January 12, 1975. Schubert cells (cont.)

Type of a Schubert cell: Let $Y = Y_{d_1, \dots, d_{\mu-1}}(V)$ $\dim V = n$, let $B$ be a Borel, $C$ an orbit of $B$ in $Y$. If $(V_p)$ is the flag belonging to $B$, then $C$ is the subset of $Y$ consisting of $0 < F_1 < \dots < F_\mu$ $\quad \dim F_j = d_j$ such that

$$\dim (F_j \cap V_p) = \operatorname{card}\{a \le p \mid \alpha(a) \le j\}$$

where $\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, \dots, \mu\}$ is a map $\ni$ $d_j = \operatorname{card} \alpha^{-1}\{1, \dots, j\}$.

Def: The map $\alpha$ is the type of the Schubert cell $C$.

I must verify that $\alpha$ is independent of the choice of $B$. Let $P = \{g \mid gC = C\}$ be the normalizer of $C$. I have seen that the different Borels of which $C$ is an orbit are those Borels contained in $P$. Given $B' \subset P$ I can choose $T \subset B \cap B'$ in which case $B' = \sigma B \sigma^{-1}$ for some $\sigma \in$ Weyl group of $P$. I have seen that $P$ is generated by $B$ and those simple reflections $s_i$ such that $\alpha s_i = \alpha$; the Weyl group of $P$ is generated by these $s_i$. Thus $\alpha$ is not changed by any $\sigma$ in the Weyl group of $P$. QED.

[redacted].

Remark: Any two Schubert cell of type $\alpha$ are conjugate under $GL(V)$. For [redacted] any two Borels are conjugate, and Schubert cells [redacted] for the same Borel are characterized by the function $\alpha$.

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Consider $Y_{12\cdots d}(V)$. The type of a [redacted] cell in this case can be identified with an embedding $\{1,2,\dots,d\} \hookrightarrow \{1,\dots,n\}$, $j \mapsto \alpha^{-1}(j)$; or if I want, a subset [redacted] $\{a_1<\cdots<a_d\}$ of $\{1,\dots,n\}$ together with an ordering on this subset given by

$$\alpha^{-1}(\rho j) = a_j \quad \text{some } \rho\in\Sigma_d$$

The roots for this cell are

[redacted]

$$\{(a,p) \mid a<p,\ \alpha(a)>\alpha(p)\}$$

To analyze inclusions among cells in $Y_{12\cdots d}(V)$, fix $T\subset B$, whence we have $V = L_1\oplus\cdots\oplus L_n$, $B$ stabilizing $L_1\oplus\cdots\oplus L_p$ for each $p$. Any other $B'\supset T$ is of the form $\sigma^{-1}B\sigma$ for some $\sigma\in\Sigma_n$ and I recall

$$\mathrm{Roots}(\sigma^{-1}B\sigma) = \{(i,j) \mid \sigma i\leq \sigma j\}$$

Denote by $\varepsilon : \{1,..,d\} \hookrightarrow \{1,..,n\}$ the embedding with $\varepsilon(j) = j$, $1 \le j \le d$, and by $\alpha_\varepsilon : \{1,..,n\} \twoheadrightarrow \{1,..,d+1\}$ the map with $\alpha_\varepsilon(a) = \begin{cases} a & 1 \le a \le d \\ d+1 & a \ge d+1 \end{cases}$. For each $\sigma \in \Sigma_n$ one gets an embedding $\sigma\varepsilon$ and a corresponding epimorphism $\alpha_\varepsilon \sigma^{-1}$.

Each $\alpha$ can be identified with a $T$-fixpt $\bar{\alpha}$ of $Y$, and one has $\sigma\bar{\alpha} = \overline{\alpha\sigma^{-1}}$. I've seen already that

$$\mathrm{Roots}(B\cdot\bar{\alpha}) = \{(i,j) \mid i<j,\ \alpha(i) > \alpha(j)\}$$

hence $$\mathrm{Roots}(\sigma^{-1}B\sigma\cdot\bar{\alpha}) = \{(i,j) \mid \sigma i < \sigma j,\ \alpha(i) > \alpha(j)\}.$$

So I am now interested in the map

$$\sigma \longmapsto \mathrm{Roots}(\sigma^{-1}B\sigma\cdot\bar{\alpha}_\varepsilon) = \{(i,j) \mid \sigma i < \sigma j,\ \alpha_\varepsilon(i) > \alpha_\varepsilon(j)\}$$
$$= R_\sigma \cap R_{-\alpha_\varepsilon}$$

since $\sigma^{-1}B\sigma\cdot\bar{\alpha}_\varepsilon = \sigma^{-1}\cdot(B\sigma\bar{\alpha}_\varepsilon) = \sigma^{-1}B\overline{\alpha_\varepsilon\sigma^{-1}}$, it follows that

$$\mathrm{Type}(\sigma^{-1}B\sigma\bar{\alpha}_\varepsilon) = \alpha_\varepsilon\sigma^{-1} \overset{\text{corres.}}{\longleftrightarrow} \sigma\varepsilon$$

So I understand the composition

$$\begin{array}{ccccc} (T\text{-Borels}) & \longrightarrow & (T\text{-cells} \ni \alpha_\varepsilon) & \longrightarrow\!\!\!\!\rightarrow & (\text{types}) \\ (\sigma) & \longmapsto & (\sigma^{-1}B\sigma\cdot\bar{\alpha}_\varepsilon) & \longmapsto & \alpha_\varepsilon\sigma^{-1} \end{array}$$

The fibres of the first map depend upon the normalizer of the Schubert cell. The fibres of the composite map are cosets $\Sigma_d\sigma$, where $\Sigma_d$ = Weyl group of stabizer of $\bar{\alpha}_\varepsilon$.

Suppose we have an inclusion of F cells $\ni \bar{\alpha}_\varepsilon$

$$\sigma^{-1}B\sigma\bar{\alpha}_\varepsilon \subset \tau^{-1}B\tau\bar{\alpha}_\varepsilon$$

$$(*) \qquad R_\sigma \cap R_{-\alpha_\varepsilon} \subset R_\tau \cap R_{-\alpha_\varepsilon}$$

Now $R_{-\alpha_\varepsilon} = \{ i,j \mid \alpha_\varepsilon(i) > \alpha_\varepsilon(j)\}$ is a disjoint union

$$= \{(i,j) \mid 1 \le j < i \le d\} \amalg \{(i,j) \mid 1 \le j \le d < i \le n\}.$$

So $(*)$ implies $\{(i,j) \mid 1 \le j < i \le d,\ \sigma j > \sigma i\} \subset \{(i,j) \mid {1 \le j < i \le d \atop \tau j > \tau i}\}$.

In other words the order on $\{1,\dots,d\}$ induced by $\sigma$ comes before that induced by $\tau$. Summarizing

<u>Assertion</u>: If one has an inclusion of Schubert cells $C \subset C'$ in $Y_{1,\dots,d}$ and the types of $C, C'$ corres. to embeddings $\delta_1, \delta_2 : \{1,\dots,d\} \hookrightarrow \{1,\dots,n\}$, then the ordering of $\{1,\dots,d\}$ induced by $\delta_1$ precedes that induced by $\delta_2$.

Suppose next I have an inclusion of cells $C \subset C'$ whose types corresp. to embeddings inducing the same order on $\{1, \dots, d\}$. As above we will assume $C, C'$ are the cells $\sigma^{-1}B\sigma\bar{\alpha}_\varepsilon$, $\tau^{-1}B\tau\alpha_\varepsilon$. Since we are assuming $\sigma, \tau$ induce the same ordering we know $R_\sigma, R_\tau$ have same intersection with $\{(i,j) \mid 1 \leq j < i \leq d\}$, so the remaining information on the inclusion is

$$(+) \quad 1 \leq j \leq d < i \leq n \implies (\sigma i < \sigma j \implies \tau i < \tau j).$$

$B$ corresponds to the flag $p \mapsto V_p = \sum_{a \leq p} k e_a$, hence $\sigma^{-1}B\sigma$ is the stabilizer of the flag

$$p \mapsto \sigma^{-1}V_p = \sum_{a \leq p} k e_{\sigma^{-1}a} = \sum_{\sigma(a) \leq p} k e_a$$

Hence

$$\sigma^{-1}B\sigma.\bar{\alpha} = \left\{ F_1 < \cdots < F_d \;\middle|\; \dim(F_j \cap \sigma^{-1}V_p) = \operatorname{Card}\left\{ a \;\middle|\; \begin{matrix} \sigma a \leq p \\ \alpha a \leq j \end{matrix} \right\} \right\}$$

$(\alpha = \alpha_\varepsilon)$

In this cell we use the subspaces where the jumps take place namely the points $\sigma a, \sigma a - 1$ where $a = 1, \dots, d$. In my old notation this cell would have been denoted

$$C(\sigma^{-1}V_{\sigma 1 - 1}, \sigma^{-1}V_{\sigma 1}; \dots; \sigma^{-1}V_{\sigma d - 1}, \sigma^{-1}V_{\sigma d})$$

What I want to show is that under the assumption that $\sigma 1, \dots, \sigma d$ and $\tau 1, \dots, \tau d$ are in order and $C \subset C'$, I want to show

$$(**) \qquad (\sigma^{-1}V_{\sigma j-1}, \sigma^{-1}V_{\sigma j}) \leq (\tau^{-1}V_{\tau j-1}, \tau^{-1}V_{\tau j})$$

for each $j = 1, \ldots, d$. But

$$\sigma^{-1}V_{\sigma j} = \sum_{\sigma a \leq p} k e_a$$

hence (**) amounts to three conditions:

i) $\sigma^{-1}V_{\sigma j-1} \subset \tau^{-1}V_{\tau j-1}$ $\qquad \sigma a < \sigma j, \ 1 \leq j \leq d \implies \tau a < \tau j$

ii) $\sigma^{-1}V_{\sigma j} \subset \tau^{-1}V_{\tau j}$ $\qquad \leq \qquad \leq$

iii) the unique $T$-line in $\sigma^{-1}V_{\sigma j}$ not in $\sigma^{-1}V_{\sigma j-1}$ is also ———— $\tau^{-1}V_{\tau j}$ ———— $\tau V_{\tau j-1}$

iii) is clear because the line in question is $k\sigma^{-1}e_{\sigma j} = ke_j = k\tau^{-1}e_{\tau j}$. i) and ii) are equivalents.

Thus I have only to show

$$\sigma a < \sigma j, \ 1 \leq j \leq d \implies \tau a < \tau j.$$

This is clear if $1 \leq a \leq d$ for we are assuming $\sigma, \tau$ give the same order on $1 \leq j \leq d$. If $a > d$, it is the extra part of the hypothesis (see (+) on previous page).

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Next I want to consider the case where $\sigma \leq \tau$, $\ell(\sigma) + 1 = \ell(\tau)$

<u>Review:</u> Let $\sigma, \tau \in \Sigma_n$ and suppose that $R_\sigma = \{(i,j) \mid \sigma i < \sigma j\}$ differs from $R_\tau$ by the reversal of a single pair $(a,b)$:

$$R_\sigma - (a,b) = R_\tau - (b,a)$$

Then I claim that $\sigma a + 1 = \sigma b$. In effect, if $\sigma a < \sigma c < \sigma b$, then $(a,c) \in R_\tau, (c,b) \in R_\tau \Rightarrow (a,b) \in R_\tau$ a contradiction.

$$\begin{array}{ccc} \{(i,j) \mid \sigma i < \sigma j\} - (a,b) & = & \{(i,j) \mid \tau i < \tau j\} - (b,a) \\ \Big\downarrow \wr \, \sigma & & \Big\downarrow \wr \, \sigma \\ \{(p,q) \mid p < q\} - (\sigma a, \sigma b) & = & \{(p,q) \mid \tau\sigma^{-1} p < \tau\sigma^{-1} q\} - (\sigma b, \sigma a) \end{array}$$

So I see that $\tau\sigma^{-1} = s_{\sigma a}$. Thus

$$\boxed{R_\sigma - (a,b) = R_\tau - (b,a) \implies \sigma b = \sigma a + 1, \quad \tau = s_{\sigma a}\sigma}$$

$$\boxed{R_\sigma - (\sigma^{-1}(i), \sigma^{-1}(i+1)) = R_{s_i\sigma} - (\sigma^{-1}(i+1), \sigma^{-1}(i))}$$

So now compare $\{(i,j) \mid \sigma i < \sigma j, \ \alpha i > \alpha j\}$ $= R_\sigma \cap R_{-\alpha}$ with the corresponding thing for $\tau = s_i \sigma$. The pair $(\sigma^{-1}(i), \sigma^{-1}(i+1))$ gets reversed so we see

$$R_{s_i\sigma} \cap R_{-\alpha} = \begin{cases} R_\sigma \cap R_{-\alpha} + (\sigma^{-1}(i+1), \sigma^{-1}(i)) & \text{if } \alpha(\sigma^{-1}(i)) < \alpha\sigma^{-1}(i+1) \\ R_\sigma \cap R_{-\alpha} & \text{"} \quad = \\ R_\sigma \cap R_{-\alpha} - (\sigma^{-1}i, \sigma^{-1}(i+1)) & \text{"} \quad \alpha(\sigma^{-1}i) > \alpha\sigma^{-1}(i+1) \end{cases}$$

Write this as follows

$$R_{s_i\sigma} \cap R_{-\alpha} = R_\sigma \cap R_{-\alpha} - (\sigma^{-1}i, \sigma^{-1}(i+1)) \quad \text{if} \quad \alpha(\sigma^{-1}i) > \alpha(\sigma^{-1}(i+1)).$$

with obvious modifications in the other cases.

Consequence: Choose $s_{i_1}, s_{i_2}, \ldots, s_{i_p}$ so that

$$R_\sigma \cap R_{-\alpha} > R_{s_{i_1}\sigma} \cap R_{-\alpha} > \cdots\cdots > R_{s_{i_p}\cdots s_{i_1}\sigma} \cap R_{-\alpha} = \phi.$$

Now $R_\tau \cap R_{-\alpha} = \phi$ means $\tau i < \tau j \Rightarrow \alpha i \leq \alpha j$, i.e. $\alpha\tau^{-1}$ is monotone.

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Now go back to examples.

$Y_{12}(V)$. Consider two cells $C \subsetneq C'$ of types $\alpha, \alpha'$. Take case $\alpha^{-1}(1) < \alpha^{-1}(2)$, $\alpha'^{-1}(1) > \alpha'^{-1}(2)$. Let $C = \sigma^{-1}B\sigma\alpha_\varepsilon$, $C' = \tau^{-1}B\tau\alpha_\varepsilon$, whence $\alpha = \alpha_\varepsilon\sigma^{-1}$, $\alpha' = \alpha_\varepsilon\tau^{-1}$. $\alpha^{-1}(j) = (\alpha_\varepsilon\sigma^{-1})^{-1}(j) = \sigma\alpha_\varepsilon^{-1}(j)$ $= \sigma j$. Thus I have $\sigma 1 < \sigma 2$, $\tau 1 > \tau 2$. My assumption $C \subset C'$ says $\{i, j \mid \sigma i < \sigma j, \alpha_\varepsilon i > \alpha_\varepsilon j\} \subset$ same for $\tau$.

The first thing to show is

$$(\sigma^{-1}V_{\sigma 1-1}, \sigma^{-1}V_{\sigma 1}) \leqslant (\tau^{-1}V_{\tau 1-1}, \tau^{-1}V_{\tau 1}).$$

It suffices to show [redacted] $\sigma p < \sigma 1 \Rightarrow \tau p < \tau 1$.
But $\sigma p < \sigma 1 \Rightarrow \alpha(\sigma p) = 3 > \alpha(\sigma 1) = 1$, so by hyp. $\tau p < \tau 1$.

So next I want to establish the following picture

$$\begin{tikzcd}
 & & & \tau^{-1}V_{\tau 2} \arrow[d, phantom, "\vee"] \\
\sigma^{-1}V_{\sigma 2} \arrow[r, equal] \arrow[d, phantom, "\vee"] \arrow[dr, phantom, "\mathrm{tr.}"] & \sigma^{-1}V_{\sigma 2} \arrow[r, dashed, no head] \arrow[d, phantom, "\vee"] & B \arrow[ur, dashed, no head] \arrow[d, phantom, "\vee"] \arrow[r, phantom, "\mathrm{tr.}"] & \tau^{-1}V_{\tau 2-1} \arrow[dl, dashed, no head] \arrow[d, phantom, "\cup"] \\
\sigma^{-1}V_{\sigma 2-1} \arrow[r, equal] \arrow[d, phantom, "\cup"] \arrow[dr, phantom, "\mathrm{tr.}"] & \sigma^{-1}V_{\sigma 2-1} \arrow[dl, dashed, no head] \arrow[d, phantom, "\vee"] \arrow[r, phantom, "\mathrm{rev.}"] & \tau^{-1}V_{\tau 2} \arrow[r, equal] \arrow[d, phantom, "\vee"] & \tau^{-1}V_{\tau 2} \arrow[d, phantom, "\vee"] \\
\sigma^{-1}V_{\sigma 1} \arrow[d, phantom, "\vee"] & A \arrow[dl, dashed, no head] \arrow[r, dashed, no head] & \tau^{-1}V_{\tau 2-1} \arrow[r, equal] & \tau^{-1}V_{\tau 2-1} \\
\sigma^{-1}V_{\sigma 1-1} & & &
\end{tikzcd}$$

Reversal means this sort of Schubert cell pair:
We start with ~~[illegible]~~ a bicart square

$$\begin{tikzcd}
 & W \\
W_1 \arrow[ur, phantom, "\cup"] & & W_2 \arrow[ul, phantom, "\supset"] \\
 & W_0 \arrow[ul, phantom, "\supset"] \arrow[ur, phantom, "\cup"]
\end{tikzcd}$$

with sides of codim 1. Then we get an inclusion

$$\left\{ (F_1 < F_2) \,\middle|\, \begin{array}{l} F_1 \xrightarrow{\sim} W_1/W_0 \\ F_2/F_1 \xrightarrow{\sim} W/W_1 \end{array} \right\} \subset \left\{ (L_1 < L_1 \oplus L_2) \,\middle|\, \begin{array}{l} L_1 \xrightarrow{\sim} W/W_2 \\ L_2 \xrightarrow{\sim} W_2/W_0 \end{array} \right\}$$

i.e. $C(W_0, W_1; W_1, W) \subset C(W_2, W; W_0, W_2)$

Review your ancient analysis of the poset of Schubert cells in $Y_2(V)$. Call this poset $J$.

2 kinds of cells. First type of cells are of the form $C(V_p, V_{p+2}) = \{ F \mid F \oplus V_p = V_{p+2} \}$. The sub poset of $J$ consisting of these cells I call $J^1$. I know it is isom. to the set of layers $(W_1, W_2)$ in $V$ of codim 2 under the relation $(W_1, W_2) \le (W_1', W_2') \iff W_i \subset W_i'$ and $W_2/W_1 \simeq W_2'/W_1'$, and I know this last poset is a class space for $GL_2$.

2nd kind of cells are of the form

$$C(V_{p-1} < V_p < V_{q-1} \le V_q) = \{ F / 0 = F \cap V_{p-1} < F \cap V_p = F \cap V_{q-1} < F \cap V_q = F \}.$$

I denote by $J^2$ the poset consisting of flags $W_1 \subsetneq W_2 \subsetneq W_3 \subsetneq W_4$ such that $\dim W_2/W_1 = \dim W_4/W_3 = 1$ with $(W_1 \quad W_4) \le (W_1' \quad W_4') \iff W_i \subset W_i'$ and $(W_1, W_2) \le (W_1', W_2')$ and $(W_3, W_4) \le (W_3', W_4')$ in the sense of perspectivity. I know $J^2 \sim (BGL_1)^2$.

Let $J^{12}$ = subposet of $J^2$ consisting of $(W_1, W_2, W_3, W_4)$ such that $W_2 = W_3$. It can also be described as the cofibred category over $J^1$ associated to the functor $(W_1, W_2) \longmapsto \mathbb{P}(W_2/W_1)$. I know $J^{12} \sim B(B_2)$.

So we get the squares

$$\begin{tikzcd} J^{12} \arrow[r] \arrow[d, hook] & J^1 \arrow[d, hook] \\ J^2 \arrow[r] & J \end{tikzcd}$$

and ~~[illegible]~~ the point to prove is that this square is cocartesian.

Carry out a similar analysis for $Y_{12}$.
Put $J =$ [redacted] poset of Schubert cells in $Y_{12}$.

2 kinds of cells. First type of cells are of the form

$$C(V_{p-1}, V_p ; V_{q-1}, V_q) = \left\{ (F_1, F_2) \,\middle|\, \begin{array}{l} F_1 \oplus V_{p-1} = V_p \\ F_2 \cap V_p = F_1 \\ F_2 \cap V_{q-1} < F_2 \cap V_q = F_2 \end{array} \right\}$$

where $V_{p-1} < V_p \subset V_{q-1} < V_q$. Call $J^1$ the poset of cells of this form. It is isomorphic to the $J^2$ considered above, hence $J^1 \sim (BGL_1)^2$.

2nd kind of cells are of the form

$$C(V_{p-1}, V_p; V_{q-1}, V_q) = \left\{ (F_1, F_2) \;\middle|\; \begin{array}{l} F_1 \oplus V_{p-1} = V_p \\ 0 = F_2 \cap V_{q-1} < F_2 \cap V_q = F_2 \cap V_{p-1} < F_2 \cap V_p = F_2 \end{array} \right\}$$

where $V_{q-1} < V_q \subset V_{p-1} < V_p$. Call the poset of cells of this form $J^2$; it is isomorphic to the $J^2$ considered above, so it has the homotopy type of $(BGL_1)^2$.

So we have inclusions which are mutual complements

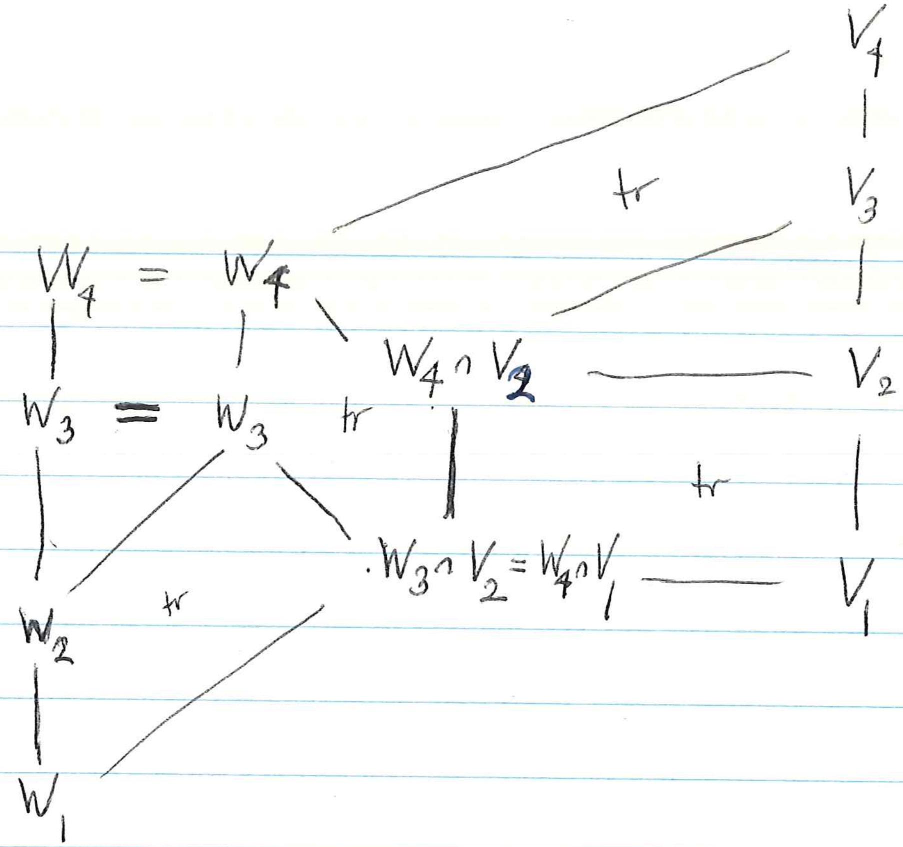
$$\begin{tikzcd} J^1 \arrow[dr, phantom, "\subset" description] & & J^2 \arrow[dl, phantom, "\supset" description] \\ & J & \end{tikzcd} \qquad J^2 = J - J^1$$

Moreover I know that $C \subset C'$, $C \in J^2$, $C' \in J^1$ is impossible, whence $J^1$ is closed under specializing and $J^2$ under generalizing ($\therefore J^2$ is "open").

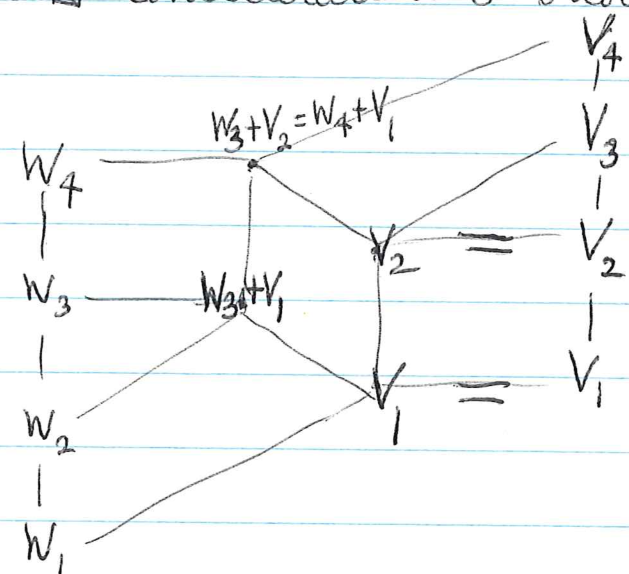
So what I need is for each $C$ in $J^2$ to know the homotopy type of $J^1/C$. So I want to recall the picture of an inclusion

$$C(W_1, W_2; W_3, W_4) \subset C(V_3, V_4; V_1, V_2)$$

where $W_1 < W_2 \subset W_2 < W_4$ and $V_1 < V_2 \subset V_3 < V_4$.




By dual considerations I should have the picture




Thus I have chosen a line $H/V_1$ in $V_4/V_1$ mapping isom to $V_4/V_3$ and so factored my inclusion into

$$C(W_1, W_2; W_3, W_4) \subset C(V_1, H; H, H+V_2) \subset C(V_3, V_4; V_1, V_2) = C$$

Therefore $J^1/C$ is h. eq. to the set of such lines $H/V_1$.

Guess: $J^1/J^2$ is h. eq. to the cat. $J^{12}$ consisting of layers $(W_1, W_2)$ of codim 2 with a decomposition $W_2/W_1 = L_1 \oplus L_2$. We have a functor $J^{12} \longrightarrow J^1/J^2$ sending $W_1 \overset{H_1}{\underset{H_2}{\rightrightarrows}} W_2$ into $C(W_1, H_1; H_1, W_2) \subset C(H_2, W_2; W_1, H_2)$ and my analysis of before ought to show this is a homotopy equivalence. So it would seem that I get a homotopy-cocartesian square

$$\begin{tikzcd} J^{12} \arrow[r] \arrow[d] & J^2 \arrow[d] \\ J^1 \arrow[r] & J \end{tikzcd}$$

since $J^{1,2}$, $J^1$ and $J^2$ will all have the h-type of $(BGL_1)^2$, so will $J$.

January 16, 1975.

$S(Y_{12}(V))$ = poset of Schubert cells in $Y_{12}(V)$ where dim $V$ is countable. There is a functor

$$f : S(Y_{12}(V)) \longrightarrow S(Y_1(V))$$

induced by the map $Y_{12}(V) \longrightarrow Y_1(V)$ sending $(F_1 < F_2) \mapsto F_1$. This functor sends the cell

$$C(V_{a-1}, V_a ; V_{b-1}, V_b) = \left\{ (L_1 \subset L_1 \oplus L_2) \;\middle|\; \begin{array}{l} L_1 \in \mathbb{P}V_a - \mathbb{P}V_{a-1} \\ L_2 \in \mathbb{P}V_b - \mathbb{P}V_{b-1} \end{array} \right\}$$

into the cell $C(V_{a-1}, V_a) = \mathbb{P}V_a - \mathbb{P}V_{a-1}$ in $Y_1(V)$.

Consider the fibres of $f$. Fix a codim 1 layer $W' < W$ in $V$. Then $f^{-1}C(W', W)$ consists of two kinds of cells.

$C(W', W ; Y', Y)$ $\quad Y \subset W'$

$C(W', W ; Z', Z)$ $\quad W \subset Z'$

which are unrelated. Hence

$$f^{-1} C(W', W) = S(\mathbb{P}W') \times S(\mathbb{P}(V/W))$$

Given a cell $C$ in $Y_{12}(V)$, for each $(F_1 < F_2) \in C$ we can consider those lines $L_2$ complementary to $F_1$. As $F_1 < F_2 > L_2$ vary, the possible $L_2$ form a subset in

$Y_1(V)$. If $C = C(V_{a-1}, V_a; V_{b-1}, V_b)$ with $V_a \subset V_{b-1}$, then this subset in $Y_1(V)$ is $C(V_{b-1}, V_b)$. But if if $V_b \subset V_{a-1}$, we seem to get $(\mathbb{P}V_b - \mathbb{P}V_{b-1}) \cup (\mathbb{P}V_a - \mathbb{P}V_{a-1})$ which unfortunately is not a cell.

<u>Basic Question</u>: Fix a codim. 1 layer $(Z';Z)$ and consider all codim 1 layers "independent" of this one, by which I mean a codim 1 layer $(W';W)$ such that $Z/Z'$ appears in $W'$ or in $V/W$, i.e.

$$\frac{Z \cap W}{Z' \cap W + Z \cap W'} = 0$$

i.e.

$$\begin{array}{ccc} Z' \cap W & \text{——} & Z \cap W \\ | & & | \\ Z' \cap W' & \text{——} & Z \cap W' \end{array}$$

is transversal. Examples:

1) If $W \subset Z'$ or $Z \subset W'$.

2) suppose $Z' = 0$, whence $Z$ is a line. In this case $(W';W)$ is independent of $(0, Z)$ iff $Z + W' < Z + W$. Thus either $Z \subset W'$ or $Z \not\subset W$, i.e $Z \notin C(W';W)$. This set is closed under specializing.

2) $Z'$ is a hyperplane, then $(W';W)$ is independent of $(Z';V)$ if either $W \subset Z'$ or $W' \not\subset Z'$, i.e. $C(W';W) \not\subset C(Z';V)$. This set is closed under generalizing.

The question is whether the poset of Schubert cells in $\mathbb{P}V$ independent of a fixed one $(Z';Z)$ is a classifying space for $k^*$.

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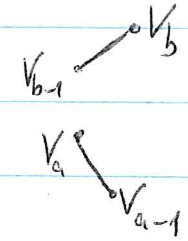
Model for cells in $Y/_{12}$. Three strata cats:

$J_1$ will consist of $V_{a-1} < V_a \subset V_{b-1} < V_b$ ordered so that each layer pair moves by perspectivity. I will denote a typical object of $J_1$ by the symbol  specifically to denote the quotients:

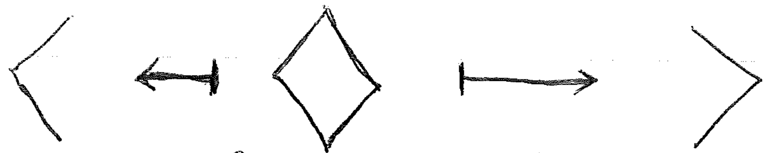



$J_2$ will be the same category as $J_1$ by its objects will be denoted

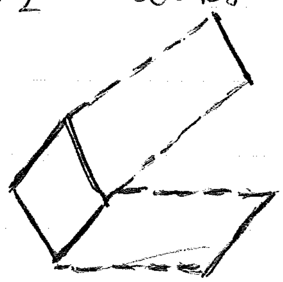
$J_3$ will consist of diamonds
$$\begin{array}{ccc} & V_a & \\ V_{a-1} & & V'_{a-1} \\ & V_{a-2} & \end{array}$$
such that

$(V_{a-2}, V_a)$ moves by perspectivity.

Now form a new category $C$ whose objects those of $J_1, J_2, J_3$. We have functors



and we define $\mathcal{C}$ to be the cofibred category over $1 > 3 < 2$ with these functors as cobase change. This means that a map from an object of $J_3$ to one of $J_2$ looks like



Now I can define functors $\mathcal{C} \longrightarrow Sch(Y_{12})$, $\mathcal{C} \rightrightarrows Sch(Y_1)$ as follows. [redacted] To define the former we already use the embeddings $J_1, J_2 \subset Sch(Y_{12})$ we have. We send $\Diamond$ to $c(<)$; the canonical arrow $\Diamond \to <$ goes to the identity, while the canonical arrow $\Diamond \to >$ goes to the inclusion of $c(<)$ in $c(>)$ associated to $\Diamond$.

One functor $\mathcal{C} \longrightarrow Sch(Y_1)$ looks at the cell [redacted] $\setminus$, the other functor looks at the cell $/$.

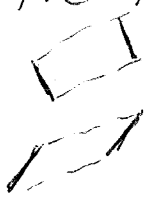
<u>Example:</u> Suppose $\dim(V) = 2$. Then $J_1$ is the set of [redacted] points in $V$, $J_2$ is the set of complements of points, and $J_{12}$ the set of independent lines. So the category $\mathcal{C}$ is the poset [redacted] of simplices in

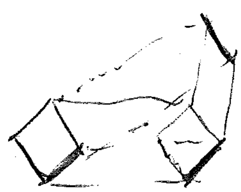
the poset of Schubert cells in $PV$.

From this example it would seem better to make $C$ a fibred cat over $1 < 3 > 2$. Hence there should be canonical arrows $< \longrightarrow \Diamond \longleftarrow >$, and the functor to $Sch(Y_{12})$ should send $\Diamond \mapsto >$.

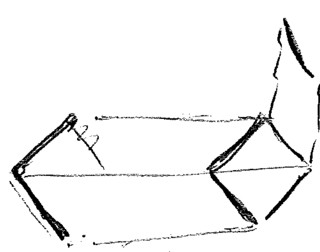
<u>Problem:</u> Show that $f : C \longrightarrow Sch(Y_{12})$ is a heq.

Look at $f/C$. If $C$ is of type 1 i.e. $\begin{smallmatrix}/\\ \backslash\end{smallmatrix}$ then any $C' \leq C$ is also of type 1, hence it is clear that $f/C$ is the same as $J_1/C$ which has the final object $C$. So suppose $C$ is of type 2. i.e. $\begin{smallmatrix}\backslash\\ /\end{smallmatrix}$, that $f(x) \leq C$ and let us consider the three possibilities for $x$.

$x = \begin{smallmatrix}\backslash\\ /\end{smallmatrix}$, then we have  so $x \leq C$ in $J_2$.

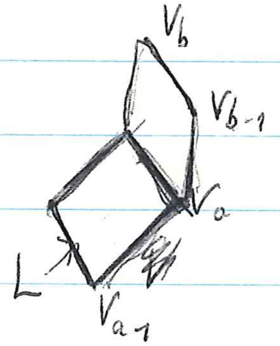
$x = \Diamond$, then we have , hence $x \leq y$ in $J_3$

where $y$ is a unique $\Diamond$ with $/$ same as bottom edge of $C$

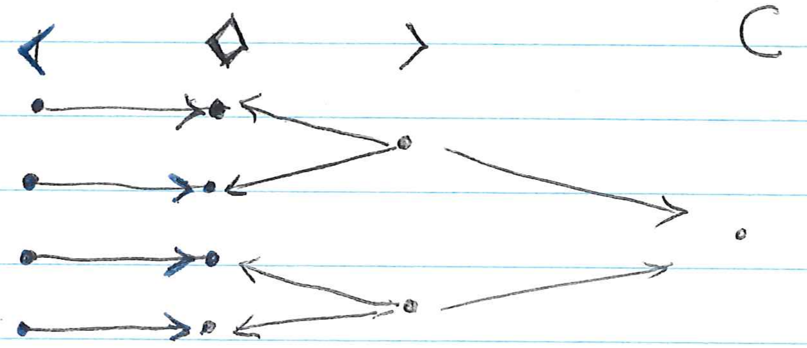
$x = <$, then we have 

so we get $x \leq x'$ where $x'$ is the left side of a! diamond with bottom right edge part of $C$.

It seems therefore that we can deform $f/C$ into the full subcategory consisting of $C' \leq C$ same bottom edge all diamonds with the same bottom right edge as the bottom of $C$, and all left halves of such diamonds. Suppose $C = C(V_{b-1}, V_b; V_{a-1}, V_a)$ $V_{a-1} < V_a \subset V_{b-1} < V_b$. Such diamonds may be identified with lines $L$ in $V_b/V_{a-1}$ mapping isom onto $V_b/V_{b-1}$




as well as the left sides.




So the proof might consist of introducing this category for each $C$.

<u>Old question</u>: Given $C \subset C'$ an inclusion of cells in a partial flag manifold, can it be lifted to an inclusion of cells of the same dimension in the full flag manifold?

<u>Example</u>: Suppose we take two cells in $Y_2(V)$ ~~of the form~~

$$C = C(Y_1 < Y_2 < Y_3 < Y_4) \subset C(Z_1 < Z_2 < Z_3 < Z_4) = C'$$

such that a reversal takes place, meaning that the inclusion comes from the inclusion

$$C(Y_1, Y_2; Y_3, Y_4) \subset C(Z_3, Z_4; Z_1; Z_2)$$

in $Y_{12}(V)$. Observe that over any cell of $Y_2(V)$ of the form $C(Y_1 < Y_2 < Y_3 < Y_4)$ there is a unique cell in $Y_{12}(V)$ over it of the same dimension, and the only other cell over it is of dimension 1 higher. Now if I could find $\tilde{C} \subset \tilde{C}'$ in $Y_{12\cdots n}(V)$, such that $\tilde{C} \simeq C$, $\tilde{C}' \simeq C'$, then their images in $Y_{12}(V)$ would furnish a contradiction.

# Pictures of Schubert cells.

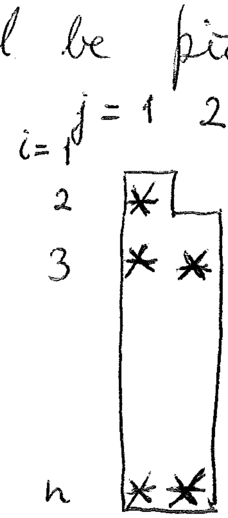
Consider $Y_{12}(V)$. I will fix a maximal torus $T$ and consider those Schubert cells normalized by $T$ which pass thru a given pt. fixed by $T$. [redacted] The $T$-fixpt will be of the form $\bar{\alpha}$ where $\alpha : \Lambda \to \{1,2,3\}$ is such that $\alpha^{-1}(1)$, $\alpha^{-1}(2)$ have card $1$. [redacted] Choose a point of $Y_{12\cdots n}(V)$ over $\bar{\alpha}$ fixed by $T$. This amounts to ordering $\Lambda$ so that $\alpha$ is monotone. Identify $\Lambda$ with $\{1,\dots,n\}$, and let $B$ be the corres. Borel. The different Schubert cells normalized by $T$ passing thru $\bar{\alpha}$ are of the form $\sigma^{-1}B\sigma \cdot \bar{\alpha}$. I can identify the open cell norm. by $T$ around ~~$\alpha$~~ $\bar{\alpha}$ with the unipotent group $N_\alpha$ whose roots are $\{(i,j) \mid \alpha i > \alpha j\}$. Then

$$N_\alpha \cap \sigma^{-1}B\sigma \xrightarrow{\sim} \sigma^{-1}B\sigma \cdot \bar{\alpha}$$

and

$$\text{Roots}\,(N_\alpha \cap \sigma^{-1}B\sigma) = \{(i,j) \mid \alpha i > \alpha j,\ \sigma i < \sigma j\}.$$

$N_\alpha$ will be pictured as matrices (+ id matrix)



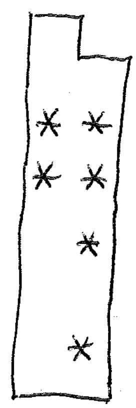
So we want pictures of the subgroups of $N_\alpha$ arising as Schubert cells.

Case 1: $\sigma 1 < \sigma 2$. In this case

$$R = \mathrm{Roots}(N_\alpha \cap \sigma^{-1} B \sigma) = \begin{matrix} i,1 & \sigma i < \sigma 1 \\ i,2 & \sigma i < \sigma 2 \end{matrix}$$

So if $(i,1) \in R \Rightarrow (i,2) \in R$.

So we get the picture



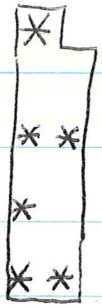
Conversely such a subset occurs – one chooses $\sigma$ to order the set

$$\{i \mid (i,1) \in R\},\ 1,\ \{i \mid (i,2) \in R, (i,1) \notin R\},\ 2,\ \text{rest of } i$$

Case 2: $\sigma 1 > \sigma 2$. In this case

$$R = \begin{cases} (i,1), & i > 1, \quad \sigma i < \sigma 1 \\ (i,2), & i > 2 \quad \sigma i < \sigma 2 \end{cases} \qquad \text{includes } (2,1)$$

and $\sigma i < \sigma 2 \Rightarrow \sigma i < \sigma 1$, so we get the picture



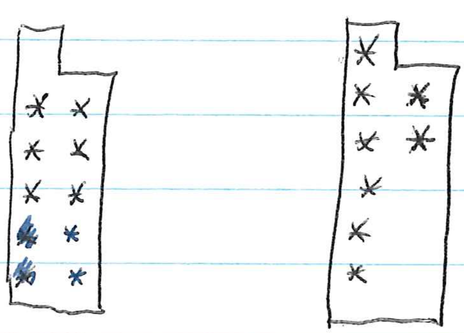
closed under $\leftarrow$

Next suppose we want the Schubert cell to be covered by a cell in $Y_{1..n}$ of same dimension passing thru the B-fixpt. Thus we want $\sigma$ such that

$$\{(i,j) \mid \sigma i < \sigma j \,,\ i > j\} = \{(i,j) \mid \sigma i < \sigma j \,,\ \alpha i > \alpha j\}$$

($\supset$ always true).

Since $i > j \Longrightarrow \alpha i \geq \alpha j$ with equality only if $\alpha i = \alpha j = 3$, This means that the $\sigma$ ordering & usual ordering ordering agree on $\{3, .., n\}$. This implies that our cells take the shape



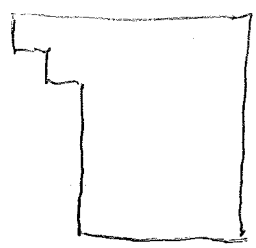
i.e. the positions form segments.

<u>Review</u>. $V = ke_1 + \cdots + ke_n$, $T$ standard max. torus, $B$ the Borel preserving $V_p = ke_1 + \cdots + ke_p$. $\bar\alpha \in Y_{12}(V)$ the point represented by the map $\alpha : \{1, \ldots, n\} \longrightarrow \{1,2,3\}$, $\alpha^{-1}(1) = 1$, $\alpha^{-1}(2) = 2$. Thus

$$\bar\alpha : \qquad ke_1 < ke_1 + ke_2 .$$

A Schubert cell normalized by $T$ containing $\bar\alpha$ is of the form $\sigma^{-1}B\sigma \cdot \bar\alpha$.

Stabilizer of $\bar\alpha$ :



$$N_\alpha \doteq \begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ \vdots & * & 1 & & \\ \vdots & \vdots & & \ddots & \\ * & * & & & 1 \end{pmatrix}$$

maps $\simeq$ open $T$-cell containing $\bar\alpha$.

$$\mathrm{Roots}(N_\alpha) = \{(i,j) \mid \alpha(i) > \alpha(j)\} = \left\{ \begin{matrix} (i,1) & i > 1 \\ (i,2) & i > 2 \end{matrix} \right\}$$

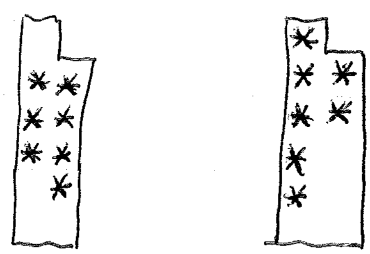
$$\mathrm{Roots}(N_\alpha \cap \sigma^{-1}B\sigma) = \left\{ (i,j) \;\middle|\; \begin{matrix} \alpha(i) > \alpha(j) \\ \sigma i < \sigma j \end{matrix} \right\}$$

$$= \begin{cases} (i,1) & i > 1 \qquad \sigma i < \sigma 1 \\ (i,2) & i > 2 \qquad \sigma i < \sigma 2 \end{cases}$$

Assume the cell $\sigma^{-1}B\sigma\cdot\bar{\alpha} \xleftarrow{\sim} N_\alpha \cap \sigma^{-1}B\sigma$ lifts to a $T$-cell in $Y_{12-n}(V)$ containing the point $\varepsilon$ fixed by $B$, and of the same dimension. This means

$$\{(i,j) \mid i>j,\ \sigma i<\sigma j\} = \{(i,j) \mid \alpha i > \alpha j,\ \sigma i < \sigma j\}$$

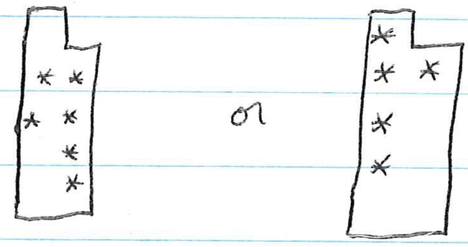
Since $i>j \Rightarrow \alpha i \geq \alpha j$ with equality iff $\alpha i = \alpha j = 3$, i.e. $i,j \in \{3,\dots,n\}$, this happens iff $i \lessdot j$ and $\sigma i \not< \sigma j$ are the same on $\{3,\dots,n\}$. Thus $\mathrm{Roots}(N_\alpha \cap \sigma^{-1}B\sigma)$ consists of $(i,1)$ $\begin{cases} i=2 & \text{if } \sigma 2<\sigma 1 \\ \text{and all } i\geq 3 & \text{with } i \leq \text{largest } i \end{cases}$ $\ni \sigma i < \sigma 1$, and similarly for $(i,2)$. Thus we get the pictures



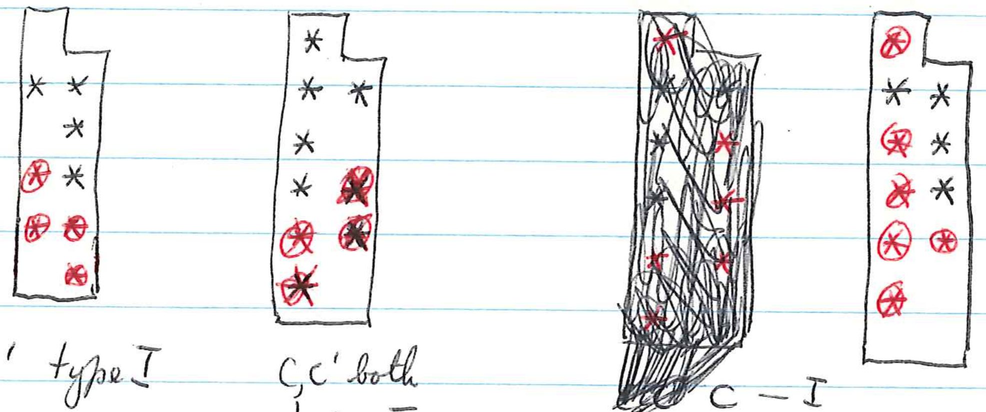
At least among these cells, $\sigma$ is determined by the cells in the following sense. Given a $T$-cell $C$ containing $\bar{\alpha}$ in $Y_{12}(V)$, there is ~~[illegible]~~ at most one $T$-cell $\tilde{C}$ in $Y_{12-n}(V)$ isomorphic to $C$ containing $\varepsilon$.

Question 1. Given $C \subset C'$ two cells in $Y_{12}(V)$ can I ~~[illegible]~~ find isom. cells $\tilde{C} \subset \tilde{C}'$ in $Y_{12\dots n}(V)$?

To do this I can ~~[illegible]~~ assume $C, C'$ normalized by $T$ and $\bar{\alpha} \in C$. In addition I can assume $\exists\ \tilde{C} \xrightarrow{\sim} C$ with $\varepsilon \in \tilde{C}$, whence $C$ appears



I must now consider the roots in $C'$ but not in $C$.

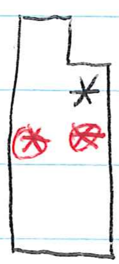


$C, C'$ type I

$C, C'$ both type II

$C$ — I
$C'$ — II

<u>Example</u>: Suppose we have an inclusion of two I-cells



Then this can not be lifted isomorphically to $Y_{1\ldots n}(V)$. In effect if one has a lifting to $\tilde{C} \subset \tilde{C}'$, then the unique $T$ fixpt. in $\tilde{C}$ amounts to an ordering of $3, \ldots, n$. which will arrange both cells as vertical strips:

Can't do both at the same time.

This example occurs with $n = 4$, specifically with an inclusion

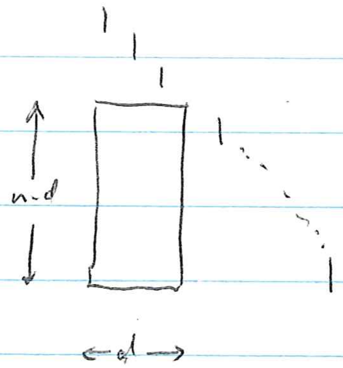
$$C(V_0, V_1; V_2, V_3) \subset C(W_1, W_2; W_3, W_4)$$

So one sees that even in $Y_{12}(V)$ there are Schubert cell inclusions which cannot be lifted isomorphically to the full flag manifold.

<u>Question</u>: Consider the simplicial complex whose simplices are chains of Schubert cells $C_0 \subset \cdots \subset C_n$ in $Y_{\bullet}$ which can be lifted isomorphically to the full flag manifold. Does this have the same homotopy type as the poset of Schubert cells?

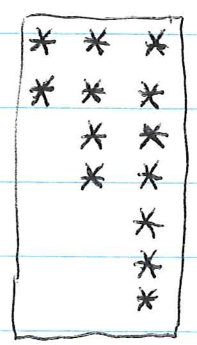
Next describe $T$-cells in $Y_d(V)$. Here $\alpha : \{1, \dots, n\} \to \{1,2\}$ has $\alpha^{-1}\{1\} = \{1, \dots, d\}$.

$N_\alpha$ :



$$\mathrm{Roots}(N_\alpha) = \left\{ (i,j) \,\middle|\, \begin{matrix} 1 \le j \le d \\ d < i \le n \end{matrix} \right\}.$$

Typical cell has roots $\left\{ (i,j) \,\middle|\, \begin{matrix} j \le d < i \\ \sigma i < \sigma j \end{matrix} \right\}$. For each $j$ the set of $i$ such that $(i,j) \in R$ must ~~strictly~~ be monotone for some ordering of $\{1, \dots, d\}$. So after arranging $\sigma$ so that $\sigma 1 < \sigma 2 < \dots < \sigma d$, $\sigma(d+1) < \dots < \sigma(n)$ the cell appears



Conversely, suppose that $\sigma$ is a perm. such that
$$\left\{ (i,j) \,\middle|\, \begin{matrix} \alpha i > \alpha j \\ \sigma i < \sigma j \end{matrix} \right\} = \left\{ (i,j) \,\middle|\, \begin{matrix} i > j \\ \sigma i < \sigma j \end{matrix} \right\}$$
Now one always has $\subset$. [redacted] Also $i > j \Rightarrow \alpha i \ge \alpha j$ with equality iff $1 \le i,j \le d$, or $d < i,j \le$ [redacted] $n$. Thus on these intervals $\sigma$ must preserve the ordering so it is a shuffle.

Consider T-cells $C$ in $Y_{d_1,\dots,d_{\mu-1}}$ containing $\bar{\alpha}$, which can be lifted isomorphically to T-cells $\tilde{C}$ in $Y_{1\dots n}$ containing a given T-pt. $\varepsilon$ over $\bar{\alpha}$. $\tilde{C} = \sigma^{-1}B\sigma.\varepsilon$ where

$$\{(i,j) \mid i>j,\ \sigma i<\sigma j\} = \{(i,j) \mid \alpha i > \alpha j,\ \sigma i < \sigma j\}$$

$(\supset$ always$)$

$i>j \Rightarrow \alpha i \geq \alpha j$ with equality iff $i,j$ belong to the same $i$ fibre. So $\sigma$ has to agree with $<$ on these fibres. This means $\sigma$ is uniquely determined if it exists (for every $i>j$ we know whether $\sigma i < \sigma j$ or $\sigma i > \sigma j$).

A simpler way to see this is as follows. $T$ being given a nbd of $\varepsilon$ has a coordinate system given by roots, part of which get killed under the map to $\alpha$. If we have a T-cell containing $\bar{\alpha}$ we can lift its roots and so construct a $\tilde{C} \xrightarrow{\sim} C$. If $\tilde{C}$ is a Schubert cell it is obviously the unique one thru $\varepsilon$ isom. to $C$.

Question: Is the poset of T-cells $\tilde{C}$ in $Y_{1\dots n}$ containing $\varepsilon$ and such that $\tilde{C}$ is isomorphic to its image in $Y_{d_1,\dots,d_{\mu-1}}$ isomorphic to the poset of "types" for cells in $Y_{d_1\dots d_{\mu-1}}$?

Recall that the type of the cell $\sigma^{-1}B\sigma\cdot\bar{\alpha}$ is the map $\alpha\sigma^{-1} : \{1,\dots,n\} \longrightarrow \{1,\dots,\mu\}$. Thus to give the type of a cell is to give the sets $(\alpha\sigma^{-1})^{-1}\{j\}$. Choosing $\sigma$ to preserve the natural ordering ordering on these fibres we get a ! T-cell containing $\bar{\alpha}$ lifting isomorphically to a T-cell thru $\varepsilon$.

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<u>Review</u>: Let $M$ be a free $k[t]$-module of rank $n$. I recall that we obtained a contractible simplicial complex $K$ as follows: A vertex of $K$ consists of a flag $0 = M_0 < M_1 < \cdots < M_p = M$ in $M$ such that each $M_i/M_{i-1}$ is free, together with unimodular subspaces $V_i \subset M_i/M_{i-1}$ (i.e. $k[t]\otimes_k V_i \xrightarrow{\sim} M_i/M_{i-1}$). ~~[illegible]~~ Say that one vertex $\{M_i, V_i\}$ refines $\{M_j', V_j'\}$ if the flag $\{M_i\}$ refines $\{M_j'\}$, and if for each $j$, the filtration of $M_j'/M_{j-1}'$ induced by the $M_i$ is compatible with $V_j'$ in the following sense

Let
$$M_{j-1}' = M_{i_{j-1}}$$
$$M_j' = M_{i_j}$$

~~[illegible]~~
$$V_{j,a}' = V_j' \cap M_a/M_{i_{j-1}} \qquad i_{j-1} \le a \le i_j$$

Then I want ~~[illegible]~~
$$V_a \cong V_{j,a}'/V_{j,a-1}' \qquad i_{j-1} < a \le i_j$$
in
$$M_a/M_{a-1}$$

Thus I want a filtration of $V_j'$ yielding the $M_i$ $i_{j-1} \le i \le i_j$, as well as $V_i$.

So this puts a partial ordering on the vertices of $K$, and the simplices are the chains.

Associate to a vertex $x$ of $K$ the sequence of numbers $\dim M_1, \dots, \dim M_{p-1}$. Then we get a subset $d(x)$ of $\{1, \dots, n-1\}$. We see $K$ sits over the product $\{0,1\}^n$, equipped with product order, $0<1$.

Notice that given two vertices $x, y$ of $K$ with $x \leq y$ ($x$ refined by $y$), then the set of $z$, $x \leq z \leq y$ may be identified with subsets $\sigma$, $d(x) \leq \sigma \leq d(y)$. This means that the fibre of $K$ over a simplex of $\{0,1\}^n$ depends only on the endpts. Strata:

Model I conjecture for the homotopy type of cells in $Y_d(V)$ goes as follows. Given a subset of $\{1, .., d-1\}$ say $0=a_0 < a_1 < \dots < a_{p-1} < a_p = d$ I will consider sequences of spaces $V_1, \dots, V_p$ with $\dim(V_i) = a_i - a_{i-1}$. These can vary up to isoms.

Given an inclusion of subsets $\sigma \subset \tau$ then a map from a $\sigma$ sequence $V_i$ to a $\tau$-sequence $V'_j$ consists of filtration on the $\sigma$ sequence and an isom of the associated graded with the $\tau$-sequence. Again a chain $\{V_i\} \longrightarrow \{\ \} \longrightarrow \dots \longrightarrow \{V'_j\}$

is determined by the map $\{v_i\} \longrightarrow \{v_j'\}$ and the chain of subsets. So what is essential to me is what sits over 1-simplices $\sigma \subset \tau$.

<u>Note</u>: Given a point $(t_1, \dots, t_n) \in [0,1]^n$ let $0 < \lambda_1 < \dots\dots < \lambda_p < 1$ be the distinct elements $\neq 0, 1$ of $\{t_1, \dots, t_n\}$, and let $\sigma_j = \{i \mid t_i \leq \lambda_j\}$, whence we get a chain of subsets

$$\sigma_0 < \sigma_1 < \dots\dots < \sigma_p$$

such that $t_i = \lambda_j$ for $i \in \sigma_j - \sigma_{j-1}$. This chain of subsets is the open simplex of $[0,1]^n$ to which the point $(t_1, \dots, t_n)$ belongs. Assume we are interested only in $\sigma_0 = \{i \mid t_i = 0\}$ and $\sigma_p = \{i \mid t_i < 1\}$. Then the different strata show up when one considers for each $i$ whether $t_i = 0$, $0 < t_i < 1$, or $t_i = 1$. Thus when $n = d-1$, there are a total of $3^{d-1}$ strata.