Notes on Buildings, symm. spaces, etc.

2nd part: K^σ action on k^-.

Let K be a compact connected Lie group with Lie algebra k. Let σ be an involution of K and let K^σ be the fixed subgroup. The Lie algebra k splits into $\pm$ eigenspaces under σ:

$$k = k^+ \oplus k^-$$

and k^+ is the Lie algebra of K^σ. In the following we will study the action of K^σ on k^- induced by the adjoint action of K on k.

Let $(\,,\,)$ be an invariant inner product on k. An element η of k^- is perpendicular to the orbit $K^\sigma\xi$ in k^- iff $\forall X \in k^+$

$$0 = ([X,\xi],\eta) = (X,[\xi,\eta])$$

But $\eta, \xi \in k^- \Longrightarrow [\eta,\xi] \in k^+$, so $[\eta,\xi] = 0$. Hence:

Prop. 1: The normal space to the orbit $K^\sigma\xi$ in k^- is $k^-_\xi = \{\eta \in k^- \mid [\xi,\eta] = 0\}$.

Suppose ξ is generic in the sense that K^σ_ξ acts trivially on the normal space k^-_ξ. If $\eta_1, \eta_2 \in k^-_\xi$, then $[\eta_1,\eta_2] \in k^+_\xi$, so $[\eta_2,[\eta_1,\eta_2]] = 0$ as $\mathrm{Lie}(K^\sigma_\xi) = k^+_\xi$ acts trivially on k^-_ξ. Thus

$$([\eta_1, \eta_2], [\eta_1, \eta_2]) = (\eta_1, [\eta_2, [\eta_1, \eta_2]]) = 0$$

and so $[\eta_1, \eta_2] = 0$. Thus

<u>Prop. 2</u>: If ξ is a generic element of $\mathfrak{k}^-$, then $\mathfrak{k}^-_\xi$ is abelian. Consequently $\mathfrak{k}^-_\xi$ is normal to the K^σ-orbit of any of its points (by prop. 1).

The preceding proof shows that

$$\xi \text{ generic} \Longrightarrow [\mathfrak{k}^+_\xi, \mathfrak{k}^-_\xi] = 0 \Rightarrow \mathfrak{k}^-_\xi \text{ is abelian.}$$

In this case K_ξ, which is connected (page 8, part 1), acts trivially on $\mathfrak{k}^-_\xi$.

<u>Remark</u>: The fact that normally to each generic orbit there is a totally geodesic submanifold normal to the orbit at each of its point is a special feature of the action which is roughly the variational completeness of Bott-Samelson.

Next take up conjugacy theorems. Fix ξ such that $\mathfrak{k}^-_\xi$ is abelian, and consider the function on $K^\sigma\eta$ given by

$$f(k\eta) = |k\eta - \xi|^2 = \underbrace{|\eta|^2 + |\xi|^2}_{\text{const.}} - 2(k\eta, \xi)$$

Then η is a critical point of this function if for all $X \in \mathfrak{k}^+$

$$0 = ([X, \eta], \xi) = (X, [\eta, \xi])$$

i.e. iff $\eta \in k_\xi^-$. Since $K^\sigma\eta$ is compact there is a critical point, so

<u>Prop. 3</u>: Every K^σ orbit on k^- meets k_ξ^-.

Let N^- be the normalizer of k_ξ^- in K^σ. As K_ξ is the centralizer of k_ξ^- in K, K_ξ^σ is the cent. of k_ξ^- in K^σ. Put

$$W^- = N^-/K_\xi^\sigma$$

for the group of autos. of k_ξ^- produced by elements of K^σ. The following is proved as in part 1, prop. 3'.

<u>Prop. 3'</u>: $K^\sigma\eta \cap k_\xi^-$ is a W^- orbit in k_ξ^-. Thus

$$K^\sigma \backslash k^- \xleftarrow{\sim} W^- \backslash k_\xi^- .$$

Next we consider the root space decomposition of k with respect to the abelian space k_ξ^-:

$$k = k_\xi \oplus \sum_{\alpha \in \Phi^+}{}' k^\alpha$$

Here Φ^+ is a finite subset of $\alpha \in \mathrm{Hom}(k_\xi^-, \mathbb{R})$ such that $\alpha(\xi) > 0$, and k^α is isomorphic to a direct sum of copies of $\mathbb{C}$ with $x \in k_\xi$ acting by $i\alpha(x)$. ~~Thus k^α has a unique complex structure such that $ad(x)$ = mult. by $i\alpha(x)$ on k^α:~~

~~$[x, v] = \alpha(x)\, iv \qquad v \in k^\alpha,\ x \in k_\xi^-$~~

4.

Since $\sigma[x,v] = -[x,\sigma v]$, if V is stable under $\mathrm{ad}(k_{\mathfrak{q}}^{-})$, so is σV; moreover if $\exists\ \theta : \mathbb{C}^n \xrightarrow{\sim} V$ with

$[x, \theta z] = \theta(\alpha(x) i z)$, then $\exists\ \sigma\theta : \mathbb{C}^n \to \sigma V$

$$[x, \sigma\theta z] = -\sigma[x, \theta z] = \sigma\theta(-\alpha(x) i z)$$

Hence there exists $\sigma\theta\tau : \mathbb{C}^n \xrightarrow{\sim} \sigma V$ $\qquad \tau z = \bar{z}$

such that
$$[x, \sigma\theta\tau z] = \sigma\theta(-\alpha(x)\, i\bar{z}) = \sigma\theta\tau(\alpha(x) z).$$

Thus one sees that k^α is stable under σ. Moreover, if $x \in k$ is chosen so that $\alpha(x) = 1$, whence the complex structure on k^α is given by

$$iv = [x, v]$$

then $\sigma(iv) = [-x, \sigma v] = -iv$, so that σ is a conjugation for the complex structure. This yields immediately

$$k^\alpha \simeq (k^\alpha)^+ \otimes_{\mathbb{R}} \mathbb{C}.$$

(Maybe a better derivation of the preceding is to use the torus $S = \exp(\mathfrak{k}_\xi^-)$, and to decompose $\mathfrak{k}$ according to the irred. repns. of the generalized dihedral group $\{\sigma\} \ltimes S$. This shows directly that is a complex space with σ a conjugation.)

The preceding construction could be done for an abelian subspace A of $\mathfrak{k}^-$ with ξ replaced by any element x not in a root hyperplane. If A is a maximal abelian subspace of $\mathfrak{k}^-$, then $\mathfrak{k}_x^- = \mathfrak{k}_A^- = A$, so on moving x into $\mathfrak{k}_\xi^-$ we get:

<u>Prop. 4</u>: All maximal abelian subspaces of $\mathfrak{k}^-$ are conjugate.

As in part 1, we can use the root space decomposition to calculate the Hessian of $f(k\eta, \xi) = |k\eta - \xi|^2$ at a critical point $\eta \in \mathfrak{k}_\xi$. We get for the Hessian

$$-\left((\operatorname{ad} X)^2 \eta, \xi\right) = \left([\eta, X], [\xi, X]\right)$$

$$= \sum_{\alpha \in \Psi^+} \alpha(\eta)\,\alpha(\xi)\,|pr^\alpha(X)|^2$$

where X ranges over $\mathfrak{k}^+ \ominus \mathfrak{k}^+_\eta$ which is the tangent space to $K^\sigma \eta$ at η. Since

$$\mathfrak{k}^+ \ominus \mathfrak{k}^+_\eta = \sum_{\alpha \in \Psi^+,\ \alpha(\eta) \neq 0} (\mathfrak{k}^\alpha)^+ .$$

we see this Hessian is non-degenerate with index:

$$\lambda_\xi(\eta) = \sum_{\substack{\alpha \in \Sigma^+ \\ \alpha(\eta) < 0}} \dim (k^\alpha)^+$$

where $\dim (k^\alpha)^+ = \frac{1}{2} \dim k^\alpha$.

So:

<u>Prop. 5</u>: Assuming ξ such that k_ξ is abelian the function f on $K^\sigma\xi$, $\xi \in k^-$ has critical points where the orbit $K^\sigma\xi$ intersects k_ξ^-. Each critical point η is non-degenerate and its index is $\lambda_\xi(\eta)$.

Let E be a maximal abelian subspace in k containing k_ξ^-. As $E \subset k_\xi = k_\xi^+ \oplus k_\xi^-$ we have $E = (E \cap k_\xi^+) \oplus k_\xi^-$, hence E is stable under σ. $E^+ = E \cap k_\xi^+$ is a maximal abelian subspace of k_ξ^+, and $E^- = k_\xi^-$. If

$$k = E \oplus \sum_{\beta \in \Phi^+} k^\beta$$

is the root space decomposition of k with respect to E, then the root space decomposition of k with respect to E^- is obtained by restricting the $\beta \in \Phi^+$ to linear functions on E^-. Thus the root hyperplanes of E^- are the intersection of the root hyperplanes of E

with E^- which do not contain E^-.

Let C^- denote the chambre of E^- containing ξ and let C be a chamber of E containing ξ. Then $C^- \subset C$ because the line joining ξ to any point of C^- doesn't cross any hyperplanes. In fact $C^- = C \cap k^-$ since if $x \in C \cap k^-$, the line joining x to ξ would lie in k^- and would not cross a root hyperplane so would be in C^-.

We suppose Φ^+ chosen to consist of $\alpha \in \Phi$ which are positive on C. One has

$$k_\xi = E \oplus \sum_{\substack{\beta \in \Phi^+ \\ \beta(\xi)=0}} k^\beta$$

$$k^\alpha = \sum_{\substack{\beta \in \Phi^+ \\ \beta | E^- = \alpha}} k^\beta \qquad \alpha \in \Phi^+.$$

Suppose we now deduce the Morse theory consequences of Prop. 5. We have that an orbit $K^\sigma \xi$ has the homotopy type of CW cx. will cells indexed by $W^- \xi = K^\sigma \xi \cap E^-$ (assuming $\xi \in E^-$), and that the dimension of the cell corresponding to $\eta \in W^- \xi$ is $i_\xi(\eta)$. If η is a point of index 0 then $\eta \in C^-$ and conversely. But $C^- \subset C$ and we know that no two points of C are K-conjugate, hence no two points of C^- are K^σ-conj.

Thus we get:

<u>Prop. 6</u>: Each K^σ-orbit on k^- intersects C^- in exactly one point, namely where the function f is minimum. Consequently

$$K^\sigma \backslash k^- \cong W^- \backslash E^- \xleftarrow{\sim} C^-$$

i.e. C^- is a fundamental domain for K^σ on k^- and W^- on E^-.

Because there is exactly one 0-cell in the CW complex homotopy equivalent to $K^\sigma \xi$ we see this orbit is connected. So:

<u>Prop 7</u>: Each K^σ-orbit is connected.

Because $C^- \subset C$ it is clear that if two points of k^- are K-conjugate they are K^σ-conjugate. If $\eta \in k^-$, then $(K\eta) \cap k^-$ is the set of points of k^- which are K-conjugate to η. Thus we have

<u>Prop. 8</u>: If $\eta \in k^-$, then $\quad K^\sigma \eta = (K\eta) \cap k^-$.

(This result can be reformulated:

$$K^\sigma / K^\sigma_\eta \xrightarrow{\sim} (K/K_\eta)^\sigma \quad .)$$

As in the first section we can prove:

<u>Prop. 9:</u> W^- is a reflection group on E^-, (the reflections associated to root hyperplanes: $\alpha = 0$ $\alpha \in \Phi$).

So the theory developed in the first part: reduced decompositions, simple reflections, etc. applies to W^-.

Let us relate W and W^-. Let W^σ be the subgroup of W commuting with σ (as autos. of E). If $w \in W^\sigma$, then w preserves the eigenspaces of σ, hence $wE^- = E^-$. Conversely if $wE^- = E^-$, then w also preserves the orthogonal complement E^+, so w commutes with σ. Thus $W^\sigma = \{w \in W \mid wE^- = E^-\}$.

~~Let $W_{E^-} = \{w \mid w = id$ on $E^-\}$, and let W_ξ be the stabilizer of ξ; $W_{E^-} \subset W_\xi$. But if $x \in N = Norm_K(E)$ centralizes ξ, then $x \in K_\xi$ which acts trivially on E^-, so $W_{E^-} = W_\xi$.~~

~~Let $y \in N^- = Norm_{K^\sigma}(E^-)$. Then $y.E$ is a max. abelian subspace of K containing E^-. I~~

Let $\gamma \in W^-$. On γC^- no α in Ψ changes sign, hence on γC^- no β in Φ changes sign. Let wC be a chambre containing $\gamma\xi$. If $\beta \in \Phi$ is positive on wC and <0 somewhere on γC^-, then β would be <0 at $\gamma\xi$, because $\gamma\xi$ is an "interior" point of γC^-, and this contradicts the fact that $\gamma\xi \in wC$. Thus $\beta \geq 0$ on $wC \Rightarrow \beta \geq 0$ on γC^- so $\gamma C^- \subset wC$. If $x \in C^-$ then γx, wx are two elements of the chambre wC which are K-conjugate. Therefore $\gamma x = wx$ for all $x \in C^-$, and hence as both are linear transfs. $\gamma x = wx$ for all $x \in E^-$. In particular w preserves E^- so $w \in W^\sigma$. It is clear w is unique modulo the centralizer W_{E^-} of E^-.

Conversely if $w \in W^\sigma$, then wC^- is contained in E^- and cut by no hyperplanes, so a similar argument shows wC^- is contained in a chambre γC^-. If $\gamma x = w'x$ for all $x \in E^-$ as above we have $wC^- \subset w'C^-$, so $w = w'$ on C^-, hence on E^-. Thus $w|E^- = \gamma$, and we obtain:

<u>Prop. 10</u>: One has an isom. $W^\sigma / W_{E^-} \xrightarrow{\sim} W^-$ given by restricting w in W^σ to E^-.

Example: $K = U(p+q)$. Let σ be the inner auto. by an element $\tilde{\sigma}$ of order 2 with p eigenvalues -1 and q eigenvalues $+1$. I suppose $p \leq q$ (otherwise replace $\tilde{\sigma}$ by $-\tilde{\sigma}$). $\tilde{\sigma}$ is conjugate to $\begin{pmatrix} -I_p & \\ & I_q \end{pmatrix}$ so $K/K^\sigma \simeq U(p+q)/U(p)\times U(q) =$ complex Grassmannian.

For calculations it is easiest to work with $\tilde{\sigma}$ the matrix

$$\begin{array}{c} \updownarrow 2p \\ \updownarrow n-2p \end{array} \begin{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & & & & \\ & \ddots & & & \\ & & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \qquad p+q=n.$$

Suppose to begin with that $p=q=1$. $\mathfrak{k}$ consists of skew-hermitian matrices $\begin{pmatrix} ia & \beta \\ -\bar{\beta} & ic \end{pmatrix}$.

$$\tilde{\sigma}\begin{pmatrix} ia & \beta \\ -\bar{\beta} & ic \end{pmatrix}\tilde{\sigma} = \begin{pmatrix} ic & -\bar{\beta} \\ \beta & ia \end{pmatrix}$$

So

$$\mathfrak{k}^+ : \begin{pmatrix} ia & ib \\ +ib & ia \end{pmatrix}$$

$$\mathfrak{k}^- : \begin{pmatrix} ia & b \\ -b & -ia \end{pmatrix}$$

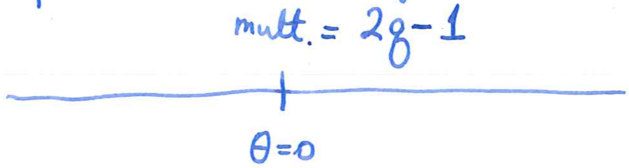
I take $\xi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ whence $k_\xi : \begin{pmatrix} ia & \\ & ic \end{pmatrix}$ and $k^- : \begin{pmatrix} ia & \\ & -ia \end{pmatrix}$ is abelian. k has the root $\begin{pmatrix} i\lambda_1 & \\ & i\lambda_2 \end{pmatrix} \longmapsto \lambda_1 - \lambda_2$, and its restriction to k_ξ^- is $\begin{pmatrix} i\theta & \\ & -i\theta \end{pmatrix} \longmapsto 2\theta$. The maximal reversed torus $\exp(k_\xi)$ is: $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ and the unique positive root is 2θ; its multiplicity is 1.

In the general case we take ξ to be the diagonal matrix $(i\xi_1, -i\xi_1, \dots, i\xi_p, -i\xi_p, 0, \dots)$ which is in k^-. I suppose $\xi_1 > \dots > \xi_p > 0$, whence k_ξ is diagonals $(i\lambda_1, \dots, i\lambda_{2p}) \oplus u_{n-2p}$, and k_ξ^- is the abelian spaces of diagonals $(i\theta_1, -i\theta_1, \dots, i\theta_p, -i\theta_p, 0 \dots, 0)$. We extend $k_\xi^- = E^-$ to the diagonal Cartan subalg. of u_n, whose roots are $\lambda_i - \lambda_j$, $i \neq j$. Calculating the restrictions to E^- we get the roots:

$\pm 2\theta_i$	$1 \le i \le p$	mult. 1
$\pm(\theta_i - \theta_j)$	$1 \le i < j \le p$	mult. 2
$\pm(\theta_i + \theta_j)$	$1 \le i < j \le p$	mult 2.
$\pm \theta_i$		mult $2(q-p)$

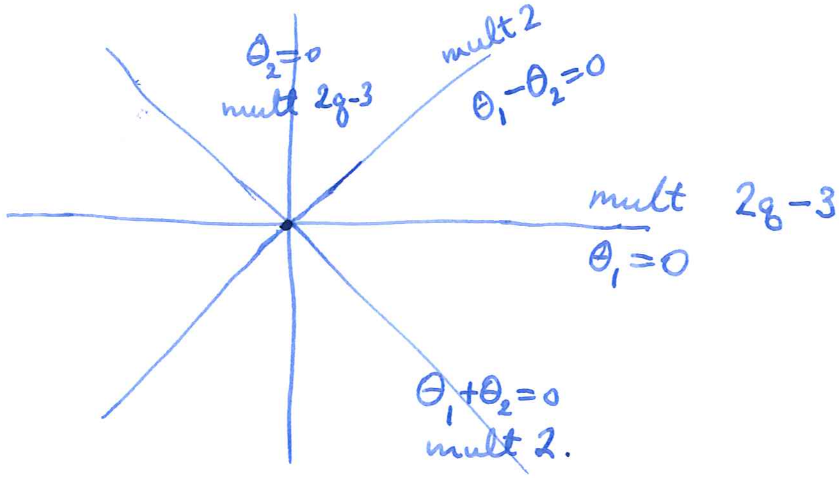
The Weyl group permutes $\theta_1, \dots, \theta_p$ and changes signs: $W^- = \Sigma_p \ltimes (\mathbb{Z}/2)^p$. The fundamental chambre is $\theta_1 \geq \cdots \geq \theta_p \geq 0$. This root system is C_p if $p=q$ and $B_p \cup C_p$ if $p<q$.

For $p=1$, the infinitesmal diagram is:



which corresponds to the fact that $U(1+q)/U(1)\times U(q) = \mathbb{C}P^q$ and the unit sphere about a point is S^{2q-1}.

For $p=2$, the infinitesimal diagram is



I want to take up this example from a more geometric point of view. Again $K = U(p+q)$ and σ is conjugation by $\tilde{s}: \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix}$, so that $K^\sigma = U(p)\times U(q)$. In this $\mathfrak{k}^-$ consists of matrices $\left(\begin{array}{c|c} 0 & B \\ \hline -B^* & 0 \end{array}\right)$ where B is a $p \times q$ complex matrix. The action of $\left(\begin{array}{c|c} A & \\ \hline & C \end{array}\right) \in U(p)\times U(q)$ on $\left(\begin{array}{c|c} 0 & B \\ \hline -B^* & 0 \end{array}\right)$ sends B to ABC^{-1}. Thus the K^σ action on $\mathfrak{k}^-$

is just the obvious action of $U(p)\times U(q)$ on $\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p)$.

Orbit structure: Given $B: \mathbb{C}^q \to \mathbb{C}^p$ we associate the quadratic form $\|Bx\|^2 = (B^*Bx, x)$ on $\mathbb{C}^q$. Under the action of $U(q)$ it is conjugate to a diagonal form $|Px|^2$ where P is the diagonal matrix with entries $(\lambda_1, \ldots, \lambda_q)$ $\lambda_1 \geq \cdots \geq \lambda_q$. Say we replace B by BC^{-1}, $C \in U(q)$ so that $B^*B = P^2$, hence working in the orthogonal complement to $\operatorname{Ker} B$, we have that BP^{-1} is a unitary embedding of $\mathbb{C}^q/\operatorname{Ker} B$ into $\mathbb{C}^p$. We can therefore find a unitary A in $U(p)$ such that ABP^{-1} sends e_i to e_i for $1 \leq i \leq r$ and the rest of the e_i $r < i \leq q$ to zero, where r is the largest such that $\lambda_r > 0$. Thus we see that B is conjugate under the $U(p)\times U(q)$-action to the matrix

$$\begin{array}{c c} p & \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \\ & q \end{array}$$

I suppose $p \leq q$. The above calculation amounts to the previous result that a chambre C for the $U(p)\times U(q)$ action on $\mathfrak{k}^-$ is given by $\theta_1 \geq \cdots \geq \theta_p \geq 0$ in $\mathbb{R}^p$.

It's clear that E^- is to be identified with diagonal $p\times q$ matrices

$$\left(\begin{array}{ccc|c} \theta_1 & & & \\ & \ddots & & 0 \\ & & \theta_p & \end{array}\right)$$

The stabilizer of a regular element : $\theta_1 > \cdots > \theta_p > 0$ in the chambre C is $(S^1)^p \times [(S^1)^p \times U(q-p)] \subset U(p) \times U(q)$. This stabilizer goes up if either θ_i becomes equal to θ_{i+1} or if $\theta_p = 0$. The walls of the fundamental chambre C are:

$$\theta_1 - \theta_2 = 0, \ \dots, \ \theta_{p-1} - \theta_p = 0, \ \theta_p = 0$$

hence the Weyl group is generated by $\theta_p \to -\theta_p$ and flipping θ_i, θ_{i+1}, so $W = \Sigma(p) \ltimes (\mathbb{Z}/2)^p$. The root hyperplanes are $\theta_i = 0$, $\theta_i - \theta_j = 0$, $\theta_i + \theta_j = 0$. Mult. of simple roots are

$$\theta_i - \theta_{i+1} : \quad \dim \frac{U(2)}{S^{1\,2}} = 2$$

pairs A, A^{-1} mod. those with A diag.

$$\theta_p : \quad \dim \frac{S^1 \times U(q-p+1)}{\Delta S^1 \times U(q-p)} = 2(q-p) + 1$$

On $\mathfrak{U}(p+q)$ we use the usual inner product $\mathrm{tr}\, XY^*$; this gives* $|B|^2 = 2\,\mathrm{tr}\, BB^* = 2 \sum |b_{ij}|^2$ on k^-. So using this inner product we get Morse functions on the orbits of $U(p) \times U(q)$. Here is a specific example:

Suppose $p = q$ and take the orbit of $B = I$ which is isomorphic to $U(p)$. Let Λ be a diagonal

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* Because B corresponds to $\left(\begin{array}{c|c} 0 & B \\ \hline -B^* & 0 \end{array}\right)$.

$p \times q$ matrix with eigenvalues $\lambda_1 > \dots > \lambda_p > 0$. Then the Morse function in question on $U(p)$ sends $A \in U(p)$ to $A \cdot I = A$ in $Hom(\mathbb{C}^p, \mathbb{C}^p)$ and then to minus the inner product of A with Λ which is

$$-(A, \Lambda) = -2\, tr(A^*\Lambda + \Lambda^* A)$$

A is a critical point of this function $\Longleftrightarrow$ for all skew-hermitian B we have

$$\begin{gathered} tr\left((AB)^*\Lambda + \Lambda(AB)\right) = 0 \\ \| \\ tr\left(-BA^*\Lambda + \Lambda AB\right) = tr\left(B(\Lambda A - A^*\Lambda)\right). \end{gathered}$$

Thus A is critical iff $\Lambda A - A^*\Lambda = 0$

~~hermitian?? Note? ΛA is hermitian~~ $\Longleftrightarrow (\Lambda A)(\Lambda A)^* = \Lambda A A^* \Lambda = \Lambda^2$

$\Longleftrightarrow$ ΛA is a square root of Λ^2 i.e.

$$\Lambda A = \begin{pmatrix} \pm\lambda_1 & & \\ & \ddots & \\ & & \pm\lambda_p \end{pmatrix}$$

It follows A is a diagonal matrix with ± 1 entries, hence there are 2^p critical points as there should be.

Some comments on classification of pairs (K, σ):

The <u>symmetric space</u> associated to the pair (K, σ) is the homogeneous space $M = K/K^\sigma$. It is a Riemannian manifold with Riemannian structure inherited from the inner product on k, and K acts as isometries. In addition at each point $m \in M$ there is an isometry σ_m of M such that $\sigma_m = -id$ on the tangent space at m. (σ_m is induced by σ if $m = e \cdot K^\sigma$, in general σ_m is induced by the involution $x \sigma x^{-1}$ if $m = x K^\sigma$.)

It is possible to find conditions under which the pair (K, σ) can be recovered from the Riemannian manifold M as follows: K is the connected component of the group of isometries of M, and σ is conjugation by the symmetry around the basepoint eK^σ of M. (An obvious necessary condition is that K act faithfully, i.e. the intersection of the conjugates of K^σ is 1.) In fact it appears that the intrinsic structure on M to work with is the connection associated to the Riemannian structure. (Thus a symmetric space is a connected manifold with affine connection having for each point m a global isometry σ_m reversing geodesics thru m.)

So if the good object is the symmetric space, we want to regard two involutions σ, σ' on K equivalent if the homogeneous spaces K/K^σ and $K/K^{\sigma'}$ are isomorphic. This is the same as the isotropy groups being conjugate. Since K is connected, K^σ determines σ ($\sigma = +1$ on $\mathfrak{k}^\sigma$ and -1 on the orthogonal complement.) Thus $\exists$ $k \in K$ such that the inner auto by k transforms σ into σ', i.e.

$$k\,\sigma(k^{-1} x k)\,k^{-1} = \sigma' x$$

or

$$p\,\sigma(x)\,p^{-1} = \sigma'(x)$$

where $p = k\,\sigma(k)^{-1} \in R_\sigma = \{ y \in K \mid \sigma y = y^{-1} \}$. On the other hand if $\sigma'(x) = y\sigma(x)y^{-1}$, then for σ' to be an involution means

$$x = \sigma'(\sigma'(x)) = y \cdot \sigma y \, x \, \sigma y^{-1} y^{-1}$$

i.e. $y \cdot \sigma y \in$ center of K. In particular $y \in R_\sigma$ $\Longrightarrow$ $\sigma' = y\sigma(\;)y^{-1}$ is an involution.

Let σ be fixed, and let K act on itself by $x * y = x\,y\,\sigma x^{-1}$. Then $K/K^\sigma \simeq$ orbit of e. Also R_σ is stable under this action. Using the exponential map $\exp: \mathfrak{k} \to K$ one sees that near the identity R_σ coincides with $\exp(\mathfrak{k}^-)$. So we have

$$\exp(\mathfrak{k}^-) \subset \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ K/K^\sigma}}{\{ k\sigma(k)^{-1} \mid k \in K \}} \subset R_\sigma .$$

$K * 1$ is closed in R_σ. It is also open as it contains the open set $R_\sigma \cap U$, U a small ball around O, and because K acts transitively. Thus we conclude:

<u>Prop. 11:</u> $K * 1 = \{k\,\sigma(k)^{-1}\}$ is the identity component of $R_\sigma = \{y \in K \mid \sigma y = y^{-1}\}$.

Thus we can identify K/K^σ with $R_\sigma^\circ = K * 1$. From geodesic theory one knows that $\exp: \mathfrak{k}/\mathfrak{k}^+ \to K/K^\sigma$ is onto, hence

$$\exp(\mathfrak{k}^-) = K * 1.$$

On $K*1$, the K^σ-action coincides with ordinary conjugation.

The K^σ-action on $\mathfrak{k}^-$ is in some sense the infinitesimal version of the K^σ-action on $M = K/K^\sigma$.

We can summarize the discussion on page 18 as follows:

<u>Prop. 12:</u> An involution σ' of K is equivalent to σ iff $\sigma'(x) = k\,\sigma(x)\,k^{-1}$ with $k \in K * 1 = \exp(\mathfrak{k}^-)$.

~~If $y \in R_\sigma$, then $\sigma'(x) = y\,\sigma(x)\,y^{-1}$ is an involution (σ' is derived from σ but not necessarily equivalent to σ).~~

$\sigma'(x) = y\,\sigma(x)\,y^{-1}$ is an involution iff $y\,\sigma y \in$ center of K, e.g. if $y \in R_\sigma$.

Example: Take $\sigma = id$, whence R_σ = elements of order 2. In case $K = U(n)$, the different symmetric spaces obtained are the Grassmannians. ($y^2 \in$ center $\Rightarrow y$ has 2 eigenvalues.)

Prop. 13: If $\sigma' = y\sigma(\ )y^{-1}$ with $y \in \tilde{R}_\sigma$, then the corresponding symmetric space is $K * y$.

Because $x * y = xy\sigma(x)^{-1} = y$ iff $y\sigma(x)y^{-1} = x$.

Conclusion: Starting from an involution σ, we can generate other symmetric spaces by taking the orbits of R_σ under the $*$ action. In fact one can take orbits of $\tilde{R}_\sigma = \{ y \mid y \cdot \sigma y \in \text{center}(K) \}$.

Note: If $y \cdot \sigma y = z \in \text{Center}(K) \Rightarrow \sigma y = y^{-1} z$ commutes with y $\Rightarrow \sigma z = z \Rightarrow z \in K^\sigma \cap \text{Center}(K)$. This is a normal subgroup of K^σ, so if K acts faithfully on its symmetric space it would be trivial, hence $R_\sigma = \tilde{R}_\sigma$.

Example: Take $K = U_n$ with σ = complex conjugation. If $y \in \tilde{R}_\sigma$, i.e. $y \cdot \sigma y = \lambda \cdot id$, then $\lambda \cdot id \in K^\sigma$ so $\lambda = \pm 1$. But if we interpret elements as transfs. of $\mathbb{C}^n$, then $\sigma(y) = \sigma \circ y \circ \sigma^{-1}$ where σ = conj. on $\mathbb{C}^n$. Thus our involution becomes

$$x \longmapsto y\,\sigma(x)\,y^{-1} = (y\sigma) \circ x \circ (y\sigma)^{-1}.$$

But $y\sigma$ is an anti-linear transf. of $\mathbb{C}^n$ such that $(y\sigma)\circ(y\sigma) = y \cdot \sigma y = \lambda$. If $\lambda = 1$, $y\sigma$ defines a real structure on $\mathbb{C}^n$, whereas if $\lambda = -1$, $y\sigma$ defines a quaternion structures. These structures form single orbits under K, so the symmetric spaces are

$$U_n/O_n \qquad U_n/Sp([\tfrac{n}{2}]) \quad \text{(note } n \text{ has to be even).}$$

Rank 1 symmetric spaces: Rank 1 means that $\dim E^- = 1$, hence all roots are proportional. Let α be a root and let $V \subset k^\alpha$ be a minimal subspace invariant under $Ad(E^-)_\Lambda$ and σ, so V is $\simeq \mathbb{C}$ with $x \in E^-$ acting as $i\,\alpha(x)$. Then $[V,V]$ is a quotient of $\Lambda^2 V$, hence $\dim [V,V] \leq 1$, which forces $[V,V] \subset k_\xi$ as this space in $Ad(E^-)$ invariant, and outside k_ξ invariant subspaces have complex structures. Under σ, V has eigenvalues $+1, -1$, hence $[V,V] \subset E^-$. Thus $E^- \oplus V$ is a 3-diml subalgebra of k stable under σ; $[V,V] = E^-$ otherwise one would contradict E^- being a maximal abelian subspace. It should be easy to see that $E^- \oplus V \simeq su_2$. Then repn. theory for su_2 forces the roots to be multiples of $\frac{\alpha}{2}$. Applied to any root, this means that we have:

<u>Prop. 14</u>: For a rank 1 situation, let α be the smallest root in Φ^+. Then the only other possible positive root is 2α.

It is known by classification theory that the rank 1 symmetric spaces are the proj. spaces

$$\mathbb{R}P^n, \ \mathbb{C}P^n, \ \mathbb{H}P^n$$

and the projective plane over the octonions. In the first case only α is a root, in the others 2α is a root and $(\mathfrak{k}^{2\alpha})^- \oplus E^-$ is essentially the field $\mathbb{C}, \mathbb{H}, \mathbb{O}$ in question.