January 1, 1975. Closures of Schubert cells.

Studying Schubert cells in $Y_{1\cdots d} = \{(F_1 < \cdots < F_d)\}$. Given $\alpha : \{1,\cdots,n\} \to \{1,\cdots,d+1\}$ such that card $\alpha^{-1}(j) = 1$ for $j = 1,\cdots,d$ we get a Schubert cell

$$C_\alpha = \{(F_1 < \cdots < F_d) \mid \dim F_j \cap V_p = \alpha_{jp}\}$$

$$\alpha_{jp} = \text{card}\,\{a \le p \mid \alpha(a) \le j\}.$$

For example let $d=2$. Then $\{\alpha^{-1}(1), \alpha^{-1}(2)\} = \{a_1, a_2\}$ where $1 \le a_1 < a_2 \le n$. There are two cells - depending on whether $\alpha^{-1}(1) = a_1$ or a_2.

First consider the case where $\alpha^{-1}(1) = a_1 < \alpha^{-1}(2) = a_2$, whence C_α consists of $F_1 < F_2$ such that

$$F_1 \cap V_p \quad \text{jumps at} \quad p = \alpha^{-1}(1) = a_1$$
$$F_2 \cap V_p \quad \text{———} \quad p = a_1, a_2$$

Example: $F_1 = L_1$, $F_2 = L_1 \oplus L_2$ where $L_1 \in \mathbb{P}V_{a_1} - \mathbb{P}V_{a_1}$ and $L_2 \in \mathbb{P}V_{a_2} - \mathbb{P}V_{a_2 - 1}$. Any such element of C_α can be described in this way, and L_2 is uniquely determined if it is $\perp$ to L_1.

Suppose we have $F_1 < F_2$ in $Y_{1,2}$ such that

$$(*) \qquad \dim(F_1 \cap V_{a_1}) \ge 1$$
$$\dim(F_2 \cap V_{a_1}) \ge 1, \quad \dim(F_2 \cap V_{a_2}) \ge 2.$$

Then $L_1' = F_1$ is the limit of $L_1 \in \mathbb{P}V_{a_1} - \mathbb{P}V_{a_1-1}$, and we have find L_2' s.t. $F_2 = F_1 \oplus L_2'$ and L_2' is the limit of $L_2 \in \mathbb{P}V_{a_2} - \mathbb{P}V_{a_1}$. Thus the closure of C_α is described by $(*)$.

Next suppose $\alpha^{-1}(1) = a_2 > \alpha^{-1}(2) = a_1$. Then An element of C_α is of the form $(L_2, L_1 \oplus L_2)$ where $L_1 \in \mathbb{P}V_{a_1} - \mathbb{P}V_{a_1-1}$, and $L_2 \in \mathbb{P}V_{a_2} - \mathbb{P}V_{a_2-1}$, and L_1, L_2 are uniquely determined.

Suppose $(F_1 < F_2) \in Y_{12}$ is such that

$$\dim(F_1 \cap V_{a_2}) \geq 1 \qquad F_1 \subset V_{a_2}$$

$$\dim(F_2 \cap V_{a_1}) \geq 1, \ \dim(F_2 \cap V_{a_2}) \geq 2 \qquad \text{i.e. } F_2 \subset V_{a_2}$$

Pick $L_1' \subset F_2 \cap V_{a_1}$; I can perturb F_1 within F_2 so that it is $\neq L_1'$. Then put $L_2' = F_1$. Now perturb L_1' to $L_1 \in \mathbb{P}V_{a_1} - \mathbb{P}V_{a_1-1}$ and L_2' to $L_2 \in \mathbb{P}V_{a_2} - \mathbb{P}V_{a_2-1}$, whence we see $(F_1 < F_2)$ can be approximated arbitrarily closely by elements of C_α.

Conjecture: Think of $\alpha : \{1, \dots, n\} \longrightarrow \{1, \dots, d+1\}$ as being a chain of subsets

$$\alpha^{-1}\{1\} < \alpha^{-1}\{1,2\} < \dots < \alpha^{-1}\{1, \dots, j\}.$$

Then $C_\beta \subseteq \overline{C_\alpha}$ iff

$$\beta^{-1}(1) \leq \alpha^{-1}(1)$$

$$\beta^{-1}\{1,2\} \leq \alpha^{-1}\{1,2\}$$ in the sense that

if $\beta^{-1}\{1,2\} = \{b_1, b_2\}$ with $b_1 < b_2$, and similarly $\alpha^{-1}\{1,2\} = \{a_1, a_2\}$ with $a_1 < a_2$, then $b_j \leq a_j$.

etc.

Show C_α dense in $\tilde{C}_\alpha$. Follow the proof that $\tilde{C}_\alpha$ is non-singular, where one chooses F_{jp}, being given $F_{j,p-1}$ and $F_{j+1,p}$. Choose $F_{d1}, F_{d2}, \cdots, F_{dn}$, then $F_{d-1,1}, F_{d-1,2}, \cdots, F_{d-2,n}$, etc.
Condition on F_{dp} is that

$$F_{d,p-1} \subset F_{d,p} \subset V_p$$

F_{dp} is fixed except if $\alpha(p) \leq d$, in which case, one generically has that $F_{d,p} \not\subset V_{p-1}$, i.e. $F_{d,p}/F_{d,p-1} \xrightarrow{\sim} V_p/V_{p-1}$.
Condition on F_{jp} is

$$F_{j,p-1} \subset F_{j,p} \subset F_{j+1,p}$$

F_{jp} is fixed except if $\alpha(p) \leq j$ and $\exists\, a \leq p$ $\alpha(a) = j+1$. i.e. if $\alpha^{-1}(j+1) < p$ and $\alpha(p) < j+1$. Generically we will have $F_{jp}/F_{j,p-1} \xrightarrow{\sim} F_{j+1,p}/F_{j+1,p-1}$.

Suppose all F_{jp} are chosen generically; then

$$F_{j,p-1} = F_{j,p} \cap F_{j+1,p-1}$$

so its clear we will have $F_{jp} = F_{jn} \cap V_p$, hence we will haves that F_{jp} is in the image of C_α in $\tilde{C}_\alpha$. Now C_α is dense in $\tilde{C}_\alpha$ because

<u>Lemma</u>: Given a map $f: X \to Y$ with fiber $\mathbb{P}^1$, let $U \subset X$ be the complement of a section, let $V \subset Y$ be open and dense. Then $U \cap f^{-1}(V)$ is open and dense.

Put $\overline{C}_\alpha = \{(F_1 < \cdots < F_d) \in Y_{1..d} \mid \dim F_j \cap V_p \geq \alpha_{jp}\}$. Claim $\tilde{C}_\alpha \longrightarrow Y_{1..d}$ has image $\overline{C}_\alpha$. In effect given $F_1 < \cdots < F_d$ in $\overline{C}_\alpha$ let us consider the problem of constructing F_{jp} with $F_{jp} \subset F_j \cap V_p$. Assume $F_{j,p-1}$ and $F_{j-1,p}$ have been found, and lets consider the problem of finding F_{jp}

$F_j \cap V_p$

$$\begin{array}{ccc} F_{j,p-1} & \subset & F_{jp} \\ \cup & & \cup \\ F_{j-1,p-1} & \subset & F_{j-1,p} \end{array}$$

The only case we could have trouble is when
$F_{j-1,p-1} < F_{j-1,p} \iff \alpha(p) < j$ and
$F_{j-1,p-1} < F_{j,p-1} \iff \alpha^{-1}(j) < p$

But then $\dim(F_j \cap V_p) \geq \alpha_{jp} = 1 + \alpha_{j-1,p} = 2 + \alpha_{j-1,p-1}$

Hence $\dim(F_{j,p-1} + F_{j-1,p}) \leq \alpha_{jp}$, and so one can find $F_{j,p}$.

<u>Corollary.</u> $\bar{C}_\alpha$ = closure of C_α.

Suppose next that we are presented with $(F_1 < \cdots < F_d)$ in $\bar{C}_\alpha$ with $(F_1 < \cdots < F_d) \in C_\beta$.

$$\beta_{jp} = \dim(F_j \cap V_p) \geq \alpha_{jp}.$$

Then the jumps in $F_j \cap V_p$ come before those of any member of C_α. i.e.

$$\beta^{-1}\{0, \dots, j\} \leq \alpha^{-1}\{0, 1, \dots, j\}$$

So we see the conjecture on page 2 is true.

<u>General case</u>: Given $\alpha : \{1,\dots,n\} \twoheadrightarrow \{1,\dots,\mu\}$ with $\operatorname{card} \alpha^{-1}\{0,\dots,j\} = d_j$, we have the Schubert cell C_α in $Y_{d_1,\dots,d_{\mu-1}}$ consisting of $(F_0 < \dots < F_\mu)$ such that $\dim\left(F_j \cap V_p\right) = \alpha_{jp} = \operatorname{card}\{a \leq p \mid \alpha(a) \leq j\}$. Put

$$\overline{C}_\alpha = \left\{ (F_0 < \dots < F_\mu) \in Y_{d_1,\dots,d_{\mu-1}} \;\middle|\; \dim\left(F_j \cap V_p\right) \geq \alpha_{jp} \right\}.$$

It is clear that $\overline{C}_\alpha$ is closed in $Y_{d_1,\dots,d_{\mu-1}}$ containing C_α; we will see below that $\overline{C}_\alpha =$ closure of C_α.

Put $\widetilde{C}_\alpha$ for the set of systems of subspaces F_{jp}, $0 \leq j \leq \mu$, $0 \leq p \leq n$, with $\dim(F_{jp}) = \alpha_{jp}$, which are monotone in j and p, and such that $F_{\mu p} = V_p$. $\widetilde{C}_\alpha$ is a closed subvariety of a product of Grassmannians, hence it is a complete variety. By sending (F_{jp}) to $(F_{0n} < F_{1n} < \dots < F_{\mu n})$ we get a map from $\widetilde{C}_\alpha$ to $Y_{d_1,\dots,d_{\mu-1}}$, whose image is contained in $\overline{C}_\alpha$ (because $F_{jp} \subset F_{jn} \cap V_p$). We can lift $C_\alpha \subset \overline{C}_\alpha$ to a map $C_\alpha \to \widetilde{C}_\alpha$ by sending (F_j) to $(F_j \cap V_p)$. An element (F_{jp}) of $\widetilde{C}_\alpha$ comes from something in C_α iff the square

$$\begin{array}{ccc} F_{j,p-1} & \subset & F_{jp} \\ \cup & & \cup \\ F_{j-1,p-1} & \subset & F_{j-1,p} \end{array}$$

is cartesian for $1 \leq j \leq \mu$, $1 \leq p \leq n$.

<u>Assertion:</u> $\tilde{C}_\alpha$ maps onto $\overline{C}_\alpha$.

<u>Proof:</u> Given (F_j) in $\overline{C}_\alpha$ we construct $(F_{jp}) \in \tilde{C}_\alpha$ such that $F_{jp} \subset F_j \cap V_p$, whence $(F_{jn}) = (F_j)$.
We construct F_{jp} assuming F_{ab} has been found for $(a,b) < (j,p)$ in the product ordering. We must find F_{jp} of dimension α_{jp} such that

$$F_{j,p-1} + F_{j-1,p} \subset F_{jp} \subset F_j \cap V_p .$$

Since $\dim(F_j \cap V_p) \geq \alpha_{jp}$, and $F_{j,p-1} + F_{j-1,p} \subset F_j \cap V_{p-1} + F_{j-1} \cap V_p \subset F_j \cap V_p$, it suffices to see that $F_{j,p-1} + F_{j-1,p}$ has dimension $\leq \alpha_{jp}$. But

$$\begin{aligned}\dim(F_{j,p-1} + F_{j-1,p}) &= \dim F_{j,p-1} + \dim F_{j-1,p} - \dim F_{j,p-1} \cap F_{j-1,p} \\ &\leq \alpha_{j,p-1} + \alpha_{j-1,p} - \underset{\substack{\shortparallel \\ \alpha_{j-1,p-1}}}{\dim F_{j-1,p-1}}\end{aligned}$$

and

$$\begin{aligned}\alpha_{jp} - \alpha_{j,p-1} &= \operatorname{card}\{a \leq p \mid \alpha(a) \leq j\} - \operatorname{card}\{a \leq p-1 \mid \alpha(a) \leq j\} \\ &= \begin{cases} 0 & \alpha(p) > j \\ 1 & \alpha(p) \leq j \end{cases} \\ &\geq \alpha_{j-1,p} - \alpha_{j-1,p-1}\end{aligned}$$

QED.

<u>Assertion:</u> $\tilde{C}_\alpha$ is non-singular and C_α is dense in $\tilde{C}_\alpha$.

<u>Proof:</u> I will construct a tower of projective bundles ending with $\tilde{C}_\alpha$. Order the pairs (j,p), $0 \le j \le \mu$, $0 \le p \le n$ by saying $(j,p) \lesssim (a,b)$ if either $j > a$ or if $j = a$ and $p \lesssim b$. Let X_{ab} be the variety of systems (F_{jp}) defined for $(j,p) \lesssim (a,b)$, satisfying the same requirements as an element of $\tilde{C}_\alpha$:

$$\dim(F_{jp}) = \alpha_{jp} \qquad F_{\mu p} = V_p$$

$$F_{j,p-1} \subset F_{j,p} \subset F_{j+1,p} \qquad \forall\ (j,p) \lesssim (a,b)$$

and let X'_{ab} be the subset consisting of those such that the square

$$\begin{array}{ccc} F_{j+1,p-1} & \subset & F_{j+1,p} \\ \cup & & \cup \\ F_{j,p-1} & \subset & F_{j,p} \end{array}$$

is cartesian for each $(j,p) \lesssim (a,b)$.

The fibre of the map

$$X_{jp} \longrightarrow X_{j,p-1} \qquad \begin{array}{l} 1 \le p \le n \\ 1 \le j < \mu \end{array}$$

is the set of new choices for $F_{j,p}$. As

$$\dim F_{j,p}/F_{j,p-1} = \begin{cases} 0 & \alpha(p) > j \\ 1 & \alpha(p) \le j \end{cases}$$

the fibre will either be a point, whence $X_{jp} = X_{j,p-1}$; or any line in $F_{j+1,p}/F_{j,p-1}$, whence X_{jp} is the projective bundle over $X_{j,p-1}$, of the vector bundle with fibre $F_{j+1,p}/F_{j,p-1}$.

$$\text{rel. dim}\,(X_{jp}/X_{j,p-1}) = \begin{cases} 0 & \text{if } \alpha(p) > j \\ \text{card}\,\{a<p \mid \alpha(a) \text{ [redacted] } = j+1\} & \text{if } \alpha(p) \leq j \end{cases}$$

X'_{jp} is the intersection of the inverse image of $X'_{j,p-1}$ with the complement of the hyperplane subbundle of X_{jp} ~~[illegible]~~ whose fibre consists of lines in the hyperplanes [redacted] $F_{j+1,p-1}/F_{j,p-1} \subset F_{j+1,p}/F_{j,p-1}$. Thus X'_{jp} is an affine space bundle over X'_{jp-1}, and it is clear X'_{jp} is dense in X_{jp}. Because:

<u>Lemma</u>: If $X \xrightarrow{\pi} Y$ is a projective bundle $X = PE$, if $Z \subset X$ is the hyperplane fibre bundle given by $E' \subset E$ of codim 1, and if V is a dense open set in Y, then $\pi^{-1}(V) \cap (X - Z)$ is open + dense in X.

Better: finite intersection of <u>open + dense sets</u> is open and dense

To finish, note that $X_{j0} = X_{j+1,n}$ and $X_{\mu 0} = \cdots = X_{\mu n} = pt$, $X_{1n} = C_\alpha$, so the assertion on page 8 is proved.

Cor: $\overline{C_\alpha}$ = closure of C_α.

Because C_α is dense in $\tilde{C}_\alpha$ which maps onto $\overline{C_\alpha}$, and $\overline{C_\alpha}$ is closed.

Definition: Let $\{a_1, \dots, a_j\}$, $\{b_1, \dots, b_j\}$ be two subsets of $\{1, \dots, n\}$ of card j, arranged in order: $a_1 < \dots < a_j$, $b_1 < \dots < b_j$. Write $\{a_1, \dots, a_j\} \leq \{b_1, \dots, b_j\}$, if $a_i \leq b_i$ for each $i = 1, \dots, j$.

Assertion: Let $\beta : \{1, \dots, n\} \longrightarrow \{1, \dots, \mu\}$ also have $\operatorname{card}(\beta^{-1}\{1, \dots, j\}) = d_j$. Then $C_\beta \subset \overline{C_\alpha}$, iff for each $j = 1, \dots, \mu$, $\beta^{-1}\{1, \dots, j\} \leq \alpha^{-1}\{1, \dots, j\}$. (denote this $\beta \leq \alpha$.)

Proof: If $(F_1 < \dots < F_\mu) \in C_\beta$, then

$$\dim(F_j \cap V_p) = \beta_{jp} = \operatorname{card}\{a \leq p \mid \beta(a) \leq j\}.$$

jumps at $p \in \beta^{-1}\{1, \dots, j\}$. The condition $\beta^{-1}\{1, \dots, j\} \leq \alpha^{-1}\{1, \dots, j\}$ is equivalent to $\beta_{jp} \geq \alpha_{jp}$, i.e. β-jumps occur before α jumps. But we've seen $\overline{C_\alpha}$ consists of (F_j) with $\dim(F_j \cap V_p) \geq \alpha_{jp}$.

(Thus $\beta \leq \alpha \implies C_\beta \subset \overline{C_\alpha} \implies \beta \leq \alpha$).

Next I want to try to compute the class of $\overline{C}_\alpha$ in $H^*(Y_{\underline{d}})$.

Introduce Z_α = variety of (F_{jp}) of dim α_{jp} as before but where F_{jp} is not required to be $\subset V_p$. Then $\overline{C}_\alpha$ = those (F_{jp}) in Z_α such that $F_{\mu p} \supseteq V_p$. I map Z_α to $Y_{\underline{d}}$ by $(F_{jp}) \longmapsto (F_{jn})$. This map is smooth because it factors into projective bundles as follows. ~~Order (j,p) by saying $(j,p) \leq (a,b)$ if $p < b$ or if $p = b$ and $j \geq a$. Thus we start with $(\mu,1)$ go to $(1,1)$, then $(\mu,2)$ to $(1,2)$~~

Order (j,p) by saying $(j,p) \leq (a,b)$ if $p > b$ or if $p = b$ and $j \leq a$. Thus we will construct a typical element of Z_α by choosing $F_{1,n-1}, \ldots, F_{\mu,n-1}$, then $F_{1,n-2} \ldots\ldots F_{\mu,n-2}$, etc. Let X_{ab} be the variety of such systems (F_{jp}) defined for $(j,p) \leq (a,b)$; forgetting F_{jp} gives a map

$$X_{jp} \longrightarrow X_{j-1,p}$$

whose fibre is the F_{jp} of dimension α_{jp} fitting into

$$\begin{array}{ccc} F_{jp} & \subset & F_{j,p+1} \\ \cup & & \\ F_{j-1,p} & & \end{array} \qquad \text{Suppose } 1 \le j \le \mu,\ 1 \le p < n$$

As $\dim F_{j,p+1}/F_{j,p} = \begin{cases} 0 & \alpha(p+1) > j \\ 1 & \alpha(p+1) < j \end{cases}$, we see that

$X_{jp} = X_{j-1,p}$ if $\alpha(p+1) > j$, and if $\alpha(p+1) < j$, then

$$X_{jp} = \mathbb{P}\left(\mathcal{F}_{j,p+1}/\mathcal{F}_{j-1,p}\right) \text{ over } X_{j-1,p}$$

so in this case

$$\dim X_{jp}/X_{j-1,p} = \operatorname{card}\{a \le p \mid \alpha(a) = j\}.$$

To simplify following I consider $Y_{1,2,\ldots,d}(V_d)$. Thus I want $\mu = d = n$, whence α is a permutation. I will put

$$z_{jp} = e\left(\mathcal{F}_{j,p}/\mathcal{F}_{j,p-1}{}^{\vee}\right) \qquad \text{if } \alpha(p) \le j$$

$$\xi_{jp} = e\left(\mathcal{F}_{jp}/\mathcal{F}_{j-1,p}{}^{\vee}\right) \qquad \text{if } \alpha^{-1}(j) \le p.$$

When X_{jp} has rel dim 1 over $X_{j-1,p}$, one has $\alpha(p+1) < j$ and $\alpha^{-1}(j) < p+1$, i.e. the pairs $\alpha^{-1}(j) < p+1$ is reversed by α. In this case we have

$$\begin{array}{ccc} F_{jp} & < & F_{j,p+1} \\ \vee & & \vee \\ F_{j-1,p} & < & F_{j-1,p+1} \end{array}$$

and integrating over the fibres of $X_{jp} \longrightarrow X_{j-1,p}$ will replace $z_{j,p+1}$ and $\xi_{j,p}$ by $\xi_{j,p+1}$ and $z_{j-1,p+1}$.

Suppose I have a 2-plane bundle E over Y which is an extension

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

and I form over $X = \mathbb{P}E$ the sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}_X \otimes_Y E \longrightarrow Q \longrightarrow 0.$$

Then $H^*(X)$ is a free $H^*(Y)$ module with basis $1, z_1 = e(\mathcal{O}(1))$. Put $z_2 = e(Q^\vee)$, $y_i = e(L_i^\vee)$. Suppose I want to integrate some polynomial $f(z_1, z_2) \in H^*(X)$ over the fibres of $X \xrightarrow{\pi} Y$. I can write

$$f(z_1,z_2) = \frac{f(z_1,z_2) + f(z_2,z_1)}{2} + \frac{f(z_1,z_2) - f(z_1,z_2)}{(z_1 - z_2)} \left(\frac{z_1 - z_2}{2}\right)$$

$$= \underbrace{\qquad\qquad}_{a} + \underbrace{\qquad\qquad}_{b} \left(\frac{z_1 - z_2}{2}\right)$$

Now a, b will be polynomials in the elem. symm. fns of z_1, z_2, hence will be in $H^*(Y)$. Since

$\pi_*(1) = 0$, $\pi_*\left(\frac{z_1 - z_2}{2}\right) = \frac{1 + (-1)}{2} = 1$. It follows that

$$\pi_* f(z_1, z_2) = b = \frac{f(y_1, y_2) - f(y_2, y_1)}{y_1 - y_2}.$$

Example: A monomial $z_1^\mu z_2^\nu$ with $\mu \geq \nu$ integrates to

$$\pi_*\left(z_1^\mu z_2^\nu\right) = \frac{y_1^\mu y_2^\nu - y_1^\nu y_2^\mu}{y_1 - y_2}$$

$$= (y_1 y_2)^\nu \left(y_1^{\mu-\nu-1} + \cdots + y_2^{\mu-\nu-1}\right)$$

$$= y_1^{\mu-1} y_2^\nu + y_1^{\mu-2} y_2^{\nu+1} + \cdots + y_1^\nu y_2^{\mu-1}$$

With these formulas I should be able to work out formulas for the classes belonging to the Schubert cells in $Y_{1,2,3}$.

I will take the cases of α of the form

$\alpha^{-1}\{1,2,3\}$	$a_1 < a_2 < a_3$
$\alpha^{-1}\{1,2\}$	$a_2 < a_3$
$\alpha^{-1}\{1\}$	a_3

and I will simplify my notation, by denoting an element of Z_α by:

$$\begin{array}{ccccccc}
0 & < & F_{31} & < & F_{32} & < & F_{33} \\
 & & \vee & & \vee & & \vee \\
 & & 0 & & F_{22} & < & F_{23} \\
 & & & & & & \vee \\
 & & & & & & F_{13}
\end{array}$$

I start with the element $z_{31}^{n-a_1} z_{32}^{n-a_2} z_{33}^{n-a_3}$ which I want to integrate.

Remove z_{31}:

integrate over F_{31}. $z_{31} = \xi_{31}$

$$x_1 = \xi_{31} \qquad y_1 = z_{22}$$

$$x_2 = z_{32} \qquad y_2 = \xi_{32}$$

$n-a_1 > n-a_2$ $\qquad \xi_{31}^{n-a_1} z_{32}^{n-a_2} z_{33}^{n-a_3} = x_1^{n-a_1} x_2^{n-a_2} z_3^{n-a_3}$

$$\pi_x\left(x_1^{n-a_1} x_2^{n-a_2} z_{33}^{n-a_3}\right) = \left(y_1^{n-a_1-1} y_2^{n-a_2} + \cdots + y_1^{n-a_2} y_2^{n-a_1-1}\right) z_{33}^{n-a_3}$$

$$= z_{22}^{n-a_1-1} \xi_{32}^{n-a_2} z_{33}^{n-a_3} + \cdots\cdots + z_{22}^{n-a_2} \cdot \xi_{32}^{n-a_1-1} z_{33}^{n-a_3}$$

$$= \sum_{\nu=0}^{b_1-b_2-1} z_{22}^{b_1-1-\nu} \xi_{32}^{b_2+\nu} z_{33}^{b_3} \qquad b_i = n-a_i$$

Next remove z_{22} : $\quad z_{22} \longmapsto \xi_{22}$

$$\sum_{\substack{\nu_1+\nu_2=b_1+b_2-1\\ b_2\le \nu_i\le b_1-1}} \xi_{22}^{\nu_1}\,\xi_{32}^{\nu_2}\,z_{33}^{b_3}$$

Next we integrate out F_{32}.

$$x_1 = \xi_{32} \qquad y_1 = z_{23}$$
$$x_2 = z_{33} \qquad y_2 = \xi_{33}$$

So we get

$$\sum_{\substack{\nu_1+\nu_2=b_1+b_2-1\\ b_2\le \nu_i\le b_1-1}}\ \sum_{\substack{\rho_1+\rho_2=\nu_2+b_3-1\\ b_3\le \rho_i\le \nu_2-1}} \xi_{22}^{\nu_1}\,z_{23}^{\rho_1}\,\xi_{33}^{\rho_2}$$

Next integrate out F_{22}

$$x_1 = \xi_{22} \qquad y_1 = z_{13} = \xi_{13}$$
$$x_2 = z_{23} \qquad y_2 = \xi_{23}$$

$$\sum_{\substack{\nu_1+\nu_2=b_1+b_2-1\\ b_2\le \nu_i\le b_1-1}}\ \sum_{\substack{\rho_1+\rho_2=\nu_2+b_3-1\\ b_3\le \rho_i\le \nu_2-1}}\ \sum_{\substack{\tau_1+\tau_2=\nu_1+\rho_1-1\\ \min(\nu_1,\rho_1)\le \tau_i\le \max(\nu_1,\rho_1)-1}} \xi_{13}^{\tau_1}\,\xi_{23}^{\tau_2}\,\xi_{33}^{\rho_2}$$

In this sum there are three independent indices, yet the degree of the monomials is $(b_1-2)+(b_2-1)+b_3$. Hence this expression will be of the form

$$\sum_{|\tau|=(b_1-2)+(b_2-1)+b_3} a_\tau\, \xi_1^{\tau_1}\xi_2^{\tau_2}\xi_3^{\tau_3} \qquad \xi_i = \xi_{i3}$$

where the $a_{\tau_1\cdots\tau_3}$ are non-negative integers. So it's clear that the formula I am after will be similar to the <u>Weyl character formula</u>.

Homology classes might be simpler. Review case of the Grassmannian: Y_d.

<u>The problem</u>: In $H_*(Y_d)$, I have homology classes associated to the Schubert cells, over which I can integrate coh. classes. The Schubert cells in Y_d are the image of Schubert cells C_α in $Y_{1\cdots d}$ in which $\alpha^{-1}(1)<\cdots<\alpha^{-1}(d)$. For such a cell I compute its homology class as follows:

Let $a_i = \alpha^{-1}(i)$ so that $a_1<\cdots<a_d$. I embed $Y_{1\cdots d}$ inside of $(\mathbb{P}V)^d$ as the set of $(L_1,\dots,L_d)$

where $L_i \perp L_j$. $\quad \overline{C}_\alpha = \{(L_1,\dots,L_d) \in Y_{1,\dots,d} \mid L_i \subset V_{a_i}\}$.

$$= Y_{1,\dots,d} \cap \left(\mathbb{P}V_{a_1} \times \cdots \times \mathbb{P}V_{a_d}\right),$$

and this intersection is transversal of the sort that

$$\text{h-class of } \overline{C}_\alpha = \text{coh. class of } Y_{1,\dots,d} \cap b_{a_1-1} \otimes \cdots \otimes b_{a_d-1}$$

where b_i = basis element of $H_{2i}(\mathbb{P}V)$. Now we've seen that

$$\text{coh. class of } Y_{1,\dots,d} = \prod_{i<j}(\xi_j - \xi_i) = \det\begin{bmatrix} 1 & \xi_1 & \cdots & \xi_1^{d-1} \\ & & & \\ 1 & \xi_d & \cdots & \xi_d^{d-1} \end{bmatrix}$$

hence

$$= \sum_\sigma (-1)^\sigma \xi_{\sigma^{-1}(2)} \, \xi^2_{\sigma^{-1}(3)} \cdots \xi^{d-1}_{\sigma^{-1}(d)}$$

$$= \sum_\sigma (-1)^\sigma \xi_1^{\sigma(1)-1} \, \xi_2^{\sigma(2)-1} \cdots \xi_d^{\sigma(d)-1}$$

Thus

$$\boxed{\begin{array}{l} \text{h-class of } \overline{C}_\alpha = \sum_\sigma (-1)^\sigma b_{a_1-\sigma(1)} \otimes \cdots\cdots \otimes b_{a_d-\sigma(d)} \\ \text{if } \alpha^{-1}(i) = a_i \text{ is in order} \end{array}}$$

Example: In Y_{12} take the Schubert cell described by $a_1 < a_2$. Its homology class is

$$b_{a_1-1} \otimes b_{a_2-2} - b_{a_1-2} \otimes b_{a_2-1}.$$

Now I want to describe in Y_{12} the homology class belonging to C_α, where $\alpha^{-1}(2) = a_1 < \alpha^{-1}(1) = a_2$. I have ~~seen~~ that $\overline{C_\alpha}$ = inverse image of the ~~class~~ ~~is~~ cycle $\overline{C_\gamma}$ in Y_2 described by $a_1 < a_2$ = image of the cycle $\overline{C_\beta}$ in Y_{12} where $\beta^{-1}(1) = a_1$, $\beta^{-1}(2) = a_2$. So I want to compute the image

$$b_{a_1-1} \otimes b_{a_2-2} - b_{a_1-2} \otimes b_{a_1-1}$$

$$\cap$$

under $$H_*(Y_{12}) \xrightarrow{f_*} H_*(Y_2) \xrightarrow{f^*} H_*(Y_{12})$$

But recall that we've seen that in cohomology

$$\begin{tikzcd}
H^*(Y_{12}) \arrow[r, "f_*"] \arrow[drr, bend right, "id - \sigma"'] & H^*(Y_2) \arrow[r, "f^*"] & H^*(Y_{12}) \arrow[d, "\xi_1 - \xi_2"] \\
& & H^*(Y_{12})
\end{tikzcd}$$

commutes. (This ~~way~~ was the formula

$$f^* f_*(a(\xi_1, \xi_2)) = \frac{a(\xi_1, \xi_2) - a(\xi_2, \xi_1)}{\xi_1 - \xi_2}$$
)

Hence we have that

$$\begin{tikzcd}
H_*(Y_{12}) & H_*(Y_2) \arrow[l, "f^*"'] & H_*(Y_{12}) \arrow[l, "f_*"'] \\
 & & H^*(Y_{12}) \arrow[u, "(\xi_1-\xi_2)\cap"'] \arrow[ull, "id-\sigma"]
\end{tikzcd}$$

commutes. But $\xi_1 = \xi\otimes 1$, $\xi_2 = 1\otimes\xi$ and

$$b_{a_1-1}\otimes b_{a_2-2} - b_{a_1-2}\otimes b_{a_1-1} = (\underbrace{1\otimes\xi - \xi\otimes 1}_{\xi_2-\xi_1})(b_{a_1-1}\otimes b_{a_2-1})$$

Thus

$$f^* f_*(\text{class } C_\beta) = -b_{a_1-1}\otimes b_{a_2-1} + b_{a_2-1}\otimes b_{a_1-1}$$

So it would seem that

$$\boxed{h\text{-class } C_\alpha = b_{a_2-1}\otimes b_{a_1-1} - b_{a_1-1}\otimes b_{a_2-1}}$$

It would seem reasenable that all the Schubert cells in $Y_{1,\dots,d}$ [redacted] are given by van der Mande determinants.

Jan 4, 1975.

New derivation of the formulas for homology classes in Y_{12} belonging to Schubert cells:

I.) Fundamental class: $\dim(V) = n$, $b_i \in H_{2i}(\mathbb{P}V)$ standard generator = class assoc. to $\mathbb{P}V_{i+1} \subset \mathbb{P}V$. I can embed $Y_{12}(V)$ into $(\mathbb{P}V)^2$ as pairs (L_1, L_2) of perpendiculare lines, hence I get a homology class in $H_*((\mathbb{P}V)^2)$ which is the image of the fundamental class of $Y_{12}(V)$, the sign being determined by the cx structure on $Y_{12}(V)$.

~~Let Z be the manifold~~ An element of Y_{12} is a flag $(F_1 < F_2)$, whence we have canon. bdles $\mathcal{F}_1, \mathcal{F}_2$ on Y_{12}. Let $Z = \mathbb{P}\mathcal{F}_2$; an element of Z is a triple $(F_1 < F_2 > L)$, so we get a map $\mathbb{P}\mathcal{F}_2 \to (\mathbb{P}V)^2$ sending $(F_1 < F_2 > L)$ to (F_1, L).

$$\begin{tikzcd} \mathbb{P}\mathcal{F}_2 \arrow[r, "g"] \arrow[d, "f"] & (\mathbb{P}V)^2 \\ Y_{12} \arrow[u, bend left, dashed, "s"] \arrow[ur, dashed, "\lambda"'] & \end{tikzcd}$$

Here s is the section given by $s(F_1 < F_2) = (F_1 < F_2 > L)$ where L = orth complement of F_1 in F_2. (Observe s is not holomorphic, however it is an embedding with a complex structure on its normal bundle.

Given $\gamma \in H^*(\mathbb{P}V)^2$, I want to pull it back to

Y_{12} and integrate; this functional on $H^*(Y_{12})$ gives me the element of $H_*(Y_{12})$ I want.

$$\int_{Y_{12}} s^* g^* \gamma = \int_{\mathbb{P}\mathcal{F}_2} s_* 1 \cdot g^* \gamma$$

Now the image of s is where the map of complex bundles

$$\mathcal{O}(-1) \subset f^*\mathcal{F}_2 \xrightarrow[\text{orthogonal projection}]{} f^*\mathcal{F}_1$$

vanishes; in fact the corresponding section of $\mathcal{O}(1) \otimes f^*\mathcal{F}_1$ is transversal to the zero section in each fibre, and the normal bundles match. (Point is that we work C^∞ horizontally so that $\mathcal{F}_1$ is a complex quotient bundle of $\mathcal{F}_2$ and s is the corresponding section). Thus

$$s_* 1 = c_1\left(\mathcal{O}(1) \otimes f^*\mathcal{F}_1\right) \qquad \mathcal{O}(-1) = g^*\mathcal{L}_2$$

$$= g^*(\xi_2 - \xi_1) \qquad \mathcal{F}_1 = g^*\mathcal{L}_1$$

where $\xi_i = pr_i^*(\xi)$ $\qquad pr_i : (\mathbb{P}V)^2 \longrightarrow \mathbb{P}V$.

Hence

$$\int_{\mathbb{P}\mathcal{F}_2} s_* 1 \cdot g^* \gamma = \int_{(\mathbb{P}V)^2} g_* 1 \cdot (\xi_2 - \xi_1)\gamma$$

But g is birational – it is the blowup of $(\mathbb{P}V)^2$ along the diagonal. So we find

$$\int_{i_*[Y_{12}]} \gamma = \int_{(\mathbb{P}V)^2} (\xi_2 - \xi_1)\gamma$$

$$= \int_{(\xi_2 - \xi_1)\cap[\mathbb{P}V]} \gamma$$

$\therefore$

$$i_*[Y_{12}] = \underset{\substack{\parallel \\ 1\otimes\xi - \xi\otimes 1}}{(\xi_2 - \xi_1)} \cap (b_{n-1}\otimes b_{n-1})$$

$$= b_{n-1}\otimes b_{n-2} - b_{n-2}\otimes b_{n-1}$$

II) Schubert cycles described by $\alpha : \begin{bmatrix} a_1 < a_2 \\ \phantom{a_1 <\ } a_2 \end{bmatrix}$

Recall $\tilde{C}_\alpha$ = a resolution of $\bar{C}_\alpha$ consists of

$$\begin{matrix} L < F_2 \\ \vee \\ F_1 \end{matrix} \quad \text{such that} \quad L \subset V_{a_1},\ F_2 \subset V_{a_2}$$

If $a_2 = n$, $a_1 = n-1$, then $\bar{C}_\alpha = Y_{12}$ so I know what the class is by I. Moreover the map

$$\tilde{C}_{\begin{bmatrix} n-1 & n \\ & n \end{bmatrix}} \longrightarrow \{(L < F_2) \mid L \in V_{n-1}\}$$

$$\begin{matrix} L < F_2 \\ \vee \\ F_1 \end{matrix} \longmapsto (L < F_2)$$

II). Schubert cycles $\alpha : \begin{pmatrix} a_1 < a_2 \\ a_1 \end{pmatrix}$.

Such a cycle $\bar{C}_\alpha$ is a non-singular intersection of divisors. Thus the condition $F_1 \subset V_{a_1}$ is the vanishing of the canonical map $\mathcal{F}_1 \to \mathcal{O} \otimes V/V_{a_1}$ transveral to zero, so described by $\xi_1^{n-a_1}$. Once on this subset the condition $F_2 \subset V_{a_2}$ is described by the vanisling of the canon. map $\mathcal{F}_2/\mathcal{F}_1 \to \mathcal{O} \otimes V/V_{a_2}$, described by the class $\xi_2^{n-a_2}$. Thus the homology class of $\bar{C}_\alpha$ is

$$\xi_1^{n-a_1} \xi_2^{n-a_2} \cap [b_{n-1} \otimes b_{n-2} - b_{n-2} \otimes b_{n-1}]$$

$$= b_{a_1-1} \otimes b_{a_2-2} - b_{a_1-2} \otimes b_{a_2-1}$$

III) Schubert cycles $\alpha : \begin{pmatrix} a_1 < a_2 \\ \quad a_2 \end{pmatrix}$. Put $\beta = \begin{pmatrix} a_1 - 1 < a_2 \\ \quad a_2 \end{pmatrix}$ and note $\tilde{C}_\beta$ is the divisor in $\tilde{C}_\alpha$ where $L \to V_{a_1}/V_{a_1-1}$ vanishes. But

$$\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \mathcal{F}_2/\mathcal{F}_1) \otimes (\mathcal{F}_2/\mathcal{F}_1)^\vee$$

$$e(\mathcal{L}^\vee) = e(\mathrm{Hom}(\mathcal{L}, \mathcal{F}_2/\mathcal{F}_1)) * e(\mathcal{F}_2/\mathcal{F}_1)^\vee$$

Now the can. map $L \to F_2/F_1$ vanishes transversally where $L = F_1$, which is the cycle $\bar{C}_{\begin{pmatrix} a_1 < a_2 \\ a_1 \end{pmatrix}}$

which is a divisor in $\tilde{C}_\alpha$. Thus I see the divisors are linear equivalent:

$$\tilde{C}_\beta \sim \tilde{C}_{\left(\begin{smallmatrix} a_1 < a_2 \\ a_1 & \end{smallmatrix}\right)} + \xi_2 \cap \tilde{C}_\alpha$$

so integrating down to Y_{12} I get

$$(*) \qquad \left[\bar{C}_{\left(\begin{smallmatrix} a_1-1 & a_2 \\ & a_2 \end{smallmatrix}\right)}\right] = \xi_2 \cap \left[\bar{C}_{\left(\begin{smallmatrix} a_1 & a_2 \\ & a_2 \end{smallmatrix}\right)}\right] + \left[\bar{C}_{\left(\begin{smallmatrix} a_1 & a_2 \\ a_1 & \end{smallmatrix}\right)}\right]$$

which can be used by downward induction on a_1 starting from

$$\bar{C}_{\left(\begin{smallmatrix} a_2-1 & a_2 \\ & a_2 \end{smallmatrix}\right)} = \{ F_1 < F_2 \mid F_2 \subset V_{a_2} \}$$

which has the homology class

$$b_{a_2-1} \otimes b_{a_2-2} - b_{a_2-2} \otimes b_{a_2-1}$$

Check earlier formula with $(*)$.

$$\xi_2 \cap \left[\bar{C}_{\left(\begin{smallmatrix} a_1 & a_2 \\ & a_2 \end{smallmatrix}\right)}\right] = \xi_2 \cap \left(b_{a_2-1} \otimes b_{a_1-1} - b_{a_1-1} \otimes b_{a_2-1}\right)$$

$$= b_{a_2-1} \otimes b_{a_1-2} - b_{a_1-1} \otimes b_{a_2-2}$$

$$\left[\bar{C}_{\left(\begin{smallmatrix} a_1 & a_2 \\ a_1 & \end{smallmatrix}\right)}\right] = b_{a_1-1} \otimes b_{a_2-2} - b_{a_1-2} \otimes b_{a_2-1}$$

$$\left[\bar{C}_{\left(\begin{smallmatrix} a_1-1 & a_2 \\ & a_2 \end{smallmatrix}\right)}\right] = b_{a_2-1} \otimes b_{a_1-2} - b_{a_1-2} \otimes b_{a_2-1}$$

January 6, 1975

New approach to Schubert classes in Y_{12}.

Suppose given $\alpha : \begin{pmatrix} a_1 & a_2 \\ & a_2 \end{pmatrix}$ as usual. I resolve $\overline{C}_\alpha$ by $\widetilde{C}_\alpha$ which consists of $\begin{matrix} L < F_2 \\ \vee \\ F_1 \end{matrix}$ with $L \subset V_{a_1}$, $F_2 \subset V_{a_2}$. Embed $\widetilde{C}_\alpha$ inside of $(\mathbb{P}V)^3$ by sending $(L < F_2 > F_1)$ to the lines

$$L_1 = F_1, \quad L_2 = F_2 \ominus F_1, \quad L_3 = L$$

I propose to compute the cohomology class of $\widetilde{C}_\alpha$ in $H^*((\mathbb{P}V)^3)$, and then to integrate via $pr_{12} : (\mathbb{P}V)^3 \longrightarrow (\mathbb{P}V)^2$ so as to get the image of the class of $\overline{C}_\alpha$ inside $H^*((\mathbb{P}V)^2)$.

~~[illegible]~~ Condition: $L_1 \subset V_{a_2}$. ~~[illegible]~~ The subset of $(\mathbb{P}V)^3$ where this is satisfied is a submanifold whose coh. class in $\xi_1^{n-a_2}$.

+ Condition: $L_2 \perp L_1$, $L_2 \subset V_{a_2}$. Have subm: W of class

$$(\xi_1 \xi_2)^{n-a_2}(\xi_2 - \xi_1)$$

+ Condition $L_3 \subset F_2$. On W we have the vector bundle $\tilde{V}/\mathcal{F}_2$, $(\tilde{V} = \mathcal{O}_W \otimes V)$, and a map $\mathcal{L}_3 \longrightarrow \tilde{V}/\mathcal{F}_2$ transversal to zero. Thus if we add this condition we get the class $i_* \, e(\mathrm{Hom}(\mathcal{L}_3, \tilde{V}/\mathcal{F}_2))$ where $i : W \hookrightarrow (\mathbb{P}V)^3$. But we compute:

$$e(\mathcal{L}_3^\vee \otimes \tilde{V}/\mathcal{F}_2) = \xi_3^{n-2} + \xi_3^{n-3} c_1(\tilde{V}/\mathcal{F}_2) + \cdots + c_{n-2}(\tilde{V}/\mathcal{F}_2)$$

and $$c_t(\tilde{V}/\mathcal{F}_2) = \frac{1}{c_t(\mathcal{L}_1 \oplus \mathcal{L}_2)} = \frac{1}{(1-t\xi_1)(1-t\xi_2)}$$

$$= \sum_a t^a\left(\xi_1^a + \cdots + \xi_2^a\right)$$

where the ξ_j here are i^* of ξ_j in $H^*((\mathbb{P}V)^3)$.
Thus

$$i_* e(\mathcal{L}_3^\vee \otimes \tilde{V}/\mathcal{F}_2) = i_* 1 \cdot \left(\xi_3^{n-2} + \xi_3^{n-3}(\xi_1+\xi_2) + \xi_3^{n-4}(\xi_1^2+\xi_1\xi_2+\xi_2^2) + \right.$$

$$= (\xi_1\xi_2)^{n-a_2}\left[(\xi_2-\xi_1)\xi_3^{n-2} + (\xi_2^2-\xi_1^2)\xi_3^{n-3} + (\xi_2^3-\xi_1^3)\xi_3^{n-4} + \cdots\right.$$

Call X the variety obtained so far. On X we have $\mathcal{L}_3 \longrightarrow (V_{a_2}/V_{a_1})^\sim$ transversal to zero. (X may be identified with $L < F_2 > F_1 \ni F_2 \subset V_{a_2}$, and I know the map $L < F_2 > F_1 \longmapsto L \in \mathbb{P}V_{a_2}$ is smooth). So I find the coh. class of $\vec{C}_\alpha$ in $H^*(\mathbb{P}V^3)$ is

$$(\xi_1\xi_2)^{n-a_2}\xi_3^{a_2-a_1}\left[(\xi_2-\xi_1)\xi_3^{n-2} + \cdots\right]$$

Integrating over L_3 means we take coeff of ξ_3^{n-1} which is

$$(\xi_1\xi_2)^{n-a_2}\left(\xi_2^{a_2-a_1} - \xi_1^{a_2-a_1}\right)$$

$$= \xi_1^{n-a_2}\xi_2^{n-a_1} - \xi_1^{n-a_1}\xi_2^{n-a_2}$$

which capped with $b_{n-1}\otimes b_{n-1}$ gives old formulas

$$b_{a_2-1}\otimes b_{a_1-1} - b_{a_1-1}\otimes b_{a_2-1}.$$

Let X be a man. over which we have $\mathcal{L}, \mathcal{L}_1, .., \mathcal{L}_k \subset \tilde{V}$, let $Y \overset{i}{\subset} X$ be where $\mathcal{L}_i \perp \mathcal{L}_j$; assume these conditions are transversal so that

$$i_* 1 = \prod_{i<j} (\xi_j - \xi_i)$$

and also that on Y, the map $\mathcal{L} \longrightarrow \tilde{V}/\mathcal{L}_1 \oplus \cdot \oplus \mathcal{L}_k$ is transversal, so that if Z is where this vanishes, then

$$i_* [Z] = i_* e(\mathcal{L}^\vee \otimes \tilde{V}/\mathcal{L}_1 \oplus \cdot\cdot \oplus \mathcal{L}_k)$$

Now

$$e(\mathcal{L}^\vee \otimes \tilde{V}/\mathcal{L}_1 \oplus \cdot\cdot \oplus \mathcal{L}_k) = T^{n-k} c_{T^{-1}}(\tilde{V}/\mathcal{L}_1 \oplus \cdot \oplus \mathcal{L}_k)\Big|_{T = e(\mathcal{L}^\vee) = z}$$

$$= T^{n-k} \prod_{i=1}^{k} \frac{1}{1 - T^{-1}\xi_i} \Big|_{T=z}$$

$$= \frac{T^n}{(T-\xi_1)\cdots(T-\xi_k)} \Big|_{T=z}$$

(Note the polys. $T - \xi_j$ are non-zero-div. in $H^*(X)[T]$ so this quotient is in the total quotient ring – actually it is a poly.). I will use the notation

$$\frac{z^n}{(z-\xi_1)\cdots(z-\xi_k)}$$ for

$$\therefore \quad i_*[Z] = \prod_{i<j} (\xi_j - \xi_i) \cdot \frac{z^n}{(z-\xi_1)\cdot\cdot(z-\xi_k)}$$

Let's try to find $\left[\bar{C}_{\left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ & a_2 & a_3 \\ & & a_3 \end{smallmatrix}\right)}\right]$.

First describe Z:

$$\begin{array}{ccc} F_{13} & F_{23} & F_3 \\ & F_{12} & F_2 \\ & & F_1 \; * \end{array} \qquad \begin{array}{ccc} 6 & 5 & 3 \\ & 4 & 2 \\ & & 1 \end{array}$$

$L_1 = F_1$, $L_2 = F_2 \ominus F_1$, $L_3 = F_3 \ominus F_3$, $L_4 = F_{23}$, $L_5 = F_{23} \ominus F_{12}$, $L_6 = F_{13}$

$$(\xi_2 - \xi_1)(\xi_3 - \xi_1)(\xi_3 - \xi_2)$$

$$\cdot \frac{\xi_4^{n}}{(\xi_4 - \xi_1)(\xi_4 - \xi_2)} (\xi_5 - \xi_4) \frac{\xi_5^{n}}{(\xi_5 - \xi_1)(\xi_5 - \xi_2)(\xi_5 - \xi_3)}$$

$$\cdot \frac{\xi_6^{n}}{(\xi_6 - \xi_4)(\xi_6 - \xi_5)} \cdot (\xi_1 \xi_2 \xi_3)^{n - a_3} (\xi_4 \xi_5)^{a_3 - a_2} \xi_6^{a_1}$$

In this product you want the coefficient of $(\xi_4 \xi_5 \xi_6)^{n-1}$

I can collapse: $\displaystyle\int_{\xi_6} \frac{\xi_6^{n+a_1}(\xi_5 - \xi_4)}{(\xi_6 - \xi_4)(\xi_6 - \xi_5)} = \int_{\xi_6} \frac{\xi_6^{n+a_1}}{\xi_6 - \xi_5} - \frac{\xi_6^{n+a_1}}{\xi_6 - \xi_4}$

$$= \xi_5^{a_1} - \xi_4^{a_1}$$

Identity:

$$\frac{(b-a)(c-a)(c-b)}{(x-a)(x-b)(x-c)} = \frac{c-b}{x-a} + \frac{a-c}{x-b} + \frac{b-a}{x-c}$$

two rational functions same poles + residues, so their difference is a constant which is zero, since both have value 0 at ∞.

So I get

$$\frac{(\xi_2-\xi_1)(\xi_3-\xi_1)(\xi_3-\xi_2)}{(\xi_5-\xi_1)(\xi_5-\xi_3)(\xi_5-\xi_3)} \xi_5^{n+a_3-a_2}(\xi_5^{a_1}-\xi_4^{a_1})$$

$$= \xi_5^{n+a_3-a_2}(\xi_5^{a_1}-\xi_4^{a_1})\left[\frac{\xi_3-\xi_2}{\xi_5-\xi_1} + \frac{\xi_1-\xi_3}{\xi_5-\xi_2} + \frac{\xi_2-\xi_1}{\xi_5-\xi_3}\right]$$

$$= (\xi_3-\xi_2)(\xi_1^{a_3-a_2+a_1} - \xi_4^{a_1}\xi_1^{a_3-a_2}) + (\xi_1-\xi_3)(\xi_2^{a_3-a_2+a_1} - \xi_4^{a_1}\xi_2^{a_3-a_2})$$
$$+ (\xi_2-\xi_1)(\xi_3^{a_3-a_2+a_1} - \xi_4^{a_1}\xi_3^{a_3-a_2})$$

This is to be multiplied by $\dfrac{\xi_4^{n+a_3-a_2}(\xi_1\xi_2\xi_3)^{n-a_3}}{(\xi_4-\xi_1)(\xi_4-\xi_2)}$

yuck!

Classify Schubert cells in Y_{123} and closure relations. 6 kinds of cells. Assume $\alpha^{-1}\{1,2,3\} = \{a_1 < a_2 < a_3\}$

$$\dim = (a_1 - 1) + (a_2 - 2) + (a_3 - 3) + \varepsilon$$

(123) $\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{bmatrix}$ $\varepsilon = 0$

(213) $\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ & \bullet & \end{bmatrix}$ $\varepsilon = 1$

(132) $\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & & \bullet \\ \bullet & & \end{bmatrix}$ $\varepsilon = 1$

(231) $\begin{bmatrix} \bullet & \bullet & \bullet \\ & \bullet & \bullet \\ & \bullet & \end{bmatrix}$ $\varepsilon = 2$

(312) $\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & & \bullet \\ & & \bullet \end{bmatrix}$ $\varepsilon = 2$

(321) $\begin{bmatrix} \bullet & \bullet & \bullet \\ & \bullet & \bullet \\ & & \bullet \end{bmatrix}$ $\varepsilon = 3$

Rule – to compute dimension one circles points (j, p) such that $\alpha(p) \geq j$ and $\exists\, a < p$, $\alpha(a) = j+1$. One counts the number of such "a".

January 7, 1975

Given two filtrations F_j, $j \in J$; V_p, $p \in J$ of a module, we have the Schreier isom

$$\begin{array}{c} gr_j(V_p/V_{p-1}) = \dfrac{F_j \cap V_p + V_{p-1}}{F_{j-1} \cap V_p + V_{p-1}} \\ \uparrow \wr \\ \dfrac{F_j \cap V_p}{F_{j-1} \cap V_p + F_j \cap V_{p-1}} \\ \downarrow \wr \\ gr_p(F_j/F_{j-1}) = \dfrac{F_j \cap V_p + F_{j-1}}{F_j \cap V_{p-1} + F_{j-1}} \end{array}$$

Assume $\forall p$ that $F_j \supset V_p$ for j large and $F_j \subset V_p$ for j small, and that V_p/V_{p-1} is simple. Then for each p, $\exists$ unique $j = \alpha(p)$ such that $gr_j(V_p/V_{p-1}) \neq 0$, so we obtain a mapping $\alpha : P \longrightarrow J$.

Ex. 1. V_p $0 \le p \le n$ a full flag in a vector space $V_n = V$ of dimension n. Suppose B is the Borel subgroup of $GL(V)$ fixing this flag. The function α associated to F_j, $j \in J$ is an invariant of the B-orbit of this flag. For each p let $x_p \in F_{\alpha(p)} \cap V_p$ have a non-zero image in $gr_{\alpha(p)}(V_p/V_{p-1}) \simeq gr_p(F_{\alpha(p)}/F_{\alpha(p)-1})$. Then $\{x_p \mid \alpha(p)=j\}$ map to a basis of F_j/F_{j-1}, whence

$$F_j = \sum_{\alpha(p)\leq j} k x_p$$

Also $V_p = kx_p + V_{p-1}$ so we can transform the basis $\{x_1,\dots,x_n\}$ to the fixed basis $\{e_1,\dots,e_n\}$ by an element of B. This shows (F_j) is in the B-orbit of the T-fixed flag

$$(F^\alpha)_j = \sum_{\alpha(p)\leq j} k e_p$$

It follows that the B-orbits are classified by maps $\alpha : \{1,\dots,n\} \to J$, and that each B-orbit has a unique T-fixpt. (Note also as $B = B^u T$, that B^u act trans. on each orbit)

To compute the dimension of the orbit C_α take $\dim(B^u) - \dim(B^u \cap \text{stab. of } F^\alpha)$.

$$\begin{array}{ccc} \dim(B^u) & - & \dim(B^u \cap \text{stab. of } F^\alpha) \\ \| & & \| \\ \text{card}\,\{(i\leq j)\} & - & \sum_p{}' \dim(F_{\alpha(p)} \cap V_p) \\ & & \| \\ & & \text{card}\,\{a \leq p \mid \alpha(a) \leq \alpha(p)\}. \end{array}$$

$$\begin{array}{ccc} F_{\alpha(p)} \cap V_p & \twoheadrightarrow & V_p/V_{p-1} \\ \Downarrow & & \\ x_p & \longmapsto & \neq 0. \end{array}$$

$$= \text{card}\,\{(i<j) \mid \alpha(i) > \alpha(j)\}.$$

Ex. 2. $F \supset R \to R/R\pi = k$ usual d.v.r., $V = F^n$.
The fixed V-filtration will be a chain of lattices

$$V_{(a,r)} = \pi^{-a}\left(Re_1 + \cdots + Re_r + R\pi e_{r+1} + \cdots + R\pi e_n\right)$$

for each $a \in \mathbb{Z}$, $0 < r \leq n$. The other F-filtration will be obtained as follows. Start with a chain of lattices

$$L_0 < \cdots < L_g \qquad \text{with} \qquad L_g \subset \pi^{-1} L_0$$

and put

$$F_{(b,s)} = \pi^{-b} L_s$$

for each $b \in \mathbb{Z}$, $0 \leq s \leq g$. Then there is a unique pair $j = (b,s)$ such that the quotient $V_{(a,r)}/V_{(a,r-1)}$ appears in F_j/F_{j-1}, and this defines the function $\alpha(a,r)$. ($j-1$ is the predecessor of $j = (b,s)$, i.e. $(b, s-1)$ if $s \geq 1$ and $(b-1, g)$ if $s = 0$). Since multiplication by π^t preserves the F, V chains with a shift, we see

$$\alpha : \mathbb{Z} \times \{1, \ldots, n\} \longrightarrow \mathbb{Z} \times \{0, \ldots, g\}$$

is equivariant for the $\mathbb{Z}$-action, hence α is determined by the values $\alpha(0,1), \ldots, \alpha(0,n)$. Next we choose $x_p \in F_{\alpha(p)} \cap V_p$ so as to be non-zero in V_p/V_{p-1}. In view of the equivariance, we can choose x_p for $p = (0,1), \ldots, (0,n)$ and put $x_{(a,r)} = \pi^{-a} x_{(0,r)}$.

~~$\alpha(0,k) = (d_k, \sigma(k))$ where $\sigma: \{1,\dots,n\} \to \{0,\dots,q\}$.~~
~~To simplify, suppose first that $q=0$, whence $F_{j-1} = \pi F_j$. We know the images of x_p with $\alpha(p)=j$ form a basis for F_j/F_{j-1}. But~~

~~$$p=(a,r) \Rightarrow \alpha(p) = (a+d_r, 0)$$~~

~~$$\alpha(p)=j \Rightarrow$$~~

Suppose $q=0$ to simplify in which case $J = \mathbb{Z}$ and $F_j = \pi^{-j} L_0$. Let $\alpha: \mathbb{Z}\times\{1,\dots,n\} \to \mathbb{Z}$ is given by $\alpha(a,r) = +a+d_r$. Put $x_r = x_{(0,r)}$ in which case $\bar{x}_1,\dots,\bar{x}_n$ is a base for $R^n/\pi R^n$ consistent with the given flag. We know $L_0/\pi L_0 = F_0/F_{-1}$ has a basis given by the images of x_p, $\alpha(p)=0$ i.e. $\pi^{-a} x_r$ with $-a+d_r=0$. Thus $L_0/\pi L_0$ has the basis $\pi^{-d_r} x_r$, so in fact L_0 has the R-basis $\pi^{-d_r} x_r$.

In the general case $\alpha(a,r) = (a+d_r, \mu(r))$ where $\mu: \{1,\dots,n\} \to \{0,\dots,q\}$, and $F(0,s)/F(0,s-1)$ will have as basis the images of the $x_{(a,r)}$ with $(a+d_r, \mu(r)) = (0,s)$. Thus L_s/L_{s-1} will have the k-basis $\pi^{-d_r}x_r$ $\quad r\in\mu^{-1}\{s\}$. It follows that L_s has the R-basis

$$\pi^{-d_r} x_r \qquad 0 \le \mu(r) \le s$$

$$\pi^{-d_r+1} x_r \qquad s < \mu(r) \le q$$

Next I want to consider the closure of these cells on the space of vertices. For each sequence of integers $u(1), \dots, u(n)$ I get a cell C_u which consists of all lattices L such that the quotient $V_{(a,r)}/V_{(a,r-1)}$ appears in $F_{\alpha(a,r)}/F_{\alpha(a,r)-1}$ where $\alpha(a,r) = a + u(r)$

$$F_j = \pi^{-j}L.$$

Thus $u(r)$ = least j such that $\pi^{-j}L + V_{(0,r-1)} \supset V_{(0,r)}$

$$u(r) = \text{least } j \text{ such that } e_r \in \underbrace{Re_1 + \dots + Re_{r-1} + \pi R^n}_{V_{(0,r-1)}} + \pi^{-j}L$$

Suppose I fix integer m', m'' such that the $u(r)$ are in the interval $[m', m'']$. Then

$$\pi^{-m''}R^n \subset L \subset \pi^{-m'}R$$

because L has basis $\pi^{-d_r}x_r$. Recall

$$\dim(F_j \cap V_p) = \text{card}\{a \le p \mid \alpha(a) \le j\}$$

in the sense that the x_a with $a \le p$ and $\alpha(a) \le j$ form a basis for $F_j \cap V_p$. Now suppose I fix L between $\pi^{-m'}R^n$, $\pi^{-m''}R^n$. Then it is clear that by counting the dimensions of $L \cap \pi^{-a}V_{(0,r)}$, or better, the indices, I will be able to read off the sequence

d_h.

$$\begin{aligned}\chi(L \cap V_p) &= \\ &= \dim\left(L \cap V_p / \pi^N V_0\right) - \dim\left(V_0 / \pi^N V_0\right) \\ &= \operatorname{card}\left\{-N < a \leq p \mid \alpha(a) \leq 0\right\} - n.N\end{aligned}$$

So χ is an increasing function of p constant with value $\chi(L)$ for p large, and equal to $\chi(V_p)$ for p small.

Conjecture is that L' is in the closure of the L orbit iff $\chi(L' \cap V_p) \geq \chi(L \cap V_p)$ for all p.

January 8, 1975 Inclusion relations between Schubert cells (cont.).

Let $X = G/B$ be the full flag manifold. I will give a new proof that any chain $C_0 \subset \cdots \subset C_q$ of Schubert cells in X is ~~fixed~~ normalized by some max. torus. It suffices to show that if we have $x \in C' \subset C$ where C is an open cell, then the unique maximal torus T ~~fixing~~ normalizing x and C also ~~fixes~~ normalizes C'. Choose a max. torus T_1 normalizing C and C', and let $y \in C'$ be the unique T_1 fixpt. If $y = x$, then $T_1 = T$ and we win. Otherwise I want to find g such that $gC = C$, $gC' = C'$ and $gy = x$, whence $gT_1g^{-1} = T$ normalizes C'.

Thus I am reduced to showing that if C' is a Schubert cell contained in an open Schubert cell C, and if B, B' are the assoc. Borels, then $B \cap B'$ acts transitively on C. Let $T \subset B \cap B'$ be a maximal torus, ~~and let us identify X with G/B', so that $C' = B'\sigma B'/B'$ for some $\sigma \in N(T)/T$~~ let $x \in X$ have the stabilizer B', and let $\sigma \in W(T) = N(T)/T$ be the unique element such that $B'\sigma x = C'$. Thus $C' \simeq B'/B' \cap \sigma B' \sigma^{-1} \simeq B'^u / B'^u \cap \sigma B'^u \sigma^{-1}$ $\simeq B'^u \cap \sigma B'^u_- \sigma^{-1}$, where B'_- is the Borel opposite to B' wrt T. But the orbit of $\sigma B'^u_- \sigma^{-1}$ thru σx is the open cell C, hence $B = \sigma B'^u \sigma^{-1}$. Thus we in fact get:

Lemma: If C' is a Schubert cell in X contained in an open Schubert cell C, and if B, B' are the Borels which are the normalizers of C and C' respectively, then $B^u \cap B'^u$ acts simply-transitively on C'.

Suppose we have a partial flag manifold Y and a Schubert cell $C \subset Y$. Then $P = \{g \mid gC = C\}$ is parabolic and C is an orbit for any of the Borels $B \subset P$. In effect if $C = Bc$, and if B' is another Borel, choose a $T \subset B \cap B'$; then C is a union of B'-orbits each with a single fixpt; as C has only one T-fixpt because it is a B-orbit, C must be a B'-orbit.

I know that if C is a B-orbit, then there is exactly one B-orbit $\tilde{C}$ in X mapping onto C, and s.t. $\dim \tilde{C} = \dim C$. Since $\tilde{C}$ determines B, it amounts to the same thing to choose a B such that C is a B-orbit, or to give a cell $\tilde{C}$ in X mapping isom. onto C.

Question: Given $C' \subset C$ in Y, and $\tilde{C} \subset X$ such that $f: \tilde{C} \xrightarrow{\sim} C$ where $f: X \to Y$ is the canon. map. Is $f^{-1}(C') \cap \tilde{C}$ a Schubert cell in X?

Given a max. torus T in G one gets a family of 1-parameter subgroups in G called <u>roots</u>. Say $T =$ diagonal matrices in GL_n. Then for each pair $i \neq j$, $1 \leq i, j \leq n$ one has the 1-parameter subgroup (homom. $G_a \to G$)

$$\lambda \longmapsto e_{ij}^{\lambda} = 1 + \lambda E_{ij}$$

I will be interested in subgroups H of G normalized by T, hence whose Lie algebra $\mathfrak{h}$ will be a subalg. of $\mathfrak{g}$ stable under $Ad(T)$, meaning $\mathfrak{h}$ is spanned by E_{ij}. If H is unipotent, then H is contained in a Borel B, which means none of the diagonal E_{ii} occur. So the roots of H will be a subset R of pairs (i,j) $i \neq j$ such that

$$(i,j) \in R \implies (j,i) \notin R \quad \text{and} \quad i \neq j$$

$$(i,j), (j,k) \in R \implies (i,k) \in R.$$

which means that R is the set of pairs i, j such that $i < j$ for some partial ordering on $\{1, .., n\}$.

<u>Assertion</u>: $\exists$ one-one corresp. between conn. unipotent subgroups U of G normalized by T and partial orderings R on $\{1, .., n\}$. The roots of U are the

pairs (i,j) such that $i<j$ for the partial ordering.

The statement that U is contained in a Borel corresponds to the fact that any partial ordering R can be refined to a linear one. ~~In effect suppose R is the relation $<$, for some p. ordering, and a,b is a pair $a \neq b$ such that $(a,b),(b,a) \notin R$. Let~~

$$\cancel{R' = R \cup \{(x,y) \mid x \leq_R a,\ b \leq_R y\}.}$$

~~Verify R' is a partial ordering.~~

To see this, I must count the elements: $x_1, x_2, \ldots$ of my poset so that $x_i < x_j \Longrightarrow i<j$. I can do this by taking x_1 to be a minimal element of the poset (call it J), the x_2 = a minimal element of $J - \{x_1\}$, etc.

This argument also shows that if I have a pair (a,b) such that $a \neq b$ and $(a,b) \notin R$, then I can arrange that b comes before a in the linear ordering. In effect, I start the ~~partial ordering~~ counting within $Z = \{x \mid x \leq b\}$. Thus if x_1 is minimal in Z, x_2 is minimal in $Z - \{x_1\}$, etc. then x_1 is minimal in J, x_2 ——— $J - \{x_1\}$, etc.

It follows that the given partial ordering on J is the intersection of the linear orderings refining it.

<u>Problem:</u> Given a Schubert cell C in Y normalized by T, classify the Borels normalizing C and containing T.

Call Λ the set of axes of T. The unique T-fixpt. of C gives us a map $\alpha : \Lambda \twoheadrightarrow \{1, .., \mu\}$ such that $\operatorname{card} \alpha^{-1}\{1, ..., j\} = d_j$, where $Y = Y_{d_1, .., d_\mu}$. Let $B \supset T$ normalize C. B gives me an ident. $\Lambda = \{1, .., n\}$. Let $P = \{g \mid gC = C\}$. Then $P \supset B$ is parabolic hence it is generated by B and those σ_i (transp $i, i+1$) that it contains. Now any $\sigma \in W(T)$ permutes the T-fixpts on Y, hence the condition that $\sigma \in P$ is simply that $\alpha\sigma^{-1} = \alpha$. ?NO So P is the parabolic containing P generated by the transpositions σ_i such that $\alpha(i) = \alpha(i+1)$.

So from C and T, I get the map α and a linear preordering on Λ = axes of T, whose equivalence classes are contained in the fibres of α. Moreover consecutive classes for the lin. preordering have different α values. Thus I get the picture:

$$\begin{tikzcd} \Lambda \arrow[r, two heads, "\alpha"] \arrow[d, "\beta"'] & \{1, \ldots, \mu\} \\ \{1, \ldots, \nu\} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

φ satisfies: $\varphi(j) \neq \varphi(j+1) \qquad 1 \leq j < \nu$.
The possible Borel subgroups B in P are in 1-1 correspondence with refinements of β to a linear ordering.

<u>Assertion:</u> Let C', C'' be Schubert cells in X with stabilizers B', B'' such that $C' \cap C'' \neq \phi$. Then $C'' \subset C' \iff B' \cap B''$ acts transitively on C''.

Assume $C'' \subset C'$.

<u>Proof.</u> Choose an open cell C with $C' \subset C$; let $B =$ norm. of C. I have seen that $B' \cap B^u$ acts simply-transitively on C', and the same for $B'' \cap B^u$ on C''. Fixing a point of C, whence we get comm. diag.

$$\begin{tikzcd} B' \cap B^u \arrow[r, phantom, "\subset"] \arrow[d, "\wr"] & B^u \arrow[r, phantom, "\supset"] \arrow[d, "\wr"] & B'' \cap B^u \arrow[d, "\wr"] \\ C' \arrow[r, phantom, "\subset"] & C \arrow[r, phantom, "\supset"] & C'' \end{tikzcd}$$

whence $C'' \subset C' \implies B'' \cap B^u \subset B' \cap B^u \implies (B'' \cap B') \cap B^u = (B'' \cap B^u)$ acts trans. on C''. $\implies B' \cap B''$ acts trans. on C''

Conversely if $x \in C' \cap C''$ and $C'' = (B' \cap B'')x$, then $C'' \subset B'x = C'$.

~~Remark: $C' \cap C'' \neq \emptyset \Rightarrow B' \cap B''$ acts trans. on the intersection $C' \cap C''$. For choose T normalizing both C', C'' whence the unique T-fixpt x lies in $C' \cap C''$~~ NO. ~~If C is the unique open cell norm. by T containing x, then~~

~~$$C' \cong B' \cap B^u$$~~
~~$$C'' \cong B'' \cap B^u \quad \text{in } C \cong B \cap B^u$$~~

~~so $C' \cap C'' \cong (B' \cap B'') \cap B^u$, so $B' \cap B''$ acts transitively.~~

If T is given, and also a T-fixpt $x \in X$, then we can visualize the unipotent subgroups U contained in B^u, B opposite to B_x, as follows. Since $B^u \xrightarrow{\sim} C$ = open T-inv. cell containing x, each such U may be identified with its orbit Ux. The T-inv. Schubert cells C' passing through x are the orbits $(B' \cap B^u)x$, where B' runs over the different Borel subgroups containing T.

I've seen that U corresp. to a partial ordering on Λ = axes of T, which is refined by the linear ordering given by B. Use B to identify Λ with $\{1, .., n\}$. Then U will correspond to a partial ordering on $\{1, .., n\}$ refined by the usual ordering. The U's describing Schubert cells are partial orderings given by intersection the B ordering with a B'-ordering, i.e. of the form

$R = \{(i<j) \mid (\sigma i < \sigma j)\}$ for some perm. σ

Such an R has the property that the complement.

$$R' = \{(i<j) \mid (i,j) \notin R\}$$

is also a partial ordering.

~~Assertion: Let R be a partial ordering of $\{1,\dots,n\}$ such that $(a,b) \in R \Rightarrow a<b$. Thus R is a subset~~

~~Assertion: Let $R \subset \{(a,b) \mid 1 \le a < b \le n\}$ be a transitive relation, and suppose R' is also let transitive, where $R' = \{(a,b) \mid 1 \le a < b \le n, (a,b) \notin R\}$~~

<u>Prop</u>: Let $R \subset \{(a,b) \mid 1 \le a < b \le n\}$ be a transitive relation, and let $R' = \{(a,b) \mid 1 \le a < b \le n,\ (a,b) \notin R\}$ be the complement. Then R is of the form

$$R = \{(a,b) \mid 1 \le a < b \le n,\ \sigma a < \sigma b\}$$

for some perm. σ of $\{1,\dots,n\}$ iff R' is transitive.

<u>Proof</u>: Have seen $\Longrightarrow$.

Conversely assume R and R' transitive, and define a new ordering $\prec$ by

$$a \prec b \quad \text{iff} \quad \begin{array}{l} a<b \quad \text{and} \quad (a,b) \in R \quad \text{or} \\ b<a \quad \text{and} \quad (b,a) \in R' \end{array}$$

Better: $a \prec b$ iff $(a,b) \in R$ or $(b,a) \in R'$.

~~[illegible]~~

If $a \neq b$, there are four cases which are disjoint

$$\left.\begin{array}{l}(a,b) \in R \\ (b,a) \in R \\ (a,b) \in R' \\ (b,a) \in R'\end{array}\right. \qquad \begin{array}{l} \Rightarrow a \prec b \\ \\ \Rightarrow b \prec a\end{array} \qquad \text{not both}$$

Transitivity: $a \prec b \prec c$. Then a, b, c are distinct and there are 6 cases

$a < b < c$ $\quad (a,b) \in R,\ (b,c) \in R \Longrightarrow (a,c) \in R \Rightarrow a \prec c$

$b < a < c$ $\quad (b,a) \in R',\ (b,c) \in R,\ \begin{cases}(a,c) \in R \\ (a,c) \in R' \Rightarrow (b,c) \in R' \ * \end{cases} \Rightarrow a \prec c$

$a < c < b$ $\quad (a,b) \in R,\ (c,b) \in R',\ \left|\begin{array}{l}(a,c) \in R \\ (a,c) \in R' \Longrightarrow (a,b) \in R' \ *\end{array}\right. \Rightarrow a \prec c$

$c < a < b$ $\quad (a,b) \in R,\ (c,b) \in R' \left|\begin{array}{l}(c,a) \in R \Rightarrow (c,b) \in R \ * \\ (c,a) \in R'\end{array}\right. \Rightarrow a \prec c$

$b < c < a$ $\quad (b,a) \in R',\ (b,c) \in R \left|\begin{array}{l}(c,a) \in R \Rightarrow (b,a) \in R \ * \\ (c,a) \in R'\end{array}\right. \Rightarrow a \prec c$

$c < b < a$ $\quad (b,a) \in R',\ (c,b) \in R' \Rightarrow (c,a) \in R' \Rightarrow a \prec c$.

<u>Example</u>: Take $n=3$, $R = \{(1,3)\}$, $R' = \{(1,2),(2,3)\}$ which means that $U = \begin{pmatrix}1 & 0 & * \\ & 1 & 0 \\ & & 1\end{pmatrix}$ is not the intersection of B and a permuted Borel. The six subgroups that do occur are indicated

$$\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix} \begin{pmatrix}1 & * & \\ & 1 & \\ & & 1\end{pmatrix} \begin{pmatrix}1 & & \\ & 1 & * \\ & & 1\end{pmatrix} \begin{pmatrix}1 & * & * \\ & 1 & \\ & & 1\end{pmatrix} \begin{pmatrix}1 & & * \\ & 1 & * \\ & & 1\end{pmatrix} \begin{pmatrix}1 & * & * \\ & 1 & * \\ & & 1\end{pmatrix}$$

Good way to state the criterion is that $U \subset B^u$ will correspond to a Schubert cell iff the complementary U' is a subgroup. Thus the complement of

$$\begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 1 & * & 0 \\ & 1 & * \\ & & 1 \end{pmatrix}$$

which is not a subgroup.

Jan. 9, 1975.

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \qquad Y = Y_{d_1,\dots,d_{\mu-1}} = \{ (F_1 < \dots < F_{\mu-1} \le V) \mid \dim F_i = d_i \}. \qquad d_\mu = n$$

$y \in Y$ a point fixed by max. torus T

y may be ident. with a map $\alpha : \Lambda \twoheadrightarrow \{1,\dots,\mu\}$ such that $\operatorname{card} \alpha^{-1}\{1,\dots,j\} = d_j$. $\Lambda \equiv$ axes of T (= roots for stand. repn of G on V). Let P_α = stabilizer of y in G. ~~Let B be a Borel subgp containing T, whence we get an ordering for Λ.~~ P_α is the subgroup of G consisting of matrices a_{ij} which are zero for $\alpha(i) > \alpha(j)$.

Let B be a Borel containing T, whence Λ gets a linear ordering. $B_y \cong B^u / B^u \cap P_\alpha$ and $B^u \cap P_\alpha$ is the unipotent group with roots (i,j) such that $i \underset{B}{<} j$ and $\alpha(i) \le \alpha(j)$. Denote by

$$N_\alpha = \text{unip. group with roots } (i,j) \quad \alpha(i) > \alpha(j)$$

Then $B^u \cap N_\alpha$ [redacted] acts simply-transitively on By

$$B^u \cap N_\alpha = \text{unip. gp roots } i,j \ni i \underset{B}{<} j,\ \alpha(i) > \alpha(j).$$

Choosing B so that $B \supset N_\alpha$ one sees that $N_\alpha \cong$ open nbd. of y.

I see that any Schubert cell containing y and normalized by T [redacted] is the orbit thru y of a conn. subgroup U of N_α [redacted] of the form $B^u \cap N_\alpha$ for some Borel B. Let R be the roots of U. Then $R \subset \{(i,j) \mid \alpha(i) > \alpha(j)\}$ is a subset such that $\exists$ an ordering [redacted] $i \underset{B}{<} j$ on Λ such that $R = \{(i,j) \mid i \underset{B}{<} j,\ \alpha(i) > \alpha(j)\}$. Hence if we put

$$R' = \{(i,j) \mid i \underset{B}{>} j \quad \alpha(i) > \alpha(j)\}$$

$$R' = \text{comp. of } R \text{ in } R(N_\alpha) = \{(i,j) \mid \alpha(i) > \alpha(j)\}$$

then we see that both R and R' are transitive.

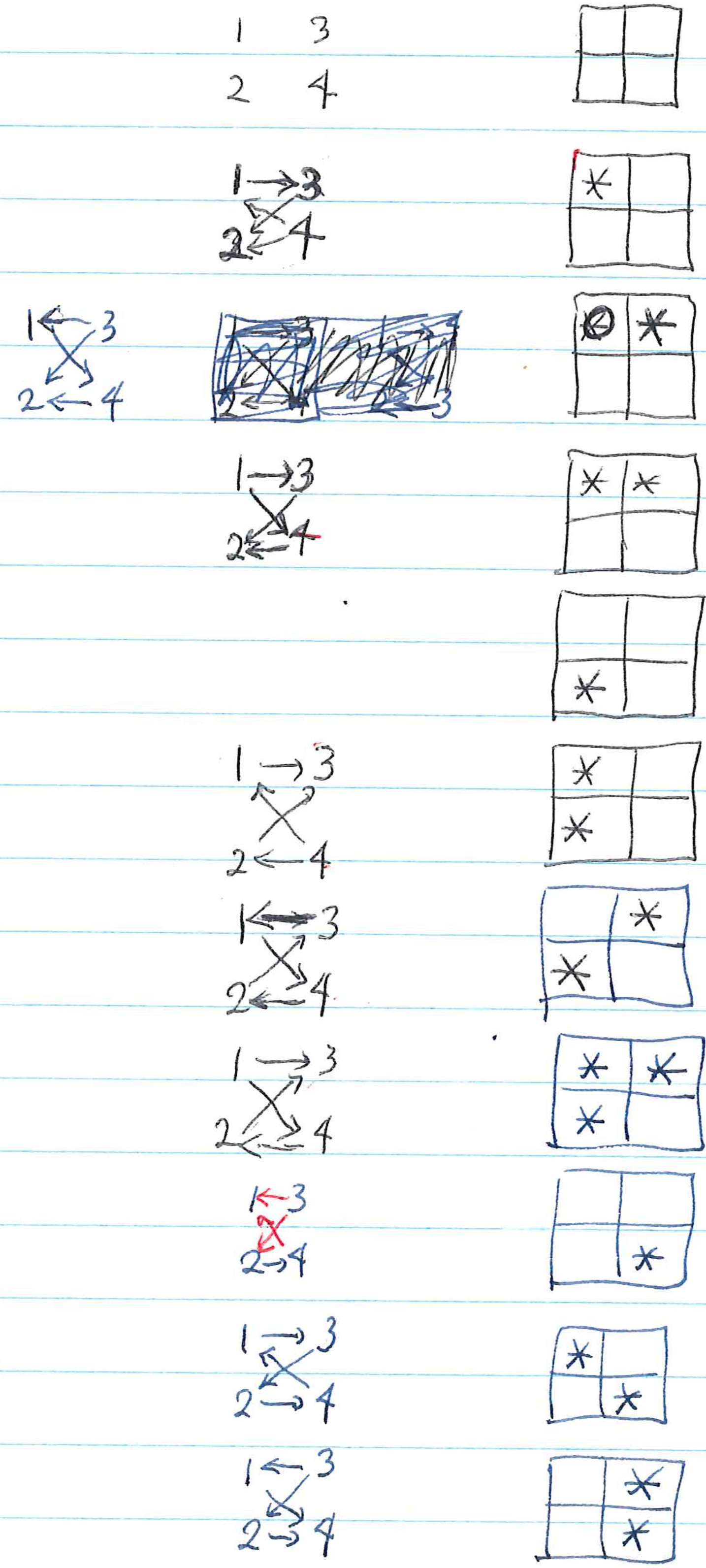
<u>Example:</u> Grassmannian. Here $\alpha : \{1, \ldots, p, p+1, \ldots, p+q\} \to \{1,2\}$ map $\{1, \ldots, p\} \mapsto \{1\}$, $\{p+1, \ldots, q\} \mapsto \{2\}$. There is no transivity to verify, so that R and R' could be complementary subsets of $\{(i,j) \mid i \leq p < j\}$. But for these to be Schubert cells it would be essential that the relation indicated by:

$$\begin{tikzcd}[row sep=small] i_1 \arrow[dr, "R"] & \\ & j \\ i_2 \arrow[ur, "R'"'] & \end{tikzcd}$$

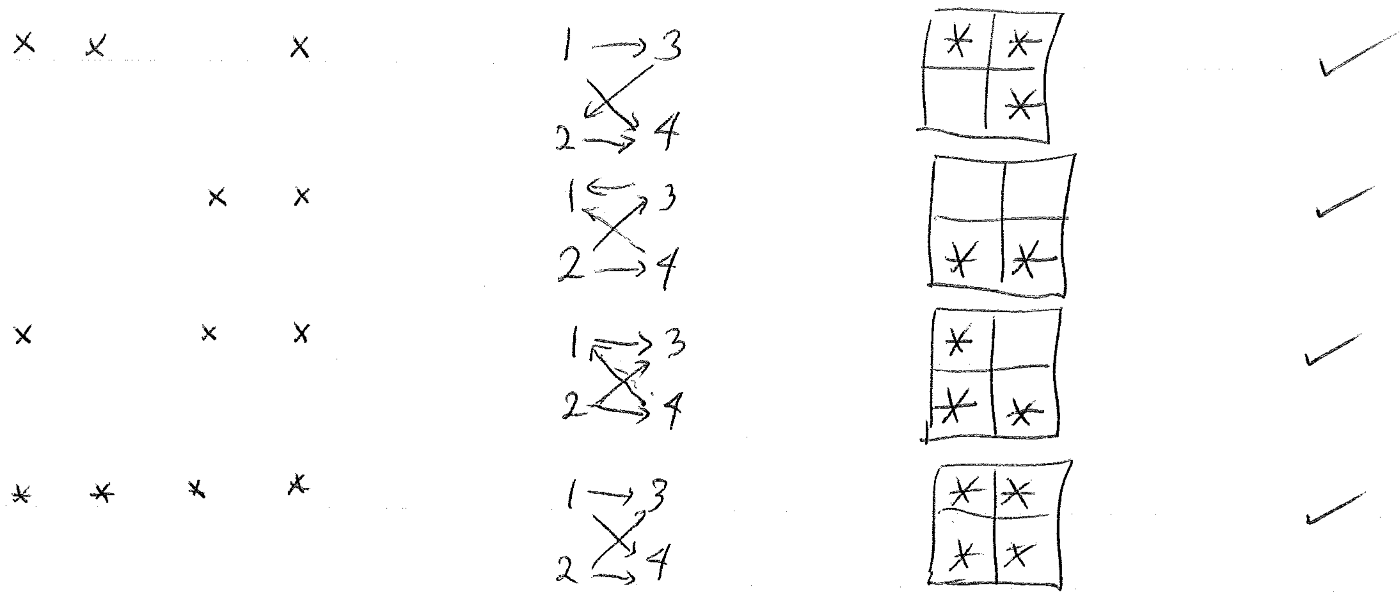
be refinable to a [redacted] linear ordering on $\{1, \ldots, p\}$.

e.g. let's classify for 2 planes in 4 space; 4 possible pairs (i,j) $1 \le i \le 2 < j \le 4$, total of 16 relations R.

Order ~~(1,2) < (1,3) < (2,3) < (2,4)~~

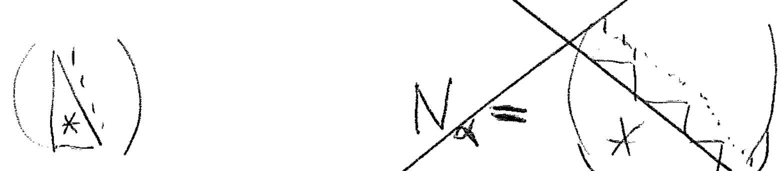


(13)	(14)	(23)	(24)	Schubert all?
				YES ✓
x				YES ✓
	x			YES ✓
x	x			YES ✓
		x		✓
x		x		✓
	x	x		NO
x	x	x		✓
			x	✓
x			x	NO
	x		x	✓



Suppose now ~~[illegible]~~ I consider the open Schubert cell C thru α. It is the orbit of N_α which has roots (i,j) $\alpha(i) > \alpha(j)$. It is the orbit of any B (containing T) such that $B \supset N_\alpha$ i.e. $\alpha(i) > \alpha(j) \Longrightarrow i \underset{B}{<} j$. Suppose such a B has been fixed whence I can identify $\Lambda = \{1,\dots,n\}$ ~~[illegible]~~ such that B = usual B_-.

B : $N_\alpha =$



Thus $\alpha(i) > \alpha(j) \Longrightarrow i > j$ for usual ordering on $\{1,\dots,n\}$. Different permutations σ preserving $\alpha^{-1}(j)$ corresponds to the different cells of X mapping to C. In particular we have the unique minimal orbit corresp. to the identity, and the open cell ~~[illegible]~~ which corresp. to the σ reversing the ordering

~~on the fibres $\alpha^{-1}(j)$.~~

I have fixed T and $y \in Y$, whence I get α. I am interested in the poset of T-invariant cells in Y containing y. Pick an $x \in X$ over y, ~~the cells of x will be independent~~ whence we get a $B = B_x$, hence an ordering $\Lambda = \{1, \ldots, n\}$ so that

$$B = \begin{pmatrix} \diagdown & * \\ & \diagdown \end{pmatrix} \qquad N_\alpha = \begin{pmatrix} L & 0 \\ * & L \end{pmatrix} + I$$

The open cell through x will be the orbit of B_-. So I am interested now in the map

$$\begin{array}{ccc} \left\{ \begin{matrix} T\text{-Cells in } X \\ \text{containing } x \end{matrix} \right\} & \longrightarrow & \left\{ \begin{matrix} T\text{-cells in } Y \\ \text{containing } y \end{matrix} \right\} \\ \| & & \| \\ J & \xrightarrow{f} & J' \end{array}$$

induced by $f : X \longrightarrow Y$. This is a surjective map of posets.

I understand the poset J. An element of J may be identified with a permutation σ. Specifically, $\sigma \longmapsto \sigma^{-1} B \sigma \cdot x$; better I want the cell to swept out by the unipotent subgroup with roots $\{(i > j) \mid \sigma i < \sigma j\}$. e.g. to $\sigma = id$ one gets a point, and to $\sigma = \text{Cox}$, one gets the whole B_-. And $\sigma \leq \tau$ in the sense of the partial ordering iff

[redacted] $\ell(\sigma) + \ell(\tau\sigma^{-1}) = \ell(\tau)$.
(In effect $\sigma^{-1}B\sigma B \subset \tau^{-1}B\tau B \iff B\tau\sigma^{-1}B\sigma B = B\tau B$ which happens iff $\ell(\tau\sigma^{-1}) + \ell(\sigma) = \ell(\tau)$.)

Given a cell C in J' I want to understand all of the cells in J mapping to it. For example if C is the open cell corresponding to $\{(i,j) \mid \alpha(i) > \alpha(j)\}$, then the cells in J mapping to C correspond to Borels $B' \supset N_\alpha$, i.e. to [redacted] permutations σ such that $\alpha(i) > \alpha(j) \implies \sigma(i) < \sigma(j)$. Thus σ is any refinement of the partial ordering corresp. to N_α, i.e. σ is some permutation on each $\alpha^{-1}(j)$. There are unique largest + smallest $\tilde{C}$ in J covering C.

Start with C in J and let me [redacted] review the calculation of the normalizer P of C. By assumption C is the orbit $\sigma^{-1}B\sigma \cdot y$ for some perm. σ. P being parabolic is generated by $\sigma^{-1}B\sigma$ and those $\sigma^{-1}s_\varepsilon\sigma$ it contains.

$$N_\alpha \cap \sigma^{-1}B\sigma \xrightarrow{\sim} C$$

Now $\sigma^{-1}s_\varepsilon\sigma$ has to shift y to some other T-fixpt of C, hence $\sigma^{-1}s_\varepsilon\sigma(y) = (y)$ or $\alpha \circ (\sigma^{-1}s_\varepsilon\sigma) = \alpha$.

Conversely suppose $\alpha \circ \sigma^{-1} s_\varepsilon \sigma = \alpha$, whence $\sigma^{-1} s_\varepsilon \sigma$ will preserve N_α.

$$\mathrm{Roots}\left(N_\alpha \cap \sigma^{-1} B \sigma\right) = \{(i,j) \mid \sigma i < \sigma j,\ \alpha i > \alpha j\}$$

$$\mathrm{Roots}\left(N_\alpha \cap \sigma^{-1} s_\varepsilon B s_\varepsilon \sigma\right) = \{(i,j) \mid s_\varepsilon \sigma i < s_\varepsilon \sigma j,\ \alpha i > \alpha j\}$$

To show these are equal it suffices to show

$$\{(a,b) \mid a < b,\ \alpha\sigma^{-1} a > \alpha\sigma^{-1} b\}$$

$$\{(a,b) \mid s_\varepsilon a < s_\varepsilon b,\ \alpha\sigma^{-1} a > \alpha\sigma^{-1} b\}$$

are the same. But the only difference is the condition $s_\varepsilon a < s_\varepsilon b$ versus $a < b$; these sets are the same except for $(a,b) = (\varepsilon, \varepsilon+1)$, and $\alpha\sigma^{-1}(\varepsilon) = \alpha\sigma^{-1}(\varepsilon+1)$, so this exception doesn't count in the presence of the condition $\alpha\sigma^{-1}(a) < \alpha\sigma^{-1}(b)$.

(This rectifies the mistake on page 11.)

For each Borel subgroup B' of P, we get a Schubert cell $B'x$ mapping to C. The different σ such that $\sigma^{-1} B \sigma \subset P$ form a coset for the Weyl group of P.

Change notation. Let C be the orbit of B through $y \in Y$, y given by $\alpha : \{1, .., n\} \longrightarrow \{1, .., \mu\}$ B being standard Borel whose roots are $i<j$. Thus

$$C \simeq N_\alpha \cap B \qquad \text{Roots}(N_\alpha \cap B) = \{(i,j) \mid i<j,\ \alpha i > \alpha j\}$$

The normalizer P of C is the parabolic group cont. B generated by transp. s_i where $\alpha(i) = \alpha(i+1)$. Thus I collapse maximal intervals in the fibres of α obtaining a factorization

$$\begin{tikzcd} \{1, .., n\} \arrow[r, "\alpha"] \arrow[d, "\eta"'] & \{1, .., \mu\} \\ \{1, .., m\} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $\varphi(j) \neq \varphi(j+1)$, and η is monotone and surjective; η corresp. to P.

To lift C to a T-cell $\tilde{C}$ in X, I choose a Borel B' contained in P ($\& \ T \subset B'$) (this amounts to linearly ordering each of the fibres of η), and I choose a T-fix pt x of X over y (this amounts to linearly ordering each of the fibres of α. Note that $\tilde{C}$ determines both B' and x. Let $B' = \sigma^{-1} B \sigma$, $B_x = \rho^{-1} B \rho$.

Then $\quad \dim \tilde{C} = \dim B' \cap B^u_{x,-} = \dim \sigma^{-1} B^u \sigma \cap \rho^{-1} B^u_- \rho$

<u>Problem</u>: Given a max. torus T, a T-fixpt y in $Y =$ (some partial flag manifold), and a T-cell C containing y, describe the T-cells in X with image C.

Suppose C is the orbit of a Borel B; identify $\Lambda =$ axes of T with $\{1, \ldots, n\}$ so that $\text{Roots}(B) = \{(i,j) \mid i<j\}$. Then y is the T-flag described by $\alpha : \{1, \ldots, n\} \twoheadrightarrow \{1, \ldots, \mu\}$. Let P be the normalizer of C; I know P is gen. by B and the transpositions s_i such that $\alpha(i) = \alpha(i+1)$. Thus if I let η be the monotone surjective map collapsing those intervals of $\{1, \ldots, n\}$ collapsed by α, I get

$$\begin{tikzcd} \{1, \ldots, n\} \arrow[r, two heads, "\alpha"] \arrow[d, "\eta"'] & \{1, \ldots, \mu\} \\ \{1, \ldots, m\} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $\text{Roots}(P) = \{(i,j) \mid \eta i \le \eta j\}$; also $\varphi(j) \neq \varphi(j+1)$.

To lift C to a T-cell $\tilde{C}$ in X I have to choose a Borel $B' < P$ and a T-fixpoint x over y, i.e. a T-Borel B'' fixing y.

$$B' = \sigma^{-1} B \sigma \qquad \text{where} \quad \sigma i < \sigma j \implies \eta i \le \eta j$$

$$B'' = \tau^{-1} B \tau \qquad \text{where} \quad \tau i < \tau j \implies \alpha i \le \alpha j$$

Thus the possible choices for $\tilde{C}$ are in one-one correspondence with pairs (σ, τ) satisfying the above

2 conditions. $\tilde{C} = B'x = \sigma^{-1}B\sigma\tau^{-1}B/B$ has dimension

$$l(\sigma\tau^{-1}) = \text{card}\,\{i<j \mid \sigma\tau^{-1}i > \sigma\tau^{-1}j\}$$

Observe that the choice of τ amounts to linearly-ordering each of the fibres of α; the possible choices for τ form a torsor under the subgroup W_α of W consisting of permutations normalizing the fibres of α; this is the Weyl group of the stabilizer of y. Similarly the choice of σ amounts to linearly-ordering each of the fibres of η; the possible σ's form the subgroup W_η of permutations normalizing the fibres of η. (It's clear therefore that the choice of the point x over y has more significant than the choice of Borel $B' \subset P$.)

Take $\sigma = id$, i.e. we fix B of which C is a B-orbit. Of all the possible τ such that $\alpha\tau^{-1}$ is monotone, there is one of least length and one of largest length. Namely τ gives us an isom of $\{1,\dots,d_1\}$ with $\alpha^{-1}(1)$, $\{d_1+1,\dots,d_2\}$ with $\alpha^{-1}(2)$, etc. If we preserve the ordering on $\alpha^{-1}(j)$ we get the τ of least length, and if the ordering gets reversed, we get the τ of greatest length. The least τ has the property that its Schubert cell $\tilde{C}$ has the same dimension as C.

I can now answer the old question: Given $C \subset C'$ in Y and a lifting $\tilde{C}'$ of C' such that $\tilde{C}' \xrightarrow{\sim} C'$, is $f^{-1}(C) \cap \tilde{C}'$ a Schubert cell? Notice that $f^{-1}(C) \cap \tilde{C}' \xrightarrow{\sim} C$. Choose a situation where $P = B$ (e.g. $Y = G_d(V_{2d})$ with $\alpha^{-1}\{1\} = \{1, 3, \dots, 2d-1\}$), and choose for C' the open T-cell containing the center of C (T has been fixed). I know there is a unique cell $\tilde{C}$ such that $\tilde{C} \xrightarrow{\sim} C$. Hence there will be a unique T-fixpt x over λ such that a T-cell through x will map isom. onto C. But for C' we have many choices for $\tilde{C}'$; in fact the possible choices for a $\tilde{C}'$ of the same dimension form a torsor for the Weyl group of the normalizer of C'.

<u>Assertion</u>: Let C be a Schubert cell in Y normalized by the maximal torus T, let P be the normalizer of C. Then the set of T-cells $\tilde{C}$ in X such that $\tilde{C} \xrightarrow{\sim} C$ is a torsor for the Weyl group of P wrt T. Better each T-Borel B in P has a unique orbit $\tilde{C}$ in X such that $\tilde{C} \xrightarrow{\sim} C$.

Assertion: Let C and C' be Schubert cells in Y with normalizers P, P' resp. If $C \subset C'$ then $P \cap P'$ acts transitively on C. Conversely if $C \cap C' \neq \emptyset$, and if $P \cap P'$ acts transitively on C, then $C \subset C'$.

Proof: Converse trivial: If $x \in C \cap C'$, then $C = (P \cap P)'x \subset P'x = C'$.

Suppose then $C \subset C'$, choose $T \subset P \cap P'$, let y be the unique T-fixpt of C, $\alpha : \Lambda \to \{1, \dots, \mu\}$ the associated map, and let C'' be the open T-cell containing y, whence $N_\alpha \xrightarrow[\sim]{\cdot y} C''$. I have seen that $P \cap N_\alpha$ acts simp. trans. on C, resp. for $P' \cap N_\alpha$ on C'. Since

$$\begin{array}{ccccc} P \cap N_\alpha & \subset & N_\alpha & \supset & P' \cap N_\alpha \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ C & \subset & C'' & \supset & C' \end{array}$$

and $C \subset C'$ it follows that $P \cap N_\alpha \subset P' \cap N_\alpha$. In particular $(P \cap P') \cap N_\alpha = (P \cap N_\alpha)$ so $P_y \supset (P \cap P')y \supset (N_\alpha \cap P)y = C$. done.

Example of a chain of Schubert cells in a Grassmannian which is not normalized by a maximal torus T. Work in the Grassmannian of 2-planes in a large space V. I recall that the Schubert cells in $Y_2(V)$ are of two types.

1) Given subspaces $V_p < V_{p+2}$ of dims $p, p+2$ resp. I have the cell

$$C(V_p < V_{p+2}) = \{ A \subset Y_2(V) \mid A \oplus V_p = V_{p+2} \}.$$

2) Given subspaces $V_p < V_{p+1} < V_q < V_{q+1}$ I have the cell

$$C(V_p, V_{p+1}, V_q, V_{q+1}) = \{ A \mid 0 = A \cap V_p < A \cap V_{p+1} = A \cap V_q < A \cap V_{q+1} = A \}.$$

Construct the chain starting with

$$\underset{\substack{\shortparallel \\ C_0}}{C(V_0, V_1, V_2, V_3)} \subset \underset{\substack{\shortparallel \\ C_1}}{C(W_1, V_3)} \leftarrow \text{open cell in } Y_2(V_3).$$

Now put $C(W_1, V_3) \subset C(W_1, W_2, W_3, W_4)^{\shortparallel\, C_2}$

$$\begin{tikzcd}
 & V_3 \arrow[r, phantom, "="] & V_3 \arrow[r, dash] \arrow[dr, phantom, "\text{trans.}"] & W_4 \\
V_1 \arrow[r, dash] \arrow[dr, phantom, "\text{trans}"] & V_2 \arrow[u, dash] \arrow[r, phantom, "\text{trans}"] & W_2 \arrow[u, dash] \arrow[r, dash] & W_3 \arrow[u, dash] \\
0 \arrow[u, dash] \arrow[r, dash] & W_1 \arrow[u, dash] \arrow[r, phantom, "="] & W_1 \arrow[u, dash] & Z_2
\end{tikzcd}$$

The point is that V_3/W_1 is canonically a direct sum of

V_2/W_1 and W_2/W_1. Next we repeat the construction

$$C(W_1, W_2, W_3, W_4) \subset \underbrace{C(Z_2, W_4)}_{C_3} \subset \underbrace{C(Z_2, Z_3, Z_4, Z_5)}_{= C_4}$$

$$\begin{array}{ccccc} W_4 & = & W_4 & - & Z_5 \\ | & & | & & | \\ W_3 & & Z_3 & - & Z_4 \\ | & & | & & \\ Z_2 & = & Z_2 & & \end{array}$$

but we choose Z_3/Z_2 to be different from the lines V_2/W_1, W_2/W_1 brought up via the isomorphism.

$$V_3/W_1 \simeq W_4/Z_2$$

So it's now clear that we can find no maximal torus normalizing the cells C_0, C_2, C_4 simultaneously.

V infinite. Let's review old analysis of the Schubert cells in $Y_2(V)$. There were two types i.e. $C(V_p, V_{p+2})$ and $C(V_p, V_{p+1}, V_q, V_{q+1})$ where $p+1 < q$.

<u>Question</u>: What can you say about the homotopy type of the poset of Schubert cells in $Y_{11}(V)$.

First describe the cells for a fixed B given by a flag (V_p). We get a cell for each ordered pair of distinct integers ≥ 1 (a, b) namely

$$C(a,b) = \{ (L_1 < L_1 \oplus L_2) \mid L_1 \in \mathbb{P}V_a - \mathbb{P}V_{a-1},\ L_2 \in \mathbb{P}V_b - \mathbb{P}V_{b-1} \}.$$

If $a > b$, L_1 and L_2 are uniquely determined, but if $a < b$, then L_2 is uniquely modulo L_1. Clearly this cell depends only upon $V_{a-1} < V_a$, $V_{b-1} < V_b$. Let a_1, a_2 denote pairs $\ni$ $a_1 < a_2$. Then for each (a_1, a_2) and $V_{a_1-1} < V_{a_1} \subset V_{a_2-1} < V_{a_2}$ I have two Schubert cells associated with the same normalizer.

Suppose we have an inclusion $C \subset C'$. Then looking at F_1 we find $(V_{a-1}, V_a) \leq (V'_{a'-1}, V'_{a'})$. Consider 4 cases.

$a < b$, $a' < b'$. Fix $L_1 \in \mathbb{P}V_a - \mathbb{P}V_{a-1} \subset \mathbb{P}V'_{a'} - \mathbb{P}V'_{a'-1}$ and let L_2 vary over $\mathbb{P}V_b - \mathbb{P}V_{b-1}$. Then

$$L_1 \oplus L_2 = L_1 \oplus L_2' \qquad \text{with } L_2' \in \mathbb{P}V'_{b'} - \mathbb{P}V'_{b'-1}$$

So in particular $V_b \subset V'_{b'}$. Observe that because L_2' is unique modulo L_1 and any other choice for L_2' will be outside of $V'_{b'-1} \supset V'_{a'}$, we could take $L_2' = L_2$.

$\therefore$ $(V_{b-1}, V_b) \leq (V'_{b'-1}, V'_{b'})$.

$a > b$, $a' > b'$. Here L_i, L_i' are uniquely determined so again $(V_{b-1}, V_b) \leq (V'_{b'-1}, V'_{b'})$

$a > b$, $a' < b'$. ~~Again I can choose~~ We have $(V_{a-1}, V_a) \leq (V'_{a'-1}, V'_{a'})$ hence V_{b-1} and V_b are contained in $V'_{b'-1}$. ~~But all we can always find a~~ So this case is impossible.

$a < b$, ~~$a' > b'$~~ $a' > b'$. Recall that there exists a torus normalizing the spaces V_i, V_i'. Fix $F_1 = L_1$. For any $F_1 < F_2 \in C$, I know $F_2 \cap V'_{b'} = L_2'$ is a line in $\mathbb{P}V'_{b'} - \mathbb{P}V'_{b'-1}$. Hence

$$\dim\{F_2\} = b-2 \leq \dim\{L_2'\} \leq \dim(V_b \cap V'_{b'}) - 1.$$

so it follows that $V_b \cap V'_{b'}$ is a hyperplane in V_b not containing V_a (as $L_1 \notin V'_{a'-1} \supset V'_{b'}$). If L_2 is the unique fixed line in V_b outside of V_{b-1}, ~~then~~ and L_1 is the same for V_a, V_{a-1}, then $L_2' = (L_1 + L_2) \cap V_{b'}$ has to be L_2. This tells me that

$$V_b \cap V'_{b'-1} = V_{b-1} \cap V'_{b'}$$

as both have $L_2 = L_2'$ for T-complement. ~~Since $V_b \not\subset V'_{b'}$~~ ~~we have $V_b \cap V'_{b'} \subset V_a \subset V_b$, and so $L_1 \not\subset V_b \cap V'_{b'}$~~. Now L_1 which is the unique T-comp for V_{a-1} in V_a is not is $V'_{b'}$, hence it must the unique T-comp. for

$V_b \cap V'_{b'}$ in V_a; thus $(V_a, V_{a-1}) \leq (V_b \cap V'_{b'}, V_b)$. This gives use the following picture

$$
\begin{array}{ccccccccc}
 & & & & & & & & V_{a'} \\
 & & & & \subset & & & & \vee \\
V_b & = & V_b & & & & \text{trans.} & & V_{a'-1} \\
\vee & & \swarrow & \searrow & & & \subset & & \cup \\
V_{b-1} & = & V_{b-1} & & & V_b \cap V'_{b'} & \subset & & V_{b'} \\
\cup & \subset & & \searrow & \swarrow & & \text{tr.} & & \vee \\
V_a & \text{tr.} & & V_{b-1} \cap V'_{b'} = V_b \cap V'_{b'-1} & & & \subset & & V'_{b'-1} \\
\vee & \subset & & & & & & & \\
V_{a-1} & & & & & & & &
\end{array}
$$

Generalization: Consider $Y_{12..d}$ and a T-fixpt $\alpha : \Lambda \longrightarrow \{1, \ldots, d+1\}$, card $\alpha^{-1}(j) = 1$ for $1 \leq j \leq d$. For each σ we can consider the orbit of $\sigma^{-1} B \sigma$ thru α. It gives us the T-cell thru α with roots

$$\{(i,j) \mid \sigma i < \sigma j,\ \alpha i > \alpha j\} = R_\sigma \cap \underline{R}_\alpha$$

Now I will order Λ so that $\alpha^{-1}(j) = j$ for $1 \leq j \leq d$, whence

$$\underline{R}_\alpha = \left\{ (i,j) \;\middle|\; \begin{matrix} 1 \leq j \leq d \\ i > j \end{matrix} \right\}$$

$$= \{(i,j) \mid 1 \leq j < i \leq d\} \amalg \{(i,j) \mid 1 \leq j \leq d < i\}.$$

So assume now that I have an inclusion of T-cells then α given by perms. σ, τ:

$$R_{-\alpha} \cap R_\sigma \subset R_{-\alpha} \cap R_\tau$$

In particular we find that

$$\{(i,j) \mid 1 \le j < i \le d,\ \sigma i < \sigma j\} \subset \{(i,j) \mid 1 \le j < i \le d,\ \tau i < \tau j\}.$$

which means that if I consider the orderings produced on $\{1, \dots, d\}$ by σ, τ, I must have $\sigma \le \tau$.

I can think of the type of a Schubert cell in $Y_{1,\dots,d}$ as being a sequence of integers $a_1 < \dots < a_d$ together with a permutation of $1, \dots, d$. (The $a_1 < \dots < a_d$ give the image of the cell in Y_d; the permutation compares the two filtrations on a typical d-plane F_d.)

I seem ~~to~~ have a different notion of type of Schubert cell. For example the type of a Schubert cell in $Y_{12\ldots d}$ is an embedding $\{1, \dots, d\} \hookrightarrow \{1, \dots, n\}$.