Feb. 1, 1976

[redacted]

[redacted]

Let A be a regular local ring of dimension d, $X = \operatorname{Spec} A$, $U = \operatorname{Spec} A - \{m\}$, $j : U \to X$ the inclusion. M a finite type A-module

$$0 \to H^0_m(M) \to M \to j_*j^*M \to H^1_m(M) \to 0$$

$$R^q j_* j^*M \xrightarrow{\sim} H^{q+1}_m(M) \qquad q \geq 1.$$

Thus $M \xrightarrow{\sim} j_*j^*M \iff H^0_m(M) = H^1_m(M) = 0$

Recall depth(M) = least p ∋ $H^p_m(M) \neq 0$

or $\operatorname{Ext}^p(A/m, M) \neq 0$

or ∃ $t_1, \ldots, t_p \in m$ regular seq. for M.

Thus $M \xrightarrow{\sim} j_*j^*M \iff$ depth$(M) \geq 2$. Recall also since A is regular that

$$\text{depth}(M) + \text{proj.dim } A = \dim A$$

and the local duality thm.

$$H^p_m(M) = \operatorname{Hom}(\operatorname{Ext}^{d-p}(M, A), \underset{\substack{\parallel \\ \text{dualizing module}}}{H^n_m(A)})$$

If $\dim A = 2$, then $\operatorname{depth}(M) \geq 2 \Rightarrow \operatorname{depth} M = 2$ so M is projective over A. Thus any fin. type A module M which is of the form j_*F for some $\mathcal{O}_U$ module ($\Rightarrow$ can take F to be j^*M) is free over A.

I want to have conditions on a $\mathcal{O}_U$-module F (quasi-coherent) which guarantee that j_*F is of finite type over A. This should be true for any vector bundle F, and more generally if the depth of F at the points of U is big enough.

Proof for a vector bundle. $j_*F^\vee$ is quasi-coherent hence it is an ind. limit of its fin. type submodules M_i $\Rightarrow$ $F^\vee = \varinjlim j^*M_i$ and since $F^\vee$ is coherent $\Rightarrow$ $F^\vee = j^*M_i$ $\Rightarrow$ can find $A^n \longrightarrow j_*F^\vee$ such that the corresp. map $\mathcal{O}_U^n \twoheadrightarrow F^\vee$ is onto. But then taking duals we get an injection $F \hookrightarrow \mathcal{O}_U^n$

hence $j_*F \hookrightarrow A^n$ and so j_*F is of finite type. (We've used that $j_*(\mathcal{O}_U) = A$ i.e. $\operatorname{depth} A \geq 2$.)

Note that if $M = j_*F$ is of depth ≥ 2 and locally free over U, then the modules $\operatorname{Ext}^q(M, A)$ for $q \neq 0$ are of finite length. Hence also the cohomology groups $H^q_m(M)$ are of finite length for $q < d$, zero for $q = 0, 1$.

The problem to examine is the following. I am given a vector bundle N over A_z where z is a non-zero element of A, say part of a system of parameters. I would like to find an extension of N to a vector bundle on A.

Question: Suppose I give an extension of N to the generic point of $z = 0$. Precisely if $\mathfrak{p} = Az$, so that $A_\mathfrak{p}$ is a d.v.r., I give a lattice Λ for $A_\mathfrak{p}$ in the vector space $N_\mathfrak{p}$ (note $A_{z,\mathfrak{p}} = A_{(0)}$ = quotient field of A). Does this determine an extension of N to a coherent sheaf on A.

The idea is the following: Let M be an A-submodule of N which is finitely generated and such that $M_z \simeq N$. Assume also that $M_\mathfrak{p} = \Lambda$ as $A_\mathfrak{p}$ submodules of $N_\mathfrak{p}$. Replace M by j_*j^*M; this doesn't affect it over A_z or $A_\mathfrak{p}$. Then it should be the case that

$$0 \longrightarrow M \longrightarrow N \times \Lambda \longrightarrow (N_\mathfrak{p} = \Lambda_z)$$

is exact, whence Λ determines M.

Problem: A reg. local of $\dim = d \geq 2$, K its quotient field. Suppose for each p of ht 1 in A I give a lattice Λ_p for A_p in K^n such that $\Lambda_p = A_p^n$ for almost all p. The problem is to show that $M = \cap \Lambda_p$ is a finite type A-module such that $M_p = \Lambda_p$ for all p of height, and to characterize the modules M so obtained.

If $d = 2$, then $\operatorname{Spec} A - \{m\}$ is a regular scheme of dim. $= 1$, hence the family of Λ_p is the same as a vector bundle E over $U = \operatorname{Spec} A - \{m\}$ equipped with an embedding into K^n. Then $M = \Gamma(U, E)$ is a free finite type A-module, for its depth is ≥ 2.

In general because most Λ_p are equal to A_p^n we can find an element $f \neq 0$ in A such that $\Lambda_p \subset \frac{1}{f} A_p^n \subset K^n$ for all p of ht. 1. Thus

$$M \subset \cap \tfrac{1}{f} A_p^n = \tfrac{1}{f} \cap A_p^n = \tfrac{1}{f} A^n$$

and so M will be finitely generated over A. Since each element of K is in almost all A_p, each element of K^n is in almost all Λ_p so by defn. of M we have an exact sequence

$$0 \longrightarrow M \longrightarrow K^n \longrightarrow \oplus K^n/\Lambda_p$$

Localizing with respect to a given p then shows (since $(A_q)_p = K$ for $q \neq p$, q of ht. 1) that $M_p = \Lambda_p$ for all p.

Note that the modules K, K/A_p are injective, hence so are K^n, K^n/Λ_p. Thus

$$0 \longrightarrow M \longrightarrow K^n \longrightarrow \bigoplus K^n/\Lambda_p$$

is the beginning of an injective resolution for M. So if q is a prime ideal of ht ≥ 2, we have

$$\mathrm{Hom}\,(A/q, K^n) = 0$$

$$\mathrm{Hom}\,(A/q, K^n/\Lambda_p) = 0$$

so therefore

$$\mathrm{Hom}(A/q, M) = \mathrm{Ext}^1(A/q, M) = 0$$

which implies that

$$\mathrm{depth}_q(M) \geq 2 \qquad \text{if} \quad ht(q) \geq 2.$$

More generally if Z is any closed subset of Spec$A$ and if F is a finite type A-module with support Z, then

$$\mathrm{Ext}^i_A(F, M) = 0 \qquad i = 0, 1$$

i.e. $\mathrm{depth}_Z(M) \geq 2$ if $\mathrm{cod}(Z) \geq 2$, i.e. $\dim(Z) \leq d-2$

Next thing to note is that conversely if $\mathrm{depth}_q(M) \geq 2$ at points of codim ≥ 2, then for any closed set Z of codimension ≥ 2 we have

$$\begin{tikzcd} 0 \arrow[r] & H^0_Z(M) \arrow[r] \arrow[d, equal] & M \arrow[r] & \Gamma(X-Z, M) \arrow[r] & H^1_Z(M) \arrow[r] \arrow[d, equal] & 0 \\ & 0 & & & 0 & \end{tikzcd}$$

so it should be the case that M is determined by M_p for $ht(p) = 1$.

[redacted] (Use the Cousin [redacted] complex construction. This gives us a resolution

$$0 \longrightarrow M \longrightarrow M \otimes K \longrightarrow \bigoplus_{ht(p)=1} H^1_p(M) \longrightarrow$$ [redacted] $$Q \longrightarrow 0$$

where [redacted] $\operatorname{codim}(Q) \geq 2$.)

Claim now I can prove that $\operatorname{depth}_q(M) \geq 2$ for $\operatorname{cod}(q) \geq 2$ [redacted] implies

$$0 \to M \to M \otimes K \to \bigoplus_{ht(p)=1} M \otimes K / M \otimes A_p$$

is exact. First of all, M is [redacted] locally free near points of codim 2, hence at points of codim [redacted] ≤ 2, hence it is torsion-free. Next suppose $\xi \in M \otimes K$ lies in $M \otimes A_p$ for all p of ht 1. [redacted] The set V of points q in Spec A where $\xi \in M_q$ is open (where ξ vanishes in [redacted] $M \otimes K/M$) and it includes all points of codim ≤ 1. [redacted] Thus $\operatorname{Cod}(X - V) \geq 2$ and as $M \xrightarrow{\sim} \Gamma(V, M)$ one must have $\xi \in M_q$ for all q.

Next let N be a finite type A-module, and $M = \operatorname{Hom}(N, A)$. Since

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(N, A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(N, K) \arrow[r] \arrow[d, equal] & \bigoplus_{ht(p)=1} \mathrm{Hom}(N, K/A_p) \\
& M \arrow[r] & M \otimes K &
\end{tikzcd}$$

exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(N, A_p) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(N, K) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(N, K/A_p) \\
& M_p \arrow[r] & M \otimes K &
\end{tikzcd}$$

exact

One sees that

$$0 \longrightarrow M \longrightarrow M \otimes K \longrightarrow \bigoplus_{ht(p)=1} M \otimes K / M_p$$

is exact, so one sees that M is in the good class. Thus M reflexive $\Rightarrow$ M good. Conversely if M is in the good class, the map

$$M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, A), A)$$

will be an isomorphism, for it becomes an isomorphism at all the codim 1 points. So

<u>Proposition:</u> A reg. local ring of dim ≥ 2, M f.t. A-mod. TFAE:

i) $0 \longrightarrow M \longrightarrow M \otimes K \longrightarrow \bigoplus_{ht(p)=1} M \otimes K / M \otimes A_p$ exact

ii) $\mathrm{depth}_q(M) \geq 2$ if $\mathrm{cod}(q) \geq 2$

iii) M reflexive: $M \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Hom}(M, A), A)$

2 February, 1976:

A local ring. Does Max(Spec $A[T]$) have dimension 1 ?

What are maximal ideals in $A[T]$ like?

Let $\mathfrak{m}$ be a maximal ideal in $A[T]$. $A \cap \mathfrak{m} = \mathfrak{p}$ is a prime ideal in A. Suppose $\mathfrak{p} = 0$. Then if t = image of T in $A[T]/\mathfrak{m} = K$ we have

$$K = A[t]$$

where A is a domain. Let F be the quotient field of A. Then $K = F[t]$, so t has to be algebraic over F, hence K is a finite extension of F.

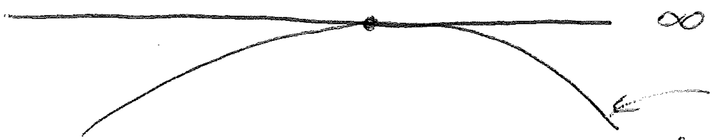
Let $d = [K:F]$. First suppose $d = 1$, where F is obtained by adjoining $t \in F$ to A. $t = b/a$ where $b, a \in A - \{0\}$. $\therefore F = A[\frac{1}{a}]$. Thus for any $x \in A - \{0\}$, $a^n \frac{1}{x} = y \in A$ for n large so $xy = a^n$. ?

~~Geometric picture: Take a point x of $\mathbb{P}^1_A$ which is in the open set $\mathbb{A}^1_A$ which means that $z \in T^{-1}$ is not in the maximal ideal at x. $z \in \Gamma(\mathbb{P}^1_A, \infty\text{section}, \mathcal{O})$. I think of $\mathcal{O}_x$ as being a~~

Geometric picture. I can think of primes $\mathfrak{p}$ in $A[T]$

as being points x in $\mathbb{P}^1_A$ not contained in the infinity section. $\mathfrak{p}$ is a maximal ideal iff all points $y \in \overline{\{x\}} - \{x\}$ are in the infinity section. Assume $\mathfrak{p}$ maximal, then x is not in the zero section, ~~so~~ we can identify $\mathfrak{p}$ with a prime ideal $\mathfrak{q}$ in $A[Z] = A[T^{-1}]$ such that $z \notin \mathfrak{q}$ and such that z belongs to any prime ideal $\mathfrak{q}' > \mathfrak{q}$.

It is clear that Max Spec $A[T]$ is not of dimension 1 because sitting inside m Spec$(A[T])$ are lots of irreducible sets arising from curves on subvarieties approaching ∞.



this is a surface, now take all curves transversal to the intersection of the curve with ∞.

Idea: [redacted] Suppose that S is a divisor in the smooth variety X, [redacted] and that N is a vector bundle on $X-S$ which locally extends to X. Then I felt that one gets [redacted] a homotopy obstruction to a global extension. But a basic fact is that the homotopy type of X in the Zariski topology is trivial, so maybe this problem admits a solution.

[redacted] $X = \mathrm{Spec}(A) = X_{f_0} \cup X_{f_1}$. Given M_i over A_{f_i} $\ni$ $(M_i)_z = N_{f_i}$. I can assume if I want to that $M_0 \simeq A^r_{f_0}$. [redacted] On the overlap we have two lattices. [redacted] Consider the case where $M_1 \simeq A^r_{f_1}$. Then [redacted] I can assume always that $(M_0)_{f_1} \subset (M_1)_{f_0}$, so I obtain a map

$$A^r_{f_0 f_1} \hookrightarrow A^r_{f_0 f_1}$$

with cokernel killed by z^e some e. In fact the determinant of this matrix should be z^e where e is the length of the cokernel at the generic pt of z. Conversely such a matrix will give me an extension [redacted] problem of the type under consideration.

Horrocks paper:

A reg. local ring of dim $d \geq 2$. E vector bundle over $Y = \operatorname{Spec} A - \{\mathfrak{m}\}$, $M = \Gamma(Y, E)$. Thus M is of depth ≥ 2 (proj dim $\leq n-2$) and locally free off $\mathfrak{m}$. [redacted] Form a minimal resolution of M:

$$0 \longrightarrow F_s \longrightarrow \cdots\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Then

$$0 \longrightarrow M^\vee \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow \cdots \longrightarrow F_s^\vee \longrightarrow 0$$

is a complex with homology groups $\operatorname{Ext}^i(M, A)$ [redacted] $1 \leq i \leq d-2$ which are of finite length. The claim is that this complex determines M up to a direct factor with a free module.

e.g. take $d = 3$, whence we have

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

$$(*) \qquad 0 \longrightarrow M^\vee \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow \operatorname{Ext}^1(M, A) \longrightarrow 0$$

exact

suppose M has no free direct summands, i.e. $\forall$ $\lambda \in F^\vee$, $\lambda(M)$ is contained in $\mathfrak{m}$, or

$$\begin{tikzcd} \underset{\check{M}}{\operatorname{Hom}(M, A)} \arrow[r] \arrow[d, hook] & \operatorname{Hom}(M, k) \arrow[d, "\wr"] \\ \operatorname{Hom}(F_0, A) \arrow[r] \arrow[d, phantom, "\parallel"] & \operatorname{Hom}(F_0, k) \arrow[d, phantom, ""] \\ F_0^\vee & F_0^\vee \otimes k \end{tikzcd}$$

is 0

i.e. $\check{M} \longrightarrow \check{F}_0 \otimes k$ is zero. Then $(*)$ is [redacted] part of a minimal resolution for the finite length module $\mathrm{Ext}^1(M, A)$. Thus [redacted] there is a 1-1 correspondence between bundles on Y without $\mathcal{O}_Y$ factors and torsion A-modules for $d=3$.

The question now is to allow the bundle E on Y to vary on the divisor Z and to see how this affects the associated torsion module. So suppose we have an exact sequence of sheaves over Y

$$0 \longrightarrow E' \longrightarrow E \longrightarrow T \longrightarrow 0$$

where E and E' are vector bundles on Y and where T is a [redacted] $\mathcal{O}_Y$ module of [redacted] projective dim. 1 having support in $Z \cap Y$.

$$0 \longrightarrow j_*E' \longrightarrow j_*E \longrightarrow j_*T \longrightarrow R^1j_*(E') \longrightarrow R^1j_*(E) \longrightarrow$$

[redacted] Z [redacted] $= \mathrm{Spec}(A/zA)$ is the spectrum of a regular local ring of dimension 2. Assume T is killed by z. Then it should be the case that T is locally free over $\mathcal{O}_{Z \cap Y}$, hence trivial; which implies that

$$j_*(T) \cong (A/zA)^r$$

$R^2j_*(E') \simeq H^3_m(M')$, $R^2j_*(E) \simeq H^3_m(M)$. [redacted]

Consider just the sequence

$$0 \longrightarrow E \xrightarrow{z} E \longrightarrow E/zE \longrightarrow 0$$

Since $R^1 j_*(E) = H^2_m(M)$, we know that multiplying by z will kill elements of $H^2_m(M)$, so consequently we know that M doesnot map onto $j_*(E/zE)$.

Suppose $zE \subset E' \subset E$, and suppose E' is better than E in the sense that the length of $H^2_m(M')$ is smaller.

$$H^2_m(M) \longrightarrow H^2_m(M') \longrightarrow H^2_m(M).$$

~~It should be true that things would~~

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "z"] & M \arrow[r] & j_*(E/zE) \arrow[r] \arrow[d] & R^1 j_*(E) \arrow[r, "z"] \arrow[d, equal] & R^1 j_*(E) \\
 & & 0 \arrow[r] & H^1_m(M/zM) \arrow[r] \arrow[d] & H^2_m(M) \arrow[r, "z"] & H^2_m(M) \\
 & & & 0 & &
\end{tikzcd}$$

I know that $j_*(E/zE) \simeq (A/zA)^r$. Suppose $E' \subset E$ such that $j_*(E/E') = A/zA$. Then

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & j_*(E/E') \arrow[r] \arrow[d] & R^1 j_*(E') \arrow[r] \arrow[d, equal] & R^1 j_*(E) \arrow[d, equal] \\
 & & 0 \arrow[r] & H^1_m(M/M') \arrow[r] & H^1_m(M') \arrow[r] & H^1_m(M)
\end{tikzcd}$$

Question: Can I arrange for E' to be better than E.

[illegible]

Basic exact sequence:

$$0 \longrightarrow M/zM \longrightarrow j_*(E/zE) \longrightarrow \operatorname{Ker}\{H^2_m(M) \circlearrowright z\} \longrightarrow 0$$

When can I find an epimorphism $M/zM \twoheadrightarrow A/zA$? If I can find such an epim., I get $M' \subset M$ such that $M/M' \simeq A/zA$, hence $H^1_m(M/M') = 0$ so $H^2_m(M') \hookrightarrow H^2_m(M)$, and this map cannot be an isomorphism if M has no free factors. Thus M' is an improvement over M.

Conversely suppose $M' \subset M$ with depth $M' \geq 2$, and assume $H^2_m(M') \subset H^2_m(M)$. Then from

$$\begin{tikzcd}
 & & 0 \arrow[d, phantom, "\parallel"] & & & \\
 & H^1_m(M') \arrow[r] & H^1_m(M) \arrow[r] & H^1_m(M/M') \arrow[r] & H^2_m(M') \arrow[r] & H^2_m(M) \\
0 \arrow[r] & H^0_m(M') \arrow[r] & H^0_m(M) \arrow[r] \arrow[d, phantom, "\parallel"] & H^0_m(M/M') \arrow[ul, bend right] & & \\
 & & 0 & & &
\end{tikzcd}$$

we see that the depth of M/M' is ≥ 2, which means it is of projective dimension 1 over A. If z kills $N = M/M'$, then writing $N = F_0/F_1$ we have

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_1^A(A/z, N) \longrightarrow F_1/zF_1 \longrightarrow F_0/zF_0 \longrightarrow N \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_1^A(A/z, N) \longrightarrow N \xrightarrow[0]{z} N \xrightarrow{=} N/zN \longrightarrow 0$$

hence we get an exact sequence

$$0 \longrightarrow N \longrightarrow F_1/zF_1 \longrightarrow F_0/zF_0 \longrightarrow N \longrightarrow 0$$

showing N is projective over A/zA when A/zA is regular.

So we consider this question. Go back to the basic sequence. Then $j_*(E/zE) \simeq (A/zA)^r$. We can find a surj. $M/zM \to A/zA \iff M/zM \to (A/zA)^r \otimes k = (A/m)^r$ is non-zero, i.e.

$$\dim \mathrm{Ker}\{H^2_m(M) \circlearrowleft z\} \otimes_A k < r$$

Side question: One sees from the basic exact sequence that if a torsion A-module T arises from a bundle E over Y, then for any $z \in m - m^2$

$$\dim \mathrm{Ker}\{T \circlearrowleft z\} \otimes_A k \leq \mathrm{rank}(E).$$

~~Because Serre's thm. implies we can assume T is of dim 2~~

I should check this. Given E, one has that E/zE is a vector bundle over $Y \cap Z = \mathrm{Spec}\{A/zA\} - \{m\}$, hence $\Gamma(Y\cap Z, E/zE) = \underset{\overset{\|}{j_*(E/zE)}}{\Gamma(Y, E/zE)}$ is $\simeq (A/zA)^r$ because $\dim(A/zA) = 2$. Thus

$$0 \longrightarrow j_*E \xrightarrow{z} j_*E \longrightarrow j_*(E/zE) \longrightarrow R^1j_*(E) \xrightarrow{z} R^1j_*(E)$$

is exact, and we get an epim

$$j_*(E/zE) \twoheadrightarrow \mathrm{Ker}\{z \text{ on } R^1j_*(E)\} = {}_z(R^1j_*(E))$$

It follows that ${}_z[R^1j_*(E)]$ has $\leq r$ generators, i.e.

$$\dim {}_z(R^1j_*(E)) \otimes_A k \leq \mathrm{rank}(E).$$

Now for the problem I am concerned with, I start with a bundle over A_z which is a regular ring of dimension 2, so by Serre's thm. I can suppose the rank of $M = 2$.

In fact I start over A_z with the cokernel N of a unimodular vector say

$$0 \longrightarrow A_z \longrightarrow A_z^3 \longrightarrow N \longrightarrow 0$$

which I can then extend to a sequence

$$0 \longrightarrow A \longrightarrow A^3 \longrightarrow M \longrightarrow 0$$

Therefore

$$0 \longrightarrow M^\vee \longrightarrow A^3 \longrightarrow A \longrightarrow \mathrm{Ext}^1(M,A) \longrightarrow 0$$

so if I replace M by $M^\vee$ I am looking at an ideal in A primary for m with 3 generators. But $\dim(A) = 3$ so these three generators form a regular sequence!

Check this: A_z is regular of dimension 2, and $K_0(A_z) \twoheadleftarrow K_0(A) = \mathbb{Z}$. Thus any bundle N over A_z is stably-free. But if $\mathrm{rank}(N) \geq DS(A_z) \leq 2+1=3$, then N **is** free. So I can suppose $\mathrm{rank}(N) \leq 2$ and $A_z \oplus N$ is free. Choose $A_z \oplus N \simeq A_z^3$, whence we get an exact ~~sequence~~

~~$0 \to A_z \to A_z^3 \to N \to 0$~~

$$0 \longrightarrow N \longrightarrow A_z^3 \longrightarrow A_z \longrightarrow 0$$

Clearing denominators we can suppose this is the localization with respect to z of a sequence

$$0 \longrightarrow M \longrightarrow A^3 \xrightarrow{u} A$$

~~The image of u is an ideal I in A such that $I_z = A_z$, hence $A_z{}^m \subset I \subset A$ for some m. $I \underset{A}{\otimes} A_{(z)} = z^k A_{(z)}$ so $I \subset z^k A_{(z)} \cap A = z^k A$. Thus replacing A by $z^{-k} A$, we can suppose $I \otimes A_{(z)} = A_{(z)}$. It follows that $I_p = A_p$ at all primes p not containing z and also for $p = (z)$~~

It follows that M is of depth ≥ 2, hence it is locally free at points of Y.

Different approach. Start with a unimodular vector in A_2^3 and lift it to a vector $A \to A^3$ in A^3 say (a_1, a_2, a_3). Then define M to be the cokernel

$$0 \longrightarrow A \longrightarrow A^3 \longrightarrow M \longrightarrow 0$$

whence the projective dim of M is ≤ 1, and $M_z = N$. If m is a torsion element of M, then because N is torsion-free, we know that $z^k m = 0$ for some k. Thus if $(b_1, b_3, b_3) \in A^3$ maps to m, $(z^k b_1, z^k b_2, z^k b_3) = (fa_1, fa_2, fa_3)$, so as I can assume that z does not divide some a_i, it follows that z^k divides f, then $(b_1, b_2, b_3) = g(a_1, a_2, a_3)$ so $m = 0$. Thus M is torsion-free and of proj. dimension ≤ 1, hence of depth ≥ 2. This does not seem to imply that M is of depth ≥ 2 at points of dimension 1, so I don't seem to get that the ideal ~~generated by~~ $\text{Im}\{A^3 \longrightarrow A\}$ is generated by a regular sequence.

Horrock's induction idea:

k field. Let P be a projective module over R_n $R_n = k[t_1, \ldots, t_n] = R_{n-1}[t]$ where $R_{n-1} = k[t_1, \ldots, t_{n-1}]$ and $t = t_n$. To prove P is free I need to extend P to $\mathbb{P}^1_{R_{n-1}}$, so I have the problem of extending the bundle $P[t^{-1}]$ over $R_{n-1}[t, t^{-1}]$ to $R_{n-1}[t^{-1}]$. But let $S = k[t] - 0$. Then $S^{-1} R_n = k(t) \otimes_k R_{n-1} = k(t)[t_1, \ldots, t_{n-1}]$

So by induction hypothesis $S^{-1}P$ is free over $S^{-1}R_n$.

~~It follows that for some non-zero monic polynomial $f \in k[t] - 0$, then P_f is free over $(R_n)_f = R_{n-1}[t][f^{-1}]$. So P is free if I remove from $\mathbb{A}^1_{R_{n-1}}$ the constant~~

It follows that P_f is free over $(R_n)_f$ for some $f \in k[t] - 0$. Now consider the open covering $\mathbb{P}^1_k = U \cup V$, where $U = \mathbb{A}^1 = \mathbb{P}^1 - \infty$ and where $V = \mathbb{A}^1 - \text{zeros of } f \cup \infty = \text{Spec } k[t]_f \cup \infty$. Then

$$U \cap V = \text{Spec } k[t]_f .$$

~~[illegible]~~ Let $p : \mathbb{P}^1_{R_{n-1}} \longrightarrow \mathbb{P}^1_k$ be the canonical map. We've seen that P which is a bundle on $p^{-1}(U)$ becomes trivial on $p^{-1}(U \cap V)$, hence [redacted] $P|_{p^{-1}(U \cap V)}$ extends to a bundle on $p^{-1}(V)$. Thus by glueing we get a bundle on $\mathbb{P}^1_{R_{n-1}}$, which restricts to P, etc.

February 8, 1976

Serre's paper relating bundles being free to complete intersections.

Prop. 1: M an A f.p. module of proj. dim ≤ 1.

$$0 \longrightarrow A \longrightarrow E_\xi \longrightarrow M \longrightarrow 0$$

the extension defined by an element $\xi \in \mathrm{Ext}^1(M,A)$. Then E_ξ is projective iff ξ generates $\mathrm{Ext}^1(M,A)$ as an A-module.

Lemma: If E is an f.p. A-module of proj. dim. ≤ 1, then E is proj. $\Longleftrightarrow$ $\mathrm{Ext}^1(E,A)=0$.

Proof: One has $0 \longrightarrow P \longrightarrow A^q \longrightarrow E \longrightarrow 0$ with P f.t. projective. $\mathrm{Ext}^1(E,P)=0$ because P is a direct factor of A^p.

Proof of Prop. E_ξ is f.p of proj. dim ≤ 1

$$\begin{tikzcd}
0 \arrow[r, "\delta"'] & \mathrm{Hom}(M,A) \arrow[r] & \mathrm{Hom}(E_\xi,A) \arrow[r] & \mathrm{Hom}(A,A) \arrow[dll, bend left] \\
& \mathrm{Ext}^1(M,A) \arrow[r] & \mathrm{Ext}^1(E_\xi,A) \arrow[r] & 0
\end{tikzcd}$$

$\mathrm{Im}\,\delta$ = A-submodule gen. by ξ. Now use lemma.

Prop. 2: assume $\mathrm{Pic}(A)=0$ M f.p of proj. dim ≤ 1 and of rank r. Then

(i) M generated by $r+1$ elements

$\Downarrow$

(ii) $\mathrm{Ext}^1(M,A)$ monogenic A-module

and the reciprocal is true if every vector bundle of rank $r+1$ is free

<u>Proof</u>: If M gen. by $r+1$ elts we have

$$0 \longrightarrow L \longrightarrow A^{r+1} \longrightarrow M \longrightarrow 0$$

where L is proj. of rank 1, so $L \simeq A$ as $\mathrm{Pic}(A) = 0$. Thus the element in $\mathrm{Ext}^1(M, A)$ represented by the above extension generates this Ext by Prop. 1.

Conversely given $\xi \in \mathrm{Ext}^1(M, A)$ a generator, we get an extension

$$0 \longrightarrow A \longrightarrow E_\xi \longrightarrow M \longrightarrow 0$$

with E_ξ proj. by Prop. 1. As E_ξ has rank $r+1$, $E_\xi \simeq A^{r+1}$ by hypothesis.

Suppose A is regular of dim 2. Let P be a vector bundle of rank 2 over A. Assume I can find a section s of P which is transversal to the zero section. This means that if I form

$$0 \longrightarrow Q \longrightarrow P^\vee \xrightarrow{s} A \longrightarrow A/I \longrightarrow 0$$

then A/I is the direct sum of the residue fields at a finite set of closed points. ~~Since A/I has support of codim ≥ 2 we have~~

~~$Q \simeq Q \otimes A \simeq A^2 P^\vee$~~,

Now $Q \simeq A$ if I assume $\mathrm{Pic}(A) = 0$. So $P^\vee$ is therefore given by an extension

$$0 \longrightarrow A \longrightarrow P^{\vee} \longrightarrow I \longrightarrow 0$$

hence by an element $\xi \in \mathrm{Ext}^1(I, A) = \mathrm{Ext}^2(A/I, A)$ $\simeq \bigoplus_i A/m_i$ if $A/I = \bigoplus A/m_i$. ~~[illegible]~~ By Prop. 2 I know that E_ξ is proj. $\Longleftrightarrow$ the image of ξ in each $\mathrm{Ext}^2(A/m_i, A)$ is $\neq 0$.

Assume that $A^* \twoheadrightarrow (A/I)^*$. ~~[illegible]~~ As extensions

$$0 \longrightarrow A \rightarrow E_\xi \rightarrow I \rightarrow 0$$

with E_ξ projective ~~will be such that~~ correspond to generators of $\mathrm{Ext}^1(I, A) = A/I$, one sees that the various projective E_ξ are isomorphic. So we see the following:

<u>Prop</u>: ~~Under the assumption that~~ Let A be a regular ring of dimension 2, let P be ~~projective~~ f.t. projective A module of rank 2 having a section s transversal to zero. If

(i) $\mathrm{Pic}(A) = 0$

(ii) $A^* \twoheadrightarrow \prod_{i=1}^{r} (A/m_i)^*$ $\quad \{m_1, \ldots, m_r\} =$ zero set of s ~~[illegible]~~

(iii) $\bigcap m_i$ generated by 2 elts ______________.

then P is free.

(This won't lead to a characterization of regular rings of dim 2 $\ni$ every vector bundle is trivial. e.g. $k[X, Y]$ has too few units.)

February 9, 1975

Let A be a 2-dim'l reg. ~~ring domain~~ local ring $z \in m - m^2$, M a torsion-free f.t. A-module. I know the double dual $P = (M^{\vee})^{\vee}$ of M is a free module and P/M is of finite length. I want a constructive procedure (analogous to blow-up) which starting from M will produce P; this should use z somehow. It's clear I have to introduce other modules $M' \subset M_z$.

Question 1: Does ∃ an $M' \subset M$ such that M/M' and P/M' are free over A/zA?

If so then $z(P/M') = 0$ so $zP \subset M$, i.e. z [redacted] kills $P/M = H^1_m(M)$. Conversely if $zP \subset M$, then we have $zP \subset M \subset P$ so ∃ an exact seq

$$0 \longrightarrow M/zP \longrightarrow P/zP \longrightarrow P/M \longrightarrow 0$$

which shows that M/zP is torsion-free over the d.v.r. A/zA, hence M/zP is free. ∴ Take $M' = zP$. Note that we also have

$$\begin{tikzcd} 0 \arrow[r] & P/M \arrow[r] & z^{-1}M/M \arrow[r] & z^{-1}M/P \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\ & & & z^{-1}(M/zP) & \end{tikzcd}$$

hence [redacted] P/M is the torsion submodule of [redacted] $z^{-1}M/M$.

Question 2: [redacted] Consider the operation T which [redacted] associates to M the module $T(M) \supset M$ such that $T(M)/M$ = torsion submodule of $z^{-1}M/M$. If this operation is iterated, does it yield P?

~~[redacted] If $T(M) = M$, then M/zM is torsion-free over A/zA, hence M is flat over A by the local flatness criterion, so $M = P$. [redacted] But I know we can find a chain of modules from M to P [redacted] with quotients $\simeq k$. So let $M \subset M' \subset P$ [redacted] be such that $M'/M \simeq k$. From the exact seq.~~

$$\begin{tikzcd} 0 \arrow[r] & M'/M \arrow[r] \arrow[d, phantom, "\wr"] & z^{-1}M/M \arrow[r] & z^{-1}M'/M' \arrow[r] & z^{-1}(M'/M) \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\ & k & & & k & \end{tikzcd}$$

~~one gets~~

$$0 \to k \to (z^{-1}M/M)_{tors} \to (z^{-1}M'/M')_{tors}$$

[redacted] [redacted]

If $T(M) = M$, then M/zM is torsion-free over over the d.v.r. A/zA, hence M is flat over A by the local flatness criterion, so $M = P$. Thus $T(M) > M$ if [redacted] $M < P$, and so the process stops since P/M is of finite length.

Here's another description of the same process. $T(M)$ is the largest submodule of $z^{-1}M$ such that $T(M)/M$ is of finite length, i.e. $T(M) \subset P$. $\therefore T(M) = z^{-1}M \cap P$, which means that I am simply filtering

P/M by the submodules $\operatorname{Ker} z^n$. This makes sense even when $z \in m^2 - \{0\}$.

Original motivation for looking at T is as follows.

Question: Can you find a filtration $M = M_0 \supset M_1 \supset \cdots \supset M_n$ such that M_n is free and $M_i/M_{i+1} \simeq A/zA$?

Important observation - When you come to generalize to higher dimensions you really have to change M along $z=0$, which means I replace M by $M' \subset M$ such that M/M' has support in $\{z=0\}$ At points where M is projective we will want M' to be projective too which means that M/M' will be projective over A/zA (say $z=0$ non-singular & $z(M/M')=0$. It may be desirable to have [illegible] proj dim $(M/M') = 1$ (this condition is [illegible] better that M/M' proj. over A/zA.) So the question to ask is [illegible], maybe

Question: [illegible] Does $\exists$ $M' \subset M$ with M' projective and M/M' killed by z^m and of proj. dim. 1 over A?

This is easy. $z^m P \subset M$ for some m. Then we have the exact sequence

$$0 \longrightarrow \underset{\cap\,\mathcal{P}_0}{z^m P} \longrightarrow \underset{\cap\,\mathcal{P}_1}{M} \longrightarrow M/z^m P \longrightarrow 0$$

which implies $M/z^m P \in \mathcal{P}_1$.

Feb 10, 1976:

Question: Suppose A regular local of dimension 2, $z \in m - m^2$, P projective over A. Consider all f.t. A-modules M contained in P_z such that $M_z = P_{z}$. Make these into a [redacted] building whose simplices are chains $M_0 \subset \cdots \subset M_p$ & M_i/M_{i-1} is free over A/zA. [redacted] Is this building [redacted] contractible?

Feb 11, 1976. Suppose M is a torsion-free module over a regular local ring A of dim. 2, and $z \in m - \{0\}$ is such that M_z is projective over A_z.

Choose k so that z^k kills $H^1_m(M)$. Define M' so that $z^k M \subset M' \subset M$ and

$$H^0_m(M'/z^k M) = H^0_m(M/z^k M)$$

i.e. M' is largest so that $M'/z^k M$ has finite length. Then $H^0_m(M/M') = 0$ so $H^1_m(M') \hookrightarrow H^1_m(M)$, and also $H^1_m(M'/z^k M) = 0$ because $M'/z^k M$ has finite length, so $H^1_m(z^k M) \twoheadrightarrow H^1_m(M')$. Since the composite

$$H^1_m(z^k M) \twoheadrightarrow H^1_m(M') \hookrightarrow H^1_m(M)$$

is isom. to $z^k : H^1_m(M) \circlearrowleft$, it is zero. $\therefore$ $H^1_m(M') = 0$ and M' is projective.

Suppose now A is a regular local ring of dimension 3 and that M is a torsion-free module of proj. dim 1 such that M_z is projective over A_z. ~~Choose k so that z^k kills $H^2_m(M)$. Again let M' be the largest submodule of M containing z^kM such that M'/z^kM has finite length~~

Suppose $M' \subset M$.

$$\begin{tikzcd}
0 \arrow[r] & H^0_m(M/M') \arrow[r] & H^1_m(M') \arrow[r] & H^1_m(M) \arrow[r] & H^1_m(M/M') & \\
& H^2_m(M') \arrow[r] & H^2_m(M) \arrow[r] & H^2_m(M/M') \arrow[r] & H^3_m(M') \arrow[r] & H^3_m(M) \arrow[d] \\
& & & & & 0
\end{tikzcd}$$

Let's try to use only M' such that $H^0_m(M/M') = 0$ and $H^2_m(M/M') = 0$. Can't be done really because otherwise $H^1_m(M/M')$ would have infinite length.

So it's clear I want to choose M' such that $H^0_m(M/M') = 0$, which forces $H^1_m(M') = 0$ (since $H^1_m(M) = 0$). Thus we have

$$0 \longrightarrow H^1_m(M/M') \longrightarrow H^2_m(M') \longrightarrow H^2_m(M)$$

which means M' won't be an improvement on M unless $H^1_m(M/M') = 0$ also, which forces M/M' to have proj. dim. ≤ 1.

Question: Do there exists any non-trivial $M' < M$ such that M/M' has proj. dim. ≤ 1?

Note that if P exists then any sufficiently far down M' such that $H^0_m(M/M') = H^1_m(M/M') = 0$ will satisfy $H^2_m(M') = 0$ (for it factors thru $z^k M$) hence M' will be projective.

Suppose now $z \notin m^2$, so A/zA is regular of dim 2. Then look at M/zM. $H^0_m(M/zM) = 0$ as $H^1_m(M)=$

$$0 \longrightarrow H^1_m(M/zM) \longrightarrow H^2_m(M) \xrightarrow{z} H^2_m(M)$$

Are there any surjections $M/zM \twoheadrightarrow A/zA$?

Problem: A reg. local of dim. 3 $z \in m - m^2$. For every finite length A-module T we get a module of proj. dim 1 by resolving minimally

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow F_0 \longrightarrow T \longrightarrow 0$$

The problem is to show that M_z is free over A_z.

Let's consider the special case where $\exists$ reg. local ring B of dim. 2 such that $B \xrightarrow{\sim} A/zA$. Suppose also that z kills T.

Let

$$0 \longrightarrow M \longrightarrow A^q \longrightarrow A^p \longrightarrow T \longrightarrow 0$$

be a minimal presentation of T over A, whence

$$p = \dim\, T \otimes_A k$$
$$q = \dim\, \mathrm{Tor}_1^A(T, k).$$

Then

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] & A^q \arrow[r] \arrow[d, "z"] & A^p \arrow[r] \arrow[d, "z"] & T \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & M \arrow[r] & A^q \arrow[r] \arrow[d] & A^p \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & (A/zA)^q \arrow[r] \arrow[d] & (A/zA)^p \arrow[r] \arrow[d] & T \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Now that because B has dim 2, K is projective hence free over B, say $K = B^r$. Note that $r = q - p$ since T is a torsion B-module. Thus we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M/zM \arrow[r] & K \arrow[r] \arrow[d, equal] & T \arrow[r] & 0 \\
 & & B^{q-p} & &
\end{tikzcd}$$

This shows that K is the canonical extension of M/zM to a vector bundle over B.

<u>Question</u>: Is $q - p > \dim\, T \otimes_B k = p$?

If so, I can find a unimodular element of M/zM.

Now we have a spectral sequence

$$E^2_{pq} = \mathrm{Tor}^B_p\left(\mathrm{Tor}^A_q(T, B), k\right) \Longrightarrow \mathrm{Tor}^A_{p+q}(T, k)$$

and $\mathrm{Tor}^A_q(T, B) = \begin{cases} T & q = 0, 1 \\ 0 & q \text{ otherwise.} \end{cases}$

$$\begin{array}{|llll} \\ \\ T \otimes_B k & \mathrm{Tor}^B_1(T,k) & \mathrm{Tor}^B_2(T,k) & \\ T \otimes_B k & \mathrm{Tor}^B_1(T,k) & \mathrm{Tor}^B_2(T,k) & 0 \\ \hline \end{array}$$

so the spectral sequence gives us an exact sequence

$$\mathrm{Tor}^B_1(T,k) \to \mathrm{Tor}^A_2(T,k) \longrightarrow \mathrm{Tor}^B_2(T,k) \longrightarrow T \otimes_B k \to \mathrm{Tor}^A_1(T,k) \to \mathrm{Tor}^B_1(T,k) \to 0$$

$$0 \to \mathrm{Tor}^B_2(T,k) \xrightarrow{\sim} \mathrm{Tor}^A_3(T,k) \to 0$$

since I am assuming $\exists$ section $B \to A$ of the map $A \to B$, this sequence splits into short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & T \otimes_B k \arrow[r] & \mathrm{Tor}^A_1(T,k) \arrow[r] & \mathrm{Tor}^B_1(T,k) \arrow[r] \arrow[l, dashed, bend right] & 0 \\ & p & q & & \end{tikzcd}$$

$\therefore q - p = \dim \mathrm{Tor}^B_1(T, k)$

But [illegible] the minimal resolution of T over B is of the form

$$0 \longrightarrow B^s \longrightarrow B^r \longrightarrow B^p \longrightarrow T \longrightarrow 0$$

$$r = \dim \operatorname{Tor}_1^B(T,k)$$
$$s = \dim \operatorname{Tor}_2^B(T,k)$$

But because T has finite length, proj dim $(T/B) = 2$, ~~so~~ $s > 0$, hence $r = p + s > p$ as was to be proved.

Recall that

$$\operatorname{Tor}_i^A(T,k) = H_i(\underline{x}, T)$$

if $\underline{x}$ is a system of parameters generating m.

$$\begin{tikzcd}
0 \arrow[r] & H_0(z, H_i(x,y,T)) \arrow[r] & H_i(z,x,y,T) \arrow[r] \arrow[d, equal] & H_1(z, H_{i-1}(x,y,T)) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Tor}_i^B(T,k) \arrow[r] & \operatorname{Tor}_i^A(T,k) \arrow[r] & \operatorname{Tor}_{i-1}^B(T,k) \arrow[r] & 0
\end{tikzcd}$$

Here I am identifying [illegible] x, y with generators of the maximal ideal of B. ~~Abstractly, one has~~ ~~$m = (z, x, y)$ so that~~ This shows in general that

$$\dim \operatorname{Tor}_1^A(T,k) = \dim \operatorname{Tor}_0^B(T,k) + \dim \operatorname{Tor}_1^B(T,k)$$
$$q \qquad = \qquad p \qquad + \quad r$$

and hence that $r \doteq q - p > p$ ~~[illegible]~~ for $T \neq 0$.

But note that the question on page 7 makes no reference to z. The problem is this:

Problem: Let T be a finite length module over a regular local ring of dim(3). Is

$$\dim \mathrm{Tor}_1^A(k,T) > 2\dim\{\mathrm{Tor}_0^A(k,T)\} \ ?$$

February 13, 1976

A reg. local dim. 3, $z \in m - m^2$, F a finite length A-module,

$$0 \to M \to A^q \to A^p \to F \to 0$$

$$0 \to A^s \to A^r \to M \to 0$$

minimal resolutions. Then

$$H^0_m(M) = H^1_m(M) = 0 \qquad H^2_m(M) = F.$$

$$H^3_m(M) = D\{\mathrm{Hom}(M,A)\}$$

$$0 \to M^{\vee} \to A^r \to A^s \to H^2_m(M^{\vee}) \to 0$$

$$H^3_m(M) = M \underset{A}{\otimes} D(A) \qquad \text{because } H^3 \text{ is right exact}$$

$$D\left[H^2_m(M^{[illegible]})\right] = H^2_m(M^{\vee})$$

Yesterday we showed there was an exact sequence

$$0 \to (M/zM) \to \widetilde{(M/zM)} \to {}_zF \to 0$$

where $\widetilde{(M/zM)} = j_*j^*(M/zM)$ is the reflexive hull of M/zM over $B = A/zA$; [redacted] also one has

$$0 \to \widetilde{(M/zM)} \longrightarrow B^q \longrightarrow B^p \longrightarrow F/zF \to 0$$

Similarly one has an exact sequence

$$0 \longrightarrow (M^\vee/zM^\vee) \longrightarrow \widetilde{(M^\vee/zM^\vee)} \longrightarrow \overset{D(F)/z}{\overset{\|}{D({}_zF)}} \longrightarrow 0$$

Next note that

$$\begin{aligned} \mathrm{Hom}\left(\widetilde{(M^\vee/zM^\vee)}, A/zA\right) &= \mathrm{Hom}\,(M^\vee/zM^\vee, A/zA) \\ &= \mathrm{Hom}\,(M^\vee, A/zA) \end{aligned}$$

$$\underset{}{M^{\vee\vee}} \xrightarrow{z} \underset{\underset{M}{\|}}{M^{\vee\vee}} \longrightarrow \mathrm{Hom}\,(M^\vee, A/zA) \longrightarrow \underset{\text{finite length}}{\mathrm{Ext}^1(M^\vee, A)}$$

$$\therefore \quad \mathrm{Hom}\,(M^\vee, A/zA) = (M/zM)^\sim.$$

Thus $(M^\vee/zM^\vee)^\sim$ and $(M/zM)^\sim$ are dual

~~Now since $zM \subset M' \subset M$ and $H^0_m(M/M') = 0$~~

~~$0 \to H^0(M/M') \to H^1(M') \to H^1(M) \to H^1(M/M) \to H^2(M') \to H^2(M) \to 0$~~

Suppose M' such that $zM \subset M' \subset M$ and

$$H^0_m(M/M') = 0. \quad \Longleftrightarrow \quad H^1_m(M') = 0$$

$$\begin{tikzcd}
0 \arrow[r] & H^1_m(M/M') \arrow[r] & H^2_m(M') \arrow[r] & H^2_m(M) \arrow[r] \arrow[d, equal] & H^2_m(M/M') \arrow[r] & H^3_m(M') \arrow[r] \arrow[d, equal] & H^3_m(M) \arrow[r] \arrow[d, equal] & 0 \\
 & & & DH^2_m(M^{\vee}) & & D(M'^{\vee}) \arrow[r] & D(M^{\vee}) &
\end{tikzcd}$$

So the sequence is the dual of

$$0 \longrightarrow M^{\vee} \longrightarrow M'^{\vee} \longrightarrow \mathrm{Ext}^1(M/M', A) \longrightarrow H^2_m(M^{\vee}) \longrightarrow H^2_m(M'^{\vee})$$

$$0 \longrightarrow M'^{\vee}/M^{\vee} \longrightarrow \mathrm{Ext}^1(M/M', A) \longrightarrow H^1_m(M^{\vee}/M'^{\vee}) \longrightarrow 0$$

So its (more or less) clear that $\mathrm{Ext}^1(M/M', A) = (M'^{\vee}/M^{\vee})^{\sim}$. Thus we get the following exact sequence of finite length modules

$$\boxed{0 \to H^1_m(M/M') \to H^2_m(M') \to H^2_m(M) \to DH^1_m(M'^{\vee}/M^{\vee}) \to 0}$$

Therefore in order that M' be better than M it is necessary & sufficient that

$$\boxed{\text{length } H^1_m(M/M') < \text{length } H^1_m(M'^{\vee}/M^{\vee})}$$

Review: A reg. local of dim 3, $z \in A - 0$
M torsion-free module $\ni$ M_z is projective
and $H^1_m(M) = 0$.

Let M' be $\ni$ $M \subset M' \subset z^{-1}M$ and $\ni$
[redacted] $H^1_m(M') \xleftarrow{\sim} H^0_m(z^{-1}M/M') = 0$.

$$0 \to H^1_m(M'/M) \to H^2_m(M) \to H^2_m(M') \to DH^1_m(M^\vee/M'^\vee) \to 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'/M \arrow[r] \arrow[d] & (M'/M)^\sim \arrow[r] \arrow[d] & H^1_m(M'/M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^{-1}M/M \arrow[r] \arrow[d] & (z^{-1}M/M)^\sim \arrow[r] \arrow[d] & H^1_m(z^{-1}M/M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^{-1}M/M' \arrow[r] \arrow[d] & (z^{-1}M/M')^\sim \arrow[r] & H^1_m(z^{-1}M/M') \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Introduce

$$\chi(M \subset M') = \text{length } H^2(M) - \text{length } H^2(M')$$
$$= \text{length } H^1(M^\vee/M'^\vee) - \text{length } H^1(M'/M)$$

If want to show $\exists$ M' $M \subset M' \subset z^{-1}M$ $\ni$
$\chi(M \subset M') > 0$. [redacted]

[redacted]

I assume $z \in m - m^2$, whence $B = A/zA$ is reg. local of dimension 2. Then $(z^{-1}M/M)^\sim$ is free over B, say of rank 2 because this is the critical case. I construct M' as follows. Choose a direct factor $B \subset (z^{-1}M/M)^{\sim} = B^2$ and put $M'/M =$ intersection of B and $z^{-1}M/M$ in $(z^{-1}M/M)^{\sim}$. Then $z^{-1}M/M'$ embeds in $(z^{-1}M/M)^{\sim}/B \simeq B$, so $H^0_m(z^{-1}M/M') = 0$ and so $H^1_m(M') = 0$. we get an M' such that

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'/M \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & H^1_m(M'/M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^{-1}M/M \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & H^1_m(z^{-1}M/M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & z^{-1}M/M' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & H^1_m(z^{-1}M/M') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is exact. Now from the maps

$$H^2_m(M) \xrightarrow{u} H^2_m(M') \xrightarrow{v} H^2_m(z^{-1}M)$$

we get the long exact sequence of Ker's + Coker's

$$\begin{tikzcd}
0 \arrow[r] & H^1_m(M'/M) \arrow[r] & H^1_m(z^{-1}M/M) \arrow[r] & H^1(z^{-1}M/M') \arrow[dll, "0", bend left] \\
 & DH^1_m(M^\vee/M'^\vee) \arrow[r] & DH^1_m(M^\vee/zM^\vee) \arrow[r] & DH^1(M'^\vee/zM^\vee) \arrow[r] & 0
\end{tikzcd}$$

By construction, the connecting homomorphism is zero. So what?

I've seen that $j^*(M^\vee/zM^\vee)$ is the bundle on $\mathrm{Spec}(A/zA) - \{m\}$ dual to $j^*(M/zM)$. Now $j^*(M'/M)$ is a sub-line bundle of $j^*(z^{-1}M/M) \simeq j^*(M/zM)$ so it should be dual to the quotient line bundle $j^*(M^\vee/M'^\vee)$ of $j^*(M/zM)$. Thus

$$j^*(M^\vee/M'^\vee) \quad \text{dual to} \quad j^*(M'/M)$$

But if over B we have

$$0 \longrightarrow N \longrightarrow P \longrightarrow S \longrightarrow 0$$

with P free and S of finite length, then

$$0 \longrightarrow \mathrm{Hom}(P,B) \xrightarrow{\sim} \mathrm{Hom}(N,B) \longrightarrow \mathrm{Ext}^1(S,B)^{\overset{0}{\shortparallel}}$$

$$\mathrm{Ext}^1(N,B) \xrightarrow{\sim} \mathrm{Ext}^2(S,B)$$

~~And Ext¹(S, B) has the same length as~~ S.

Key point?:

I'm free to choose as an arbitrary sub-vector bundle

if I choose a direct factor

$$\begin{tikzcd}
(M'/M)^\sim \arrow[r, two heads] \arrow[d, phantom, "\cap"] & H^1(M'/M) \arrow[d, phantom, "\cap"] & \\
(z^{-1}M/M)^\sim \arrow[r, two heads] & H^1(z^{-1}M/M) & = {}_zH^2(M) \\
\text{dual} & \text{not dual but of same length} & \\
(M^\vee/zM^\vee)^\sim \arrow[r, two heads] \arrow[d] & H^1(M^\vee/zM^\vee) \arrow[d] & = D(H^2(M)/z) \\
(M^\vee/M'^\vee)^\sim \arrow[r, two heads] & H^1(M^\vee/M'^\vee) &
\end{tikzcd}$$

The key point is whether you can choose $(M'/M)^\sim$ so that length $H^1(M^\vee/M'^\vee) <$ length $H^1(M'/M)$.

So it seems that what we have is a rank 2 free B-module P and finite length quotients P/R, [illegible] $P^\vee/R'$ of the same length. I seek a direct factor L of P such that

$$\mathrm{Im}(L \to P/R) \quad \text{and} \quad \mathrm{Im}(L^\perp \to P^\vee/R')$$

are large.

Situation: Suppose $M = f_* E$ where $f: X \to \mathrm{Spec}(A)$ is a birational proper map, an isomorphism off $z=0$, where E is a vector bundle on X. X is the blow-up of an ideal $I = f_* \mathcal{O}(1)$, and $z \in I$ is a section of $\mathcal{O}(1)$ such that

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{z^{-1}} \mathcal{O} \longrightarrow \mathcal{O}_Y \to 0$$

defines the inverse image Y of $z=0$. Then

$$E \hookrightarrow E(1) \hookrightarrow E(2) \hookrightarrow \cdots$$

give me embeddings

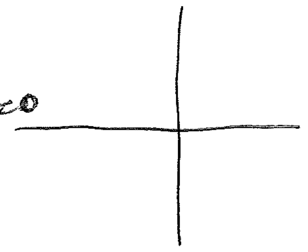
$$M = f_* E \subset f_* E(1) \subset \cdots \subset f_*(E(k)).$$

But A being regular $\Rightarrow$ E is of finite Tor dim. over A $\Rightarrow$ $Rf_*(E(k))$ is a perfect complex. But $R^1 f_*(E(k)) = 0$ for k large, $\therefore$ $f_* E(k)$ is a bundle over A. NO its necessary that E be flat over A, which is impossible

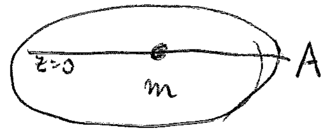
~~Example: Can you understand the case~~
~~bundles~~

A regular local of dim. 3
$X = \text{Proj}\{\bigoplus_{k\geq 0} \mathfrak{m}^k\}$ = scheme obtained by blowing up the closed point of Spec(A). Let E be a vector bundle over X and let $M = \Gamma(X, E)$. $f_* \mathcal{O}(+1) = \mathfrak{m}$, so z gives a section $\mathcal{O} \to \mathcal{O}(1)$ on X, hence maps $E \subset E(1) \subset E(2) \subset \cdots$. Is there any chance that $f_* E(k)$ is free over A for k large? NO, this would imply that E is flat over A, because E is obtained by localization from $\bigoplus_{k\geq 0} f_* E(k)$.

Let $X = \text{Proj}\{\bigoplus \mathfrak{m}^k\}$, let E be a vector bundle on X, let $M = \Gamma(X, E)$. Then M is locally-free on Spec$(A) - \{\mathfrak{m}\}$. Look at the restriction of E to ~~the~~ fibre over $\{\mathfrak{m}\}$ which is $\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)^\vee$. On the affine space complementary to $z = 0$, E becomes trivial, hence replacing E by $\mathcal{O}(m) \otimes E$, I can find $s_i \in H^0(E \otimes k)$ which form a free basis ~~on~~ the special fibre off $z = 0$. From



$$H^0(E) \longrightarrow H^0(E \otimes k) \longrightarrow H^1(E\,\mathfrak{m})$$



Since $H^1(E(n)\mathfrak{m}) = 0$ for n large, I thus ought to be able to find $s_i \in H^0(E) = M$ which form a free basis on the special fibre off $z = 0$.

?

February 16, 1976: curves.

Let C be a complete non-sing. curve over $\mathbb{C}$. There appears to be a "topological" K-theory of vector bundles over C, which is related to the discrete K-theory in some way to be understood.

Recall the exact sequence

$$\longrightarrow K_1C \longrightarrow K_1F \xrightarrow{\partial} K_0k\otimes D \longrightarrow K_0C \longrightarrow K_0F \longrightarrow 0$$

where D is the group of divisors on C and F is the function field. In the "top" K-theory $K_*^{top}(F)=0$.

An element of $K_n^{top}(C)$ is represented by a family of vector bundles over $\mathbb{C}$ parameterized by S^n. Let $E=\{E_t\}$ be a family of vector bundles on C param. by a finite complex T. Fix a basepoint ∞ of whence we get a vector bundle $\iota_\infty^* E$ over T.

$$0 \longrightarrow E\otimes \mathcal{O}(-1) \longrightarrow E \longrightarrow \iota_{\infty *}\iota_\infty^* E \longrightarrow 0$$

$$\longrightarrow \underline{\mathrm{Hom}}(\mathcal{O}\otimes \iota_\infty^* E, E) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{O}\otimes \iota_\infty^* E, \iota_{\infty *}\iota_\infty^* E) \to 0$$

So if we take global sections over $T\times S$, we get an exact seq

$$\mathrm{Hom}_{T\times S}(\mathcal{O}\underset{T}{\otimes} i_\infty^* E, E) \to \mathrm{Hom}_T(i_\infty^* E, i_\infty^* E) \to H^1(T\times S, \underline{\mathrm{Hom}}(\mathcal{O}\underset{T}{\otimes} i_\infty^* E, E\otimes\mathcal{O}(-1)))$$

$$\overset{\|}{[E\otimes\mathcal{O}(-1)]\otimes_T(i_\infty^* E)^\vee}$$

If E is sufficiently untwisted the last group should be zero, hence we can find a map

$$\mathcal{O}\otimes i_\infty^* E \longrightarrow E$$

reducing to the identity of $i_\infty^* E$. The best statement in general is that $\exists$ a map

$$\mathcal{O}(-N)\otimes i_\infty^* E \longrightarrow E$$

reducing at ∞ to the canonical isomorphism. Note that by the local flatness theorem, this map is injective and its cokernel is flat over T. ~~So~~ if we dualize we get an exact sequence

$$0 \longrightarrow E^\vee(-N) \longrightarrow \mathcal{O}\otimes(i_\infty^* E)^\vee \longrightarrow F \longrightarrow 0$$

where F is finite and flat over T.

Suppose $i_\infty^* E$ is trivialized. Then we get a map

$$T \longrightarrow \mathrm{Quot}^n_d(C-\infty) = \text{quotients of } \mathcal{O}^n_{C-\{\infty\}} \text{ of length } d.$$

where I know $\mathrm{Quot}^n_d(C-\infty)$ is a non-singular variety.

February 17, 1976:

C complete non-singular curve over $\mathbb{C}$
T finite complex
$E = \{E_t \mid t \in T\}$ is a continuous family of vector bundles of rank n over C parameterized by T.

$\mathcal{O}(1)$ = the line bundle on C associated to a fixed point ∞.

Prop: For m sufficiently large $\exists$ a map

$$\mathcal{O}^n_{T\times C} \longrightarrow E(m)$$

which is an embedding over each point $t \in T$.

Proof: Let $P_1, \dots, P_k$ be distinct points of C. Replacing E by $E(m)$, we can assume $t \mapsto \Gamma(E_t)$ is a vector bundle on T and that we have a surjection

$$\Gamma(E_t) \longrightarrow\!\!\!\!\rightarrow (E_t(P_1) \times \cdots \times E_t(P_k)).$$

Because E is a rank n, the sequences in $E_t(P_j)^n$ which form a basis in $E_t(P_j)$ form the complement of a hypersurface. Thus the sequences in $[E_t(P_1) \times \cdots \times E_t(P_k)]^n$ whose projections in $E_t(P_j)$ form a basis for E_j form the complement of a subvariety of codimension k.

Thus the set of sequences $(s_1, \ldots, s_n) \in \Gamma(E_t)^n$ which form a basis for $E_t(P_j)$ for some j is the complement of a subvariety of codim k. By [redacted] transversality, if $k > \dim(T)$ one can find a section $t \mapsto (s_{1t}, \ldots, s_{nt})$ of $t \mapsto \Gamma(E_t)^n$ such that for each t the s_{it} are ind. at some P_j. The prop. follows.

Refinement: Suppose T' is a subcomplex of T and one is given $\mathcal{O}^n \hookrightarrow E_t(m)$ for $t \in T'$. Look [redacted] inside $\Gamma(E_t(m))^n$ for the good sequences. The bad sequences form a subvariety of large codim. More precisely we can contain the bad sequences in a subvariety of codim k which [redacted] varies nicely for nearby t. Thus using transversality step-by-step over the cells of $T - T'$, one can extend a "good" section over T' to T.

Consequences: since we have $\mathcal{O}_C^n \hookrightarrow E_t(m)$ $\forall t$ we get

$$E_t^\vee \hookrightarrow \mathcal{O}_C^n(m)$$

$$\text{length}\left(\mathcal{O}_C^n(m)/E_t^\vee\right) = nm - \deg E_t^\vee = nm + \deg E_t.$$

hence a map

$$T \to \underset{nm+d}{\text{Quot}}\left(\mathcal{O}_C^n(m)\right)$$

where $d = \deg E$.

Questions:

① [redacted] Does the inclusion

$$\mathrm{Quot}_{nm+d}(\mathcal{O}^n_C(m)) \hookrightarrow \mathrm{Quot}_{n(m+1)+d}(\mathcal{O}^n_C(m+1))$$

become[redacted] increasingly connected [redacted] as m increases?

② Is $\coprod_d \varinjlim_m \mathrm{Quot}_{nm+d}(\mathcal{O}^n_C(m))$ of the same homotopy type as $(BU_n)^C$?

Example. Take $C = \mathbb{P}^1$. Then by topology I know every top bundle of rank n over $T \times C$ is given by a clutching function given by a Laurent polynomial matrix non-singular on the unit circle.* This same matrix A defines a holomorphic [redacted] bundle E over $\mathbb{P}^1$. [redacted] $s \in \Gamma(E)$ can be identified with a vector f of holom. functions in $1 \leq |z| \leq \infty$ such that Af has a holom. extension for [redacted] $0 \leq |z| \leq 1$. $E(m)$ corresp. to the matrix $z^m A$ which will be a polynomial matrix for m sufficiently large. Then we will get a map $\mathcal{O}^n \to E(m)$ which is an isomorphism near $z = \infty$.

* Assuming that the bundle is trivial when pulled [redacted] back to T

[redacted] Adele version. The basic object appears to be a pair (E, u), where E is a vector bundle of rank n over C, and where $u: \mathcal{O}^n \dashrightarrow E$ is a rational trivialization of E. I ought to be able to make a [redacted] space [redacted] out of such pairs. A point of the space is a collection of lattices [redacted] E_P in F^n for $\mathcal{O}_P$ as $P \in C$, almost all equal to $\mathcal{O}_P^n$.

$\mathrm{Quot}_{d+nm}(\mathcal{O}_C^n(m))$ classifies such pairs where $E_P \subset \mathcal{O}_P^n$ for all $P \neq \infty$, and $E_\infty \subset$ [redacted] $\pi_\infty^{-m}\mathcal{O}_\infty^n$, and [redacted] where the degree of E is d.