \begin{cases} X^H \sim pt & 0 \leq H < G \\ X^G = \phi \end{cases}$

This implies that $\Delta(G)$ contains $[X]-1$ which satisfies $\lambda_H(-[X]+1) = \begin{cases} 0 & 0 \leq H < G \\ 1 & H = G \end{cases}$

b) For G odd this is impossible, because $\Delta(G)$ has a certain structure.

If X^H is contractible or ϕ for each $H \subseteq G$, then the family $\mathcal{F}$ of H such that $X^H \sim pt$ corresponds to a division of

Spec $\{A(G)\}$; i.e. if K/H is solvable, then either H, K are both in $\mathcal{F}$ or both or not. Oliver calls such an $\mathcal{F}$ separating and he proves:

<u>Oliver's theorem 4</u>: If $\mathcal{F}$ is a separating family, then $\exists$ smooth action of G on a disk D such that

$$D^H = \phi \qquad H \notin \mathcal{F}$$

$$D^H = \text{disk} \quad H \in \mathcal{F}.$$

Notice that if X is ~~contractible~~ acyclic modulo p then so is X^P for any p-group P in G in particular $X^P \neq \phi$ for any p-subgroup. Hence there doesn't exist a contractible G-space having only only elementary abelian isotropy groups.

March 3, 1976

Let G be a finite group. Do we get an analogue of Whitehead simple homotopy theory by using G-complexes and [redacted] replacing simple homotopy equivalence with G-homotopy equivalence?

The objects should be contractible (finite) G-complexes, the morphisms G-homotopy equivalences.

Oliver's construction of attaching G-cells:

$$\begin{array}{ccc} G/H \times S^{i-1} & \xrightarrow{f} & X \\ \cap & & \cap \\ G/H \times D^{i} & \longrightarrow & Y \end{array} \qquad \text{cocart.}$$

f is determined by a map $S^{i-1} \longrightarrow X^H$. One can think of any G-polyhedron as being built up in this way. So there's a natural notion of G-complex: A CW complex on which G acts cellularly. [redacted]

It's clear that if $y \in Y$ is fixed by K, then either $y \in X^K$ or $y \in \left(G/H \times (D^i - S^{i-1})\right)^K = (G/H)^K \times e^i$. Thus we see that

$$\begin{array}{ccc} (G/H)^K \times S^{i-1} & \longrightarrow & X^K \\ \cap & & \cap \\ (G/H)^K \times D^{i} & \longrightarrow & Y^K \end{array}$$

is also cocartesian. So we get on homology

$$H_t(Y^K, X^K) = \begin{cases} 0 & t \neq i \\ \mathbb{Z}[(G/H)^K] & t = i \end{cases}$$

$$\longrightarrow H_i(Y^K) \longrightarrow \mathbb{Z}[(G/H)^K] \xrightarrow{\partial} H_{i-1}(X^K) \longrightarrow H_{i-1}(Y^K) \longrightarrow 0$$

here $i \geq 1$ say. This shows that this cell-attaching process decreases $H_{i-1}(X^K)$ and keeps $H_t(X^K)$ the same for $t < i-1$.

Suppose we look at the question of whether we can attach free G-cells to get an embedding X into a contractible complex. So we can certainly attach cells so as to embed $X \subset X'$ such that $\tilde{H}_t(X') = 0$, $t \neq n$ where $n = \max\{\dim X, 2\}$ and X' simply-connected. From then on any further attaching of free cells just adds free G-modules to $H_n(X')$. So we need to know $H_n(X')$ is stably-free in order to get a contractible Y. Since

$$0 \longrightarrow \tilde{C}_\cdot(X) \longrightarrow \tilde{C}_\cdot(X') \longrightarrow C_\cdot(X', X) \longrightarrow 0$$

and $C_\cdot(X', X)$ is made up of free $\mathbb{Z}[G]$-modules, this invariant we get depends only on $C_\cdot(X)$.

By the P.A. Smith theorem if Y is to exist, then for each prime p and p-subgroup $Q > 1$ of G we must have X^Q mod p acyclic. Conversely if this condition holds for a prime p dividing $|G|$, let G_p be a Sylow subgroup. Then

$$\bigcup_{0<Q\leq G_p} X^Q$$

is mod p acyclic in X, so as G_p-modules

$$\tilde{C}_\circ(X)_{(p)} \underset{\text{quis}}{\sim} C_\circ\left(X, \bigcup X^Q\right)_{(p)}.$$

Better

$$\tilde{C}_\circ(X')_{(p)} \underset{\text{quis}}{\sim} C_\circ\left(X', \bigcup_{0<Q\leq G_p} X'^Q\right)_{(p)} \qquad X'^Q = X^Q$$

and the last group is a complex of free $\mathbb{Z}_{(p)}[G_p]$-modules. It follows that $\tilde{H}_n(X')_{(p)}$ is $\mathbb{Z}_{(p)}[G_p]$-projective, hence $\mathbb{Z}_{(p)}[G]$-projective by transfer theory. Thus doing this for all p we see $\tilde{H}_n(X')$ is $\mathbb{Z}[G]$-projective. So

<u>Prop 1</u>: Let X be a G-complex such that for each non-trivial p-subgroup Q of G any prime p we have X^Q is mod p acyclic. Then there is an element of $\tilde{K}_0(\mathbb{Z}[G])$ which is an obstruction to embedding X into a contractible G-complex Y such that $X-Y$ is G-free. This obstruction is the class of the chain complex $\tilde{C}_\circ(X;\mathbb{Z})$ which is a perfect $\mathbb{Z}[G]$-module complex.

<u>Prop. 2</u>. If X^H is acyclic for $0 < H \leq G$, then the class of $\tilde{C}_*(X, \mathbb{Z})$ in $\tilde{K}_0(\mathbb{Z}[G])$ is zero.

<u>Proof</u>: I know $\bigcup_{0<H\leq G} X^H$ is acyclic, hence this follows from

$$0 \longrightarrow \underset{\substack{\wr \\ 0}}{\tilde{C}_*\left(\bigcup X^H\right)} \longrightarrow \tilde{C}_*(X) \longrightarrow \underset{\substack{\uparrow \\ G\text{-free}}}{\tilde{C}_*\left(X, \bigcup X^H\right)} \longrightarrow 0$$

Suppose one considers the set of all G-complexes X such that X^Q is mod p acyclic if Q is a non-trivial p-subgroup of G (all p). Restrict attention to ones with basepoint. Then one has operations of addition (wedge) negative (suspension).

Question: Let X be a G-complex. Can one find an embedding of X into a contractible G-complex Y such that all isotropy groups of $Y - X$ are p-subgroups (for different p)?

Let $\mathcal{F}$ be any family containing the family of p-subgroups of G for the different primes p. What Oliver shows is that if $\exists$ contractible G-space Z with $\chi(X^H) = \chi(Z^H)$ for all $H \notin \mathcal{F}$, then such a Y exists.

<u>Good question</u>: Let $\mathcal{F}$ be the family of p-subgroups of G for all p, let X be a G-complex. Does there exist a contractible G complex Y containing X such that $Y-X$ has isotropy groups in $\mathcal{F}$?

Recall $\Delta = \{ [Y]-1 \in A(G) \mid Y \sim pt \}$ and that

$$\begin{tikzcd} \Delta \subset \operatorname{Ker}\Big\{ A(G) \arrow[r, "\varphi"] & \prod\limits_{Q\in\mathcal{F}} \mathbb{Z} \Big\} \\ A_{\mathcal{F}}(G) = \coprod\limits_{Q\in\mathcal{F}} \mathbb{Z} \arrow[u, "\psi"] \arrow[ur, bend right, "\text{isom. over } \mathbb{Q}"'] & \end{tikzcd} \qquad \varphi = \{\lambda_Q\}$$

$$\psi(1_Q) = [G/Q]$$

Because $\mathcal{F}$ is a family one has an ideal $A_{\mathcal{F}}(G) \subset A(G)$ generated by $[G/Q]$ with $Q \in \mathcal{F}$, i.e. ~~one has~~

$$A_{\mathcal{F}}(G) = \{ [Z] \mid Z \text{ has isotropy groups in } \mathcal{F} \}$$

Now if the question above has answer yes, then

$$A(G) = \Delta + A_{\mathcal{F}}(G)$$

and this sum has to be direct since

$$A_{\mathcal{F}}(G) = \operatorname{Ker}\Big\{ A(G) \longrightarrow \prod_{Q\notin\mathcal{F}} \mathbb{Z} \Big\}$$

$$\Delta \cap A_{\mathcal{F}}(G) \subseteq \operatorname{Ker}\Big\{ A(G) \longrightarrow \prod_{Q\in\mathcal{F}} \mathbb{Z} \times \prod_{Q\notin\mathcal{F}} \mathbb{Z} \Big\} = 0.$$

So we would get an idempotent in $A(G)$ generating Δ. In fact if we take X to be empty, then we get a contractible complex Y with all isotropy groups in $\mathcal{F}$. This means that $\mathcal{F}$ is separating hence that all solvable subgroups of G are in $\mathcal{F}$, which is possible only if G is a p-group.

So ~~let's~~ see what goes wrong. Let X be a G complex, and let's try to enlarge it using cells $G/H \times e^i$ where $H \in \mathcal{F}$. I can assume that the homology of X is concentrated in dimension n if I want. Consider a maximal Q in $\mathcal{F}$ such that X^Q is not mod p acyclic, where p is the prime associated to Q. Then $N(Q)$ acts on X^Q so the thing to try maybe is to show that we can add things to X^Q.

Suppose that X^Q were mod p acyclic for all Q in $\mathcal{F}$ with $Q \neq 1$. Then I have an obstruction in $\tilde{K}_0(\mathbb{Z}[G])$. How to remove?

Here is Oliver's basic induction step:

Suppose $f: X \longrightarrow Y$ is a mod p homology isomorphism where Y has a basepoint y and all isotropy groups in $Y-y$ are p-groups. Take a maximal isotropy group H of $Y-y$ and attach cells $G/H \times e^i$ to $X^{(H)} = GX^H$ (note $X^H \to Y^H$ is a $H_*(\,, \mathbb{Z}/p)$ isom by the Smith theory) and attach the same cells to $Y^{(H)}$ to get $f: X_1 \to Y_1$ a $H_*(\; , \mathbb{Z}/p)$ isom such that $H_*(Y_1^H, \mathbb{Z}/p)$ is concentrated in one dimension n. Since $N(H)/H$ acts freely on $Y_1^H - y$, this forces $H_n(X_1^H, \mathbb{Z}/p) \simeq H_n(Y_1^H, \mathbb{Z}/p)$ to be stably free over NH/H, hence one can attach more $G/H \times e^i$ to $X_1^{(H)}$ and $Y_1^{(H)}$ to get another $\mathbb{Z}/p$-homology isom. $f: X_2 \to Y_2$ such that X_2^H and Y_2^H are $\mathbb{Z}/p$-acyclic. Then

$$Y_2^{(H)} = G \times_{N(H)} Y^H / G \times_{N(H)} \{y\}$$

is $\mathbb{Z}_p$-acyclic, so we can replace Y_2 by $Y_2 / Y_2^{(H)}$ to get a Y_3 with fewer orbit types than Y. Now use induction and you end up with an $X \subseteq X^\wedge$ with $X^\wedge$ $\mathbb{Z}_p$-acyclic and all isotropy gps of $X^\wedge - X$ equal to isotropy groups of $Y - y$.

March 5, 1976

Problem: Find a periodic K-theory $\hat{K}_*$ together with a map $K_* \longrightarrow \hat{K}_*$ which [redacted] is an isom. in high degrees. What does the fibre theory look like? Some kind of K-homology?

Suppose one [redacted] considers a finite field $\mathbb{F}_q$. Then a "periodic" version can exist only if q is [redacted] inverted. This is an opinion based on the long exact sequence

$$\cdots \longrightarrow K_i(\mathbb{F}_q) \longrightarrow K_iU(pt) \xrightarrow{\Psi^q - 1} KU_i(pt) \longrightarrow \cdots\cdots$$

It is clearly necessary that we invert q in order to obtain Adams operations in negative degrees. Calculate

$$\begin{aligned}
\hat{K}_i(\mathbb{F}_q) &= K_i(\mathbb{F}_q) && i > 0 \\
&= \mathbb{Z}[\tfrac{1}{q}] && i = 0, -1 \\
&= K_{-i-2}(\mathbb{F}_q) && i < -1
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow \mathbb{Z}[\tfrac{1}{q}] \xrightarrow{0} \mathbb{Z}[\tfrac{1}{q}] \to \hat{K}_{-1}\mathbb{F}_q \\
&\longrightarrow 0 \longrightarrow 0 \longrightarrow \hat{K}_{-2}\mathbb{F}_q \\
&\longrightarrow \mathbb{Z}[\tfrac{1}{q}] \xrightarrow{q^{-1}-1} \mathbb{Z}[\tfrac{1}{q}] \longrightarrow \hat{K}_{-3}\mathbb{F}_q
\end{aligned}$$

$$\begin{array}{ccccccc} \mathbb{Z}/q^2-1 & 0 & \mathbb{Z}/q-1 & \mathbb{Z}[\frac{1}{q}] & \mathbb{Z}[\frac{1}{q}] & 0 & \mathbb{Z}/q-1 \\ & & & 0 & -1 & -2 & \end{array}$$

This is what one would expect from a Atiyah-Hirz. spectral sequence.

$$\begin{array}{cccccccc} \mathbb{Z}/q^2-1 & 0 & \mathbb{Z}/q-1 & \mathbb{Z} & 0 & 0 & 0 & 0 \\ \downarrow s & & \downarrow s & \downarrow s & & & & \\ \mathbb{Z}/q^2-1 & 0 & \mathbb{Z}/q-1 & \mathbb{Z}[\frac{1}{q}] & \mathbb{Z}[\frac{1}{q}] & 0 & \mathbb{Z}/q-1 & 0 \\ & & & & & & & \\ 0 & 0 & 0 & \mathbb{Q}_p/\mathbb{Z}_p & \mathbb{Z}[\frac{1}{q}] & 0 & \mathbb{Z}/q-1 & 0 \end{array}$$

Try to construct $\hat{K}$ the way one does with Tate cohomology as some sort of inductive limit

Lundell construction. Bott maps.

$$\Sigma_1 U_n \hookrightarrow \mathrm{Grass}_n(\mathbb{C}^{2n}) \hookrightarrow \Omega U_{2n}$$

gives a map $\Sigma^2 U_n \longrightarrow U_{2n}$. Lundell shows this map factors through a map

$$\Sigma^2 U_n \longrightarrow U_{n+1}.$$

Thus he gets a commutative diagram

$$\begin{tikzcd} \Sigma^2 U_n \arrow[r] \arrow[d] & U_{n+1} \arrow[d] \\ \Sigma^2 U \arrow[r] & U \end{tikzcd}$$

One has

$$K^q(X) = \varinjlim_n \left[\Sigma^{-q+2n} X, BU\right]$$
$$= \varinjlim_n \left[\Sigma^{-q+2n+1} X, U\right]$$

Define

$$K_L^q(X) = \varinjlim_n \left[\Sigma^{-q+2n+1} X, U_n\right]$$

and note that if $X = pt$, then for q positive it is an isomorphism:

$$K_L^q(pt) \xrightarrow{\sim} K^q(pt) \qquad q \geq 2$$

For $q = 2$, we get $\varinjlim \pi_{2n-1}(U_n) = \mathbb{Z}$

$q = +1$, we get $\varinjlim \pi_{2n}(U_n) = \varinjlim \mathbb{Z}/n! = \mathbb{Q}/\mathbb{Z}$?

So we have to review:

$$\begin{tikzcd} 0 \arrow[d] & & & & \\ \pi_{2n+1}(U_{n+1}) \arrow[r] \arrow[d, "\cong"] & \pi_{2n+1}(S^{2n+1}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_{2n}(U_n) \arrow[r] & \pi_{2n}(U_{n+1}) \arrow[r] & \pi_{2n}(S^{2n+1}) \arrow[d, phantom, "\shortparallel"] \\ \pi_{2n+1}(U_{n+2}) = \mathbb{Z} \arrow[d] & \mathbb{Z} & & & 0 \\ \pi_{2n+1}(S^{2n+3}) = 0 & & & & \end{tikzcd}$$

and we know $U_{n+1} \underset{\mathbb{Q}}{\sim} S^1 \times S^3 \times \cdots \times S^{2n+1}$

so we see that $\pi_{2n}(U_n)$ is cyclic.

We can interpret elements of $\pi_{2n+1}(U_{n+1}) = \pi_{2n+2}(BU_{n+1})$

as $(n+1)$-dimensional bundles over S^{2n+2}. The map $\pi_{2n+1}(U_{n+1}) \to \pi_{2n+1}(S^{2n+1})$ is probably the Euler class of this bundle. Bott proved the Euler class was $n!$ so

$$\pi_{2n}(U_n) = \mathbb{Z}/n!\,.$$

Now according to Lundell the homotopy groups of K_L^0 are

$$\begin{array}{cccccc}
 & & & \overset{\deg 0}{\downarrow} & & \\
\mathbb{Q}/\mathbb{Z} & 0 & \mathbb{Q}/\mathbb{Z} & 0 & \mathbb{Q}/\mathbb{Z} & \mathbb{Z} \\
 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
 & \mathbb{Q} & & \mathbb{Q} & &
\end{array}$$

and that the cofibre is the connected theory associated to $K^0 \otimes \mathbb{Q}$, i.e.

$$F^0(X) = \prod_{i \geq 0} H^{2i}(X, \mathbb{Q})$$

$$F^{2j}(X) = \prod_{i \geq j} H^{2i}(X, \mathbb{Q}) \qquad \text{etc.}$$

Serre's example: First let us consider the universal example of a bundle E of rank n such that $E + 1$ is trivialized. Thus I wish to consider over a ring A the set of pairs (s, p) where

$$A \underset{s}{\overset{p}{\rightleftarrows}} A^{n+1} \qquad ps = id_A$$

Locally for the Zariski topology on A, $\operatorname{Ker} p$ is $\simeq A^n$, so GL_{n+1} acts transitively on the set of pairs (s,p) locally. Thus the universal example occurs over

$$GL_{n+1}/GL_n$$

which has coordinate ring $\mathbb{Z}[X_1,\dots,X_{n+1},Y_1,\dots,Y_{n+1}]/(\sum X_iY_i=1)$.
By topology

$$GL_{n+1}/GL_n \sim U_{n+1}/U_n = S^{2n+1}.$$

Furthermore $GL_{n+1} \longrightarrow GL_{n+1}/GL_n$ is $\sim$ the projection map:

$$U_{n+1} \longrightarrow S^{2n+1}.$$

This is known not to have a section for $n \geq 2$. The principal bundle $U_n \to U_{n+1} \to S^{2n+1}$ is classified by the generator of $\pi_{2n}(U_n) \simeq \mathbb{Z}/n!$.

I can map $\pi_{2n}(U_n) \longrightarrow \pi_{2n+2}(U_{n+1})$

~~to an n-dimensional E bundle over S^{2n+1}~~

how?

Let $\mathcal{F}$ be a family of subgroups of a finite group G. Let S be a finite G-set such that $\mathcal{F}$ is the set of isotropy groups of S. Let us consider the functor

$$i : (S, G) \longrightarrow (pt, G).$$

Then we have a standard resolution

$$\longrightarrow (i_! i^*)^2 \mathbb{Z} \longrightarrow i_! i^* \mathbb{Z} \longrightarrow \mathbb{Z}$$

Let $J_S(n)$ denote the truncated complex

$$0 \longrightarrow (i_! i^*)^n \mathbb{Z} \longrightarrow \cdots\cdots \longrightarrow i_! i^* \mathbb{Z} \longrightarrow 0$$

and let $\tilde{J}_S(n)$ denote the complex

$$0 \longrightarrow (i_! i^*)^n \longrightarrow \cdots \longrightarrow i_! i^* \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

so that we have an exact sequence of complexes

$$0 \to \mathbb{Z} \to \tilde{J}_S(n) \longrightarrow \Sigma J_S(n) \longrightarrow 0$$

Let M be a complex of G-modules which is bounded below (cochain like). Since we have the formula

$$(i_! i^* M) = i_! i^* \mathbb{Z} \otimes M$$

for all G-modules M, it is clear that the

$$J_S(n) \otimes M \longrightarrow M \longrightarrow \tilde{J}_S(n) \otimes M \longrightarrow \Sigma \bar{J}_S \otimes M$$

one sees M is a retract of $J_S(n) \otimes M$.

<u>Prop. 2</u>: Let M be a complex satisfying the equivalent condition of Prop. 1. Then for any M'

$$[M, M'] = \lim [M, J_S(n) \otimes M']$$

complex $J_S(n) \otimes M$ has a filtration with quotients of the form $i_! N$.

<u>Proposition 1.</u> TFAE

(i) The ind-object $n \mapsto \tilde{J}_S(n) \otimes M$ in $D^+(G)$ is essentially zero.

(ii) M is a retract of a complex which has a filtration with quotients of the form $i_! N$

(iii) $J_S(n) \otimes M \to M$ has a section.

(ii) $\Rightarrow$ (i) Suffices to show that $\tilde{J}_S(n) \otimes i_! N = i_! (i^* \tilde{J}_S(n) \otimes N)$ is essentially zero, which is clear because $i^* \tilde{J}_S(n)$ is homotopy-equivalent to a complex concentrated in degree n.

(i) $\Rightarrow$ (ii) If $M \longrightarrow \tilde{J}_S(n) \otimes M$ is the zero map, then from the triangle

$$J_S(n) \otimes M \longrightarrow M \longrightarrow \tilde{J}_S(n) \otimes M \longrightarrow \Sigma J_S \otimes M$$

one sees M is a retract of $J_S(n) \otimes M$.

<u>Prop. 2</u>: Let M be a complex satisfying the equivalent condition of Prop. 1. Then for any M'

$$[M, M'] = \varinjlim [M, J_S(n) \otimes M']$$

<u>Proof</u>: It suffices

to show $\varinjlim_{n} [M, \tilde{J}_S(n) \otimes M'] = 0$, and to do this for $M = i_! N$. Since

$$[i_! N, \tilde{J}_S(n) \otimes M'] = [N, i^*(\tilde{J}_S(n) \otimes M')]$$

and $i^* \tilde{J}_S(n)$ is essentially zero one wins.

So what we've done is to define in Prop. 1 a subcategory $D^+_S(G)$ of $D^+(G)$ closed under extensions and to show that for any M there is an ind-object $n \mapsto J_S(n) \otimes M$ in $D^+_S(G)$ which is universal for maps of objects of $D^+_S(G)$ to M.

Let me now wor