Nov. 3, 1975

pgs 1-7 given to Hiller

Let V be [redacted] a vector space over a field K [redacted] of countable infinite dimension. Form a poset consisting of subspaces W of V such that $\dim W = \dim V/W = \infty$, with $W_1 < W_2$ iff $W_1 \subset W_2$ and $\dim(W_2/W_1) = \infty$. Is X contractible?

[redacted]

[redacted] Again let F be a finite subset of X. Pick a maximal subset $\{x_1, \ldots, x_\ell\}$ of F such that $y = x_1 \cap \cdots \cap x_\ell$ has infinite dimension. Then for all other x in F, $x \cap y$ is finite-dimensional and so we can shrink y successively still keeping it infinite dimensional until we get z in X such that for all $x \in F$ either $z \cap x =$ [redacted] 0 or [redacted] $z \subset x$. Now [redacted] let z' be a subspace of z of infinite dim + codim in z.

I consider sending x in F to $x + z'$. If $x \cap z = 0$, then [redacted] $x+z/x+z' \simeq z/z'$ is inf. diml. so $x + z' \in X$. Also $x < x+z' >$ [redacted] z'. On the other hand if [redacted] $z \subset x$, then $x + z' = x \in F$ and $x = x+z' > z'$ because $z/z' \subset x/z'$. Finally if $x_1 < x_2$ we want to show that $x_1 + z' < x_2 + z'$. This is clear if either $z \subset x_1$ or if $z \cap x_2 = 0$. If $z \subset x_2$ and $z \cap x_1 = 0$, then $x_1 + z' < x_2 = x_2 + z'$ because $x_2/x_1+z' \supset x_1 \oplus z / x_1 \oplus z' \simeq z/z'$ is infinite dimensional.

Therefore X is contractible.

Given a p-simplex $x_0 < x_1 < \cdots < x_p$ in X there exists a decomposition $V = V_0 \oplus \cdots \oplus V_{p+1}$ such that $x_j = V_0 \oplus \cdots \oplus V_j$, where V_i is infinite-dimensional, hence isomorphic to V. Thus $G = \mathrm{Aut}(V)$ acts transitively on the p-simplices, the stabilizer of a p-simplex being a group:

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$$

with $p+1$ blocks in the diagonal positions, and where the entries come from the ring $\mathrm{End}(V)$. So now make the usual assumptions that guarantee the unipotent radical doesn't contribute to homology:

a) homology with coefficients in $\mathbb{Q}$

or b) homology with coefficients in $\mathbb{F}_\ell$ where $\ell^{-1} \in K$.

Then the spectral sequence becomes

$$E^1_{qp} = H_q(G^{p+1}) \Longrightarrow 0$$

and as before this implies $\tilde{H}_*(G) = 0$.

Fix a vector space M over K, and consider the groupoid $\mathcal{E}_M$ consisting of exact sequences of K-vector spaces:

$$0 \longrightarrow V \longrightarrow E \longrightarrow M \longrightarrow 0$$

with $\dim(V)$ countable infinite; morphisms are isomorphisms over M. On $\mathcal{E}_M$ we have a product functor

$$(E_1, E_2) \longmapsto E_1 \times_M E_2$$

which induced a product on $H_*(\mathcal{E}_M)$ which is commutative and associative. Note that $\mathcal{E}_M$ is equivalent to the group

$$Aut(M \oplus V/M) = \begin{bmatrix} id_M & 0 \\ Hom(M,V) & Aut\,V \end{bmatrix}$$

In addition we have an infinite sum functor Σ defined as follows. Given $E \longrightarrow M$, consider inside of $E \times_M E \times_M E \times_M \cdots\cdots$ the subspace formed of sequences $(e_1, e_2, e_3, \cdots\cdots)$ such that $\{e_1, e_2, e_3, \ldots\ldots\}$ is finite. Call this subspace $(E/M)^{(\infty)}$, whence we have an exact sequence

$$0 \longrightarrow V^{(\infty)} \longrightarrow (E/M)^{(\infty)} \longrightarrow M \longrightarrow 0$$

which we define to be $\Sigma(E/M)$. So the K-theory of $\mathcal{E}_M$ is trivial. This means that the embedding

$$\begin{bmatrix} id_M & 0 & 0 \\ Hom(M,V) & Aut(V) & 0 \\ 0 & 0 & id_V \end{bmatrix} \subset \begin{bmatrix} id_M & & \\ Hom(M,V) & \multicolumn{2}{c}{\multirow{2}{*}{Aut(V \oplus V)}} \\ Hom(M,V) & & \end{bmatrix}$$

should induce the zero map on $\tilde{H}_*$.

It seems necessary to review stability for a field. Let V_0 be a subspace of V and let $P(V,V_0)$ be the poset of subspaces $W < V$ such that $W + V_0 = V$. According to Lusztig this complex has the homotopy type of a bouquet of spheres of its dimension (which is $\dim(V_0)-1$. It follows that we get a Lusztig sequence.

$$\cdots \longrightarrow \bigoplus_{\substack{W+V_0=V \\ \dim(W\cap V_0)=1}} J(W, W\cap V_0) \longrightarrow \bigoplus_{\substack{W+V_0=V \\ \dim W\cap V_0=0}} \mathbb{Z} \longrightarrow \mathbb{Z}$$

So let me consider the analogous infinite situation. I want to understand the map on homology induced by the inclusion

$$\operatorname{Aut}(V_0) \cong \begin{bmatrix} id_M & 0 \\ 0 & \operatorname{Aut} V_0 \end{bmatrix} \subset \begin{bmatrix} id_M & \\ * & \operatorname{Aut} V_0 \end{bmatrix} = \operatorname{Aut}(M \oplus V_0/M).$$

So I will want to make the group $\begin{bmatrix} id_M & \\ * & \operatorname{Aut} V_0 \end{bmatrix}$ act on a poset whose minimal elements are the subspaces W of V such that $W \oplus V_0 = V$. (I have changed notation from $0 \to V \to E \to M \to 0$ to $0 \to V_0 \to V \to M \to 0$).

Thus in the infinite situation let Y be the poset consisting of subspaces W of V such that $W + V_0 = V$, such that $W \cap V_0$ is of inf. codim in V_0, and such that $\dim(W \cap V_0) = 0$ or ∞

$$\begin{tikzcd}[column sep=small] & V \arrow[dl, dash] \arrow[dr, dash] & \\ W \arrow[dr, dash] & & V_0 \arrow[dl, dash] \\ & W \cap V_0 \arrow[d, dash] & \\ & 0 & \end{tikzcd}$$

Define $W_1 < W_2$ if $W_1 \subset W_2$ and $\dim(W_2/W_1) = \infty$.

Put $G = GL(V/M)$ for the group of autos. of V inducing the identity on M. Does G act transitively on the simplices of Y?

November 5, 1975.

Let $\mathcal{S}$ be a groupoid with product functor $\perp : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$. Assume $\mathcal{S}$ has two iso. classes: $0, N$ such that $0 \perp N \simeq N$, $N \perp N \simeq N$. I want to consider the poset X consisting of isos.

$$\alpha : N \perp N \xrightarrow{\sim} N$$

with $\alpha < \alpha'$ iff $\exists \gamma, \gamma' \ni$

$$\begin{tikzcd}
N \perp N \arrow[d, dash, "\wr"', "\gamma' \perp id"] \arrow[drr, "\alpha'"] & & \\
(N \perp N) \perp N \arrow[d, equal] & & N \\
N \perp (N \perp N) & & \\
N \perp N \arrow[u, dash, "\wr", "id \perp \gamma"'] \arrow[uurr, "\alpha"'] & &
\end{tikzcd}$$

commutes. Assume γ, γ' uniquely determined essentially because $\perp$ is faithful.

Better description. Consider the simplicial groupoid, which in degree p is $\mathcal{S}^{p+2}$

$$\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \mathcal{S} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \mathcal{S} \times \mathcal{S} \times \mathcal{S} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{S} \times \mathcal{S}$$

There is an augmentation to $\mathcal{S}$ so I can form the fibre simplicial set over an object N. Thus a p-simplex in X is a partitioning of N into $(p+1)$-pieces:

$$M_0 \perp \cdots\cdots \perp M_{p+1} \simeq N$$

Assume X is contractible. Does it follow that I get a stability thm:

$$e_\circ : H_*(\mathrm{Aut}\, N) \xrightarrow{\sim} H_*(\mathrm{Aut}(N)) \ ?$$

By letting $G = \mathrm{Aut}\, N$ act on X I get a spectral sequence

$$E^1_{pq} = H_q(G^{p+1}) \Longrightarrow 0$$

Can I produce a relative spectral sequence?

$$\cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Let G' be the image of $G \hookrightarrow G$ obtained from $N + N \xrightarrow[\sim]{\alpha} N$ trivial action on the first factor. Let X' be the corresponding subcomplex of X; it consists of partitions of the second factor $N + N \simeq N$. Then I get a map $(G', X') \longrightarrow (G, X)$.

Next I have to compute the relative terms.

$X_0 = G/G'' \times G'$ $\qquad$ $X_0' = G'/(G')'' \times (G')'$?

Redo: Start with $N' \subset N$, specifically given by $N' \perp N'' = N$. Let $G' = \mathrm{Aut}(N')$ be viewed as a subgroup of $G = \mathrm{Aut}(N)$. We can also view $X(N')$ as a [redacted] complex of $X(N)$. In effect given

$$N_0 \perp \cdots \perp N_{p+1} = N'$$

we send it to the [redacted] partition

$$N_0 \perp \cdots \perp N_p \perp (N_{p+1} \perp N'') = N.$$

(This corresponds to the obvious map of the building of V' into the building of V when $V' \subset V$.)

[redacted]

Fix a p-simplex in $X(N')$,

$$N_0' \perp \cdots \perp N_{p+1}' = N'.$$

[redacted] and let $G'_{(p+2)}$ denote its stabilizer. Then

$$G'/G'_{(p+2)} \xrightarrow{\sim} X(N')_p$$

The image of this simplex in $X(N)$ is $N_0 \perp \cdots \perp N_{p+1} = N$ where $N_0' = N_0, \ldots, N_p' = N_p$, $N_{p+1} = N_{p+1}' \perp N''$. Denote the stabilizer in G of this by $G_{(p+2)}$ so that

$$G/G_{(p+2)} \xrightarrow{\sim} X(N)_p$$

Then the inclusion $X(N')_p \subset X(N)_p$ correspond to the map $G'/G'_{(p+2)} \longrightarrow G/G_{(p+1)}$ induced by the inclusions

$$G' \subset G$$

$$G'_{(p+2)} \subset G_{(p+2)}.$$

Now

$$\begin{array}{rcl} G'_{(p+2)} & = & \mathrm{Aut}(N'_0) \times \cdots \times \mathrm{Aut}(N'_{p+1}) \\ & & \quad \Vert \\ G_{(p+2)} & = & \mathrm{Aut}(N_0) \times \cdots \times \mathrm{Aut}(N'_{p+1} \perp N'') \end{array}$$

and so we see that

$$(G_{(p+2)}, G'_{(p+2)}) \simeq G^{p+1} \times (G, G').$$

Therefore the relative spectral sequence should have the E^1-term

$$E^1_{pq} = H_q(G^p \times (G, G')) \Longrightarrow 0$$

If I know that $H_q(G, G') = 0$ for $q < r$, then

$$E^1_{pr} = H_r(G^p \times (G, G')) = \bigoplus_{i+j=r} H_i(G^p) \otimes H_j(G, G')$$

$$= H_0(G^p) \otimes H_r(G, G') \quad = H_r(G, G')$$

So you have to compute $d_1 : E'_{1r} \to E'_{0r}$ and show that it's zero.

$$\begin{tikzcd}
G \times G \times G \arrow[r, shift left, "\perp \times id"] \arrow[r, shift right, "id \times \perp"'] & G \times G \arrow[r, "\perp"] & G \\
G \times G \times G \arrow[u, "id \times id \times (\perp N)"] \arrow[r, shift left] \arrow[r, shift right] & G \times G \arrow[u, "id \times \perp N"] \arrow[r] & G \arrow[u, "\perp N"]
\end{tikzcd}$$

This induces

$$\begin{tikzcd}
G \times G \times (G, G') \arrow[r, shift left, "\perp \times id"] \arrow[r, shift right, "id \times \perp"'] & G \times (G, G') \arrow[r, "\perp"] & (G, G')
\end{tikzcd}$$

and so what happens for H_r is that this is the same as for the sub-gadget

$$\begin{tikzcd}
H_r(G, G') \arrow[r, shift left, "id"] \arrow[r, shift right, "N\perp"'] & H_r(G, G') \arrow[r, "N\perp"] & H_r(G, G')
\end{tikzcd}.$$

$(N\perp)$ is an idempotent operator, and we know it's surjective, hence it must be the identity. But then from:

$$\begin{tikzcd}
H_r(G') \arrow[r] \arrow[d] & H_r(G) \arrow[r] \arrow[d] & H_r(G, G') \arrow[r] \arrow[d] & H_{r-1}(G') \arrow[r] \arrow[d, "\text{same}"] & H_r(G) \\
H_r(G') \arrow[r] \arrow[d] & H_r(G) \arrow[r] \arrow[d, "\text{same}"'] & H_r(G, G') \arrow[r] \arrow[d] & H_{r-1}(G) \arrow[r] & {} \\
H_r(G') \arrow[r] & H_r(G) \arrow[r] & H_r(G, G') & &
\end{tikzcd}$$

we see $(N\perp)^2 = 0$ which concludes the proof.

I will now attempt to make the above stability proof geometric. Let [redacted] M be an associative monoid (without 1) of the homotopy type BG and let $M' = pt \amalg M$ be the associated monoid with 1. We can form the simplicial space

(*)

$$\cdots\ M \times (M')^2 \times M \;\substack{\textstyle\rightarrow\\[-4pt]\textstyle\rightarrow\\[-4pt]\textstyle\rightarrow}\; M \times M' \times M \;\rightrightarrows\; M \times M$$

which is obtained in the usual way by letting M' act the the left + right on M. By hypothesis on the contractibility of X the augmentation $M \times M \longrightarrow M$ given by $\perp$ gives a homotopy equivalence of (*) with M.

Let a be the basepoint of M. Right multiplication by a [redacted] furnishes an embedding of (*) into itself, so we can form the quotient

(**)

$$\cdots\ M \times (M')^2 \times M/Ma \;\substack{\textstyle\rightarrow\\[-4pt]\textstyle\rightarrow\\[-4pt]\textstyle\rightarrow}\; M \times M' \times M/M_a \rightrightarrows M \times M/M_a$$

which will be heq. to M/Ma via the evident augmentation. So assume $\tilde{H}_*(M/Ma)$ begins in dim h. From the homology spectral sequence I [redacted] get an exact sequence

$$\begin{tikzcd}
H_r(M \times M' \times M/Ma) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & H_r(M \times M/M_a) \arrow[r] \arrow[d, "\cong"] & H_r(M/M_a) \arrow[r] & 0 \\
H_r(M/Ma) \arrow[r, shift left, "id"] \arrow[r, shift right, "a\cdot"'] & H_r(M/Ma) \arrow[ur, "a\cdot"'] & &
\end{tikzcd}$$

so I can argue as before.

~~October~~ November 7, 1975:

Notation: $\mathcal{N}$ = the groupoid of countable infinite sets and bijections with $\sqcup$ operation, $\mathcal{N}' = \{\phi\} \sqcup \mathcal{N}$. $\mathcal{M}$ = a connected groupoid with product operation $\perp$ which is associative + commutative, $\mathcal{M}' = \underset{\{0\}}{pt} \sqcup \mathcal{M}$. I suppose I am given a functor

$$\theta : \mathcal{N}' \longrightarrow \mathcal{M}'$$

compatible with products such that $\theta(\mathcal{N}) \subset \mathcal{M}$.

Example: Take $\mathcal{M}$ to be countable infinite sets with morphisms defined to be isos. modulo finite sets.

I want to establish stability for $\mathcal{M}$. Let L be an object of $\mathcal{M}$. What is the unimodular complex in this situation?

I construct the following poset $X(L)$. An element is a "mono" $\theta(N) \rightarrowtail L$. More precisely it is an isomorphism $\theta(N) \perp L'' \xrightarrow{\sim} L$ modulo isos. of L'' (I continue to assume $\perp$ faithful). $\alpha : \theta(N) \rightarrowtail L$ is $>$ $\beta : \theta(N) \rightarrowtail L$ iff $\exists$ $i : N \rightarrowtail N$ such that $\alpha\,\theta(i) = \beta$. I guess I have to assume θ faithful.

Put $G = \mathrm{Aut}(L)$ $\Sigma = \mathrm{Aut}(N)$. Then if I fix a vertex $\theta(N) \perp L'' \xrightarrow{\sim} L$ and let $G'' = \mathrm{Aut}(L'')$ viewed as a subgroup of L, G acts transitively on $(p-1)$-simplices, so

$$\{(p-1)\text{-simplices}\} \simeq G/\Sigma^p \times G''$$

IS

$$\{\theta(N_0 \perp \cdots \perp N_{p-1}) \rightarrowtail L\}.$$

But this is the fibre of

$$\mathcal{N}^p \times \mathcal{M} \longrightarrow \mathcal{M}$$
$$(N_0, \dots, N_{p-1}, L_1) \longmapsto \theta(N_0) \perp \dots \perp \theta(N_{p-1}) \perp L_1$$

over L.

Hence contractibility of $X(L)$ tells us that the simplicial object with augmentation

$$\mathcal{N} \times (\mathcal{N}')^2 \times \mathcal{M} \equiv \mathcal{N} \times \mathcal{N}' \times \mathcal{M} \rightrightarrows \mathcal{N} \times \mathcal{M} \dashrightarrow \mathcal{M}$$

is contractible.

However using the fact that $\mathcal{N}$ is acyclic we can compute the E^1 term of the spectral sequence

$$E^1_{pq} = H_q(\mathcal{M}) \quad \text{for all } p.$$

Recall $d_i : \mathcal{N}_\times(\mathcal{N}')^p \times \mathcal{M} \longrightarrow \mathcal{N}_\times(\mathcal{N}')^{p-1} \times \mathcal{M}$

$$d_i(N_0, N_1, \dots, N_p, L) \longmapsto (N_0, \dots, N_i \amalg N_{i+1}, \dots, L)$$

so $d_i : E^1_{p*} \longrightarrow E^1_{p-1,*}$ is identity if $0 \leq i \leq p-1$
is mult. by e if $i = p$

Thus the E^1-term is

$$\begin{tikzcd}
{} \arrow[r] & H_*(\mathcal{M}) \arrow[r, "e"] \arrow[d, phantom, "\parallel"] & H_*(\mathcal{M}) \arrow[r, "id-e"] \arrow[d, phantom, "\parallel"] & H_*(\mathcal{M}) \arrow[r, dotted, "e"] \arrow[d, phantom, "\parallel"] & H_*(\mathcal{M}) \\
& E^1_{2*} & E^1_{1*} & E^1_{0*} &
\end{tikzcd}$$

Since e is idempotent it follows that the sequence is exact, so

$$E^2_{p*} = \begin{cases} 0 & p > 0 \\ e\,H_*(\mathcal{M}) & p = 0 \end{cases}$$

So the spectral sequence collapses and we get

$$H_*(\mathcal{M}) \xrightarrow{id-e} H_*(\mathcal{M}) \xrightarrow{e} H_*(\mathcal{M}) \longrightarrow 0$$

is exact. Algebraically this implies e is the identity.

Notice the preceding calculation of the E^2 term is completely independent of what $\mathcal{M}$ is. $\mathcal{M}$

can be any space on which $\mathcal{N}$ acts. This leads to

<u>Question:</u> Is the simplicial space

(*) $\quad \cdots \to \mathcal{N}\times\mathcal{N}'\times\mathcal{M} \rightrightarrows \mathcal{N}$[redacted]$\times\mathcal{M}$ [redacted]

[redacted] of the same <u>homology</u> as $(\mathcal{N}')^{-1}\mathcal{M}$ for a geometric reason?

<u>Possible proof.</u>

$$\begin{tikzcd}
 & & & \mathcal{M} \\
\mathcal{N}\times(\mathcal{N}')^2\times\mathcal{M} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \mathcal{N}\times\mathcal{N}'\times\mathcal{M} \arrow[r, shift left, "\perp\times id"] \arrow[r, shift right, "id\times\perp"'] \arrow[d, "pr_{23}"] & \mathcal{N}\times\mathcal{M} \arrow[ur, dashed, "heq"] \arrow[d, "pr_2"] & \\
(\mathcal{N}')^2\times\mathcal{M} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{N}'\times\mathcal{M} \arrow[r, shift left, "pr_2"] \arrow[r, shift right, "\perp"'] & \mathcal{M} \arrow[dr, dashed, "heq"] & \\
 & & & \langle\mathcal{N}';\mathcal{M}\rangle
\end{tikzcd}$$

$\leftarrow$ homology isos. because [redacted] $\tilde{H}_*(\mathcal{N})=0$

[redacted] But we know [redacted] by virtue of commutativity that on $\langle\mathcal{N}',\mathcal{M}\rangle$ the operation $M \longmapsto M\perp\theta(N)$ is homotopic to the identity. [redacted]

Next don't assume $\mathcal{N}$ is acyclic, but instead suppose $\bar{R} = H_*(\mathcal{N})$ is a ring with identity whence

$$R = H_*(\mathcal{N}') \simeq k \times \bar{R}$$

Now the E^1 term for

$$(*) \qquad \mathcal{N} \times (\mathcal{N}')^2 \times \mathcal{M} \rightrightarrows \mathcal{N} \times \mathcal{N}' \times \mathcal{M} \rightrightarrows \mathcal{N} \times \mathcal{M}$$

is

$$\bar{R} \otimes R \otimes R \otimes H_*(\mathcal{M}) \rightrightarrows \bar{R} \otimes R \otimes H_*(\mathcal{M}) \rightrightarrows \bar{R} \otimes H_*(\mathcal{M})$$

and I recognize this as a standard construction for computing Tor: So

$$E^2_{p*} = \operatorname{Tor}^R_p(\bar{R}, H_*(\mathcal{M})).$$

But $\bar{R} = eR$ is projective as an R-module, so

$$E^2_{p*} = \begin{cases} 0 & p \neq 0 \\ eH_*(\mathcal{M}) & p = 0 \end{cases}$$

So homologically at least I see that $(*)$ has the homology type of $\mathcal{N}^{-1}\mathcal{M}$.

Question again is whether the above argument can be made geometric. Possibility: Show that the two maps from $(*)$ to itself given by $x \mapsto x\cdot e$ on $\mathcal{M}$ and $y \mapsto e\cdot y$ on $\mathcal{N}$ are homotopic. Then use the fact you have stability for $\mathcal{N}$ [redacted].

<u>Question</u>: Can you relate the fact that

$$\overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} \mathcal{M}\times(\mathcal{M}') \rightrightarrows \mathcal{M}\times\mathcal{M} \dashrightarrow \mathcal{M}$$

is a heq with the space

(2) $$\overset{\rightarrow}{\rightrightarrows} \mathcal{M}'\times\mathcal{M} \rightrightarrows \mathcal{M}$$

(h-orbit of $\mathcal{M}'$ acting on $\mathcal{M}$). Is the latter contractible?

(2) is essentially the category $\langle \mathcal{M}', \mathcal{M} \rangle$ which consists of objects of $\mathcal{M}$ with $\rightarrowtail$ for morphisms. It is a monoid $\ni$ $X \perp X \leftarrowtail X$, hence I know at least that its homology is zero. In the case of countable infinite sets I know the category is contractible.

Can I use (2) to prove stability. Again I get a spectral sequence

$$E^1_{pq} = H_*(\mathcal{M}'^{p} \times (\mathcal{M}, \mathcal{M}a)) \Rightarrow 0$$

November 8, 1975

Consider the category consisting of all countable infinite sets N in which a map is either an injection $N \hookrightarrow N'$ with infinite complement, or an isomorphism. This is the category $\langle \mathcal{N}'; \mathcal{N} \rangle$. It is contractible by the cone construction

$$N \hookrightarrow N \amalg N_0 \hookleftarrow N_0 .$$

Consider also the simplicial groupoid which in degree p consists of an N in $\mathcal{N}$ equipped with a filtration

$$0 \le F_1 \le F_2 \le \cdots \le F_p < N$$

In degree 0 we get $0 < N$

In degree 1 we get $0 \le F_1 < N$ $\quad \xmapsto{d_0} 0 \le N - F_1$, $\quad \xmapsto{d_1} 0 \le N$

In degree 2 we get $0 \le F_1 \le F_2 < N$ $\quad \xmapsto{d_0} 0 \le F_2 - F_1 \le N - F_1$, $\quad \xmapsto{d_1} 0 \le F_2 < N$, $\quad \xmapsto{d_2} 0 \le F_1 < N$

$$(\mathcal{M}')^2 \times \mathcal{M} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{M}' \times \mathcal{M} \;\underset{d_1 = \perp}{\overset{d_0 = pr_2}{\rightrightarrows}}\; \mathcal{M} .$$

I have a functor from the simplicial cat. to $\langle \mathcal{M}'; \mathcal{M} \rangle$ sending $(0 \le F_1 \le \cdots \le F_p < N)$ to N. The over-fibre over Y consists of $0 \le F_1 \le \cdots \le F_p < N \le Y$?

November 8, 1975

Let $\mathcal{N}$ be the [redacted] groupoid of countable infinite sets and all isos. between them; let $\mathcal{N}' = pt \amalg \mathcal{N}$. Equip $\mathcal{N}'$ with the operation of disjoint union.

Form $\langle \mathcal{N}', \mathcal{N} \rangle$. The objects are those of $\mathcal{N}$, a morphism $N_1 \longrightarrow N_2$ is given by an isom. $N \amalg N_1 \xrightarrow{\sim} N_2$ modulo isos of $N \in \mathcal{N}'$. Thus a morphism in $\langle \mathcal{N}', \mathcal{N} \rangle$ is simply an injection whose complement is empty or infinite.

Claim $\langle \mathcal{N}', \mathcal{N} \rangle$ is contractible: Use the cone construction:

$$N \longrightarrow N \amalg N_0 \longleftarrow N_0$$

are maps in $\langle \mathcal{N}', \mathcal{N} \rangle$ and $N \longmapsto N \amalg N_0$ is a functor.

Next consider the [redacted] simplicial groupoid which in degree p consists of p-simplices

$$N_0 \to N_1 \longrightarrow \cdots \longrightarrow N_p$$

in $\langle \mathcal{N}', \mathcal{N} \rangle$ and their isomorphisms. The obvious augmentation to $\langle \mathcal{N}', \mathcal{N} \rangle$ is a homotopy equivalence; this would be true for any category. Picture of this simplicial category

$$\mathcal{N}\times(\mathcal{N}')^2 \;\substack{\xrightarrow{\perp\times id}\\ \xrightarrow{id\times\perp}\\ \xrightarrow[pr_{23}]{}}\; \mathcal{N}\times\mathcal{N}' \;\substack{\xrightarrow{\perp}\\ \xrightarrow[pr_1]{}}\; \mathcal{N}$$

$$(N_0, X_1, X_2) \qquad\qquad (N_0, X_1) \substack{\longmapsto N_0\perp X_1\\ \longmapsto N_0}$$

$$(N_0, X_1, X_2) \leftrightarrow (N_0 \to N_0\perp X_1 \to N_0\perp X_1\perp X_2)$$

$$(N_0, X_1) \leftrightarrow N_0 \to N_0 \perp X_1$$

We get a functor from this simplicial cat to itself by adding on the left a fixed object E of $\mathcal{N}$. It sends $N_0 \to \cdots \to N_p$ into $E\perp N_0 \to E\perp N_1 \to \cdots \to E\perp N_p$. This gives us a map

$$\begin{array}{ccccc}
\mathcal{N}\times(\mathcal{N}')^2 & \rightrightarrows & \mathcal{N}\times\mathcal{N}' & \rightrightarrows & \mathcal{N} \\
\downarrow{\scriptstyle (E\perp)\times id} & & \downarrow{\scriptstyle (E\perp)\times id} & & \downarrow{\scriptstyle (E\perp)} \\
\mathcal{N}\times(\mathcal{N}')^2 & \rightrightarrows & \mathcal{N}\times\mathcal{N}' & \rightrightarrows & \mathcal{N}
\end{array}$$

and hence a relative spectral sequence

$$E^1_{pq} = {}^{\text{Norm}}_{\wedge}H_q\big(\mathcal{N}\times\mathcal{N}'^p, (E\perp\mathcal{N})\times\mathcal{N}'^p\big) \Rightarrow 0$$

$$= {}^{\text{Norm}}_{\wedge}H_q\big((\mathcal{N}, E\perp\mathcal{N})\times\mathcal{N}'^p\big) = H_q\big((\mathcal{N}, E\perp\mathcal{N})\times\mathcal{N}'^p\big)$$

So now suppose $H_q(\mathcal{N}, E\perp\mathcal{N}) = 0$ for $q < n$.

whence $E^1_{pq} = 0 \quad q < r$ and we have

$$E^1_{pr} = H_r(\mathcal{N}, E\perp\mathcal{N})$$

Now
$$d_i : \mathcal{N}\times(\mathcal{N})^p \longrightarrow \mathcal{N}\times(\mathcal{N})^{p-1}$$
$$(N, X_1, \ldots, X_p) \longmapsto \begin{cases} N\perp X_1, X_2, \ldots, X_p & i=0 \\ N, X_1, \ldots X_i X_{i+1}, \ldots X_p & 1\leq i < p \\ N, X_1, \ldots, X_{p-1} & i = p \end{cases}$$

Because the iso [redacted] $H_r((\mathcal{N}, E\perp\mathcal{N})\times\mathcal{N}^p) = H_r(\mathcal{N}, E\perp\mathcal{N})$ is induced by a basepoint $pt \longrightarrow \mathcal{N}^p$, $\cdot \longmapsto \underbrace{E\perp\cdots\perp E}_{p}$ it follows that d_i on E^1_{pr} is the identity for $1\leq i\leq p$ and [redacted] for $i=0$ the map [redacted] induced by $N\longmapsto N\perp E$. Denote this by $\cdot e$. So the E^1_{*r}-term looks:

$$\underset{p=2}{H_r(\mathcal{N}, E\perp\mathcal{N})} \xrightarrow{\ \cdot e\ } \underset{p=1}{H_r(\mathcal{N}, E\perp\mathcal{N})} \xrightarrow{\ \cdot e - id\ } \underset{p=0}{H_r(\mathcal{N}, E\perp\mathcal{N})}$$

So $\cdot e$ is a projection on $H_r(\mathcal{N}, E\perp\mathcal{N})$ and $\cdot e - id$ is onto so $\cdot e = 0$. So we get nothing at all this way.

Let's try to understand H_1; try to show $\pi_1(\mathfrak{M}/\mathfrak{M}a) = 0$ geometrically. Hence we want to show that any covering of $\mathfrak{M}$ which is trivial over $\mathfrak{M}a$ is trivial. So let F be a covering, i.e. a functor from $\mathfrak{M}$ to Sets. I am assuming that if $N = N' \amalg N''$ with N', N'' infinite, then $\mathrm{Aut}(N')$ acts trivially on $F(N)$. It follows that $\mathrm{Aut}(N'')$ acts trivially also.

Now also I know that

$$\cdots \mathfrak{M} \times \mathfrak{M}' \times \mathfrak{M} \rightrightarrows \mathfrak{M} \times \mathfrak{M}$$

is heq to $\mathfrak{M}$. This means that to give a covering of $\mathfrak{M}$ is the same as giving a covering of $\mathfrak{M} \times \mathfrak{M}$ equipped with descent data. Specifically this means that if I give a functor F' on pairs N', N'' [redacted] together with isos.

$$F'(N' \amalg X, N'') \simeq F'(N', X \amalg N'')$$

satisfying some sort of transitivity, then $F'(N', N'')$ depends only on [redacted] $N' \amalg N''$. [redacted] Therefore it should follow that because $F(N)$ is acted on trivially by $\mathrm{Aut}(N') \times \mathrm{Aut}(N'')$, it is a trivial $\mathrm{Aut}(N)$-set.

Specifically the argument goes as follows: To show $g \in \mathrm{Aut}(N)$ acts trivially on $F(N)$: This depends only on the splitting $gN' \amalg gN'' = N$ which is a vertex x

in the contractible complex $X(N)$. So we choose a path $x_0, x_1, \ldots, x_n = x$ in the complex and put $x_i = g_1 \cdots g_i x_0$. So it is enough to worry about a g which gives rise to [redacted] a one-simplex in $X(N)$.

$$N = N_1 \amalg N_2 \amalg N_3$$

$$N = N_1 \amalg N_2 \amalg N_3$$

But I can choose [redacted] this isom. [redacted] to be the identity on an infinite subset of N_1.

November 10, 1975 (Jeanie is 35)

Fix A in $\mathcal{M}$. Assuming $\mathcal{M}$ is associative one has that the maps $\lambda_A(M) = A \perp M$, $\rho_A(M) = M \perp A$ commute, hence λ_A induces a map

$$\bar{\lambda}_A : \mathcal{M}/\mathcal{M}\perp A \longrightarrow \mathcal{M}/\mathcal{M}\perp A$$

where [redacted] $\mathcal{M}/\mathcal{M}\perp A$ denotes the [redacted] cone of ρ_A. I want to show that $\bar{\lambda}_A$ is null-homotopic using the commutativity isomorphism. ~~(doesn't work)~~

Recall properties of the cone. Given $f : X \to Y$, one puts $\mathrm{Cone}(f) = \mathrm{Cyl}(f)/X\times 0$ where $\mathrm{Cyl}(f) = X \times [0,1] \overset{f}{\underset{X\times 1}{\cup}} Y$.
Given

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g"'] \arrow[ur, phantom, "h"] & Y' \end{tikzcd}$$

h a homotopy $gf \Leftarrow f'g'$

one [redacted] constructs a map $\mathrm{Cyl}(f) \to \mathrm{Cyl}(f')$ as follows. First one [redacted] forms the comm. square

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "i_0"'] & X' \arrow[d, "f'"] \\ \mathrm{Cyl}(f) \arrow[r, "h+g"'] & Y' \end{tikzcd}$$

$$h+g : \begin{cases} y \longmapsto \\ (x,t) \longmapsto h_t(x) \end{cases}$$

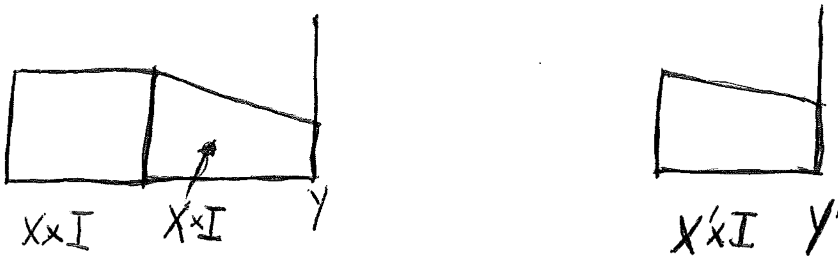
$h_0(x) = f'g'$

$h_1(x) = gf$

$X \boxed{\;\cdot\;} Y \xrightarrow{h+g} Y'$

Then one takes cylinders

$$\mathrm{Cyl}(i_0) \longrightarrow \mathrm{Cyl}(f')$$



and identifies $\mathrm{Cyl}(i_0)$ with $\mathrm{Cyl}(f)$.

If one has

$$\begin{tikzcd} X \arrow[r, "g_1'"] \arrow[d, "f"'] & X' \arrow[r, "g_2'"] \arrow[d, "f'"] \arrow[dl, phantom, "\swarrow h_1"] & X'' \arrow[d, "f''"] \arrow[dl, phantom, "\swarrow h_2"] \\ Y \arrow[r, "g_1"'] & Y' \arrow[r, "g_2"'] & Y'' \end{tikzcd}$$

then the map $\mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f') \longrightarrow \mathrm{Cyl}(f'')$ is homotopic to the map $\mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f'')$ associated to $g_{2*}h_1 + h_2 \cdot g_1'^{*} : g_2 g_1 f \Longrightarrow f'' g_2' g_1'$

I want to apply this to the following situation:

$$\begin{tikzcd} M \arrow[r, "\lambda_A"] \arrow[d, "\rho_A"'] & M \arrow[r, "id"] \arrow[d, "id"] \arrow[dl, phantom, "\swarrow h_1"] & M \arrow[d, "\rho_A"] \arrow[dl, phantom, "\swarrow h_2"] \\ M \arrow[r, "id"'] & M \arrow[r, "\lambda_A"'] & M \end{tikzcd}$$

Here h_1 is the homotopy from λ_A to ρ_A furnishes by the commutativity isom. $\theta : A \perp X \xrightarrow{\sim} X \perp A$. h_2 is furnished by the inverse of θ. What is the composition homotopy from $\rho_A \circ id \circ \lambda_A$ to $\lambda_A \circ id \circ \rho_A$.

$$(\rho_A \; id \; \lambda_A)(X) = (A \perp X) \perp A$$

$$\simeq \; (\lambda_A \; id \; \lambda_A)(X) = A \perp (A \perp X) \qquad (\text{by } h_2^{-1}{}_A)$$

$$\simeq \; (\lambda_A \; id \; \rho_A)(X) = A \perp (X \perp A) \qquad \text{by } \lambda_A \cdot h_2$$

~~However the coherence axiom for commutativity says that~~

$$\begin{tikzcd}
A \perp X \perp B \arrow[r, "\sim"] \arrow[d, "\wr"] & X \perp A \perp B \arrow[dl, "\sim"] \\
X \perp B \perp A &
\end{tikzcd}$$

~~commutes. The isomorphism I'm after is the special case when $B = A$ of~~

$$\cancel{A \perp X \perp B \simeq B \perp (A \perp X) \simeq B \perp (X \perp A)}$$

Unfortunately this automorphism of $A \perp X \perp A$ is not the identity:

$$(a, x, a') \longmapsto (a', x, a)$$

This argument does show that $(\bar{\lambda}_A)^2$ is null-homotopic, because let γ be the endo map of Cone ρ_A that we have defined above using the comm. isom. Then it's clear that γ^2 is the same as $(\bar{\lambda}_A)^2$ because the problem with the commutativity isomorphism is of order 2. On the other hand, γ is null-homotopic because γ factors thru Cone(id_M).

Since $A^2 \simeq A$ one sees that $\bar{\lambda}_A$ is idempotent $\bar{\lambda}_A^2 = \bar{\lambda}_A$ hence null-homotopic.

Program: I want to give a geometric proof that $M/M \perp A$ is contractible; the point is not to use spectral sequences but instead to see what spaces actually occur.

Return to the space

$$\overset{\to}{\underset{\to}{\to}} M \times M' \times M \rightrightarrows M \times M \dashrightarrow M$$

and form the cone on the map f_A:

$$(M \times (M')^2) \wedge (M/Ma) \Rrightarrow (M \times M') \wedge (M/Ma) \rightrightarrows M \wedge (M/Ma)$$

This space has an augmentation to M/Ma which is a heq. Introduce the skeleta of the realization. Let Y be the realization of the above simplicial space and let F_pY be its skeleta: Then

$$F_pY/F_{p-1}Y = Y_p/Y_p^{deg} \wedge \Delta(p)/\partial\Delta(p)$$
$$= (M^{p+1} \cup pt) \wedge (M/Ma) \wedge S^p$$

Assume I know that $H_*(M/Ma)$ begins in degree r; then $H_*(F_pY/F_{p-1}Y)$ begins in degree $p+r$. Hence

$$H_r(F_0Y) \twoheadrightarrow H_r(F_1Y) \xrightarrow{\sim} H_r(F_2Y) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H_r(Y)$$

But $F_0Y = (M \cup pt) \wedge (M/Ma)$ and M is connected, so $H_r(M/Ma) = H_r(F_0Y)$. So

$$H_r(M/Ma) \twoheadrightarrow H_r(Y) = H_r(M/Ma)$$

But on the other hand I know this map is zero. $\therefore$ $H_*(M/Ma)$ begins in degree $r+1$. etc.

I have a similarity in the preceding with Cech cohomology in sheaf theory. I should go over the latter.

So let X be a space. Given a presheaf F and a covering $\mathcal{U}$ of X I get a complex

$$C^*(\mathcal{U}, F)$$

whose homology groups one denotes $H^*(\mathcal{U}, F)$. Then

$$\check{H}^*(X, F) = \varinjlim_{\mathcal{U}} H^*(\mathcal{U}, F).$$

is the Cech cohomology.

What is the nature of $C^*(\mathcal{U}, F)$? A presheaf is a functor on the category of open sets. Associate to $\mathcal{U}$ the crible R consisting of open sets contained in members of $\mathcal{U}$. I claim that

$$H^*(\mathcal{U}, F) = R^* \varprojlim_{R} (F).$$

I have to check the effaceability. To put it another way, I can show that if $\{U_i \mid i \in I\}$ is a family of object covering R such that all fibre products exist: $U_{i_1} \times \cdots \times U_{i_p}$, then the nerve:

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; \coprod_{i_0, i_1 \in I} U_{i_0} \times U_{i_1} \rightrightarrows \coprod_{i_0 \in I} U_{i_0}$$

is acyclic. This is clear.

So I see that $H^*(\mathcal{U}, F)$ is [redacted] just the cohomology of the presheaf F pulled back to the subcat. $R(\mathcal{U})$ of $Open(X)$.

$$R(\mathcal{U}) \xhookrightarrow{i} Open(X)$$

$$R(\mathcal{U})^\wedge \underset{i^*}{\overset{i_!}{\rightleftarrows}} Open(X)^\wedge$$

$$(i_! F)(V) = \varinjlim_{\substack{U \in R(\mathcal{U}) \\ \ni V \subset U}} F(U)$$

$$= \begin{cases} 0 & V \notin R(\mathcal{U}) \\ F(V) & V \in R(\mathcal{U}) \end{cases}$$

Since $i_!$ [redacted] is exact, it follows that i^* [redacted] preserves injectives:

<u>Lemma</u>: If F is an injective presheaf, then F restricted to any crible R is also injective.

Alternative approach to stability. Let me [redacted] consider the analogue of the unimodular complex. [redacted] Fix N in $\mathcal{N}$ and consider the set of all embeddings $u : \mathbb{N} \hookrightarrow N$ with infinite complement. Make these into a simplicial complex by calling $(u_0, \ldots, u_p)$ a simplex if $u_0\mathbb{N} \amalg \cdots \amalg u_p\mathbb{N}$ embeds in N with infinite complement [redacted] If the unimodular complex is contractible, then I get an acyclic complex

$$\cdots \longrightarrow \bigoplus_{(u_0,u_1)} \mathbb{Z} \longrightarrow \bigoplus_{u_0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

(this is not the complex of chains on the simplicial complex, but it should still be acyclic). This complex should furnish a spectral sequence

$$E^1_{pq} = H_q(G)$$

with [redacted] each d_i multiplication by a. Thus

$$d_1 = \begin{cases} \text{mult by } a \text{ if } [redacted]\, p \text{ is odd} \\ 0 \quad \text{if } p \text{ is } [redacted] \text{ even.} \end{cases}$$

[redacted] Then I can use induction: If $H_q(G) = 0$, $1 \le q < r$ then $E^2_{0,r} = \mathrm{Coker}\,\{H_r(G) \xrightarrow{a} H_r(G)\} = 0$, so a is onto.

Also $E^2_{1,n} = \mathrm{Ker}\{H_n(G) \xrightarrow{a_\cdot} H_n(G)\}$ is zero, so $a_\cdot$ is injective.

Suppose we try to prove contractibility of the unimodular complex by a variant of Kervaire-Lusztig. Let F be a finite subcomplex. I want to regard a simplex in the unimodular complex as giving me a partition of N into infinite pieces. Take all the partitions of N associated to the simplices of F and form their infimum, i.e. the finite non-empty intersections. Now throw away the finite sets in this partition. Then one gets $N = A_1 \sqcup \cdots \sqcup A_m \sqcup$ (finite set) such that for any simplex σ each A_i is a subset of one of the blocks of σ.

Let F_i be the subcomplex of $X(N)$ consisting of simplices $\alpha: \mathbb{N}^p \hookrightarrow N$ such that A_i is in the complement of this embedding. Choose $v_i: \mathbb{N} \hookrightarrow A_i$ with inf. complement for each $i = 1, \ldots, m$. Then the vertex v_i can be joined to F_i. Furthermore the simplex $\{v_{i_1}, \ldots, v_{i_p}\}$ can be joined to

$F_{i_1} \cap \dots \cap F_{i_p}$ for [redacted] $1 \le i_1 < \dots < i_p \le m$. It follows (I think) that $\bigcup_{i=1}^{m} F_i$ [redacted] can be contracted to a point in $X(N)$.

Furthermore, I claim $F \subset \bigcup_{i=1}^{m} F_i$. In effect given [redacted] $u : N^p \hookrightarrow N$, we know each A_i is contained in $\operatorname{Im} u$ or in the complement of u, and not all A_i are contained in $\operatorname{Im}(u)$ as $N - \bigcup A_i$ is finite. Hence $u \in F_i$ for some i.

Thus it is clear that the unimodular complex is contractible, and so we again get a stability result.

Let S be a set. Form the complex

$$\cdots \longrightarrow \coprod_{(s_0,s_1,s_2)} \mathbb{Z} \longrightarrow \coprod_{(s_0,s_1)} \mathbb{Z} \longrightarrow \coprod_{s_0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the sum is taken over the sequences $(s_0, \dots, s_k)$ of distinct elements of S. What is the homology of this complex? Let's use the Kervaire-Lusztig argument. Let F_s be the subcomplex formed using the set $S - \{s\}$. We try to show that any cycle z [redacted] with support in $F_1 \cup \dots \cup F_p$ is homologous to one in $F_1 \cup \dots \cup F_{p-1}$. For example let z have support in F_1. Here $S = \{1, \dots, n\}$.

Then let T_1 be the cone operator

$$T_1(s_0,\dots,s_m) = (1, s_0,\dots,s_m)$$

so that $dT_1 + T_1 d = id$. More precisely $T_1 : F_1 \to C$ and

$$dT_1(s_0,\dots,s_m) = (s_0,\dots,s_m) - \sum(-1)^i(1, s_0, \hat{s_i}, s_m)$$

$$\therefore \; dT_1 + T_1 d = \text{inclusion } F_1 \hookrightarrow C.$$

So if z is a cycle in F_1, then $dT_1 z = z$, so z is homologous to zero.

Let z be a cycle in $F_1 + \cdots + F_p$ and write

$$z = u - v$$

with $u \in F_1 + \cdots + F_{p-1}$ and v in F_p. Then $du = dv \in (F_1 + \cdots + F_{p-1}) \cap F_p$. Consider

$$d(T_p v) + T_p(dv) = v$$

Then

$$z = u - v = u - T_p(dv) - d(T_p v)$$
$$\sim u - T_p(dv)$$

so all that remains is to show $T_p(dv) \in F_1 + \cdots + F_{p-1}$, i.e. that $T_p((F_1 + \cdots + F_{p-1}) \cap F_p) \subset F_1 + \cdots + F_{p-1}$. for let $(s_0,\dots,s_k) \in F_i \cap F_p$. Then $T_p(s_0,\dots,s_k) = (p, s_0,\dots,s_k)$ still belongs to F_i. Thus everything works, and we have proved:

<u>Prop:</u> The complex

$$\to \bigoplus_{(s_1,\dots,s_q)} \mathbb{Z} \to \cdots\cdots \to \bigoplus_{(s_1,s_2)} \mathbb{Z} \to \bigoplus_{s_0} \mathbb{Z} \to \mathbb{Z} \to 0$$

where $(s_1,\dots,s_q)$ runs over sequences of distinct elements of S is acyclic except in dimension $n = \mathrm{card}(S)$.

Consider the following poset J attached to a set S. The elements of J are finite sequences $(s_1,\dots,s_q)$ $q \geq 1$ of <u>distinct</u> elements of S. In other words embeddings $u:\{1,\dots,q\} \hookrightarrow S$. [redacted] We have [redacted] $(s_1,\dots,s_q) \leq (t_1,\dots,t_p)$ iff $\exists$ $1 \leq i_1 < i_2 < \cdots < i_q \leq p$ such that $s_j = t_{i_j}$, $1 \leq j \leq q$. [redacted] In other words [redacted] $s:\{1,\dots,q\} \hookrightarrow S$ is $\leq$ $t:\{1,\dots,p\} \hookrightarrow S$ iff there exists a monotone inj map $i:\{1,\dots,q\} \hookrightarrow \{1,\dots,p\}$ such that

$$\begin{tikzcd}
\{1,\dots,q\} \arrow[rr, hook, "i"] \arrow[dr, "s"'] & & \{1,\dots,p\} \arrow[dl, "t"] \\
& S &
\end{tikzcd}$$

commutes. Note that i is unique if it exists.

[redacted]

Next note that for any element σ of J the poset $\{\tau \mid \tau < \sigma\}$ is isomorphic to the set of proper subsets of $\{1,\dots,q\}$ if $q =$ size of σ. Thus [redacted] it should be so that the complex in the proposition is the Lusztig sequence associated to the poset J, hence J should be spherical.

(But now recall what you were doing about stability for the symmetric groups. I formed a category C of pairs (S_1, S_2) of finite sets of same card such that a map $(S_1', S_2') \longrightarrow (S_1, S_2)$ consists of a pair of injections + an isomorphism between the complements. Now you wanted to see what the relative terms were if you filtered by size.

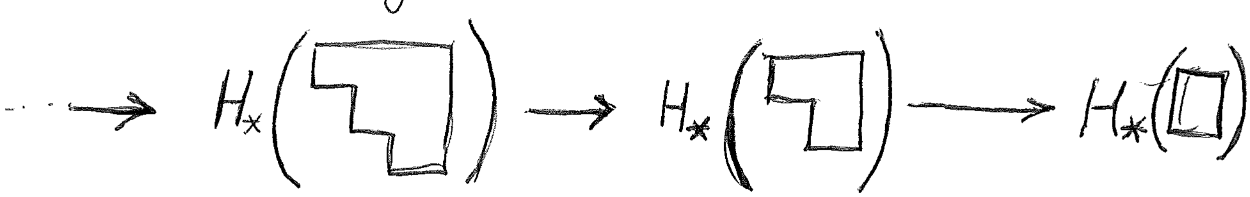
Thus fix (S,S) and you want to calculate $C/(S,S)$, which can be identified with the poset consisting of a pairs of splittings

$$S = S_1' \amalg S_1''$$
$$S = S_1'' \amalg S_2''$$

together with an isomorphism $S_1'' \simeq S_2''$. Also we want $S_2'' \simeq S_1'' \neq \phi$. Thus this poset is not the same as J above.)

<u>November 12, 1975</u>

Let V be a vector space over k of dim. $\aleph_0$. Let $X(V)$ be the poset [redacted] consisting of subspaces of inf. dim and codim, in which $W_1 < W_2$ iff $W_1 \subset W_2$ and W_2/W_1 has inf. dim. I have seen that $X(V)$ is contractible. $X(V)$ gives a spectral sequence [redacted] abutting to 0 with E^1-term



Let V_0 be an elt of $X(V)$ and let $Y(V, V_0)$ denote the subposet of $X(V)$ consisting of subspaces W such that $W + V_0 = V$ and $W \cap V_0 \in X(V_0)$.
(Thus

$$\begin{tikzcd} & V \arrow[dl, dash, "\infty"'] \arrow[dr, dash] & \\ W \arrow[dr, dash] & & V_0 \arrow[dl, dash, "\infty"] \\ & W \cap V_0 \arrow[d, dash, "\infty"] & \\ & 0 & \end{tikzcd}$$

.)

I let $\mathrm{Aut}(V \text{ over } V/V_0) = \left(\begin{array}{c|c} \mathrm{id}_{V/V_0} & \\ \hline * & \mathrm{Aut}(V_0) \end{array}\right)$ act on $Y(V,V_0)$ and I get a spectral seq. abutting to zero with E^1 term:

$$H_*\left(\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline * & * & * & * \\ \hline & & * & * \\ \hline & & & * \\ \hline\end{array}\right) \longrightarrow H_*\left(\begin{array}{|c|c|c|}\hline 1 & & \\ \hline * & * & * \\ \hline & & * \\ \hline\end{array}\right) \longrightarrow H_*\left(\begin{array}{|c|c|}\hline 1 & \\ \hline * & * \\ \hline\end{array}\right)$$

suppose I try now to [redacted] understand H_1.

$$H_1\left(\,\square\,\right) \twoheadleftarrow H_1\begin{pmatrix} \boxed{*} & 0 \\ & 1 \end{pmatrix} \oplus H_1\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

$$H_1\left(\begin{array}{c|c} 1 & \\ \hline * & * \end{array}\right) \twoheadleftarrow H_1\left(\begin{array}{c|cc} 1 & & \\ \hline * & * & 0 \\ 0 & 0 & 1 \end{array}\right) \oplus H_1\left(\begin{array}{c|cc} 1 & & \\ \hline 0 & 1 & * \\ & & * \end{array}\right)$$

Now stably I should know that

$$H_q\left(\begin{array}{c|c} 1 & 0 \\ \hline * & * \end{array}\right) \longleftarrow H_q\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline * & * & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

is the zero map, because I have ∞ sums. Thus I find that

$$H_1\left(\begin{array}{c|c} 1 & 0 \\ \hline * & * \end{array}\right) \hookleftarrow H_1\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right) \twoheadleftarrow H_1\left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 1 & * \\ & & * \end{array}\right)$$

is onto giving me [redacted] $H_1\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & * \end{array}\right) = H_*\left(\begin{array}{c|c} 1 & \\ \hline * & * \end{array}\right)$
as well as $H_1\begin{pmatrix} 1 & * \\ & * \end{pmatrix} \twoheadrightarrow H_1(*)$

$\therefore\ H_1\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \twoheadrightarrow H_1(*)$ implying $H_1(*) = H_1\left(\begin{array}{c|c} 1 & 0 \\ \hline * & * \end{array}\right) = 0.$

Therefore it seems that I get the Kuiper theorem at least over a field.

So back to stability for Σ_n. I have seen that the map

$$H_*(\Sigma_n, \Sigma_{n-1} \perp a) \xrightarrow{a\cdot} H_*(\Sigma_{n+1}, \Sigma_n \perp a)$$

when iterated has square zero. *Hypothesis:* The image of the preceding map is killed by 2. Thus if we invert 2 this map is zero. Let's see if I can establish linear stability with this hyp. NO

So I will use the complex

$$0 \to J_n \to \bigoplus_{(s_1,\dots,s_n)} \mathbb{Z} \to \cdots\cdots \to \bigoplus_{s_1} \mathbb{Z} \to \mathbb{Z} \to 0$$

on which Σ_n acts, and the corresponding one for Σ_{n+1}.

$$\text{junk} \to \text{junk} \to H_*(\Sigma_1, \Sigma_0^a) \to \cdots\cdots \to H_*(\Sigma_n, \Sigma_{n-1}^a) \to H_*(\Sigma_{n+1}, \Sigma_n^a)$$

$$\qquad\qquad\qquad n \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$$

To get $E^2_{0,r} = H_r(\Sigma_{n+1}, \Sigma_n^a) = 0$ we need to know $H_{r-1}(\Sigma_{n-1}, \Sigma_{n-2}) = 0$. Doesn't work

~~Let A be a ring let M be an A-module. The [redacted] poset of frames of M, denoted Fr(M), consists of unimodular sequences $(u_1, \ldots, u_p)$ in M with the inclusion ordering. I assume that A is such that~~

Let's axiomatize the arguments a little.

If A is a ring let $F(A)$ be the poset consisting of sequences $(a_1, \ldots, a_p)$, $p \geq 1$ in A such that there exists an element a_{p+1} such that

$$A^{p+1} \xrightarrow{\sim} A, \quad (x_1, \ldots, x_{p+1}) \mapsto \sum x_i a_i$$

Assume $F(A)$ is contractible. Then it gives [redacted] me a Lusztig sequence [redacted]

$$\longrightarrow \bigoplus_{(a_1,a_2)} \mathbb{Z} \longrightarrow \bigoplus_{a_1} \mathbb{Z} \to \mathbb{Z} \longrightarrow 0$$

and hence a spectral sequence abutting to 0 with

$$E^1_{pq} = \begin{cases} H_q\left(\begin{array}{c|c} 1 & * \\ \hline & G \end{array}\right) & p > 0 \\ H_q(G) & p = 0. \end{cases}$$

If $H_*\left(\begin{array}{c|c} 1 & * \\ \hline & G \end{array}\right) = H_*(G)$, then d_1 becomes 0 in even degrees, mult. by a in odd degrees, so we get stability easily.

November 17, 1975.

Let S be a finite set of card n. Let $Y_2(S)$ be the simplicial complex whose p-simplices are $\{v_0, \dots, v_p\}$, where v_i is a card 2 subset of S and $v_0, \dots, v_p$ are disjoint. Then $\dim Y_2(S) = \left[\frac{n}{2}\right] - 1$. I would like to show that $Y_2(S)$ has the homotopy type of a bouquet of spheres of dimension $\left[\frac{n}{2}\right]-1$.

Fix $s_0 \in S$ and let $S' = S - \{s_0\}$. One obtains $Y_2(S')$ from $Y_2(S)$ by removing simplices containing a vertex v with $v = \{s, s_0\}$, $s \in S'$. Thus

$$Y_2(S) = \bigcup_{s \in S'} Y_2(S') \underset{\text{Link}\{s,s_0\}}{\cup} \text{Cone Link}\{s,s_0\}$$

Moreover $\text{Link}\{s,s_0\} = Y_2\left(S' - \{s\}\right)$. Now if $n \overset{=2m+1}{}$ is odd, we know that $Y_2(S')$ is a bouquet of $m-1$ spheres, and $Y_2(S'-\{s\})$ is a bouquet of $(m-2)$-spheres. Thus $Y_2(S)$ will be a bouquet of $m-1 = \left[\frac{n}{2}\right]-1$ spheres. The critical case then is when $n = \text{card } S$ is even $= 2m$.

Suppose $n = 2m$. We know $Y_2(S')$ is a bouquet of $(m-2)$-spheres and also the same is true for $Y_2(S'-\{s\})$. We have

$$\overset{\overset{0}{\parallel}}{\tilde{H}_{m-1}(Y_2(S'))} \longrightarrow \tilde{H}_{m-1}(Y_2(S)) \longrightarrow \bigoplus_{s} \tilde{H}_{m-2}(Y_2(S'-s))$$

$$\xrightarrow{\partial} \tilde{H}_{m-2}(Y_2(S')) \longrightarrow \tilde{H}_{m-2}(Y_2(S)) \longrightarrow 0$$

Thus what I need to know is that ∂ is surjective. Now the preceding argument showed that

$$\tilde{H}_{m-2}(Y(\underset{2m-1}{S'})) \cong \tilde{H}_{m-2}(Y(\underset{2m-2}{S''})) \oplus \bigoplus_{s\in S''} \tilde{H}_{m-3}(Y(\underset{2m-3}{S''-s}))$$

It appears therefore that the correct inductive hypothesis involves surjectivity of

$$\bigoplus_{s\in S} \tilde{H}_{m-2}(Y_2(S-s)) \longrightarrow \tilde{H}_{m-2}(Y_2(S))$$

if card $S = 2m-1$. Granting this we see that $\tilde{H}_*(Y(S))$ is concentrated in degree $m-1$ if $n=2m$. Now suppose $n = 2m+1$, whence we have

$$0 \to \tilde{H}_{m-1}(Y_2(\underbrace{S'}_{2m})) \to \tilde{H}_{m-1}(Y_2(\underbrace{S}_{2m+1})) \to \bigoplus_{s\in S'} \tilde{H}_{m-2}(Y_2(\underbrace{S'-s}_{2m-1})) \to 0$$

Applying induction hypothesis to $Y_2(S'-s)$ we know that $\tilde{H}_{m-2}(Y_2(S'-s))$ is generated by $\tilde{H}_{m-2}(Y_2(S'-s-t))$. Thus

$$\begin{tikzcd}
\displaystyle\bigoplus_{\substack{s\in S-\{s_0\}\\ t\in S-\{s_0,s\}}} \tilde{H}_{m-2}\big(Y_2(S-\{s_0,s,t\})\big) \arrow[d, dashed] \arrow[dr] & \\
H_{m-1}\big(Y_2(S)\big) \arrow[r] & \displaystyle\bigoplus_{s\in S-s_0} \tilde{H}_{m-2}\big(Y_2(S-\{s_0,s\})\big)
\end{tikzcd}$$

Let's guess that the dotted arrow arises by means of the map

$$\Sigma' Y_2(S-\{s_0,s,t\}) \longrightarrow Y(S)$$

one obtains from the contractions using the vertices $\{s_0,s\}$ and $\{s,t\}$.

<u>November 19, 1975.</u>

This just doesn't work. If $\operatorname{card}(S)=4$, then the simplicial complex $Y_2(S)$ is a disjoint union of 1-simplices, hence it is not connected.

November 19, 1975

Recall Segal's funny idea about group-completion. He takes a free $^{(top)}$ monoid M and constructs $M[a^{-1}]$. Actually it is enough to construct $Ma^{-\infty}M$ for this has the correct homology and probably the correct fundamental group.

First step: analyze $Ma^{-1}M$. We have cocart. square

$$\begin{tikzcd}
(Ma \times M) \cup (M \times aM) \arrow[r, phantom, "\subset"] \arrow[d] & M \times M \arrow[d] \\
M \arrow[r, phantom, "\subset"] & Ma^{-1}M
\end{tikzcd}$$

Now let's find a category having the homotopy type of $Ma^{-1}M$.

Recall that if G is a group with subgroups G_1, G_2 we know how to interpret the space

$$BG_1 \overset{B(G_1 \cap G_2)}{\cup} BG_2$$

as a category, namely, as the fibred category over G with fibre the poset of left cosets for the family G_1, G_{12}, G_2 of G.

So it seems that the category I seek consists of finite sets E with autos, and pairs (E,F) with autos. A morphism $E \longrightarrow (F_1, F_2)$ should consist of a reduction of (F_1, F_2) to $(MaxM) \cup (M \times aM)$ (which means we fix an element of F_1 or F_2 or both) and an isomorphism of E with $F_1 \amalg F_2$ minus this element. This won't work very simply.

November 20, 1975

Form the category $\mathcal{C}$ consisting of triples (E, k, F) where E, F are (say sets) and k is an integer. A map $(E, k, F) \longleftarrow (E', k', F')$ consists of a pair of isos.

$$E \simeq E' \oplus A^{\mu}$$

$$F \simeq A^{\nu} \oplus F'$$

such that $\mu + k' + \nu = k$. Think of (E, k, F) as $E \oplus A^{-k} \oplus F$; $E' \oplus A^{-k'} \oplus F' = E' \oplus A^{\mu} \oplus A^{-\mu - k' - \nu} \oplus A^{\nu} \oplus F' = E \oplus A^{-k} \oplus F)$. What is the homotopy type of $\mathcal{C}$.

$$\{(E,k)\} \times \{(\ell, F)\} \longrightarrow \mathcal{C}$$

$$(E,k), (\ell, F) \longmapsto (E, k+\ell, F)$$

Here (E,k) denotes the fibred category over the ordered set $\mathbb{Z}$ associated to the functor

$$k' \leq k \longmapsto \left(E \longmapsto E \oplus A^{k-k'}\right)$$

$\{(\ell, F)\}$ is similarly defined. Call this functor f. $(E, m, F)\backslash f$ consists of $((E',k'), (\ell',F'))$ plus isos.

$$E \oplus A^{\nu} \xrightarrow{\sim} E' \qquad \nu + m + \mu = k' + \ell'$$
$$F \oplus A^{\mu} \xrightarrow{\sim} F'$$

Thus I should be able to identify $(E,m,F)\backslash f$ with (ν, μ, k', ℓ') such that $\nu + m + \mu = k' + \ell'$, with

$$(\nu_1, \mu_1, k'_1, \ell'_1) \leq (\nu_2, \mu_2, k'_2, \ell'_2)$$

meaning
$$\nu_2 - \nu_1 = k'_2 - k'_1 \geq 0$$
$$\mu_2 - \mu_1 = \ell'_2 - \ell'_1 \geq 0. \qquad \text{doesn't work.}$$

<u>Conjecture</u>: $\mathcal{C}$ heq to $\{(E,k)\} \times \{(\ell,F)\}$.

Assume this for now. Inside of $\mathcal{C}$ we have the full subcategory $\mathcal{C}_{\leq 0}$ consisting of (E,k,F) with $k \leq 0$. To (E,k,F) in $\mathcal{C}_{\leq 0}$ we can associate the object

$$E \oplus A^{-k} \oplus F$$

of M. Moreover to the arrow $(E', k', F') \longrightarrow (E, k, F)$ in $\mathcal{C}_{\leq 0}$ given by

$$E' \oplus A^{\mu} \simeq E \qquad \mu + k' + \nu = k$$
$$A^{\nu} \oplus F' \simeq F$$

[redacted]

I can associate the isomorphism

$$E' \oplus A^{-k'} \oplus F' = E' \oplus A^{\mu} \oplus A^{-\mu-k'-\nu} \oplus A^{\nu} \oplus F'$$
$$\simeq \quad E \oplus A^{-k} \oplus F$$

Thus I have a functor from $\mathcal{C}_{\leq 0}$ to $\mathcal{M}$ which carries arrows into isomorphisms.

<u>Question:</u> Fix an object M of $\mathcal{M}$ and look at $\mathcal{C}_{\leq 0}/M$. What is this?

Objects of $\mathcal{C}_{\leq 0}/M$ can be identified with decompositions of M:

$$M = E \oplus A^{[redacted]p} \oplus F$$

a[redacted] morphism $(M = E' \oplus A^{p'} \oplus F') \longrightarrow (M = E \oplus A^{p} \oplus F)$ consists of $E' \oplus A^{\mu} = E$, [redacted] $A^{\nu} \oplus F'$ [redacted] $\simeq F$, $\mu + (-p') +$ [redacted]$^{\nu} = -p$ or $\mu + p + \nu = p'$. What this gadget is is clear – it is a poset of layers in M such that the layer has some sort of additional structure, i.e. reduction to a ordered free module.

Note that if $E, F \in$ ~~[illegible]~~ point category, then $\mathcal{C}$ is $\mathbb{N} \times \mathbb{N}$ acting on $\mathbb{Z}$ which is <u>not</u> homotopy equivalent to $\mathbb{N}$ acting on $\mathbb{Z} \times \mathbb{N}$ acting on $\mathbb{Z}$

so the conjecture is wrong. However we have a functor

$$\mathcal{C} \longrightarrow \langle \mathbb{N} \times \mathbb{N}, \mathbb{Z} \rangle$$
$$(E, k, F) \longmapsto k$$

with fibres $\mathcal{M} \times \mathcal{M}$

November 23, 1975

Let M be a free simplicial monoid, and let G be the associated simplicial group. I want to construct a category realizing G.

Look at $M \cdot (M)^{-1} \subset G$. We have a canonical reduced word description of the elements of G. Elements of M are written $m = s_1 \cdots s_k$, $s_i \in S$ the generating set for M. If

$$m(m')^{-1} = s_1 \cdots s_k \, s_{k+1}^{-1} \cdots s_n^{-1}$$

can be reduced there has to be cancellation. So if we form the nerve [redacted]

$$\begin{matrix}\rightarrow\\\rightarrow\\\rightarrow\end{matrix} (M \times M) \times \Delta M \rightrightarrows M \times M$$

of the category defined by ΔM acting by right multiplication on $M \times M$, then this nerve ought to be heq to $M \cdot M^{-1}$.

Next look at $M \cdot (M)^{-1} \cdot M$. I start with $M \overset{M}{\times} M^{-1} \overset{M}{\times} M$ which suffices to describe what's happening with reduced words of the form $s_1 \cdots s_\ell \, s_{\ell+1}^{-1} \cdots s_m^{-1} \, s_{m+1} \cdots s_n$ with $\ell < m$. Then I have to work in what happens when $\ell = m$.

We have a functor

$$M \times M \longrightarrow M \overset{M}{\times} M^{-1} \overset{M}{\times} M$$

sending $(m, m') \longmapsto (m, e, m')$ and we also have the product functor $M \times M \longrightarrow M$. So the conjecture is that the diagram

$$\begin{tikzcd} M \times M \arrow[r] \arrow[d] & M \times_M M^{-1} \times_M \text{[redacted]}\, M \arrow[d] \\ M \arrow[r] & M \cdot M^{-1} \cdot M \end{tikzcd}$$

is homotopy-cocartesian.

Take $M = \mathbb{N}$.

$\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ modulo $\underset{x}{\mathbb{N}} \times \underset{y}{\mathbb{N}}$ acting by

$$(x, y) \cdot (m, n, p) = (m + x,\ x + n + y,\ p + y)$$

November 30, 1975

Here's a crucial point where algebraic and topological K-theory differ. Topologically: $K^{-1}(X)$ is the Grothendieck group formed out of couples (E, θ) where $E \in P_X$ and $\theta \in \mathrm{Aut}(E)$; one introduces relations coming from exact sequences, homotopies of the auto. θ, and also $(E, \mathrm{id}_E) \mapsto 0$.

Now suppose we were to try to do the same thing in algebraic K-theory. We form the Grothendieck group of couples (E, θ). These are the same thing as $A[T, T^{-1}]$-modules which are finite and flat over A. If A is a field this K-theory is a direct sum:

Let $F = \bar{F}$ be alg. closed. Then

$$K_*(\mathrm{Mod\,ftors}\, F[T,T^{-1}]) = \bigoplus_{\lambda \in F^\bullet} K_*(F) = \mathbb{Z}[F^\bullet] \otimes K_*(F)$$

So far we have considered only the relations coming from exact sequences among the couples (E, θ). So how might we handle homotopies?

Try $F^\bullet \overset{L}{\otimes} K_*(F)$. Ignoring uniquely divisible stuff, this is

$$\mathrm{Tor}_1(F^\bullet, K_*(F)) = \mathrm{Tor}_1(\mu_\infty, K_*(F)).$$

In degree n, it is $\mathrm{Tor}_1(F^\bullet, K_{n-1}(F)) = K_{n-1}(F)(\pm 1)$ and I want it to be something like $K_n^{-1}(F) = K_{n+1}(F)$, hence I am not very far away from periodicity.

So in some funny way what I have to do is to formulate some sort of algebraic $K_*^{-1}(A)$ which is to be constructed out of (E, θ) and satisfies $K_*^{-1}(A) \cong K_{*+1}(A)$. This should be what Vilodin + Wagoner have done. Then I will want to relate $K_*^{-1}(F)$ to $K_*(F) \overset{L}{\otimes} (F^\bullet)$ with torsion coefficients present. At this point we will get some sort of periodicity.

Now what has all this to do with topological periodicity?

Funny thing is that if I, instead of couples (E, θ), consider $P \in \mathcal{P}_{A[T,T^{-1}]}$, then the exact sequence relations give me the gps

$$K_n(A[T,T^{-1}])$$

so if I further kill the direct summand K_nA coming from $T=1$ I get

$$K_n(A[T,T^{-1}])/K_nA = K_{n-1}A \qquad \text{if } A \text{ reg.}$$

Loday's pairing:

$$\mathrm{Rep}(G, \mathcal{P}_A) \times \mathcal{P}_{A[G]} \longrightarrow \mathcal{P}_A$$

$$(V \;, M) \longmapsto V \otimes_{A[G]} M$$

Perhaps this induces a map

$$K_i(BG;A) \otimes K_j(A[G]) \longrightarrow K_{i+j}(A)$$

Take $G = \mathbb{Z}$. Then

$$K_i(B\mathbb{Z}, A) = K_{-i}(A) \oplus K_{i+1}(A)$$

$$K_i(A[T,T^{-1}]) = K_i(A) \oplus K_{i-1}(A)$$

Thus the map is probably the usual cup product.

Notice how things are backward. $A[T,T^{-1}]$ is the coordinate ring of $\mathrm{Spec}A \times \mathbb{G}_m$. Topologically

$$\begin{aligned} K_q(X \times \mathbb{C}^*) &= K_q(X \times S^1) = K_q(X) \oplus \tilde{K}_q(SX) \\ &= K_q(X) \oplus \tilde{K}^{-q}(SX) \\ &= K_q(X) \oplus \tilde{K}^{-q-1}(SX) \\ &= K_q(X) \oplus K_{q+1}(X) \end{aligned}$$

However we have

$$K_i(A[T,T^{-1}]) = K_iA \oplus K_{i-1}A.$$

Or in another direction I recall that topological connected $K_{-1}(X)$ can be defined using bundles over X equipped with a decomposition relative to S^1, or $\mathbb{C}^*$. Over $\mathbb{C}$ a bundle E + decomp. rel. to $\mathbb{G}_m$ is the same as a bundle plus an automorphism.

Idea: A bundle E with an auto θ is something like a finite module over $\mathbb{G}_m$, i.e. a sheaf over $X \times \mathbb{G}_m$ proper over X. It is like a section of some gadget $\square$ over $X \times \mathbb{G}_m$ having support proper over X. But I have seen in duality theory that the functor $f_!$ (when put into the derived category) is unusual.

Let's guess that what I am after is something over $X \times \mathbb{P}^1$ which dies canonically on $X \times 0$ and $X \times \infty$, modulo stuff with support on $X \times 1$.

$$(?)_0 \longrightarrow K_0(X \times \mathbb{P}^1) \longrightarrow K_0(X) \times K_0(X)$$

$$(?)_1 \longrightarrow K_1(X \times \mathbb{P}^1) \longrightarrow K_1(X) \times K_1(X)$$

This shows that

$$0 \longrightarrow K_1(X) \longrightarrow (?)_0 \longrightarrow K_0(X) \longrightarrow 0 .$$

More completely: $K_n(X\times\mathbb{P}^1) = K_n(X)\cdot 1 \oplus K_n(X)\cdot(\mathcal{O}(1)-1)$, and the latter factor dies on $X\times 0$ and $X\times 1$. Now inside $K_n^0(X\times\mathbb{P}^1)$ is Im $K_n(X\times 1)$ which $= K_n(X)\cdot(\mathcal{O}(1)-1)$. Thus it does seem that what we get is $K_1(X)$.

Perhaps I want to think of $K_1(X)$ as being K_0 of virtual bundles over $X\times\mathbb{G}_m$ with proper support over X.