Notes on buildings, symmetric spaces, etc.

Fourth part: G-action on $\mathfrak{p}$, Iwasawa decomposition.

Let K be a compact Lie group, G its complexification, and let $\mathfrak{k}$, $\mathfrak{g}$ be their Lie algebras, so that $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. We write $\mathfrak{p}$ for the subspace $i\mathfrak{k}$ of $\mathfrak{g}$ on which $\theta = -1$. K acts on $\mathfrak{p}$ by the adjoint action, and we now propose to extend this to a (non-linear) action of G.

Let ξ be an element of $\mathfrak{p}$, and let $e^{t\xi}$ be the corresponding 1-parameter subgroup of G. If we have given an embedding $K \subset U_m$, then the image of ξ in $\mathfrak{gl}_m$ is a hermitian matrix. Hence $e^{t\xi}$ is a matrix whose entries are $\mathbb{C}$-linear combinations of exponential functions $e^{\lambda t}$ with $\lambda \in \mathbb{R}$. Let $\mathcal{E}$ denote the ring of $\mathbb{C}$-linear combinations of these exponential functions; $\mathcal{E}$ is isomorphic to the group ring $\mathbb{C}[\mathbb{R}]$.

Let

(1) $$B_\xi = \left\{ g \in G \,\middle|\, \begin{matrix} e^{-t\xi} g e^{t\xi} \text{ converges in } G \\ \text{as } t \to +\infty \end{matrix} \right\}.$$

The function $e^{-t\xi} g e^{t\xi}$ viewed in GL_m has entries in $\mathcal{E}$, and for g to be in B_ξ means that no entry involves $e^{\lambda t}$ with $\lambda > 0$, and also that the limit matrix as $t \to +\infty$ is invertible. B_ξ is a subgp. of G.

If $g \in B_\xi$ let

$$(2) \qquad \ell(g) = \lim_{t\to+\infty} e^{-t\xi} g e^{t\xi}.$$

Then

$$e^{-s\xi}\ell(g)e^{s\xi} = \lim_{t\to+\infty} e^{-(s+t)\xi} g e^{(s+t)\xi} = \ell(g)$$

which shows $\ell(g) \in G_\xi = \{\gamma \in G \mid Ad(\gamma)\xi = \xi\}$.
Thus [redacted] we have a homomorphism $\ell : B_\xi \to G_\xi$.
Its kernel we denote

$$(3) \qquad B_\xi^u = \{g \in G \mid e^{-t\xi} g e^{t\xi} \longrightarrow 1 \text{ as } t \to +\infty\}.$$

If $\gamma \in G_\xi$, then $e^{-t\xi}\gamma e^{t\xi} = \gamma$, so $\gamma \in B_\xi$ and $\ell(\gamma) = \gamma$. Thus we have

<u>Prop. 1:</u> $B_\xi = G_\xi \ltimes B_\xi^u$.

<u>Example:</u> Let $\xi = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$ with $G = GL_m$.

If $g = (g_{ij})$, then

$$e^{-t\xi} g e^{t\xi} = \left(e^{-t(\lambda_i - \lambda_j)} g_{ij}\right).$$

Suppose $\lambda_1 = \cdots = \lambda_{a_1} > \lambda_{a_1+1} = \cdots = \lambda_{a_1+a_2} > \cdots = \lambda_{a_1+\cdots+a_r} = \lambda_m$.
[redacted] $B_\xi \ni g \iff g_{ij} = 0$ for $\lambda_i < \lambda_j$. Thus

$$(4) \qquad B_\xi = \begin{pmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix}$$

If $G \subset GL_m$, then $B_\xi(G) = G \cap B_\xi(GL_m)$, showing $B_\xi(G)$ is an algebraic subgroup of G.

Suppose next that $k \in K \cap B_\xi$. Then

$$e^{-t\xi} k \, e^{t\xi} = e^{-t\xi} e^{t \, k\cdot\xi} k$$

converges in G as $t \to +\infty$. Suppose more generally that $e^{-t\xi} e^{t\eta}$ converges in G as $t \to +\infty$ where $\xi, \eta \in \mathfrak{p}$. Applying Cartan involution θ:

$$\theta\left(e^{-t\xi} e^{t\eta}\right) = e^{t\xi} e^{-t\eta}$$

we see $e^{-t\xi} e^{t\eta}$ converges in G as $t \to -\infty$ also. But $e^{-t\xi} e^{t\eta}$ viewed in GL_m is a matrix with entries in $\mathcal{E}$, hence no entry involves $e^{\lambda t}$ with $\lambda \neq 0$. Then $e^{-t\xi} e^{t\eta}$ is constant, hence $= 1$, and therefore $\xi = \eta$. Thus we have proved:

Prop. 2: If $\xi, \eta \in \mathfrak{p}$ are such that $e^{-t\xi} e^{t\eta}$ converges in G as $t \longrightarrow +\infty$, then $\xi = \eta$. Consequently if $k \in K \cap B_\xi$, then $k \cdot \xi = \xi$, i.e:

$$K \cap B_\xi = K_\xi .$$

~~Consider G_ξ which is the complexification of [illegible]~~

Recall the Cartan decomposition for G:

(5) $$G = K \times P \,, \quad \exp : \mathfrak{p} \xrightarrow{\sim} P$$

Taking fixpts for the 1-parameter group $e^{it\xi}$ of autos. of G, we get the Cartan decomposition for G_ξ

(6) $$G_\xi = K_\xi \times P_\xi \,, \quad \exp : \mathfrak{p}_\xi \xrightarrow{\sim} P_\xi \,.$$

Thus the subset KB_ξ of G can be written:

$$\begin{aligned} KB_\xi &\leftleftarrows K \times^{K \cap B_\xi} B_\xi \\ &= K \times^{K_\xi} (G_\xi \times B_\xi^u) \\ &= K \times^{K_\xi} (K_\xi \times P_\xi \times B_\xi^u) \\ &= K \times P_\xi \times B_\xi^u \,. \end{aligned}$$

We propose now to show that $G = KB_\xi = K \times P_\xi \times B_\xi^u$. Suppose this has been established for $G = GL_n$, $K = U_n$. Consider a sequence

$$K \xrightarrow{i} K' \underset{\rho'}{\overset{\rho}{\rightrightarrows}} K''$$

with $K' = U_m$, $K'' = U_n$ as in Prop. 7 of the 3rd part. It is immediate that for ξ in $\mathfrak{p}$ one has an exact sequence

$$B^u_\xi \longrightarrow B^u_{\xi'} \rightrightarrows B^u_{\xi''}$$

where $\xi' = i(\xi)$, $\xi'' = \rho i(\xi) = \rho' i(\xi)$. The same will hold for $\mathfrak{p}_\xi$, P_ξ. Thus from $G' = K' \times P_{\xi'} \times B^u_{\xi'}$ and similarly with double primes, we can deduce by diagram chasing that the result holds for G.

Suppose $G = GL_n$, and let ξ be a hermitian matrix. To show $G = KB_\xi$ one can conjugate ξ by any element of K, hence I can suppose ξ is a diagonal matrix with entries $\lambda_1 \geq \cdots \geq \lambda_n$. Then from the formula for B_ξ established at the bottom of page 2, we see B_ξ contains the Borel B of upper triangular matrices. But then $G = KB$ results from the classical Gram-Schmidt orthogonalization process (given any basis $v_1, \dots, v_n$ there is an orthonormal basis of the form $v_j' = a_{1j} v_1 + \cdots + a_{jj} v_j$ with $a_{jj} > 0$.) So we have proved:

<u>Prop. 3:</u> (Iwasawa decomposition).

$$G = K \times^{K_\xi} B_\xi = K \times P_\xi \times B^u_\xi .$$

Strictly speaking the Iwasawa decomposition is the following special case: Take ξ to be a regular element of $\mathfrak{p}$, i.e. such that $\mathfrak{p}_\xi$ is abelian. Then with standard notation one has: $\mathfrak{p}_\xi = \mathfrak{a}$,

$P_\xi = A$, $B_\xi^u = N$ and $G = KAN$.

Comments: As

(7) $$K/K_\xi \xrightarrow{\sim} G/B_\xi$$

the algebraic variety G/B_ξ is compact. Thus B_ξ is ___parabolic___ subgroup of G.

We now use Prop. 3 to define an action of G on $\mathfrak{p}$. Given $\xi \in \mathfrak{p}$ and $g \in G$ we know from $G = KB_\xi$ that there exists k in K such that $k^{-1}g \in B_\xi$, i.e.

(8) $$\begin{array}{c} e^{-t\xi}k^{-1}g\,e^{t\xi} \quad \text{converges in } G \text{ as } t \to +\infty \\ \| \\ k\,e^{-t\,k\cdot\xi}\,g\,e^{t\xi} \end{array}$$

We define $g \cdot \xi$ to be $k \cdot \xi$; this is independent of the choice of k as k is unique up to $K \cap B_\xi = K_\xi$. Thus we extend the K action on K/K_ξ to a G-action via the isom. (7). In virtue of (8) $g \cdot \xi$ is the unique element of $\mathfrak{p}$ such that

(9) $$e^{-t(g\cdot\xi)}\,g\,e^{t\xi} \quad \text{converges as } t \to +\infty.$$

Uniqueness results from Prop. 2. Therefore

<u>Prop. 4</u>: The formula (9) defines an action of G on $\mathfrak{p}$ extending the adjoint action of K. K acts transitively on each G-orbit.

I want to give an intrinsic description of $\mathfrak{p}$ with its G-action which is independent of the choice of maximal compact subgroup K. I start with the subset I' of $\mathfrak{g}$ consisting of elements conjugate to elements of $\mathfrak{p}$. Because maximal compact subgroups of G are all conjugate it follows that I' is independent of the choice of K. If $G = GL_n$, then I' consists of matrices with real eigenvalues which are semi-simple.

To each element X of I' we associate the 1-parameter subgroup e^{tX} in G. Call two elements X, Y of I' <u>equivalent</u> if $e^{-tX}e^{tY}$ converges in G as $t \to +\infty$. This is an equivalence relation. Let I be the quotient of I' by this equivalence relation. There is an evident map $\mathfrak{p} \to I$ which we now show is bijective. First of all, it is injective by Prop. 2. Next given $X \in I'$ we know that there exists $g \in G$ such that $Ad(g^{-1})X = \xi \in \mathfrak{p}$. If $\eta = g\cdot\xi$, then

$$e^{-t g\cdot\xi} e^{tX} = e^{-t g\cdot\xi} g e^{t\xi} g^{-1}$$

converges as $t \to +\infty$ by the definition of $g\cdot\xi$; therefore η is equivalent to X.

Next note that G acts on I' via the adjoint action on $\mathfrak{g}$, and this action preserves equivalence, so one gets a G-action on I. Because $e^{-t g\cdot\xi} g e^{t\xi} g^{-1}$ converges as $t \to +\infty$, it follows that the action defined on p agrees with this adjoint action of G on I.

Suppose $X, Y \in I'$ are equivalent:

$$e^{-tY} e^{tX} \longrightarrow g \qquad \text{as } t \to +\infty$$

Then $e^{-sY} g e^{sX} = g$ all s, so $\mathrm{Ad}(g) X = Y$, and

$$g e^{-tX} g^{-1} e^{tX} \to g$$

so $g^{-1} \in B^u_X$, hence $g \in B^u_X$. Thus if X, Y are equivalent one has $Y = \mathrm{Ad}(g) X$ where $g \in B^u_X$; the converse is evident.

Let's apply this to $G = GL_n$. X is semi-simple with real eigenvalues. If these are arranged in order: $\lambda_1 > \cdots > \lambda_\ell$ and if the corresponding eigenspaces are $W_1, \ldots, W_\ell$, then B_X is the subgroup of GL_n stabilizing the flag

$$(*) \qquad 0 < W_1 < W_1 \oplus W_2 < \cdots < W_1 + \cdots + W_\ell = \mathbb{C}^n$$

(Note: $x_i \in W_i \implies e^{tX} g e^{tX} x_i = \sum_j g(x_i)_j e^{-t(\lambda_j - \lambda_i)}$. If $g \in B_X$ then $g(x_i)_j \neq 0 \implies \lambda_j \geq \lambda_i \implies i \geq j$. Thus

$$g(W_i) \subset W_i + W_{i-1} + \cdots + W_1 .)$$

If Y is equivalent to X, then $Y = gXg^{-1}$ with $g \in B_X^u$, so g stabilizes the flag and acts trivially on the quotients. Thus Y is any matrix stabilizing $(*)$ and having the same eigenvalue λ_i as X does on $W_i \oplus \cdots \oplus W_i / W_1 \oplus \cdots \oplus W_{i-1}$. Summary:

Prop. 5: Let I' be the set of elements of $\mathfrak{g}$ conjugate to elements of $\mathfrak{p}$ (call these real semi-simple elements of $\mathfrak{g}$), and let I be the quotient of I' by the equivalence relation $X \sim Y \Leftrightarrow e^{-tY} e^{tX}$ converges as $t \to +\infty$. Then $\mathfrak{p} \simeq I$ and this isomorphism commutes with the G action on $\mathfrak{p}$ and with the adjoint action of G on I.

One has $X \sim Y \Leftrightarrow Y = \mathrm{Ad}(g) X$ with $g \in B_X^u = \{ g \mid e^{-tX} g e^{tX} \to 1 \text{ as } t \to +\infty \}$. In $\mathfrak{gl}_n$ two real semi-simple matrices are equivalent iff the associated flags and eigenvalues are the same.

~~Let's discuss next continuity of the action of G on $\mathfrak{p}$. What I want to prove is that if $g_n \to g$ is a convergent sequence in G and $\xi_n \to \xi$ is a convergent sequence in $\mathfrak{p}$, then $g_n \xi_n \to g\xi$. It is evidently enough to do this for $G = GL_n$. Let p be the composite map $I' \twoheadrightarrow I \simeq \mathfrak{p}$, whence $g \cdot \xi = p(g \xi g^{-1})$. Since $g_n \xi_n g_n^{-1} \longrightarrow g \xi g^{-1}$, it is enough to prove that p is continuous, i.e. that $X_n \to X$ implies $p(X_n) \to p(X)$.~~

Structure of B^u_ξ: Let $g \in B^u_\xi$, whence $t \mapsto e^{-t\xi} g e^{t\xi}$ is a path in G ending at 1; precisely it is a continuous map of $\mathbb{R} \cup \{+\infty\}$ into G sending infinity to $\mathbf{1}$. ~~Because $\exp: \mathfrak{g} \to G$ is a local isomorphism, we can find a path X_t, $a \le t \le +\infty$, in $\mathfrak{g}$ ending at ∞ such that $\exp(X_t) = e^{-t\xi} g e^{t\xi}$ as $t \le +\infty$~~ ~~For $a \le t \le +\infty$ we~~ For t sufficiently large $X_t = \log(e^{-t\xi} g e^{t\xi})$ is defined and satisfies

$$X_{t+\varepsilon} = \log(e^{-\varepsilon\xi} e^{-t\xi} g e^{t\xi} e^{\varepsilon\xi}) = Ad(e^{-\varepsilon\xi}) X_t$$

for ε small. This forces

$$e^{-t\xi} g e^{t\xi} = \exp\left(Ad(e^{-(t-a)\xi} X_a)\right)$$

for all t as both sides are analytic and agree near a. Thus if we put $X_t = Ad(e^{-(t-a)\xi})X_a$ for all t we have

$$X_t = Ad(e^{-t\xi})X_0 \quad \text{and}$$

$$e^{-t\xi} g e^{t\xi} = \exp(Ad(e^{-t\xi})X_0)$$

Moreover as $e^{-t\xi} g e^{t\xi} \to 1$, $X_t = Ad(e^{-t\xi})X_0 \to 0$. Thus if we put $\mathfrak{b}^u_\xi = \{X \in \mathfrak{g} \mid Ad(e^{-t\xi})X_0 \to 0 \text{ as } t \to +\infty\}$ we know $\exp: \mathfrak{b}^u_\xi \to B^u_\xi$ is onto. It also has to be $\mathbf{1}$-$\mathbf{1}$, because it is 1-1 near 0 and any pair of points can be pulled into a nbd. of zero using $Ad(e^{-t\xi})$. So

<u>Prop. 6</u>: Let $\mathfrak{b}^u_\xi = \{X \in \mathfrak{g} \mid Ad(e^{-t\xi})X \to 0$ as $t \to +\infty\}$. Then $\mathfrak{b}^u_\xi = Lie(B^u_\xi)$ and $\exp : \mathfrak{b}^u_\xi \to B^u_\xi$ is a diffeomorphism.

We can also prove this first for GL_n and then taking subsets where $\rho = \rho'$.

Recall $B_\xi = G_\xi \times B^u_\xi$ where G_ξ is the centralizer of ξ. We know $Lie(G_\xi) = \{X \mid [\xi, X] = 0\}$ is the zero eigenspace for $Ad\,\xi$; denote it $\mathfrak{g}_\xi$. So,

<u>Prop. 6'</u>: Let $\mathfrak{g}_\xi$, $\mathfrak{b}_\xi$, $\mathfrak{b}^u_\xi$ denote the largest subspaces of $\mathfrak{g}$ invariant under $Ad\,\xi$ on which has eigenvalues 0, ≥ 0, > 0 resp. Then $\mathfrak{g}_\xi$, $\mathfrak{b}_\xi$, and $\mathfrak{b}^u_\xi$ are respectively the Lie algebras of G_ξ, B_ξ and B^u_ξ.

If $\xi \in \mathfrak{a}$ = an abelian subspace of $\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha$ is the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, then

$$\mathfrak{g}_\xi = \mathfrak{g}_{\mathfrak{a}} + \sum_{\alpha(\xi) = 0}' \mathfrak{g}^\alpha$$

$$\mathfrak{b}^u_\xi = \sum_{\alpha(\xi) > 0} \mathfrak{g}^\alpha \quad , \quad \mathfrak{b}_\xi = \mathfrak{g}_\xi \oplus \mathfrak{b}^u_\xi$$

Let's consider the orbit structure of I for the G-action. Suppose G connected. As G-orbits coincide with K orbits, we know from the first part of these notes that each G-orbit contains a unique point of C where C is a chambre in a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$:

$$C \xrightarrow{\sim} G\backslash I .$$

Let $\xi \in C$ and suppose G is connected. We know G_ξ is connected (it has same homotopy type as K_ξ), hence $B_\xi = G_\xi \times B_\xi^u$ is the connected subgroup of G with Lie algebra $\mathfrak{b}_\xi$. Thus the stabilizer B_ξ depends only on the positive roots of $\mathfrak{g}$ with respect to C which vanish at ξ.

Let $\alpha_1, \dots, \alpha_\ell$ be the simple positive roots. We know $C : \{ x \in \mathfrak{a} \mid \alpha_i(x) \geq 0 \quad i = 1, \dots, \ell \}$ and that $\alpha_1, \dots, \alpha_\ell$ are independent. Moreover any $\alpha \in \Phi^+$ is a linear combination $\alpha = n_1\alpha_1 + \dots + n_\ell \alpha_\ell$ with $n_i \geq 0$. Hence $\alpha(\xi) = 0 \iff (n_i > 0 \Rightarrow \alpha_i(\xi) = 0)$. Thus if we stratify C according to the subset of simple roots vanishing at a point, the stabilizers remain constant on the strata. So we get:

Assume G is connected.

Prop. 7: Let $\Sigma = \{\alpha_1, \ldots, \alpha_\ell\}$ be the simple roots of $\mathfrak{g}$ with respect to the chambre C. For each subset σ of Σ, let C_σ be the subset of C consisting of points where the α_i in σ vanish and the α_i not in σ are positive. Then $B_\xi = B_{\xi'}$ iff ξ, ξ' are in the same stratum of C.

Formula: Put $B_\sigma = B_\xi$ for $\xi \in C_\sigma$. Then

$$\mathfrak{b}_\sigma = \mathfrak{g}_0 + \sum \mathfrak{g}^\alpha$$

where α ranges over those positive roots of the form $\alpha = \sum_{\alpha_i \in \sigma} \lambda_i \alpha_i$ with $\lambda_i > 0$; (call these positive roots with support σ).

Note that $\sigma \subset \tau$, then $\overline{C_\sigma} \supset \overline{C_\tau}$ and $B_\sigma \subset B_\tau$ so there is a map $G/B_\sigma \longrightarrow G/B_\tau$. Thus we can form a space by taking

$\coprod_\sigma G/B_\sigma \times \overline{C_\sigma}$ and identifying $(i_* x, y) \sim (x, i^* y)$ for each inclusion $i : \sigma \subset \tau$. One has a continuous map

$$\coprod_\sigma G/B_\sigma \times \overline{C_\sigma} \longrightarrow I$$

because $G/B_\sigma \simeq K/K_\sigma$, $K_\sigma = B_\sigma \cap K$, and the K action is continuous. As this map is compatible with the equivalence relation one gets a map

$$(*) \qquad \coprod_\sigma G/B_\sigma \times \overline{C}_\sigma \Big/ \text{reln} \longrightarrow I$$

which one sees from the fact that both spaces sit over C (recall $K \backslash I = C$) and the fibres are the same (note each $\xi \in C$ is contained in a smallest $\overline{C}_\sigma$ and $K \cdot \xi = G/B_\sigma$.) Because I is Hausdorff the former space is Hausdorff, and so since both spaces are proper over C, it follows $(*)$ is a homeomorphism.

I claim G acts continuously on $Y = \coprod_\sigma G/B_\sigma \times \overline{C}_\sigma / \text{reln}$. We know it acts continuously on $X = \coprod_\sigma G/B_\sigma \times \overline{C}_\sigma$, and the map $X \to Y$ is proper + surjective. But a proper surjective map is a quotient map, hence $G \times X \to G \times Y$ is a quotient map, and so the map $G \times X \xrightarrow{\mu} X \to Y$ induces $G \times Y \to Y$. So we have proved:

<u>Prop. 8</u>: G acts continuously on I.

Actually the proof assumes G connected, but it suffices to do the proof for GL_n.

~~I shall now give a direct demonstration~~

Generalization to the real case:

Let K be a compact group with involution σ, G the complexification of K, and let σ be extended to G in anti-holomorphic fashion. We have seen that the decomposition

$$G = K \times P, \qquad \exp : \mathfrak{p} \xrightarrow{\sim} P$$

yields on taking σ-fixpts

$$G^\sigma = K^\sigma \times P^\sigma \qquad \exp : \mathfrak{p}^\sigma \xrightarrow{\sim} P^\sigma .$$

Recall: $\mathfrak{p} = i\mathfrak{k}$, so $\mathfrak{p}^\sigma = i\mathfrak{k}^- = \{ x \in \mathfrak{g}^\sigma \mid \theta x = -x \}$.

Now in the preceding, we can take σ-fixpts to get the following:

$$B_\xi^\sigma = \{ g \in G^\sigma \mid e^{-t\xi} g e^{t\xi} \text{ converges in } G^\sigma \text{ as } t \to +\infty \}$$

$$B_\xi^\sigma = G_\xi^\sigma \ltimes B_\xi^{u,\sigma}$$

$$G_\xi^\sigma = K_\xi^\sigma \times P_\xi^\sigma$$

$$G^\sigma = K^\sigma \times P_\xi^\sigma \times B_\xi^{u,\sigma}$$

$$\exp : \mathfrak{b}_\xi^{u,\sigma} \xrightarrow{\sim} B_\xi^{u,\sigma} \qquad \text{where}$$

$$\mathfrak{b}_\xi^{u,\sigma} = \{ X \in \mathfrak{g}^\sigma \mid \mathrm{Ad}(e^{-t\xi} X) \longrightarrow 0 \text{ as } t \to +\infty \}$$

If we take ξ to be a regular element of $\mathfrak{p}^\sigma$, that means $\mathfrak{p}_\xi^\sigma$ is abelian, in fact a maximal abelian subspace of $\mathfrak{p}^\sigma$, then

$$G^\sigma = K^\sigma \times P^\sigma_\xi \times B^{u,\sigma}_\xi$$

is the Iwasawa decomposition of G^σ.

It is pretty clear that the above notation is awkward. The following notation is more standard. Replace G^σ, K^σ by G, K, so that now G is a reductive algebraic group over $\mathbb{R}$, and K is a maximal compact subgroup. Similarly we drop σ from the rest of the notation. If we have occasion the new notation for (G, K) is (G_c, U), so we have the picture:

$$\begin{array}{ccccl} G & \subset & G_c & & \\ \cup & & \cup & \longleftarrow & \theta \text{ fixpts} \\ K & \subset & U & & \\ & \uparrow & & & \\ & \sigma \text{ fixpts.} & & & \end{array}$$

again $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\theta = -1$ on $\mathfrak{p}$, $+1$ on $\mathfrak{k}$.

Suppose now U is connected (this means G as an algebraic group is connected, not necessarily that G as a Lie group is connected, e.g. $G = \mathbb{R}^*$, $G_c = \mathbb{C}^*$.) Then I know that the K-orbits in $\mathfrak{p}$ are connected. The K-orbit of ξ is $K/K_\xi \simeq G/B_\xi$. I want to understand the natural stratification on $I \simeq \mathfrak{p}$. I know $K\backslash I \simeq W\backslash\mathfrak{a} \simeq C$ where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, and where C is a chambre in $\mathfrak{a}$. Moreover C is described by $\alpha_i(x) \geq 0$ $i = 1, \dots, l$ where $\alpha_i = 0$ are the walls of C, and $\alpha_1, \dots, \alpha_l$ are independent. Question: Does B_ξ remain constant as ξ varies over a stratum of C? Yes, because we know this is the case in G_c and B_ξ is the σ-invariant subgroup of the corresponding stabilizer in G_c.

In order to understand this point, let's go back to the $(K, G, K^\sigma, G^\sigma)$ notation. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}^\sigma$, and let the root decomp. be

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha$$

where Φ is the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. (Φ consists of real linear functions on $\mathfrak{a}$). Let ξ be a point of a chambre C of $\mathfrak{a}$. Its centralizer G_ξ is connected with

$$\mathrm{Lie}(G_\xi) = \mathfrak{g}_\xi = \mathfrak{g}_{\mathfrak{a}} + \sum_{\substack{\alpha \in \Phi \\ \alpha(\xi) = 0}} \mathfrak{g}^\alpha$$

Because $\sigma = \mathrm{id}$ on $\mathfrak{a}$, $\mathfrak{g}^\alpha$ is stable under σ, so

$$\mathrm{Lie}(G^\sigma_\xi) = \mathfrak{g}_{\mathfrak{a}}^\sigma + \sum_{\alpha(\xi) = 0} \mathfrak{g}^{\alpha, \sigma}$$

and
$$\mathrm{Lie}(B_\xi^\sigma) = \mathfrak{g}_{\mathfrak{a}}^\sigma + \sum_{\alpha(\xi)\geq 0} \mathfrak{g}^{\alpha,\sigma}.$$

It's clear from this that from the stabilizer B_ξ^σ we can recover the roots $\alpha \in \Phi^+$ which vanish on ξ. Thus we see prop. 7 (p. 13) holds also in the real case.

Suppose $\xi, \xi' \in \mathfrak{p}^\sigma$ are such that $B_\xi^\sigma = B_{\xi'}^\sigma$. Complexifying Lie algebras, we get $B_\xi = B_{\xi'}$; intersecting with K we get $K_\xi = K_{\xi'}$; it follows that $e^{it\xi}$ commutes with ξ', hence $[\xi, \xi'] = 0$. This means that ξ, ξ' are contained in a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}^\sigma$. Further the sign of any root of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is the same for ξ and ξ', hence ξ, ξ' lie in the same stratum of $\mathfrak{a}$.

<u>Lemma</u>: $P \cap B_\xi = P_\xi$. More generally if $g \in B_\xi$ is such that $\Theta g = g^{-1}$, then $g \in G_\xi$.

<u>Proof</u>: $\Theta(e^{-t\xi} g\, e^{t\xi}) = e^{t\xi} g^{-1} e^{-t\xi} = (e^{t\xi} g\, e^{-t\xi})^{-1}$ converges as $t \to \infty$. Thus $e^{t\xi} g e^{-t\xi}$ converges as $t \to \pm\infty$, so as it has entries which are linear combinations of real exponentials, it is constant, so $g \in G_\xi$.

(This lemma can be used so: $B_\xi^\sigma = B_{\xi'}^\sigma \Rightarrow e^{t\xi} \in P \cap B_{\xi'}^\sigma = P_{\xi'}^\sigma \subset G_{\xi'}^\sigma$ hence ξ, ξ' commute.)

~~*Summary*: $\xi, \xi' \in \mathfrak{p}^\sigma$ are in the same stratum $\Longleftrightarrow$ $B^\sigma_\xi = B^\sigma_{\xi'}$.~~

~~*Consequence*: Consider the orbit $G\cdot\xi \cong G/B_\xi$. We know this meets the chambre C in exactly one point, namely ξ, if we start with $\xi \in C$. The stratum of ξ consists of all points of C~~

Stratification of $\mathfrak{p}^\sigma$: Again suppose G connected, let C be a chambre in a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}^\sigma$, and let Σ be the set of simple roots. For each subset τ of Σ let C_τ be the set of points where the α_i in τ vanish and those not in τ are >0. (Thus $\tau = \phi \Rightarrow C_\tau = \{0\}$, $\tau = \Sigma \Rightarrow C_\tau = \operatorname{Int} C$). We know the stabilizer B_ξ is constant as ξ ranges over C_τ; denote this stabilizer by B_τ. Then we have a stratification:

$$\coprod_\tau G^\sigma/B^\sigma_\tau \times C_\tau \xrightarrow{\sim} \mathfrak{p}^\sigma$$

(set-theoretic isomorphism) because C is a fundamental domain for the G-action on $\mathfrak{p}^\sigma$.

Thus $\mathfrak{p}^\sigma$ is broken up into strata gC_τ, $g \in G$, $\tau \subset \Sigma$. According to the discussion on p. 18 we have:

Assertion: ξ and ξ' are in the same stratum

$\Longleftrightarrow \; B^\sigma_\xi = B^\sigma_{\xi'}$ (In fact it suffices that $\mathrm{Lie}(B^\sigma_\xi) = \mathrm{Lie}(B'_\xi)$.)

Consequence: Let $\xi \in C$ and consider the orbit $G^\sigma \cdot \xi \simeq G^\sigma / B^\sigma_\xi$. We know this orbit meets C in exactly one point, namely ξ. Since the stratum of ξ in $\mathfrak{p}^\sigma$ is the stratum of ξ in C, it follows that all the points of the orbit $G^\sigma \cdot \xi$ except ξ belong to different strata, hence their stabilizers differ from B^σ_ξ. But the stabilizer of $g\xi$ is $g B^\sigma_\xi g^{-1}$. Thus $g \notin B^\sigma_\xi \Longrightarrow g B_\xi g^{-1} \neq B^\sigma_\xi$ and we get:

<u>Cor</u>: B^σ_ξ is its own normalizer.

Rank 1 case: This means $\dim \mathfrak{a} = 1$. (One could generalize a bit and only require card $\Sigma' = 1$.) In this case the orbits for K^σ in $\mathfrak{p}^\sigma$ are the spheres around 0 because C is a ray containing zero. Thus what we have is a sphere $G^\sigma / B = K^\sigma / M$ of dimension $= \dim(\mathfrak{p}^\sigma) - 1$ and the G^σ-space $\mathfrak{p}^\sigma$ may be viewed as the open disk associated to this action of G^σ. The action of G^σ on $\mathfrak{p}^\sigma$ is evidently continuous, but probably not differentiable, because otherwise it would be linear, as it is homogeneous.