# Notes on buildings, symmetric spaces, etc.

## Fifth part: $B^u_\xi$-orbits on $\mathfrak{p}$, Bruhat decomposition, compactifying Schubert cells.

To fix the ideas, suppose $G$ is the complexification of the compact connected group $K$; put $\mathfrak{p} = i\mathfrak{k}$ as usual. Let $\xi \in \mathfrak{p}$. We propose to determine the $B^u_\xi$-orbit structure of $\mathfrak{p}$. For this purpose we consider the flow $\eta \longmapsto e^{t\xi} * \eta$ in $\mathfrak{p}$.

Suppose $e^{t\xi} * \eta = \eta$ for all $t$, i.e. $e^{t\xi} \in B_\eta$. As $e^{t\xi} \in P$ and $P \cap B_\eta = P_\eta$ (part 4, p. 18), this happens iff $[\xi, \eta] = 0$. Thus $\mathfrak{p}_\xi$ is the set of fixpts for the flow $\eta \longmapsto e^{t\xi} * \eta$.

Let's compute the tangent vector to the curve $e^{t\xi} * \eta$ at $t = 0$. Recall the $G$-action on $K\eta$ is determined by the isom. $K/K_\eta \simeq G/B_\eta$. Thus we take the image of $\xi$ in $\mathfrak{g}/\mathfrak{b}_\eta \simeq \mathfrak{k}/\mathfrak{k}_\eta$ and apply it to $\eta$ to get what we are after. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\eta$ and

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha$$

the corresponding root decomposition. Note $\theta \mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$. If $\xi = \xi_0 + \sum \xi_\alpha$ is the corresp. decomp of $\xi$, then $\theta\xi = -\xi$ translates to $\theta\xi_0 = -\xi_0$, $\theta\xi_\alpha = -\xi_{-\alpha}$.

Hence

$$\gamma = \sum_{\alpha(\eta)>0} \left(-\xi_\alpha + \xi_{-\alpha}\right)$$

is in $\mathfrak{k}$ and $\xi - \gamma \in \mathfrak{g}_{\mathfrak{a}} + \sum_{\alpha(\eta)\geq 0} \mathfrak{g}^\alpha = \mathfrak{b}_\eta$.
Consequently the tangent vector to $e^{t\xi}{}_*\eta$ at $t=0$ is

$$[\gamma, \eta] = \sum_{\alpha(\eta)>0} \left(-\alpha(\eta)\,\xi_\alpha - \alpha(\eta)\,\xi_{-\alpha}\right).$$

Thus

$$\left.\frac{d}{dt}\right|_{t=0} \left(e^{t\xi}{}_*\eta, \xi\right) = -\sum_{\alpha(\eta)>0} \alpha(\eta)\left(|\xi_\alpha|^2 + |\xi_{-\alpha}|^2\right)$$

$$\leq 0$$

with equality iff $[\xi, \eta] = 0$. Since $|e^{t\xi}{}_*\eta - \xi|^2 = |\eta|^2 + |\xi|^2 - 2(e^{t\xi}{}_*\eta, \xi)$, this means that the distance from $e^{t\xi}{}_*\eta$ to $\xi$ is strictly increasing in $t$ provided $[\xi, \eta] \neq 0$:

<u>Prop. 1</u>: If $\eta \notin \mathfrak{p}_\xi$ (i.e. $\eta$ is not stationary for the flow $e^{t\xi}{}_*$?), then the distance from $e^{t\xi}{}_*\eta$ to $\xi$ is strictly increasing in $t$.

We apply this result as follows. Let $\eta' \in \mathfrak{p}$ be such that $\eta' \notin \mathfrak{p}_\xi$, and let $S$ be the set of limit points of the curve $e^{t\xi}{}_*\eta'$ as $t \to -\infty$. $S$ is nonempty because $\mathcal{O} = G * \eta'$ is compact. Let $f$ be the function on $\mathcal{O}$ giving the distance to $\xi$. Because $t \mapsto f(e^{t\xi} * \eta')$ is strictly increasing and bounded, it has a limit $L$ at $t \to -\infty$, hence $f(S) = L$. But $S$ is stable under the flow $e^{t\xi} * (?)$, so as $f$ is constant on $S$ we can conclude $S \subset \mathfrak{p}_\xi$ from the proposition. Thus we have proved that there exists a point $\eta$ in $\mathfrak{p}_\xi$ which is a limit point of $e^{t\xi} * \eta'$ as $t \to -\infty$.

I next want to describe a nbd. of $\eta$ in $G\eta$. As $\mathfrak{g} = \mathfrak{b}^u_{-\eta} \oplus \mathfrak{b}_\eta$, the function theorem tells me that the map $B^u_{-\eta} \longrightarrow G/B_\eta \simeq G\eta$, $g \longmapsto g * \eta$ is a diffeomorphism at the identity. Hence $B^u_{-\eta} B_\eta$ is open in $G$. Now if $g \in B^u_{-\eta} \cap B_\eta$, then $e^{-t\eta} g e^{t\eta}$ converges at $t \to +\infty$ and converges to $1$ as $t \to -\infty$. Since it is, when viewed in $GL_n$, a matrix whose entries are linear combinations of real exponentials $e^{-t\eta} g e^{t\eta}$ is constant in $t$ so $g = 1$. Thus we have:

<u>Prop. 2.</u> The map $B^u_{-\eta} \longrightarrow G\eta$, $g \longmapsto g\eta$ is a diffeomorphism of $B^u_{-\eta}$ onto an open nbd. of $\eta$ in $G\eta$.

In virtue of this result and the diffeom. $\exp : \mathfrak{b}^u_{-\eta} \xrightarrow{\sim} B^u_{-\eta}$ we therefore can study what's going on near $\eta$ in $G\eta$ using $\mathfrak{b}^u_{-\eta}$. Furthermore $G\eta = G/B_\eta$ with $G$ acting by left multiplication. As $\xi$ commutes with $\eta$, the flow $e^{t\xi}$ normalizes $B^u_{-\eta}$, $B_\eta$, and the open set $B^u_{-\eta}\eta$. Thus we get an isomorphism

$$\mathfrak{b}^u_{-\eta} \xrightarrow{\sim} B^u_{-\eta}\eta \overset{\text{open}}{\subset} G\eta$$

compatible with the adjoint action of $e^{+t\xi}$ on $\mathfrak{b}^u_{-\eta}$ and the multiplication action on $G\eta$. However the action of $Ad(e^{t\xi})$ on $\mathfrak{b}^u_{-\eta}$ is simple to analyze in terms of the eigenvalues of $ad(\xi)$. One sees that only the elements of $\mathfrak{b}_\xi \cap \mathfrak{b}^u_{-\eta}$ have a limit point in $\mathfrak{b}^u_{-\eta}$ under $Ad(e^{t\xi})$ as $t \to -\infty$, and that this limit point is a unique element of $\mathfrak{g}_\xi \cap \mathfrak{b}^u_{-\eta}$. In particular the points of $\mathfrak{b}^u_{-\eta}$ having $0$ as limit point under $Ad(e^{t\xi})$ as $t \to -\infty$ are just points of $\mathfrak{b}^u_\xi \cap \mathfrak{b}^u_{-\eta}$.

So we see that if $\eta$ is a limit point of $e^{+t\xi} * \eta'$ as $t \to -\infty$, then in fact $e^{t\xi} * \eta'$ converges to $\eta$. Moreover the set of $\eta'$ with limit $\eta$ is isomorphic to $\mathfrak{b}^u_\xi \cap \mathfrak{b}^u_{-\eta}$. Consider the inclusions:

$$\mathfrak{b}^u_\xi \cap \mathfrak{b}^u_{-\eta} \xhookrightarrow[\exp]{} B^u_\xi \cap B^u_{-\eta} \subset B^u_\xi \eta \subset G\eta.$$

If $g \in B^u_\xi$, then $e^{t\xi} * (g * \eta) = (e^{t\xi} g e^{-t\xi}) * \eta \longrightarrow \eta$ as $t \to -\infty$. Thus we can conclude that the first two inclusions are isos.

So we obtain:

<u>Theorem</u>: Each $B^u_\xi$-orbit on $\mathfrak{p}$ contains a unique element of $\mathfrak{p}_\xi$:

$$\mathfrak{p}_\xi \xrightarrow{\sim} B^u_\xi \backslash \mathfrak{p}.$$

Specifically, given $\eta'$ in $\mathfrak{p}$, the limit $\eta = \lim_{t \to -\infty} e^{t\xi} * \eta'$ exists and its the unique element of $B^u_\xi \eta' \cap \mathfrak{p}_\xi$. If $\eta \in \mathfrak{p}_\xi$, then one has isoms:

$$\mathfrak{b}^u_\xi \cap \mathfrak{b}^u_{-\eta} \xrightarrow[\exp]{\sim} B^u_\xi \cap B^u_{-\eta} \xrightarrow{\sim} B^u_\xi \eta$$
$$g \longmapsto g * \eta$$

Additions: The vector field on $G\eta$ induced by the flow $e^{t\xi} *$ is not the same as the gradient of the distance-squared-to-$\xi$ function. However this vector field plays the same role as the gradient in the Morse theory.

Bruhat decomposition: Take $\xi$ to be a regular element of $\mathfrak{p}$, i.e. $\mathfrak{p}_\xi$ is maximal abelian subspace of $\mathfrak{p}$. I know from parts I and II that $K\eta \cap \mathfrak{p}_\xi$ is a $W$-orbit in $\mathfrak{p}_\xi$. If $\eta \in \mathfrak{p}_\xi$, then we have

$$G\eta = K\eta = \coprod_{w\eta \in W\eta} B^u_\xi w\eta$$

Since $B_\xi = G_\xi \ltimes B^u_\xi$, where $G_\xi$ acts trivially on $\mathfrak{p}_\xi$ we therefore get $B_\xi w\eta = B^u_\xi G_\xi w\eta = B^u_\xi w G_\xi \eta = B^u_\xi w\eta$, so

$$G/B_\eta = \coprod_{w\eta \in W\eta} B_\xi w B_\eta / B_\eta$$

or

$$G = \coprod_{w\eta \in W\eta} B_\xi w B_\eta .$$

Furthermore I know that $B_\xi w\eta \simeq B^u_\xi \cap B^u_{-w\eta}$, hence

$$B_\xi w B_\eta \cong \left(B^u_\xi \cap B^u_{-w\eta}\right) \times B_\eta$$

The Bruhat decomposition is the special case $\eta = \xi$: $B\backslash G/B \simeq W$.

Therefore the $B_\xi$-orbits $(= B^u_\xi$-orbits$)$ on $G\eta = G/B_\eta$ are indexed by points of $W\eta$. Each orbit is a cell; the orbit $B_\xi w\eta$ is isomorphic to the vector space $\mathfrak{b}^u_\xi \cap \mathfrak{b}^u_{-w\eta}$. Its dimension is the number of hyperplanes (counted with multiplicity) crossed in going from $\xi$ to $w\eta$ in $\mathfrak{p}_\xi$.

The preceding applies to the following more general situation: $G$ is the group of real points of a real algebraic group arising, as in part III, p. 18, from a compact connected Lie group with involution; ~~K is the associated maxim~~ (I think every real reductive group arises in this way). $K$ is the associated maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. $\xi$ is a regular element in $\mathfrak{p}$, $\mathfrak{a} = \mathfrak{p}_\xi$, $C$ = the chambre of $\mathfrak{a}$ containing $\xi$, $B = B_\xi$, $N$ = normalizer of $\mathfrak{a}$ in $G$, $Z = G_\xi$ = centralizer, $W = N/Z$ the Weyl group.

The theorem on page 5 says

$$(*) \qquad \mathfrak{a} \xrightarrow{\sim} B\backslash\mathfrak{p}$$

($\mathfrak{a}$ is a fundamental domain for the $B$-action), and it gives a description of the $B$-orbits.

Now I recall that $C \xrightarrow{\sim} G\backslash\mathfrak{p}$, and that the stabilizers $B_\eta$ are constant over the strata of $C$. This I saw gave rise to a stratification of $\mathfrak{p}$, preserved by the $G$-action. If $L$ is a stratum of $\mathfrak{p}$, and we pick $\eta \in L$, and $b \in B$ such that $b\eta \in \mathfrak{a}$, then $bL$ is a stratum of $C$, namely "the" stratum containing $b\eta$. This bL coincides with the stratum of $\mathfrak{a}$ containing $b\eta$, and multiplication by $b$ sets up an isomorphism of the strata $L$ and $bL$. $\therefore$

<u>Prop. 3</u>: The projection $p : \mathfrak{p} \longrightarrow \mathfrak{a}$ defined by $(*)$ preserves strata. It preserves the

specialization relation between strata.

(For the last statement, suppose $L \subset \overline{L'}$ and $bL' \subset \mathfrak{a}$. Then by continuity $bL \subset b\overline{L'} \subset \overline{bL'} \subset \mathfrak{a}$, so $p(L) \subset p(L')$.)

Let $S \subset W$ be the set of reflections thru the walls of $C$. If $s \in S$, let $C_s$ be the corresponding "panel" of $C$, and let $B_s$ be its stabilizer.

Let $C'$ be a chambre of $\mathfrak{p}$ containing $C_s$, and choose $g \in G$ such that $gC = C'$. Then $g^{-1}C_s \subset g^{-1}C' = C$, and $C_s \subset C$ and as $C$ is a $G$-fund. domain, $g \in B_s$. Thus

$$B_s/B \cong \text{chambres of } \mathfrak{p} \text{ containing } C_s.$$

Next with $C'$ as above consider $pC'$. It is a chambre of $\mathfrak{a}$ containing $C_s$. As $C_s$ is a panel of $C$, this means either $pC' = C$ or $pC' = sC$. Thus there are 2 $B$-orbits on $B_s/B$ and we get:

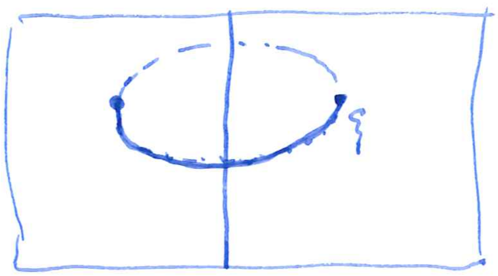
$$B_s = B \amalg BsB.$$

Fix a generic element $\eta$ of $C_s$ so that $B_s = B_\eta$. We know that any element of $\mathfrak{p}_\eta$ can be moved into $\mathfrak{a}$ by an element of $K_\eta$. If $C'$ is a chambre of $\mathfrak{p}$ containing $C_s$, and if $\xi'$ is the point of $C'$ $G$-conjugate to $\xi$, then

$\xi'$ is $K_\eta$ conjugate. This shows that

$$B_s/B \cong K_\eta\cdot\xi \cong K_\eta/K_\xi .$$

But now form inside $\mathfrak{p}_\xi$, the sphere $S_\xi$ obtained by rotating $\xi$ around the wall $H_s$ around which $s$ reflects




$K_\eta$ preserves $S_\xi$, so $S_\xi$ is a union of $K_\eta$-orbits. Working in the group $G_\eta$ we know that each $K_\eta$ orbit on $\mathfrak{p}_\eta$ intersects $\mathfrak{a}$ in an orbit for the Weyl group, which in the case of $G_\eta$ is reflection thru $H_s$. Thus $K_\eta$ is transitive on this sphere. So it's clear now that we get:

<u>Prop. 4</u>: $B_s/B$ is a sphere. It is the 1-point compactification of

$$BsB/B \cong B^u \cap sB_-^u s^{-1} \cong \mathfrak{b}^u \cap s\mathfrak{b}_-^u s^{-1}$$

(where $B_- = B_{-\xi}$).

Let us fix an element $\eta$ of $\mathfrak{a}$. We know the orbit $B\eta$ is a cell in $G\eta$ for it is isomorphic to $B^u \cap B^u_{-\eta} \simeq \mathfrak{b}^u \cap \mathfrak{b}^u_{-\eta}$. What we want to do now is to show the embedding $B\eta \subset G\eta$ extends to a map of a closed disk with interior $\cong B\eta$. This will show that the $B$-orbits of $G\eta$ are the cells for a CW complex structure. At the same time we will determine the closure of $B\eta$ in $G\eta$.

To this end, let us choose a gallery in $\mathfrak{a}$

$$C, s_1C, \ldots\ldots, s_1\cdots s_nC \qquad s_i \in S.$$

such that $\eta \in s_1\cdots s_nC$. We denote by $\Gamma(s_1,\ldots,s_n)$ the set of galleries $C_0, C_1, \ldots, C_n$ in $\mathfrak{p}$ such that $C_0 = C$ and such that $C_{i-1}$ and $C_i$ have their $s_i$-th panel in common. If we choose $g_i \in G$, $i=1,\ldots,n$, such that $C_i = g_1\cdots g_iC$, then this condition means that $C = g_{i-1}^{-1}\cdots g_1^{-1}C_{i-1}$ and $g_{i-1}^{-1}\cdots g_1^{-1}C_i = g_iC$ have the panel $C_{s_i}$ in common, that is, $g_i \in B_{s_i}$. Thus one gets an isomorphism

$$B_{s_1}\times^B\cdots\times^B B_{s_n}/B \xrightarrow{\ \sim\ } \Gamma(s_1,\ldots,s_n)$$

$$(g_1,\ldots\ldots,g_n) \longmapsto (C, g_1C, \ldots, g_1\cdots g_nC).$$

Let $\eta_0$ be the point in $C$ such that $\eta = s_1\cdots s_n\eta_0$, and ~~define the map $\Phi$ to be~~ let

$$\Phi : \Gamma(s_1,\ldots,s_n) \longrightarrow G\eta$$

be the map sending $(g_1,\ldots,g_n)$ to $g_1\cdots g_n\eta_0$ = the point of $C_n = g_1\cdots g_nC$ conjugate to $\eta$.

If $\gamma = (C_0, \dots, C_n) \in \Gamma(s_1, \dots, s_n)$, let $p\gamma = (pC_0, \dots, pC_n)$ be the image of $\gamma$ under the projection $p : \mathscr{J} \to \mathfrak{a}$. Then $p\gamma$ is a gallery in $\mathfrak{a}$, with $pC_0 = C$ and such that $pC_{i-1}$ and $pC_i$ have the $s_i$-th panel in common. Thus $pC_i = s_1' \cdots s_i' C$, where $s_i' = s_i$ or $1$. Hence $p\Phi(\gamma) = s_1' \cdots s_n' \eta_0$. Note that $B$ acts on $\Gamma(s_1, \dots, s_n)$ by left multiplication, and $\Phi$ is compatible with this action, so $\mathrm{Im}\,\Phi$ is a union of $B$-orbits. Thus we have:

<u>Prop. 5</u>: The image of $\Phi : \Gamma(s_1, \dots, s_n) \to G s_1 \cdots s_n \eta_0$ is the union of the $B$-orbits $B s_1' \cdots s_n' \eta_0$, where $(s_1', \dots, s_n')$ runs over all sequences with $s_i' = s_i$ or $1$. In other terms,

$$\mathrm{Im}\,\Phi = \bigcup_{\substack{1 \le i_1 < \dots < i_p \le n \\ 0 \le p \le n}} B s_{i_1} \cdots s_{i_p} \eta_0 .$$

Before going on, I need something which should have been developed earlier before starting galleries.

Let $\eta \in \mathfrak{a}$. We define the <u>length</u> of $\eta$ with respect to $C$ to be the number of root hyperplanes separating $\eta$ from $C$; denote it $\ell_C(\eta)$. The length of an arbitrary point $\xi$ of $\mathscr{J}$ (with respect to $C$) is defined to be the length of its image $p\xi$ in $\mathfrak{a}$. Clearly the length function is constant on each stratum of $\mathscr{J}$, and on each $B$-orbit.

Let $\eta \in \mathfrak{a}$, and let $\xi$ be an interior point of $C$. The element $\eta + \varepsilon(\xi - \eta)$ is regular for $\varepsilon$ small and $>0$; moreover any root hyperplane separating $\eta$ and $C$ separates $\eta + \varepsilon(\xi-\eta)$ for $\varepsilon$ small, and conversely. Let $wC$ be the chambre containing $\eta + \varepsilon(\xi - \eta)$; then $\eta \in wC$ and $\ell_C(\eta) = \ell_C(wC)$.

~~Note that $wC$ is the unique chambre of $\mathfrak{a}$ containing of the same length as $\eta$, for if $w'C \neq wC$ and then there is a root hyperplane~~ Let $w'C$ be a chambre containing $\eta$ of the same length as $\eta$. Since every root hyperplane separating $C$ and $\eta$ automatically separates $C$ and $w'C$, it follows that the root hyperplanes separating $C$ and $w'C$ are the same as those separating $C$ and $\eta$. Thus $w'C = wC$ for if $w'C \neq wC$, then there would exist a root hyp. separating these two and $\xi$ would be one side of it. Therefore $wC$ is the unique chambre of $\mathfrak{a}$ containing $\eta$ of the same length as $\eta$. From part I we know that $n = \ell_C(\eta) = \ell_C(wC)$ is the length of a reduced decomposition of $w$: $w = s_1 \cdots\cdots s_n$. Thus we can also describe $\ell_C(\eta)$ as the least $n$ such that $\exists\ s_1, \ldots, s_n \in S$ with $\eta \in s_1 \cdots s_n C$.

Next let us compare the $B$-orbits of $wC$ and $\eta$. ~~According to the above theorem~~ Note that there is an evident map $B \cdot wC \longrightarrow B \cdot \eta$ which associates to the chambre $b \cdot wC$ the unique point of this chambre which is conjugate to $\eta$, namely by $\eta$. According to the previous theorem $B\cdot\eta$ is acted on simply-transitively by the group $B^u \cap B^u_{-\eta}$, which is set-isomorphic to $\mathfrak{b}^u \cap \mathfrak{b}^u_{-\eta}$. Also $\mathfrak{b}^u \cap \mathfrak{b}^u_{-\eta} = \sum \mathfrak{g}^\alpha$ where $\alpha$ ranges over roots with $\alpha(\xi) > 0$, $\alpha(\eta) < 0$. It follows from the nature of $w$ that $\mathfrak{b}^u \cap \mathfrak{b}^u_{-\eta} = \mathfrak{b}^u \cap \mathfrak{b}^u_{-w\xi}$, hence $B^u \cap B^u_{-\eta} = B^u \cap B^u_{-w\xi}$, hence $B\cdot wC \xrightarrow{\sim} B\cdot\eta$.

So we have shown that any point $\eta$ of $\mathfrak{a}$ is contained in a unique chambre $wC$ of $\mathfrak{a}$ with $\ell_C(wC) = \ell_C(\eta)$, and that $B\cdot wC \xrightarrow{\sim} B\cdot\eta$. If $\zeta$ is any element of $B\eta$, say $\zeta = b\eta$, then $b\cdot wC$ is a chambre of $\mathfrak{p}$ containing $\zeta$ of the same length as $\zeta$. Let $C'$ be another chambre such that $\zeta \in C'$, $\ell_C(\zeta) = \ell_C(C')$. Let $C' = b_1\, pC'$. Then $pC'$ is a chambre of $\mathfrak{a}$ containing $p\zeta = \eta$ of the same length as $\eta$. Thus we have $pC' = wC$. But also we have $b_1\eta \in C'$ is conjugate to $b\eta = \zeta$, hence $b_1\eta = b\eta$. So from $B\cdot wC = B\eta$, we conclude that $C' = b_1 wC = b\cdot wC$, which show $b\cdot wC$ is the unique chambre containing $\zeta$ of the same length. Thus we have proved:

<u>Prop. 6:</u> Any point $\zeta$ of $\mathfrak{p}$ is contained contained in a unique chambre $C_\zeta$ with the same length with

respect to $C$ as $\mathcal{J}$. Furthermore $B \cdot C_{\mathcal{J}} \xrightarrow{\sim} B \cdot \mathcal{J}$.

Now let us return to the map $\Phi : \Gamma(s_1, \dots, s_n) \longrightarrow G\eta$, and let us assume that $n = l_C(\eta)$, that is, that $C, s_1 C, \dots, s_1 \cdots s_n C$ is a minimum gallery with $\eta \in s_1 \cdots s_n C$. Denote by $\Gamma(s_1, \dots, s_n)^*$ the subset of $\Gamma(s_1, \dots, s_n)$ consisting of galleries $(C_0, \dots, C_n)$ without (immediate) repetitions i.e $C_{i-1} \neq C_i$. Let $(C_0, \dots, C_n) \in \Gamma(s_1, \dots, s_n)^*$ and let $(C, s_1' C, \dots, s_1' \cdots s_n' C)$ be its projection in $\mathfrak{a}$. We show $s_i' = s_i$ by induction on $i$. Assuming true for $j < i$, suppose $s_i' = 1$. Then $C_{i-1}, C_i$ project to $s_1 \cdots s_{i-1} C$, hence they have the same length. Moreover $C_{i-1}, C_i$ have the same $s_i$-th panel $(= g_1 \cdots g_{i-1} C_{s_i}$ if $C_{i-1} = g_1 \cdots g_{i-1} C)$, which projects to $s_1 \cdots s_{i-1} C_{s_i}$. But because of the minimality of the gallery $(C, \dots, s_1 \cdots s_n C)$, it is clear that $l_C(s_1 \cdots s_{i-1} C) = l_C(s_1 \cdots s_{i-1} C_{s_i}) = i - 1$. Thus $C_{i-1}$, $C_i$ are two chambres containing a common panel and of the same length as this panel. By Prop. 6 $C_{i-1} = C_i$ which contradicts $(C_0, \dots, C_n) \in \Gamma(s_1, \dots, s_n)^*$. Thus $s_i' = s_i$, and so we have shown that $(C_0, \dots, C_n)$ projects to $(C, \dots, s_1 \cdots s_n C)$. In particular, $\Phi(C_0, \dots, C_n) \in B\eta$.

Because $B$ acts on $\Gamma(s_1,\dots,s_n)^*$, it is clear that $\Phi$ maps $\Gamma(s_1,\dots,s_n)^*$ onto $B\eta$. In fact the map is 1-1 for if $\Phi(C_0,\dots,C_n) = b\eta$, then $C_n$ is the ! chambre containing $b\eta$ of the same length, and $C_{n-1}$ is the ! chambre containing the $s_n$-th face of $C_n$ and of the same length as this face, etc. So we've proved:

<u>Prop. 7:</u> Let $\eta \in \mathfrak{O}\!\iota$ have length $n$ w.r.t. $C$ and let $\eta \in s_1 \cdots s_n C$. Then the map $\Phi$ induces a [redacted] bijection:

$$\Gamma(s_1,\dots,s_n)^* \xrightarrow{\sim} B\eta \quad .$$

Let us now consider $\Gamma(s_1,\dots,s_n) = B_{s_1} \times^B \cdots \times^B B_{s_n}/B$ as a space. ~~By Prop. 4, $B_s/B$ is a sphere, hence $\Gamma(s_1,\dots,s_n)$ is a fibre bundle over $B$~~ $\Gamma(s_1,\dots,s_n)$ is the fibre space over $\Gamma(s_1,\dots,s_{n-1})$ associated to the principal $B$-bundle $B_{s_1} \times^B \cdots \times^B B_{s_n}$ and the $B$-space $B_{s_n}/B$. According to Prop[redacted].4, $B_s/B$ is a sphere with 2 $B$-orbits: [redacted] a point and $BsB/B$. Thus $\Gamma(s_1,\dots,s_n)$ is a sphere bundle over $\Gamma(s_1,\dots,s_{n-1})$ having a distinguished section. The image of this section is the set of galleries $(C_0, C_1,\dots,C_n)$ such that $C_{n-1} = C_n$; let's denote it $Z_n$.

Let $Z_i$ be the subspace of $\Gamma(s_1,\dots,s_n)$ consisting of galleries with $C_{i-1} = C_i$, $i = 1,\dots,n$. We have a tower

$$\Gamma(s_1,\dots,s_n) \longrightarrow \Gamma(s_1,\dots,s_{n-1}) \longrightarrow \cdots\cdots \longrightarrow \Gamma(s_1) \longrightarrow pt$$

such that each map is a fibre bundle projection with fibre a sphere and having a distinguished section. $Z_i$ is the inverse image in $\Gamma(s_1,\dots,s_n)$ of the distinguished section of $\Gamma(s_1,\dots,s_i)$ over $\Gamma(s_1,\dots,s_{i-1})$. $Z_i$ is a submanifold of $\Gamma(s_1,\dots,s_n)$ of codimension equal to $d(s_i) = \dim(B s_i B/B)$. The $Z_i$ intersect transversally and

$$\Gamma(s_1,\dots,s_n)^* = \Gamma(s_1,\dots,s_n) - \bigcup_{i=1}^{n} Z_i = \bigcap_{i=1}^{n}\left(\Gamma(s_1\dots s_n) - Z_i\right)$$

Thus $\Gamma(s_1,\dots,s_n)^*$ is an open dense subspace of $\Gamma(s_1,\dots,s_n)$.

In the "complex case" i.e. $G$ = complexification of $K$, I know that $B_s/B$ is $\cong \mathbb{CP}^1$ (compare Part I, pg. 21), hence $\bigcup Z_i$ is a divisor with normal crossings.

Put $d(s) = \dim(B s B/B)$. This is the dimension of $\sum \mathfrak{g}^\alpha$ where $\alpha$ ranges over the roots with $\alpha(\xi) > 0$ and $\alpha | H_s = 0$. (Recall such roots are proportional, & there are at most two such roots $\alpha, 2\alpha$). $d(s)$ is the multiplicity of the root hyperplane $H_s$. The dimension of $\Gamma(s_1,\dots,s_n)$ is

$$d = \sum_{i=1}^{n} d(s_i).$$

<u>Prop. 8</u>: There exists a continuous map $\psi : D^d \longrightarrow \Gamma(s_1,\dots,s_n)$, where $D^d$ is a disk of dimension $d$ such that $\psi(\partial D^d) \subset \bigcup Z_i$ and such that $\psi$ induces a homeomorphism of $\mathrm{Int}(D^d)$ with $\Gamma(s_1,\dots,s_n)^*$.

(In other words $(\Gamma(s_1, \ldots, s_n), \cup Z_i)$ is a relative cell of dimension $d$.)

~~This will be proved by induction on $n$. To handle the induction step, I consider the following situation. Let $E$ be a fibre bundle over a space $X$ with fibre $S^a$, and let $Z \subset E$ be the image of a section of $E$ over $X$.~~ [redacted]

[redacted] Lemma. Let $E$ be a fibre bundle over $X$ with fibre $S^a$ and let $Z \subset E$ be the image of a section of $E$. Let $Y$ be a closed subspace of $X$ such that $(X, Y)$ is a relative cell of dimension $b$ ($\exists\ \varphi : D^b \to X$ with $\varphi(\partial D^b) \subset Y$ and $\mathrm{Int}\, D^b \xrightarrow{\sim} X - Y$). Then $(E,$ [redacted] $Z \cup E_Y)$ is a relative cell of dimension $a+b$.

The prop. 8 [redacted] is proved by induction on $n$ using the lemma in the induction step with $E = \Gamma(s_1, \ldots, s_n)$, $Z = Z_n$, $X = \Gamma(s_1, \ldots, s_{n-1})$, [redacted] $Y = \Gamma(s_1, \ldots, s_{n-1}) - \Gamma(s_1, \ldots, s_{n-1})^*$.

To prove the lemma, note that $E$ is associated to a principal bundle for the group of homeos. of $S^a$ leaving a basepoint $\infty$ fixed. Because $D^b$ is contractible [redacted] this principal bundle becomes trivial when pulled back to $D^b$ via $\varphi$, hence we get a homeom.

$$\Theta : \varphi^*(E, Z) \xleftarrow{\sim} D^b \times (S^a, \infty)$$

over $D^b$. Let $\lambda : D^a \longrightarrow S^a$ be the standard map such that $\lambda(\partial D^a) = \infty$, $\lambda : \mathrm{Int}(D^a) \xrightarrow{\sim} S^a - \infty$. Let $\psi$ be the map

$$D^{a+b} \cong D^b \times D^a \xrightarrow{id \times \lambda} D^b \times S^a \xrightarrow{\theta} \varphi^* E \xrightarrow{pr_2} E.$$

Then

$$\psi(\partial D^{a+b}) = \psi(\partial D^b \times D^a) \cup \psi(D^b \cup \partial D^a)$$
$$\subset E_Y \cup Z$$

and $\psi$ restricted to $\mathrm{Int}(D^{a+b})$ is the homeomorphism

$$\mathrm{Int}(D^a) \times \mathrm{Int}(D^b) \simeq \mathrm{Int}(D^b) \times (S^a - \infty) \simeq \varphi^*(E-Z)_{\mathrm{Int}(D^b)}$$
$$\simeq (E-Z)_{X-Y} = E - (Z \cup E_Y),$$

which proves the lemma.

Now let's put together the previous five or so propositions. We have seen that the orbit $G\eta_0 \simeq G/B_{\eta_0}$ is a union of orbits

$$G\eta_0 = \coprod_{w \in W/W_{\eta_0}} Bw\eta_0$$

and that each orbit is a cell. Here I suppose $\eta_0 \in C$. Look at the orbit $B\eta$ where $\eta = w\eta_0$. We can suppose $w$ chosen so that ~~the~~ length $\ell_C(wC)$ equals $\ell_C(\eta)$ (Note: $\ell_C(wC) = \ell_S(w)$ in the standard notation.) Let $w = s_1 \cdots s_n$, $n = \ell_C(\eta)$ be a reduced decomposition, and form the map

$$\Phi : \Gamma(s_1, \ldots, s_n) \longrightarrow G\eta_0$$

By propositions 7 + 8 we get a map

$$D^d \xrightarrow{\varphi} \Gamma(s_1, \ldots, s_n) \xrightarrow{\Phi} G\eta_0$$

which induces a ~~homeomorphism~~ bijection:

$$(*) \qquad \mathrm{Int}\,(D^d) \xrightarrow{\sim} \Gamma(s_1, \ldots, s_n)^* \xrightarrow{\sim} B\eta .$$

As $\Phi\varphi(D^d)$ is compact, it is the quotient space of $D^d$ by the equivalence relation defined by the map $\Phi\varphi$. Since the equivalence classes are points on $\mathrm{Int}(D^d)$, we see that in the nbd. of points of $\mathrm{Int}\,(D^d)$, $\Phi\varphi$ is a homeomorphism. Thus $(*)$ is a homeomorphism. This proves that the cell decomposition of $G\eta_0$ given by the $B$-orbits is a CW decomposition. Furthermore because $\Gamma(s_1, \ldots, s_n)^*$ is dense in $\Gamma(s_1, \ldots, s_n)$, Prop. 5 gives the closure of $B\eta$. So we get:

<u>Theorem</u>: The orbit $G\eta_0 \simeq G/B_{\eta_0}$ is a CW complex with cells $B \cdot w\eta_0$, $w \in W/W_{\eta_0}$. If $w = s_1 \cdots s_n$ with $n = l_c(w\eta_0)$, then

$$\overline{Bw\eta_0} = \bigcup_{\substack{1 \leq i_1 < \cdots < i_p \leq n \\ 0 \leq p \leq n}} B\, s_{i_1} \cdots s_{i_p} \eta_0 .$$