Jan. 23, 1975

Let V be a vector space over k. For each hyperplane $H \in \mathbb{P}(V^*)$ the complement of H, $\mathbb{P}V - H$ is an open set in $\mathbb{P}V$. I will be interested in the nerve of this covering, and the finite intersections of members of the covering.

Suppose given $H_0, H_1, \dots, H_q \in \mathbb{P}(V^*)$, whence

$$\bigcap_{i=0}^{q} \mathbb{P}V - H_i = \{ L \in \mathbb{P}V \mid L \xrightarrow{\sim} V/H_i \quad \text{all } i \}$$

Notation. $\sigma = \{H_0, \dots, H_q\}$, $U_\sigma = \bigcap_{i=0}^{q} \mathbb{P}V - H_i$.

Note we have a map

$$U_\sigma \longrightarrow \mathbb{P}\left(\prod_i (V/H_i)\right)$$

If $Z = \bigcap H_i$, then the fibre of this map over a line $\ell \subset \prod (V/H_i)$ is a torsor under $\mathrm{Hom}(\ell, Z)$, provided the fibre is non-empty. Such fibres are the Schubert cells in $\mathbb{P}V$ associated to layers $(\bigcap H_i, W)$ where $W \not\subset$ any H_i.

Put $\sigma = \{H_0, \dots, H_q\}$ and let $\emptyset \neq \tau \subset \sigma$, whence $U_\sigma \subset U_\tau$. Given a cell in U_σ assoc. to $(\bigcap_{H_i \in \sigma} H_i, W)$ where $W \not\subset H_i$, any $H_i \in \sigma$ it follows that

$$\Big(\bigcap_{H_i\in\sigma} H_i\,,\ W\Big) \leq \Big(\bigcap_{H_i\in\tau} H_i\,,\ \bigcap_{H_i\in\tau} H_i + W\Big)$$

hence the cells we've found in U_σ are contained in the cells found in U_τ.

In fact, if a cell $C(A,B) \subset U_\sigma$ then $(A,B) \leq (H_i, V)$ for all $H_i \in \sigma$, hence $A \subset \bigcap H_i$ and

$$(A,B) \leq (\cap H_i\,,\ \cap H_i + B).$$

So therefore it is clear that the cells inside U_σ we've found are the maximal Schubert cells in this intersection.

~~Now I can form a simplicial complex over the nerve of the covering $\{PV - H_i\}$ by calling simplices~~

Now I can form over the poset of simplices in the nerve of the covering $\{PV - H\}$ a fibred cat whose fibre over σ is the set of maximal cells in U_σ. Question: Is this fibred category of the same homotopy type as the poset of cells in PV?

Call the dual of the fibred category C. An object of C is a σ + a max cell γ in U_σ and a map $(\sigma, \gamma) \longrightarrow (\tau, \delta)$ is an inclusion $\tau \subset \sigma$ such that $U_\sigma \subset U_\tau$.

$$\gamma \longmapsto \delta$$

There is a functor $f: \mathcal{C} \longrightarrow \mathcal{S}(\mathbb{P}V)$ sending (σ,γ) to γ. $\gamma_0 | f$ consists of pairs $(\sigma,\gamma) \ni \gamma_0 \subset \gamma$ i.e. such that $\gamma_0 \subset U_\sigma$. Thus I have to look at all finite $\neq \phi$ subsets $\sigma \subset \mathbb{P}(V^*)$ consisting of $H \ni \gamma_0 \subset \mathbb{P}V - H$. But the poset of non-empty finite subsets of a set S is contractible if $S \neq \phi$, which is the case as every γ_0 is contained in some $\mathbb{P}V - H$.

Jan. 25, 1975

I have introduced a modification of $BGL_n(F)$. BGL_n is the classifying space of the groupoid n-diml. v.s. and their isos. The modified category will consist of different types of objects. For each decomp. $n = a_1 + a_2 + \cdots + a_p$ with a_i integers >0, the objects of this type will be a sequence of vector spaces $V_1, \ldots, V_p$ with $\dim(V_j) = a_j$.

I am replacing the category $\mathrm{Iso}(\mathcal{P}_F)$ by the category of finite sequences $(V_1, V_2, \ldots, V_p)$ of objects of $\mathcal{P}_F$ in which a morphism from $(V_1, \ldots, V_p)$ to $(W_1, \ldots, W_q)$ consists of a monotone surjective map $\eta : \{1, \ldots, p\} \to \{1, \ldots, q\}$, + a filtration of W_j indexed by $\eta^{-1}\{j\}$

$$F_i W_j$$

(if $\eta^{-1}\{j\} = [a_{j-1}+1, a_j]$ then I want

$$0 = \subset F_{a_{j-1}+1} W \subset \cdots \subset F_{a_j} W_j = W_j \Big)$$

+ isos

$$F_i W_j / F_{i-1} W_j \simeq V_i .$$

February 3, 1975:

Problem: Given a complete non-sing. curve C over k, to find a category constructed from rank 2 vector bundles over C and their isomorphisms which deserves to be called the "space" of rank 2 bundles over C. First take $C = \mathbb{P}^1$.

I believe that for k itself, the "space" of rank 2 vector spaces over k is the double mapping cylinder of

$$B\begin{pmatrix} k^\cdot & \\ & k^\cdot \end{pmatrix} \longleftarrow B\begin{pmatrix} k^\cdot & k \\ & k^\cdot \end{pmatrix} \hookrightarrow B(GL_2 k).$$

I have half-checked that this "space" is h.eq. to the poset of Schubert cells in the Grassmannian of 2 planes in an infinite diml k-vector space.

Try topologically. $\mathbb{P}^1 \sim S^2$. The space of rank 2 bundles is the space $BU_2^{S^2} = \underline{Map}(S^2, BU_2)$. Have fibration

$$\underline{Map_{pt}}(S^2, BU_2) \longrightarrow \underline{Map}(S^2, BU_2) \longrightarrow BU_2$$

$$\begin{array}{c} \| \\ Map_{pt}(S^1, U_2) \\ \| \\ \Omega U_2 \end{array}$$

Now I have a model for ΩU_2 namely the space of lattices for $k[z]$ inside $k[z,z^{-1}]^2$.

My idea is this. Try to describe the homotopy type of the poset of Schubert cells in the space of these lattices. This might suggest a candidate for the space of rank 2 bundles over $\mathbb{P}^1$.

Other possibility: $\underline{\mathrm{Map}}(S^2, BU)$ has only even diml. cells, hence it might be like an algebraic variety. In fact I can get this variety as follows. Start first with the bundle $k[z]^2$ over $\mathbb{P}^1 - \infty$ and take the space $\mathcal{L}$ of lattices at ∞. Obviously $GL_2 k$ acts on $\mathcal{L}$ thru the embedding $GL_2 k \subset GL_2 k[t]$, so I can form a fibre bundle Y over BU_2 with fibre $\mathcal{L}$. Over Y we have a canonical rank 2 bundle on $\mathbb{P}^1$ equipped with a ~~con~~retraction of the bundle covering the radial retraction of $\mathbb{P}^1 - \infty$ to 0.

Maybe should try this: My candidate for the "space" of rank 2 bundles on $\mathbb{P}^1$ maps by restriction to my candidate for the "space" of rank 2 bundles on $\mathbb{A}^1$, which contracts to the "space" of rank 2 k-vector spaces. So try to lift this homotopy and if you can, you will deform the thing you want understand into something simpler.

Recall the homotopy involved:

Nagao thm: $GL_2(k[z]) = GL_2(k) \underset{\begin{pmatrix} k^\cdot & k \\ 0 & k^\cdot \end{pmatrix}}{*} \begin{pmatrix} k^\cdot & k[z] \\ 0 & k^\cdot \end{pmatrix}$

hence

$$\begin{tikzcd}
B\begin{pmatrix} k^\cdot & k \\ & k^\cdot \end{pmatrix} \arrow[r, hook] \arrow[d, hook] & B\,GL_2(k) \arrow[d, hook] \\
B\begin{pmatrix} k^\cdot & k[t] \\ & k^\cdot \end{pmatrix} \arrow[r, hook] & B\,GL_2(k[t])
\end{tikzcd}$$

is h-cocartesian. Now I define $\widetilde{BGL}_2$ by a cocartesian square

$$\begin{tikzcd}
B\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \arrow[r, hook] \arrow[d] & BGL_2 \arrow[d] \\
B\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \arrow[r] & \widetilde{BGL}_2
\end{tikzcd}$$

at least for a PID.

Since $k[t]^\cdot = k^\cdot$, we have two h-cocartesian squares:

$$\begin{tikzcd}
B\begin{pmatrix} k^{\cdot} & k \\ & k \end{pmatrix} \arrow[r] \arrow[d] & BGL_2(k) \arrow[d] \\
B\begin{pmatrix} k^{\cdot} & k[t] \\ & k^{\cdot} \end{pmatrix} \arrow[r] \arrow[d] & BGL_2(k[t]) \arrow[d] \\
B\begin{pmatrix} k^{\cdot} & \\ & k^{\cdot} \end{pmatrix} \arrow[r] & \widetilde{BGL_2}(k[t])
\end{tikzcd}$$

so the big square is h-cocartesian, showing that $\widetilde{BGL_2}(k) \sim \widetilde{BGL_2}(k[t])$.

Recall $\widetilde{BGL_2}(A)$ is the classifying space of the category given as follows. Objects are either rank 2 bundles E or pairs (L_1, L_2) of rank 1 bundles. Maps are isos. $E \simeq E'$, $(L_1, L_2) \cong (L_1', L_2')$, and there is a map $(L_1, L_2) \longrightarrow E$ for each exact sequence

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

For $\widetilde{BGL_2}(k) \subset \widetilde{BGL_2}(k[t])$, we can consider $\widetilde{BGL_2}k$ as realized by the same category, but where thes objects are enriched by giving unimodular subspaces

Digression:

Theorem: Let
$$f(x) = f_0 + \cdots + f_m x^m$$
$$g(x) = g_0 + \cdots + g_n x^n \qquad \in k[x]$$

Put
$$R = \left|\begin{array}{cccccc} f_0 & & & g_0 & & \\ f_1 & f_0 & & g_1 & \ddots & \\ \vdots & f_1 & & \vdots & & g_0 \\ f_m & \vdots & & g_n & & g_1 \\ & \ddots & & & \ddots & \vdots \\ & & f_m & & & g_n \end{array}\right| \in k$$
$$\longleftarrow n \longrightarrow \quad \longleftarrow m \longrightarrow$$

Then $R \neq 0 \iff$ (f_m or $g_n \neq 0$ and f, g rel. prime in $k[x]$)

If $f = f_0 x_0^m + \cdots + f_m x_1^m \in S_m$, $g \in S_n$, then one has

$$S_{n-1} \oplus S_{m-1} \longrightarrow S_{n+m-1}$$
$$(u \;,\; v) \longmapsto fu + gv$$

both spaces have dim $m+n$

and R is the determinant of this map.

Also if one has

$$0 \to \mathcal{O}(-1) \longrightarrow \mathcal{O}(n-1) \oplus \mathcal{O}(m-1) \longrightarrow \mathcal{O}(n+m-1) \longrightarrow 0$$

then as $H^0(\mathcal{O}(-1)) = H^1(\mathcal{O}(-1)) = 0$, we are in the resultant situation.

Category interpretation of the ~~[illegible]~~ homotopy property on page 4.

$\widetilde{GL}_2(k[t])$:

$$\begin{tikzcd}
\begin{pmatrix} k^\cdot & k[t] \\ & k^\cdot \end{pmatrix} \arrow[r] \arrow[d] & GL_2\, k[t] & (L_1 \to E \to L_2) \arrow[r, mapsto] \arrow[d, mapsto] & E \\
\begin{pmatrix} k^\cdot & \\ & k^\cdot \end{pmatrix} & & (L_1, L_2) &
\end{tikzcd}$$

$$\uparrow$$

$$\begin{tikzcd}
\begin{pmatrix} k^\cdot & k \\ & k \end{pmatrix} \arrow[r] \arrow[d] \arrow[drr] & GL_2\, k & & \\
\begin{pmatrix} k^\cdot & k[t] \\ & k^\cdot \end{pmatrix} \arrow[d] & & \begin{pmatrix} k^\cdot & k \\ \text{[redacted]} & k \end{pmatrix} \arrow[r] \arrow[d] & GL_2\, k \\
\begin{pmatrix} k^\cdot & \\ & k^\cdot \end{pmatrix} & & \begin{pmatrix} k^\cdot & \\ & k^\cdot \end{pmatrix} &
\end{tikzcd}$$

Intermediate category thus consists of 4 object types:

$l \subset V \qquad V$

$L \subset E$

L_1, L_2

February 9, 1975

I want to establish conventions concerning lattices, Laurent polys. etc.

I recall that $GL_n(\mathbb{C}[z,z^{-1}])'$ has the homotopy type of $\Omega GL_n\mathbb{C}$; also $U_n(\mathbb{C}[z,z^{-1}])' \sim \Omega U_n$. To prove this I use the principal bundle P over GL_n with fibre $GL_n(\mathbb{C}[z,z^{-1}])'$ consisting of functions $w \mapsto A(w)$, $\mathbb{C} \longrightarrow GL_n$ of the form

$$A(w) = e^{2\pi i w T} F(z)$$

$z = e^{2\pi i w}$

$F \in \mathcal{G}' = GL(\mathbb{C}[z,z^{-1}])'$

$T \in gl_n$.

~~The point in the proof is to show that P is contractible. To do this is use the~~

~~$GL_n(\mathbb{C}[z,z^{-1}]) \quad GL_n(\mathbb{C}[z]) \quad U_n$~~

The analogous principal bundle P_0 over U_n with fibre $\mathcal{U}'$, $\mathcal{U} = U_n(\mathbb{C}[z,z^{-1}])$, consisting of A with unitary values, can be identified with the building constructed out of lattices. I want the formulas.

$\underline{n=1}$. $\mathcal{U}' = \{ z^n \mid n \in \mathbb{Z} \} \simeq \mathbb{Z}$

$$\begin{array}{ccccc} \mathcal{U}' & \longrightarrow & P_0 & \longrightarrow & U_1 \\ \wr\| & & \wr\uparrow \; e^{2\pi i w t} & & \| \\ \mathbb{Z} & \longrightarrow & \mathbb{R} \ni t & \xrightarrow{e^{2\pi i}} & S^1 \end{array}$$

Fundamental domain in $\mathbb{R}$, $0 \leq t \leq 1$.

I want to decide how to identify $\mathcal{U}'$ with lattices. The point is that an element $g \in \mathcal{G} = GL_n(\mathbb{C}[z,z^{-1}])$ gives ~~us~~ a clutching function ~~used~~ to define a bundle over $\mathbb{P}^1$ trivial off 0 and ∞. The bundle has sections (f_1, f_2) where $f_1 \in k[z]^n$, $f_2 \in k[\frac{1}{z}]^n$ $\ni$ $f_1 = g f_2$. Do it this way so that $g = z^r$ gives ~~us~~ $\mathcal{O}(r)$ which has degree r. Seems to me I want to look at the lattice at ∞ given by gR^n where $R = k[\frac{1}{z}]$.

Formulas: Embed $0 \leq T \leq 1$, T self-adjoint into the building as follows.

$$0 \longmapsto R^n$$

$$1 \longmapsto zR^n$$

Next use the identification of lattices L between R^n and zR^n with subspaces of $\mathbb{C}^n$. T determines a sequence of subspaces

$$0 \leq W_0 < W_1 < \cdots < W_p \leq V \qquad \begin{array}{l} T = t_i \text{ on } W_i \ominus W_{i-1} \\ T = 0 \text{ on } W_0 \\ 1 \text{ on } V \ominus W_p \end{array}$$

where T has eigenvalues $0 < t_1 < \cdots < t_p < 1$ not 0 or 1.

This sequence corresponds to a sequence of lattices

$$L_0 < \cdots < L_p$$

Especially noteworthy is the fact the fundamental domain cons. of T with $0 \leq T < 1$ becomes identified with lattice [redacted] chains $L_0 < \cdots < L_p$ where $L_0 = R^n$.

[redacted] Let's try again to understand lattices $L \subset F^n$. Put

[redacted]

$$V_p = Re_1 + \cdots + Re_p + R\pi e_{p+1} + \cdots + R\pi e_n \qquad 0 \leq p \leq n$$

and extend to all $p \in \mathbb{Z}$ by

$$V_{p+qn} = \pi^{-q} V_p$$

Thus $V_0 = \pi R^n$, $V_n = R^n$. [redacted] Let

$$e_{p+qn} = \pi^{-q} e_p, \text{ whence } \quad V_p / V_{p-1} \simeq k e_p$$

Suppose given a lattice L. Put

$$F_j = \pi^{-j} L \qquad j \in \mathbb{Z}.$$

As before from the Schreier isomorphism relating $\{F_j\}$ to $\{V_p\}$ I will get a function

$$\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}$$

defined by $\alpha(p) =$ least j such that $V_p = V_{p-1} + \pi^{-j} L \cap V_p$

Clearly $\alpha(p+n) = \alpha(p)+1$, so that α is determined by the values $\alpha(1), \alpha(2), \ldots, \alpha(n)$. Choose

$$x_p \in \pi^{-\alpha(p)} L \cap V_p$$

so that $x_p \in e_p + V_{p-1}$. Assume now we are in an equi-characteristic situation. Then I will be able conclude as before that

$$F_j / F_{j'} \xleftarrow{\sim} \bigoplus_{j' < \alpha(a) \leq j} k x_a$$

$\blacksquare$

Formulas:

$\mathcal{G} = GL_n(\mathbb{C}[z, z^{-1}])$, $\mathcal{G}' =$ subgp of α, $\alpha(1) = 1$

$\mathcal{U}$ = subgroup of α such that $|z| = 1 \Longrightarrow \alpha(z) \in U_n$

(equivalently if $\alpha(z) = \sum a_n z^n$, then

$$\alpha(z)^{-1} = \sum a_n^* z^{-n})$$

$\mathcal{U}' = \mathcal{U} \cap \mathcal{G}'$, $\quad \mathfrak{gl}_n = n \times n$ matrices

$\mathfrak{h}$ = hermitian $n \times n$ - matrices

Lemma: If $X, Y \in \mathfrak{gl}_n$ are such that $e^{2\pi i X} = e^{2\pi i Y}$, then

$$e^{2\pi i \omega X} e^{-2\pi i \omega Y} = F(e^{2\pi i \omega})$$

where $F(z) \in \mathcal{G}'$. If further $X, Y \in \mathfrak{h}$, then $F \in \mathcal{U}'$.

$\tilde{\mathcal{P}}$ = set of holomorphic functions $\omega \mapsto A(\omega)$ from $\mathbb{C}$ to GL_n which are of the form $A(\omega) = F(e^{2\pi i \omega}) e^{2\pi i \omega X}$ with $F \in \mathcal{G}'$, $X \in \mathfrak{gl}_n$.

$\mathcal{P}$ = subset where $F \in \mathcal{U}'$, $X \in \mathfrak{h}$.

P = set of paths $\omega \longmapsto A(\omega) \in U_n$, $0 \le \omega \le 1$ of the form $F(e^{2\pi i \omega}) e^{2\pi i \omega X}$ with $F \in \mathcal{U}'$, $X \in \mathfrak{h}$.

The lemma implies that set-theoretically P is a principal (left) $\mathcal{U}'$-bundle over U_n. One maps $P \longrightarrow U_n$ by $A \longmapsto A(1)$; thus if $A(\omega) = F(e^{2\pi i \omega})\, e^{2\pi i \omega X}$, then $A(1) = e^{2\pi i X}$. According to the lemma two elements of P A, B with $A(1) = B(1)$ can be rep.

$$A(\omega) = F(e^{2\pi i \omega})\, e^{2\pi i \omega X}$$
$$B(\omega) = G(e^{2\pi i \omega})\, e^{2\pi i \omega X}$$

whence A, B differ by $FG^{-1} \in \mathcal{U}'$; in fact they differ by a unique element of $\mathcal{U}'$ since a holom. function is determined by its values on a countable set of points.

Relate P to building. Every element of P has a unique representation $A = F(e^{2\pi i \omega})\, e^{2\pi i \omega X}$ where $0 \leq X < 1$. Let the ~~distinct~~ non-zero eigenvalues of X be $0 < \lambda_1 < \dots < \lambda_p$, let the projector on the corresp. eigenspace be E_i, and E_0 the projector on the zero eigenspace, so that

$$X = \sum_{i=0}^{p} \lambda_i E_i$$

$$e^{2\pi i X} = \sum_{i=0}^{p} e^{2\pi i \lambda_i} E_i$$

with $\lambda_0 = 0$.

To X I want to associate a simplex in the building. The vertices correspond to moving $\lambda_1, \ldots, \lambda_i$ to 0 and $\lambda_{i+1}, \ldots, \lambda_p$ to 1, whence we get the lattice

$$L_i = \left(E_0 + \cdots + E_i + zE_{i+1} + \cdots + zE_p\right) \mathbb{C}[z]^n$$

Here $0 \le i \le p$, $L_0 < L_1 < \cdots < L_p = \mathbb{C}[z]^n$.
Therefore the formula goes as follows:

~~Given $A = F(e^{2\pi i \omega}) e^{2\pi i \omega X}$, $0 \le X < 1$, $F \in \mathfrak{U}'$, I write~~

$$\cancel{X = \sum_{i=1}^{p} \lambda_i E_i \qquad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_p < 1}$$

~~where~~ $\cancel{I = \sum_{i=0}^{p} E_i}$

Given $A = F(e^{2\pi i \omega})\, e^{2\pi i \omega X} \in \mathcal{P}$ with $F \in \mathfrak{U}'$, $X \in \mathfrak{h}$, $0 \le X < 1$, I write

$$X = \sum_{i=1}^{p} \lambda_i E_i \qquad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_p < 1$$

$$E_0 = I - (E_1 + \cdots + E_p) = \text{proj. on } 0\text{-eigenspace}$$

and associate to A the p-simplex in the building with the vertices

$$L_i = F(z)\left(E_0+\cdots+E_i + zE_{i+1}+\cdots+zE_p\right)\mathbb{C}[z]^n$$

$i=0,\dots,p$. I send A to the point in this simplex with barycentric coordinates $(\lambda_1, \lambda_2-\lambda_1, \dots, \lambda_p-\lambda_{p-1}, 1-\lambda_p)$.

Conversely given a point in the building X, $\sum_{i=0}^{p} t_i L_i$ where $\pi L_p \subset L_0 < \cdots < L_p$ and $t_i \geq 0$. Then if $F \in \mathcal{U}'$ is the unique thing such that $F^{-1}L_p = \mathbb{C}[z]^n$, we can put

$$E_i = \text{proj on } F^{-1}L_i \ominus F^{-1}L_{i-1}$$

$$E_0 = \text{———}\ F^{-1}L_0 \ominus \pi F^{-1}L_p$$

whence $L_i = F(E_0+\cdots+E_i+zE_{i+1}+\cdots+zL_p)\,\mathbb{C}[z]^n$.

February 14, 1975

Cell decomposition of a flag manifold

Let V be an n-dimensional vector space over the field k with the basis $e_1, \ldots, e_n$. I will identify V with the space k^n of column vectors of length n, with e_i = column vector with entry 1 in the ith column and zero elsewhere. Then $\mathrm{Aut}(V)$ is identified with $GL_n k$ by the rule

$$g \in \mathrm{Aut}(V) \quad \text{corresp. to} \quad (g_{ij}) \in GL_n k$$
$$\text{iff} \qquad g(e_i) = \sum g_{ji}\, e_j$$

Let B be the subgroup of $GL_n(k)$ fixing the flag

$$V_p = ke_1 + \cdots + ke_p \qquad 1 \le p \le n.$$

B is the group of upper-triangular matrices with diagonal entries in $k^{\bullet}$. Let T = group of invertible diagonal matrices, B^u = upper-triangular matrices with 1's on the diagonal. B is the semi-direct product $T \ltimes B^u$.

Fix a subset σ of $\{1, \ldots, n\}$ containing n, say $\sigma = \{i_1, \ldots, i_r\}$ where $0 < i_1 < \cdots < i_r = n$. Denote by $Y_\sigma(V)$ the set of flags in V

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = V$$

such that $\dim(F_j) = i_j$. $(i_0 = 0)$.

It is easily seen that a subspace W of V is normalized (left-invariant) by T iff it is of the form $W = \bigoplus_{p \in S} k e_p$ where $S \subset \{1, \dots, n\}$. Consequently $\{F_j\}$ is a T-fixpt of $Y_\sigma(V)$ iff

$$F_j = \bigoplus_{p \in S_j} k e_p$$

where $\phi = S_0 \subset S_1 \subset \cdots \subset S_r = \{1, \dots, n\}$ is a chain of subsets with $\operatorname{card}(S_j) = i_j$. If we let $\alpha : \{1, \dots, n\} \to \{1, \dots, r\}$ be the map such that $\alpha(S_j - S_{j-1}) = j$, it is clear we get:

<u>Assertion:</u> The T-fixpts of $Y_\sigma(V)$ are in 1-1 corresp. with maps $\alpha : \{1, \dots, n\} \to \{1, \dots, r\}$ such that $\operatorname{card} \{p \mid \alpha(p) \leq j\} = i_j$. To α corresp. the flag

$$F_j = \bigoplus_{\alpha(p) \leq j} k e_p .$$

Let $\{F_j\} \in Y_\sigma(V)$. The filtration $\{F_j\}$ induces a filtration on V_p / V_{p-1} namely

$$(F_j \cap V_p + V_{p-1}) / V_{p-1} \subset V_p / V_{p-1}$$

and similarly $\{V_p\}$ induces a filtration

must exclude $k = \mathbb{F}_2$ where $T = \{1\}$.

$$(F_j \cap V_p + F_{j-1})/F_{j-1} \hookrightarrow F_j/F_{j-1}$$

on F_j/F_{j-1}. One has the canonical Schreier isomorphism:

$$\begin{array}{ccc}
gr_j^{\{F_j\}}(V_p/V_{p-1}) & = & \dfrac{F_j \cap V_p + V_{p-1}}{F_{j-1}\cap V_p + V_{p-1}} \\
 & & \uparrow \wr \\
 & & \dfrac{F_j \cap V_p}{F_{j-1}\cap V_p + F_j \cap V_{p-1}} \\
 & & \downarrow \wr \\
gr_p^{\{V_p\}}(F_j/F_{j-1}) & = & \dfrac{F_j \cap V_p + F_{j-1}}{F_j \cap V_{p-1} + F_{j-1}}
\end{array}$$

Because $\dim(V_p/V_{p-1}) = 1$, there is a unique $\alpha(p)$, $1 \leq \alpha(p) \leq r$ such that

(1) $$F_j \cap V_p + V_{p-1} = \begin{cases} V_{p-1} & j < \alpha(p) \\ V_p & j \geq \alpha(p). \end{cases}$$

~~can~~

~~Choose $x_p \in F_{\alpha(p)} \cap V_p$ so that $x_p \equiv e_p \bmod V_{p-1}$~~

~~By the Schreier isomorphism, the filtration~~

~~$F_j \cap V_p + F_{j-1}$ $\qquad 0 \leq p \leq r$~~

~~has 1 dimensional jumps precisely at those p such that $\alpha(p) = j$, and moreover~~

Thus I get a function $\alpha : \{1, \dots, n\} \to \{1, \dots, r\}$ ④
which depends only on the flags $\{F_j\}, \{V_p\}$, and hence which is an invariant of the B-orbit of $\{F_j\}$ in $Y_\sigma(V)$.

By $(*)$, I can choose $x_p \in F_{\alpha(p)} \cap V_p$ so that $x_p \equiv e_p \bmod V_{p-1}$. Using the Schreier isom, one has

$$F_j \cap V_p + F_{j-1} = \begin{cases} F_j \cap V_{p-1} + F_{j-1} & \text{if } \alpha(p) \neq j \\ kx_p \oplus \left(F_j \cap V_{p-1} + F_{j-1}\right) & \text{if } \alpha(p) = j . \end{cases}$$

It follows that

$$F_j = \bigoplus_{\alpha(p)=j} kx_p \oplus F_{j-1}$$

$$(2) \qquad F_j = \bigoplus_{\alpha(p)\leq j} kx_p$$

Since $x_p \equiv e_p \bmod (V_{p-1})$, one has

$$x_p = g_{1p} e_1 + \dots + g_{p-1,p} e_{p-1} + e_p$$

where $g = (g_{ap}) \in B^u$. Thus we see that the flag $\{F_j\}$ is in the B^u-orbit of the flag

$$\Big\{ \bigoplus_{\alpha(p)\leq j} ke_p \Big\},$$

hence we get:

<u>Assertion:</u> The B-orbits on $Y_\sigma(V)$ are in 1-1 corresp. with maps $\alpha: \{1, \dots, n\} \to \{1, \dots, r\}$ such that $\lambda_j = \operatorname{card}\{p \mid \alpha(p) \le j\}$. The correspondence is given as follows:

To α one associates the B-orbit of the T-invariant flag $\{\bigoplus_{\alpha(p)\le j} k e_p\}$.

To a flag $\{F_j\}$ one associates the function α defined by

$$\alpha(p) = \text{least } j \text{ such that } F_j \cap V_p + V_{p-1} = V_p.$$

In particular each B-orbit contains a unique T-fixpoint, and B^u acts transitively on each B-orbit.

Next analyze the choice of x_p:

$$0 \to F_{\alpha(p)} \cap V_{p-1} \longrightarrow F_{\alpha(p)} \cap V_p \longrightarrow V_p / V_{p-1} \to 0$$

Recall was any element of $F_{\alpha(p)} \cap V_p$ of the form $g_{p1} e_1 + \dots + g_{p-1,p} e_{p-1} + e_p$. Since $F_{\alpha(p)} \cap V_{p-1}$ has the basis $\{x_q \mid q < p, \ \alpha(q) \le \alpha(p)\}$, it is clear that we can alter our choice of x_p by adding a suitable element of $F_{\alpha(p)} \cap V_{p-1}$

and so arrange that $g_{ap}=0$ for $a<p$, $\alpha(a)\le\alpha(p)$, in which case x_p is uniquely determined.
So I get

<u>Assertion</u>: Given $\{F_j\}$ with the invariant α. There exists a unique matrix $g\in B^u$ such that $g_{ap}=0$ for $a<p$, $\alpha(a)\le\alpha(p)$, and

$$g\Big\{\bigoplus_{\alpha(p)\le j} ke_p\Big\} = \{F_j\}.$$

In particular the orbit $B\{\bigoplus_{\alpha(p)\le j} ke_p\}$ is set-theoretically isomorphic to an affine space of dimension equal to card $\{(a,p)\mid a<p,\ \alpha(a)>\alpha(p)\}$.

The subgroup of $g\in B^u$ with $g_{ap}=0$ for $a<p$, $\alpha(a)\ge\alpha(p)$ acts simply-transitively on the orbit $B\{\bigoplus_{\alpha(p)\le j} ke_p\}$. This is the unipotent gp. normalized by T with the roots $\{(a,p)\mid {a<p \atop \alpha(a)>\alpha(p)}\}$.

Let $T(V)$ be the Tits building of G. It is the simplicial complex associated to the poset of proper subspaces of V. We have seen above that in each B-orbit of simplices, there is exactly one simplex fixed by T. Hence

$$T(V)^T \xrightarrow{\sim} B\backslash T(V).$$

On the other hand $T(V)^T$ is the simplicial complex associated to the poset of proper subsets of $\{1,\dots,n\}$, hence

$$\left|T(V)^T\right| = \partial\Delta(n-1).$$

is a sphere of dimension $n-2$.

~~Now I can understand Serre's proof that the building is a bouquet of $(n-2)$-spheres. He orders the elements of the Weyl group according to length. This is something like a Morse function on $B\backslash T(V)$.~~

Cell decomposition in space of lattices

Let R be a discrete valuation ring with fraction field K, unique maximal ideal $\mathfrak{m} = R\pi$, residue field $k = R/\mathfrak{m}$. Let V be a vector space of dim. n over F. An R-lattice in V is by definition an R-submodule L such that if $x_1, \dots, x_n$ is a basis for V over F, then for some N we have

$$\pi^N(Rx_1 + \dots + Rx_n) \subset L \subset \pi^{-N}(Rx_1 + \dots + Rx_n).$$

Equivalently, L is fin. gen. over R and $\bigcup \pi^{-\nu} L = V$. Since R is a PID, it follows that $L = Rx_1 + \dots + Rx_n$ for some basis $x_1, \dots, x_n$ of V over K.

The building (or Tits complex) of V with respect to R, denoted $T(V;R)$, is defined to be the simplicial complex whose vertices are R-lattices L in V and whose $(r-1)$-simplices are chains of lattices

(*) $$L_1 < \dots < L_r$$

such that $\pi L_r < L_1$.

The group $G = \mathrm{Aut}(V)$ acts transitively on the set of lattices. We fix a basis $e_1, \dots, e_n$ for V over K and identify $V = K^n$, $G = GL_n F$ so that an auto. g of V is identified with the matrix (g_{ij}) given by

$$g(e_i) = \sum_{j=1}^{n} g_{ji} e_j.$$

The stabilizer of the lattice $R^n = Re_1 + \cdots + Re_n$ is $GL_n R$, hence the set of lattices in K^n ~~is~~ may be identified with $GL_n K / GL_n R$.

Consider the simplex $(*)$ and put

$$d_j = \dim_k (L_j / L_0) \qquad j = 1, \dots, r$$

where $L_0 = \pi L_r$.

Acting on $(*)$ by an element of G we can conjugate it to a simplex $(L'_1 < \cdots < L'_r)$ with $L'_r = R^n$. Since $GL_n R \twoheadrightarrow GL_n k$, we can further arrange that $(*)$ is conjugate to the simplex with $L'_j = V_{d_j}$, where

$$V_i = Re_1 + \cdots + Re_i + \pi(Re_{i+1} + \cdots + Re_n) \qquad 0 \le i \le n$$

In particular one has that the sequence $0 \le d_1 < d_2 < \cdots < d_r = n$ describes the G-orbit of the simplex $(*)$.

Let B be the subgroup of G which leaves invariant the chain $V_0 < V_1 < \cdots < V_n$. B is the group of matrices (g_{ij}) with coefficients in R such that $g_{ii} \in R^\bullet$ and $g_{ij} \in \mathfrak{m}$ for $i > j$. (Note: such a matrix is invertible mod $\mathfrak{m}$, hence invertible). I want now to determine the B-orbits on the simplices of $T(V; R)$.

Define V_p, e_p for all $p \in \mathbb{Z}$ by

$$V_{i+tn} = \pi^{-t} V_i \qquad i = 1, \dots, n, \quad t \in \mathbb{Z}$$

$$e_{i+tn} = \pi^{-t} e_i$$

whence $V_p / V_{p-1} \simeq k e_p$, V_p is invariant under $\mathcal{B}$.

Given a simplex $L_1 < \dots < L_r$, define L_j for all $j \in \mathbb{Z}$ by

$$L_{i+tr} = \pi^{-t} L_i \qquad i = 1, \dots, r, \quad t \in \mathbb{Z}.$$

By the assumption $L_0 = \pi L_r \subset L_1$, it follows that $\{L_j\}$ is a chain of lattices such that $\pi^{-1} L_j = L_{j+r}$.

Next we compare the filtrations $\{V_p\}$, $\{L_j\}$ using the Schreier isom. The layer $V_{p-1} \subset V_p$ being of codim 1, there exists ! $\alpha(p) \in \mathbb{Z}$ such that

$$L_j \cap V_p + V_{p-1} = \begin{cases} V_{p-1} & j < \alpha(p) \\ V_p & j \geq \alpha(p). \end{cases}$$

In view of the invariance of the filtration under multiplication by π^{-1}, the function $\alpha : \mathbb{Z} \to \mathbb{Z}$ satisfies

$$\alpha(p+n) = \alpha(p) + r.$$

Hence α is determined by the numbers $\alpha(1), \dots, \alpha(n)$.
α is an invariant of the $\mathcal{B}$-orbit of $\{L_j\}$.

Choose $x_p \in L_{\alpha(p)} \cap V_p$ such that $x_p \notin V_{p-1}$.

By the Schreier isom, we have

$$\frac{L_j \cap V_p + L_{j-1}}{L_j \cap V_{p-1} + L_{j-1}} \xleftarrow{\sim} \frac{L_j \cap V_p}{L_{j-1} \cap V_p + L_j \cap V_{p-1}} \xrightarrow{\sim} \frac{L_j \cap V_p + V_{p-1}}{L_{j-1} \cap V_p + V_{p-1}}$$

$$\simeq \begin{cases} 0 & j \neq \alpha(p) \\ k x_p & j = \alpha(p) \end{cases}$$

Since L_j / L_{j-1} is a vector space over k, it follows that

$$L_j / L_{j-1} \xleftarrow{\sim} \bigoplus_{\alpha(p)=j} k x_p ,$$

hence also

$$L_j / \pi L_j = L_j / L_{j-n} = \bigoplus_{j-n < \alpha(p) \leq j} k x_p$$

By Nakyama's lemma this means

$$L_j \xleftarrow{\sim} \bigoplus_{j-n < \alpha(p) \leq j} R x_p$$

Notice that once $x_1, \ldots, x_n$ have been chosen, I can ~~put~~ define

$$x_{i+tn} = \pi^{-t} x_i .$$

~~Notice also that since $V_p / V_{p-1} \xleftarrow{\sim} k x_p$, $x_1, \ldots, x_n$ is a basis for R^n over k~~ Since $V_p / V_{p-1} \xleftarrow{\sim} k x_p$ the matrix $g = (g_{ij})$ defined by

$$x_i = \sum g_{ji} e_j$$

is in the Iwahori group $\mathcal{B}$. Also $x_p = g(e_p)$ for all $p \in \mathbb{Z}$. Thus

$$\{L_j\} = g\Big\{\bigoplus_{j-r<\alpha(p)\leq j} Re_p\Big\}.$$

It follows that the function α, which is an invariant of the $\mathcal{B}$-orbit of $\{L_j\}$, determines this orbit.

Next, I want to produce a normal form for the elements of the $\mathcal{B}$-orbit described by α, that I want a normal form for the choice of x_p. (preferred choice). Let $\Lambda \subset R$ be a subset such that $\Lambda \xrightarrow{\sim} k$ under the mod π-reduction; I suppose $0 \in \Lambda$. Then every element u of K has a Laurent series expansion

$$\sum_{\nu\leq N} \lambda_\nu \pi^{-\nu}$$

with $\lambda_\nu \in \Lambda$ uniquely determined by requiring

$$u - \sum_{\mu<\nu\leq N} \lambda_\nu \pi^{-\nu} \in R\pi^{\mu}$$

for each μ.

$$u \text{ "=" } \sum_\nu \lambda_\nu \pi^{-\nu}$$

with $\lambda_\nu \in \Lambda$, $\lambda_\nu = 0$ for $\nu \gg 0$; this Laurent series is uniquely determined by requiring

$$u - \sum_{\nu > q} \lambda_\nu \pi^{-\nu} \in R\pi^q$$

for each $q \in \mathbb{Z}$.

In an analogous way each element of K^n has a ~~unique~~ Laurent series expansion

$$v \text{ "=" } \sum_{a \in \mathbb{Z}} \lambda_a e_a \qquad \lambda_a \in \Lambda, \quad \lambda_a = 0 \quad a \gg 0.$$

Now suppose given a simplex $\{L_j\}$ as above with associated invariant α. Then as $x_p \in V_p$ it will have an expansion

$$x_p \text{ "=" } \sum_{a<p} \lambda_{ap} e_a$$

<u>Assertion</u>: There is a unique choice for x_p such that $\lambda_{pp} = 1$ and

$$\lambda_{ap} = 0 \quad \text{for} \quad a < p, \ \alpha(a) \le \alpha(p).$$

Hence the B-orbit ~~[illegible]~~ with invariant α is in 1-1 correspondence with $\Lambda^{\ell(\alpha)}$, where

$$\ell(\alpha) = \sum_{i=1}^{n} \operatorname{card}\{a<i \mid \alpha(a) > \alpha(i)\}.$$

Proof: We recall

$$0 \to L_{\alpha(p)} \cap V_{p-1} \longrightarrow \overset{\substack{x_p \\ \cap}}{L_{\alpha(p)} \cap V_p} \longrightarrow V_p/V_{p-1} \to 0$$

was chosen to map onto a non-zero element of V_p/V_{p-1}. So we can choose it to map onto e_p, which means $\lambda_{pp} = 1$. We are free to modify it by any element of $L_{\alpha(p)} \cap V_{p-1}$. Now $L_{\alpha(p)} \cap V_{p-1}$ contains x_a for $a<p$, $\alpha(a) \le \alpha(p)$. Since the leading term of x_a is e_a, we can modify x_p so that $\lambda_{ap} = 0$, and if this is done in decreasing order over the set $_p = \{a<p \mid \alpha(a) \le \alpha(p)\}$, we won't change earlier coeffs. Choose m so that $\pi^m R^n \subset L_{\alpha(p)} \cap V_{p-1}$, and do this process until we have an x_p such that

for some $\lambda_{ap} \in \Lambda$

$$x_p - \sum_{\substack{a<p \\ \alpha(a)>\alpha(p)}} \lambda_{ap} e_a - e_p \in \pi^m R^n$$

Then we can take $\quad x_p = \sum_{\substack{a<p \\ \alpha(a)>\alpha(p)}} \lambda_{ap} e_a + e_p$

as asserted. This gives existence of the normal form.

If one had 2 such choices for x_p look at the first coefficient where they differ. Subtracting the two choices, we would get an element of $L_{\alpha(p)}$ with leading term $c e_a$ with $c \neq 0$ and $a<p$, $\alpha(a)>\alpha(p)$. This contradicts the fact that the leading term of any element of $L_{\alpha(p)}$ involves e_a with $\alpha(a) \leq \alpha(p)$.

So once one has uniqueness, it is clear that $\pi^{-t} x_p = x_{p+tn}$, so that $x_1, \dots, x_n$ determine the rest. It follows that we get a mapping of the B-orbit corresp. to α to the $\Lambda^{\ell(\alpha)}$, and this mapping is injective. It is onto because the $\{\lambda_{ap}\}$ give us a $g \in B$ which gives us a lattice from which we could recover the λ_{ap}.

Example: Take a single lattice $L \subset F^2$.

Case 1: $\alpha(1) \geq \alpha(2)$. Consider x_1. Here any $a < 1$ is of the form $1-2i$, $2-2i$ with $i > 0$ and

$$\alpha(1-2i) = \alpha(1) - i < \alpha(1)$$
$$\alpha(2-2i) = \alpha(2) - i < \alpha(2) \leq \alpha(1)$$

so $a < 1 \implies \alpha(a) < \alpha(1)$, so $\lambda_{a1} = 0$ for $a < 1$ and $x_1 = e_1$.

Next consider $a(2)$. Again $2-2i < 2$ $\implies \alpha(2-2i) = \alpha(2) - i < \alpha(2)$; but $1-2i < 2$

$$\alpha(1-2i) = \alpha(1) - i > \alpha(2)$$

for $i = 0, \dots, \alpha(1) - \alpha(2) - 1$. Thus

$$x_2 = (\lambda_0 + \lambda_1 \pi + \dots + \lambda_g \pi^g) e_1 + e_2$$

where $g = \alpha(1) - \alpha(2) - 1$.

So in the case $\alpha(1) \geq \alpha(2)$ the lattice L has a preferred basis of the form

$$\begin{cases} \pi^{\alpha(1)} e_1 \\ \left(\lambda_0 \pi^{\alpha(2)} + \dots + \lambda_{\alpha(1)-\alpha(2)-1} \pi^{\alpha(1)-1}\right) e_1 + \pi^{\alpha(2)} e_2 \end{cases}$$

Case 2: $\alpha(1) < \alpha(2)$.

Here $a < 2 \implies \alpha(a) < \alpha(2)$, so $\lambda_{a2} = 0$ and $x_2 = e_2$. Also a odd $< 1 \implies \lambda_{a1} = 0$.

But $2-2i < 1$ $\alpha(2-2i) = \alpha(2) - i > \alpha(1)$ for $i = 1, \dots, \alpha(2) - \alpha(1) - 1$. Thus

$$x_1 = e_1 + \left(\lambda_1 \pi + \dots + \lambda_{\alpha(2)-\alpha(1)-1} \pi^{\alpha(2)-\alpha(1)-1}\right) e_2$$

so the lattice has a preferred basis of the form

$$\begin{cases} \pi^{\alpha(1)} e_1 + \left(\lambda_1 \pi^{\alpha(1)+1} + \dots + \lambda_{\alpha(2)-\alpha(1)-1} \pi^{\alpha(2)-1}\right) e_2 \\ \pi^{\alpha(2)} e_2. \end{cases}$$

Alternative approach to B-orbits of lattices.

Suppose L is an R-lattice in F^n. Let $e_p = \pi^{-t} e_i$ if $p = i + tn$, $1 \le i \le n$ as above.

Let p_1 be the least p such that $e_p \notin \pi L$. Since $\pi e_{p_1} = e_{p_1 - n} \in \pi L$, $e_{p_1} \in L$, so I get a basis element $x^1 = e_{p_1}$ for L, i.e. Rx^1 is a direct summand of L.

~~Next consider the lattice $\bar{L} = L/Rx^1 \subseteq F^n/Rx^1$ and let p_2 be the least p~~

Let p_2 be the least p such that

$$e_p \notin \pi(L + Fe_{p_1}).$$ Clearly $p_1 < p_2$.

Then $e_{p_2} \in L + Fe_{p_1}$, hence $\exists\ x^2 \in L$ and $f \in F$ such that

$$x^2 = e_{p_2} + f e_{p_1}.$$

Moreover f is unique modulo R. (In effect if $e_{p_2}+f_i e_{p_1} \in L$, $i=1,2$, then $(f_1-f_2)e_{p_1} \in L \Rightarrow f_1-f_2 \in R$).

There is a unique rep. for f of the form $f = \sum_{i>0} \lambda_i \pi^{-i}$; if $f \neq 0$ let a be the largest $\ni \lambda_a \neq 0$. ~~[illegible]~~

$$x^2 = e_{p_2} + \sum_{i>0} \lambda_i e_{p_1+in}$$

If $p_1+an > p_2$, then $\pi^a x^2 \in \pi L$ and have leading term $\lambda_a e_{p_1}$ contradicting the fact that e_{p_1} is the first e_p outside of πL. Thus $p_1+an \leq p_2$, in fact $< p_2$ as $e_{p_2} \notin Fe_{p_1}$.

So we get a unique $x^2 \in L$ of the form

$$x^2 = e_{p_2} + \sum_{\substack{i>0 \\ p_1+in<p_2}} \lambda_i e_{p_1+in}$$

Since $x^2 \notin \pi L + Fe_{p_1}$, it follows that $Rx^1 + Rx^2$ is a direct summand of L.

One can continue this process by letting e_{p_3} be the first e_p outside of $\pi L + Fx^1 + Fx^2 = \pi L + Fe_{p_1} + Fe_{p_2}$. One will will get a unique x^3 in L of the form

$$x^3 = e_{p_3} + \sum_{\substack{p_1<a<p_3 \\ a \equiv p_1\ (n)}} \lambda_a e_a + \sum_{\substack{p_2<a<p_3 \\ a \equiv p_2\ (n)}} \lambda_a e_a$$