[redacted] October 1, 1975:

A abelian monoid, S finite set with basepoint, $A[S,*] = A^{S-\{*\}}$ regarded as a covariant functor of S. If T is a space with basepoint, put

$$A[T,*] = \varinjlim_{S\to T} A[S,*].$$

[redacted]

A point of $A[T,*]$ is a chain $\sum a_t t$ on T with coefficients in A, such that $a_* *$ is identified with $0\cdot *$. Hence any point of $A[T,*]$ can be uniquely represented [redacted] in the form $\sum_{t\in S} a_t t$ where S is a finite subset of $T-\{*\}$ and $a_t \in$ [redacted] $A-0$.

[redacted] It is clear that $A[T,*]$ is also an abelian monoid at least [redacted] set-theoretically. Also

$$(A[T,*])[T',*] = A[T\wedge T',*].$$

In effect ~~an element of the former is a~~ one has $(A[T])[T'] = A[T\times T']$ and we collapse $A[*\times T'] +$ [redacted] $A[T\times *] = A[T\vee T']$.

Take $T = S^1$. We have

$$A[S^1,*] = \coprod_{k\geq 0} (A-0)^k \times \{0<t_1<\dots<t_k<1\}.$$

Int $\Delta(k)$ to be interpreted as pt for $k=0$

$$\sum a_i t_i \longleftarrow (a_1,\dots,a_k) \times (t_1,\dots,t_k)$$

Therefore $A[S^1,*]$ is set-theoretically the geometric realization of the simplicial set:

$$\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} A\times A \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} A \rightrightarrows pt$$

in other notation:

$$\boxed{A[S^1,*] = \overline{\mathcal{B}}(A)}$$

Iterating
$$\begin{aligned} A[S^n,*] &= A[S^{n-1},*][S^1,*] \\ &= \overline{\mathcal{B}}^{n-1}(A)[S^1,*] \\ &= \overline{\mathcal{B}}\,\overline{\mathcal{B}}^{n-1}(A) = \overline{\mathcal{B}}^n(A). \end{aligned}$$

Note also that $\overline{\mathcal{B}}(\overline{\mathcal{B}}(A))$ is the geometric realization of

$$\overline{\mathcal{B}}(A)^2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \overline{\mathcal{B}}(A) \rightrightarrows pt$$

which is the geom. real. of the bisimplicial space

$$\begin{array}{ccc} (A^2)^2 & A^2 & pt \\ A^2 & A & pt \\ pt & pt & pt. \end{array}$$

[redacted]

I want to consider the following problem. Let A be an abelian monoid [redacted], say discrete. Let T be a finite simplicial complex. Then we have a space $A[T]$. In terms of the simplicial structure of T is it possible to describe nicely a category having the homotopy type of $A[T]$?

The basic idea should be that T is a disjoint union of strata, hence to a first approximation $A[T]$ is the [redacted] direct sum of copies of A for each stratum. [redacted] Thus chains $\sum a_\sigma \sigma$ ought to be the objects I seek. Next I have have to capture the topology of $A[T]$.

So what I've done so far is to [redacted] partition $A[T]$ according to the map $A[T] \longrightarrow A[\mathrm{Simp}(T)]$ induced by the projection $T \longrightarrow \mathrm{Simp}(T)$. Do I actually have a stratification of $A[T]$? Are the partitions locally closed?

Strata of $A[T]$ are indexed by chains $\sum a_\sigma \sigma$ where σ runs over the simplices of T. The stratum to which $\sum a_x x$ belongs is found by: $a_\sigma = \sum_{x\in\sigma} a_x$. If $\sum a_\sigma \sigma$ specializes to $\sum b_\tau \tau$ this means that a_σ gets split into pieces according to the different faces of σ and these contributions are added to get b_τ. Precisely, a map $\sum a_\sigma \sigma \longrightarrow \sum b_\tau \tau$ consists of a chain c on the set of inclusions in $\mathrm{Simp}(T)$ such that

$$c = \sum_{\tau\subset\sigma} c_{\tau\sigma}(\tau\sigma)$$

$$a_\sigma = \sum_{\tau\subset\sigma} c_{\tau\sigma}$$

$$b_\tau = \sum_{\sigma\supset\tau} c_{\tau\sigma}$$

Thus if $p_1 : \{\tau\subset\sigma\} \to \{\tau\}$ sends $\tau\subset\sigma$ to τ we have $(p_1)_!(c) = b$ $\quad (p_2)_!(c) = a$.

It follows that the nerve of this stratification is

$$\coprod_{\rho\subset\sigma\subset\tau} A \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \coprod_{\sigma\subset\tau} A \;\rightrightarrows\; \coprod_{\sigma} A$$

This is just the simplicial monoid of chains on the nerve of $\mathrm{Simp}(T)$ with coefficients in A.

October 3, 1975:

Let $G = GL_n(\mathbb{C})$. I have identified the building of G with the unit sphere in $Lie(U_n)$.

Let G be the complexification of a compact Lie group K. The building of G, suitably topologized, can be identified with the unit sphere in $\mathfrak{k}$. But this building has a topology as a simplicial complex whose maximal simplices correspond to chambres.

The above might be regarded as an example of a classical G-space and its discrete version.

Note that if K acts on itself by conjugation, then again one gets a discrete version

Let $U(n)$ acting on itself by conjugation. The strata are determined by the sequence of eigenvalues

$$e^{2\pi i t_1}, \ldots\ldots\ldots, e^{2\pi i t_n}$$

where $0 \leq t_1 \leq \cdots \leq t_n < 1$. Now there is no obvious way to extend this to a G-action except by using flags based on the ordering of the real numbers t_i. ~~[illegible]~~ Trouble when $t_n \uparrow 1$

$U(2)$. Strata are:

$$\begin{cases} pt \quad (t_1 = t_2 = 0), \quad pt \times \{0 < t_1 = t_2 < 1\} \\ \mathbb{P}(\mathbb{C}^2) \times \{0 = t_1 < t_2 < 1\} \\ \mathbb{P}(\mathbb{C}^2) \times \{0 < t_1 < t_2 < 1\} \end{cases}$$

$$\begin{cases} pt \times \{0 \leq t_1 = t_2 < 1\} \\ \mathbb{P}(\mathbb{C}^2) \times \{0 \leq t_1 < t_2 < 1\} \end{cases}$$

Important specializations are $t_2 \nearrow 1$. This corresponds to the map $\mathbb{P}(\mathbb{C}^2) \longrightarrow \mathbb{P}(\mathbb{C}^2)$ sending a line into its orthogonal complement.

$U(2)$ is the realization of

$$\begin{tikzcd} \mathbb{P}(\mathbb{C}^2) \arrow[r] \arrow[dr, shift left, "\perp"] \arrow[dr, shift right, "id"'] & pt \arrow[d, shift left] \arrow[d, shift right] & pt \\ & \mathbb{P}(\mathbb{C}^2) & \end{tikzcd}$$

The problem is that I can't translate this into something intrinsic because of the map $L \mapsto L^\perp$.

October 3, 1975. Cerf's paper on pseudo-isotopy IHES 39 1

$\mathcal{F}$ = space of C^∞ functions $(V\times I, V\times 0, V\times 1) \longrightarrow (I, 0, 1)$ with critical point on boundary.

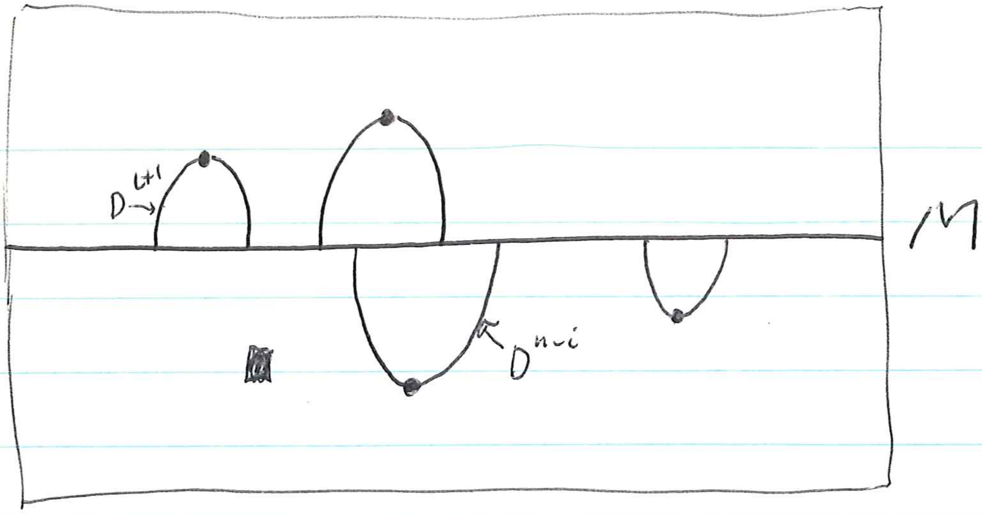
$\mathcal{F}_{i,q}$ subspace of $\mathcal{F}$ consisting of functions f with q non-degenerate critical points of index i, q non-deg. critical points of index $i+1$ such that the value of f at any critical point of index i comes before the value of any critical point of index $i+1$.

$\mathcal{F}_{i,q}$ is stratified according to how many of the critical values of index i coincide. (It is remarkably like a building). Cerf proposes to determine the nerve of $\mathcal{F}_{i,q}$.

First thing he does is to break up $\mathcal{F}_{i,q}$ into pieces according to the intermediate variety M. If $f \in \mathcal{F}_{i,q}$ then it has a (well-defined up to isotopy in $W = V\times I$) intermediate variety M. Let $\mathcal{F}_M$ be the subset of $\mathcal{F}_{i,q}$ consisting of f having M as intermediate variety.

$$H_{i+1}(W_M^+, M) \cong H_{n-i}(W_M^-, M) \cong \mathbb{Z}^q$$

bases are determined by choosing a gradient vector field and taking nappes.



$n-1 = \dim M$
$n = \dim W$

Thus we get two free $\mathbb{Z}$-modules $H_{i+1}(W_M^+, M)$ and $H_{n-i}(W_M^-, M)$ depending only on M. The function $f \in \mathcal{F}_M$ determines bases in these $\mathbb{Z}$-modules up to some unipotent subgroup. I guess he chooses bases in $H_{i+1}(W_M^+, M)$ and $H_i(W_M^-, M)$, whence to f he gets a coset in $GL_q\mathbb{Z} \times GL_q\mathbb{Z}$ for some subgroup. This gives a map

$$(*) \qquad (\text{Nerve of } \mathcal{F}_M) \longrightarrow C_q \times C_q$$

C_q is a complex formed from $GL_q(\mathbb{Z})$, Σ_q and T_q (T_q = triangular subgroup. One proves $(*)$ is a covering.

Forgetting M one gets a map

$$(**) \qquad (\text{Nerve of } \mathcal{F}_{i,q}) \longrightarrow \mathcal{A}_q$$

where $\mathcal{A}_q$ is a quotient $GL_q(\mathbb{Z}) \backslash C_q \times C_q$. Thm: $(**)$ is an isomorphism!

The basic geometric fact is:

<u>Basis thm:</u> (W, V, V') a triad of dimension n possessing a Morse function f with all critical points of index λ and on the same level; let ξ be a gradient vector field for f. Then given any basis for $H_\lambda(W, V)$, $\exists\ f', \xi'$ agreeing with (f, ξ) near $V \cup V'$, same critical points on same level, such that the nappes of ξ' from the critical points of f, when suitably oriented form the given basis.

Basic geometry: Let (W, V, V') be a triad having a Morse function with q critical points of index λ. Let $\mathcal{F}$ be the space of these Morse functions. We know then that

$$H_\lambda(V, W) \simeq \mathbb{Z}^q$$

If $f \in \mathcal{F}$, then applying f to the critical points of f, we get a positive divisor in $\mathbb{R}$ of degree q, namely the divisor of critical values. We also get a flag in $H_\lambda(V, W)$ namely

$$F_x H_\lambda(W, V) = H_\lambda(W_x, V)$$

Therefore we seem to be getting a map of $\mathcal{F}$ into [redacted] something like the space of self-adjoint matrices [redacted] whose eigenvalues [redacted] satisfy $0 < \lambda < 1$.

[redacted]

It is almost as if the critical value a is the eigenvalue for the jump:

$$H_\lambda(W_{a+\varepsilon}, V) / H_\lambda(W_{a-\varepsilon}, V) \cong H(W_{a+\varepsilon}, W_{a-\varepsilon}).$$

What it is I have to understand: Let $\mathcal{F}_{i,q}$ be the set of Morse functions $f : (W, V, V') \to (I, 0, 1)$ having q critical values of index i preceding q critical values of index $i+1$. [redacted]

Let $\mathcal{F}'_{i,q} \subset \mathcal{F}_{i,q}$ be the subset containing f such that $\frac{1}{2}$ is a regular value separating the critical points of index i and $i+1$. This should be a homotopy equivalence.

The point is that $\mathcal{F}_{i,q}$ has a natural stratification which we wish to determine. In fact

What it is I have to understand: Let $\mathcal{F}_{i,q}$ be the space of Morse functions $f:(V\times I, V\times 0, V\times 1) \longrightarrow (I, 0, 1)$ with q critical points of index i preceding q critical points of index $i+1$. This space has a natural stratification, hence one gets a poset – the nerve of the stratification. Now by using gradient vector fields one can define a canonical map from the poset of strata of $\mathcal{F}_{i,q}$ into a poset of cosets constructed using $Gl_q(\mathbb{Z})$, and the triangular group.

October 4, 1975: Cerf's paper.

V fixed, closed manifold. $\mathcal{F}_{i,q}$ = space of Morse maps $(V\times I, V\times 0, V\times 1) \longrightarrow (I, 0, 1)$ having exactly $2q$ critical points, q of which are of index i, q of which have index $i+1$, and all index i points precede the index $i+1$ points. Associated to an element $f \in \mathcal{F}_{i,q}$ is an acyclic complex which we get as follows. Let $0<a<1$ be a regular value such that critical points of index i (resp $i+1$) have level $<a$ (resp. $>a$). We know $H_*(f^{-1}[0,a], V)$ is $\simeq \mathbb{Z}^q$ in dim i and 0 elsewhere, and also that $H_*(V\times I, f^{-1}[0,a])$ is $\simeq \mathbb{Z}^q$ in dim 1 and 0 elsewhere. Thus from the exact sequence of the triple $(V \subset f^{-1}[0,a] \subset V\times I)$, we get an isomorphism

$$\begin{array}{c} H_{i+1}(V\times I, f^{-1}[0,a]) \xrightarrow[\sim]{\partial} H_i(f^{-1}[0,a], V) \\ \| \\ H_{i+1}(f^{-1}[a,1], f^{-1}\{a\}) \end{array}$$

i.e. an acyclic complex.

Furthermore suppose the critical points of index i are enumerated $c_1, \ldots, c_q$ such that $f(c_1) \leq \cdots \leq f(c_q)$. If $0 \leq b \leq a$ is a regular value, then

$$H_i(f^{-1}[0,b], V) \hookrightarrow H_i(f^{-1}[0,a], V)$$

is an admissible monomorphism. Thus we see that $H_i(f^{-1}[0,a], V)$ has a canonical filtration whose jumps correspond to the critical values of f in $[0,a)$. Moreover if b is a critical value in $(0,a)$, then the corresponding quotient

$$(*) \qquad H_i\left(f^{-1}[b+\varepsilon, b-\varepsilon], f^{-1}\{b-\varepsilon\}\right)$$

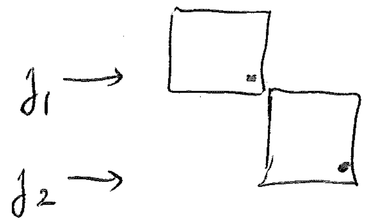
has a natural basis up to signs indexed by the critical points of index level b. Specifically if we orient descending maps from the critical points, we get a basis. So we see that $(*)$ has a canonical reduction to the group of monomial matrices $\Sigma_j \, S(\mathbb{Z}/2\mathbb{Z})$.

In a similar way $H_{i+1}(V \times I, f^{-1}[0,a])$ has a canonical filtration whose quotients admit reductions to the group of monomial matrices.

Next idea is somehow to interpret the structure produced on $H_i(f^{-1}[0,a], V)$ as the orbit of an isomorphism $\alpha : \mathbb{Z}^g \xrightarrow{\sim} H_i(f^{-1}[0,a], V)$ under a certain subgroup of $GL_g(\mathbb{Z})$. This subgroup is defined as follows. Let $J \subset \{1, \dots, g-1\}$ be the places where the sequence $f(c_1) \leq \dots \leq f(c_g)$ of critical values jumps. Then the subgroup is

$$H_J = \left\{ \begin{pmatrix} \cdot & * & * \\ & \cdot & * \\ & & \cdot \end{pmatrix} \right\}$$

the subgroup of the parabolic group with blocks having jumps at points $\{j_1, j_2, \dots, j_s\} = J$



etc. The matrices $\boxed{\cdot}$ are monomial matrices.

Similarly the structure on $H_{i+1}(V\times I, f^{-1}[0,a])$ provides us with an isom. $\alpha_+ : \mathbb{Z}^b \xrightarrow{\sim} \underline{\qquad}$, unique up to multiplication by an element of a similar subgroup $H_{J'}$ of $GL_b(\mathbb{Z})$. Thus if we take

$$\alpha_-^{-1}\,\partial\,\alpha_+ \in GL_b(\mathbb{Z})$$

we get a double coset in $H_J \backslash GL_b(\mathbb{Z}) / H_{J'}$ attached to the element $f \in \mathcal{F}_{iq}$.

October 5, 1975 Wagoner's construction:

Let (W, V, V') be a triad such that there exists a morse function $(W, V, V') \longrightarrow (I, 0, 1)$ having exactly q critical points with the same index λ. Let $\mathcal{C}$ be the space of these morse functions. Let C_i be the subspace consisting of f having $q-i$ distinct critical values:

$$C = C_0 \amalg \cdots \amalg C_{q-1}.$$

A better stratification would be as follows. For each ordered partition J of $\{1, \ldots, q\}$, (i.e.

$$\{1, \ldots, j_1\}, \{j_1+1, \ldots, j_2\}, \ldots, \{j_{s-1}, \ldots, j_s = q\}$$

so that J is the same as a subset $1 \leq j_1 < \cdots < j_{s-1} < q$ of $\{1, \ldots, q-1\}$), we can let C_J be the subspace of C whose critical values are of type J – this means that if we ordered the critical points so the values increase then

$$f(c_1) = \cdots = f(c_{j_1}) < f(c_{j_1+1}) = \cdots = f(c_{j_2}) < \cdots\cdots$$

Then $C = \coprod C_J$.

Let $\mathcal{G}$ be the group of diffeomorphisms of W; it acts to the right on C. Given a function $f \in C$ we can try to determine the codimension of its orbit infinitesimally. The map $Lie(\mathcal{G}) \longrightarrow$ tangent space to C at f can be identified with sending a vector field X into the function $Xf = \langle X, df \rangle$. This map will be onto except for the critical points, so one sees that the codimension of the orbit ought to be the number of critical points of f. Geometrically this means that the G-action will not change the critical values of f, so that [redacted] we have q directions in which to push f transversal to its G-orbit.

[redacted] Let G_I be the diffeos. of I, and let $G_I \times G$ [redacted] act on C. Here the tangent map sends (Y, X) to $w \mapsto Xf(w) + Y_{f(w)}$, so the cokernel would be the tangent spaces to the critical points, added up, modulo vectors obtained by varying the values of f at the critical points. So a critical ~~point~~ value of multiplicity r [redacted] will contribute $\mathbb{R}^r/\Delta\mathbb{R}$ to the cokernel. Thus C_J which should be [redacted] a union of open orbits of $G_I \times G$ should be of codimension $q-$ no of partitions in $J = q - \operatorname{card} \mathbf{J} - 1$ (if we regard $\mathbf{J}$ as a subset of $\{1, .., q-1\}$.)

Normal space to C_J in C at f should be as follows. [redacted] Fix a critical value a and let the critical points be $c_1, \dots, c_s$. Then we can always add a function constant in a nbd. of c_i and of support in a slightly [redacted] larger nbd.

Suppose we now have a normal disk to a stratum C_J at a point f. To simplify, suppose that $J = \phi$, so that f has critical points $c_1, \dots, c_q$ at the same level a. The disk has [redacted] dimension $q-1$. [redacted] I ought to be able to assume that [redacted] as t varies over the disk D^{q-1}, the functions f_t have the same critical points [redacted] and that only the critical values change. So [redacted] I can take D^{q-1} to be [redacted] a disk in the subspace $\{(x_1, \dots, x_q) \in \mathbb{R}^q \mid \sum_i x_i = 0\}$. ~~The question now is whether it is possible to~~ [redacted] ~~decompose the graphic into a union of q~~ [redacted] ~~sections over D^{q-1}. The answer is obviously yes because we have been assuming that the critical points don't vary.~~

~~So the procedure seems to be this. Suppose we have a family $D^k \longrightarrow C$.~~ [redacted] ~~The "graphic" of the family is a q fold covering~~

Suppose I have a family $D^k \longrightarrow C$. The critical points of the [redacted] various members of the family form a covering of degree q of D^k, which is trivial if $k \geq 2$. Thus one has a completely natural way to order the critical points and to assign orientations to the decreasing stable manifold.

Suppose we go back to Cerf's situation, where we consider $W = V \times I$ and morse functions with q critical pts of index i preceding q critical pts. of index $i+1$; $\mathcal{F}_{i,q}$ is the space of these morse fns. Suppose we have a family $D^k \longrightarrow \mathcal{F}_{i,q}$, $t \longmapsto f_t$. Then clearly we can consider over D^k the two q-fold coverings given by the critical points of f_t. So we get over $\mathcal{F}_{i,q}$ a $\Sigma_q \times \Sigma_q$ - covering $\widetilde{\mathcal{F}}_{i,q}$ whose elements are pairs (f, σ), σ an ordering on the critical points of f.

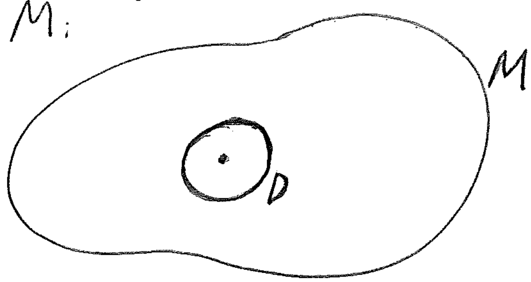
In Wagoner's situation one ought also to give, when $k=1$, not only the ordering of the critical pts. but also the orientations of the descending stable manifolds. Question: Is it possible to vary f in C in such a way that one can shift signs of the orientations? What I mean is this — can I find a family f_t with oriented critical points so that $f_0 = f_1$ but the orientations of the critical points change.

The covering in question is the following: Define an oriented critical point to be one with a given orientation on the negative eigenspace for the Hessian. This gives us over [redacted] C a covering for the group $\Sigma_q \ltimes \mathbb{Z}_2^{q}$ [redacted], which can be reduced to the subgroup [redacted] having the sign $+1$.

[redacted] Go back to $F_{i,q}$. [redacted] Then after I [redacted] lift to a covering for the group $\tilde{\Sigma}_q \times \tilde{\Sigma}_q$ I get [redacted] a map from the stratification of $\tilde{F}_{i,q}$ to the Wagoner style complex.

<u>October 10, 1975</u>

One of the applications of the h-cobordism theorem is to show that a n-manifold M which is contractible is diffeom. to D^n, (assuming $n \geq 6$ or something similar). Here's the proof: Take a small disk D around an interior point of M:



Apply the h-cobordism theorem to $M - \mathrm{Int}(D)$ to get $M - \mathrm{Int}(D) \cong \partial D \times I$, whence $M \cong D^n$.

Let's try to find a parameterized version of this theorem. Let M now be a fibre bundle over X with each fibre a contractible n-manifold (hence $\cong D^n$). Choose a section $s: X \longrightarrow \mathrm{Int}(M)$. The normal bundle to s is isomorphic to the pull-back via s of the tangent bundle along the fibres of M over X. Since any two sections are homotopic, the normal bundle to s is a vector bundle on X, well-defined up to isomorphism. Call this bundle E. We want to show M is isomorphic to the disk bundle $\mathbb{D}(E)$ of E.

We can embed $\mathbb{D}(E)$ inside M and consider the complement $W = M - \operatorname{Int} D(E)$. We want to produce a function $f : (W, \partial D(E), \partial M) \longrightarrow (I, 0, 1)$ with no critical points when restricted to each fibre of W over X.

Another approach. The type of fibre bundles M/X I am considering may be identified with torsors for the top. groupoid of contractible n-manifolds and diffeoms. Since I know any such manifolds is diffeom. to D^n, I am considering torsors for the group $\operatorname{Diff}(D^n)$. The first reduction is:

$$\operatorname{Diff}(D^n, 0) \longrightarrow \operatorname{Diff}(D^n) \longrightarrow \operatorname{Int}(D^n)$$

(the inclusion is a heq), and the second reduction is:

$$\mathcal{G}' \longrightarrow \operatorname{Diff}(D^n, 0) \longrightarrow GL_n \mathbb{R}.$$

(this corresponds to associating to M/X the bundle E). $\mathcal{G}'$ will be homotopy equivalent to the subgroup $\mathcal{G}$ of diffeos. of the annulus $S^{n-1} \times I$ which are the identity in a nbd. of $S^{n-1} \times 0$. Therefore analysis of $\operatorname{Diff}(D^n)$ reduces to the analysis of the pseudo-isotopy group $\mathcal{G}$.

Next we can let $\mathcal{G}$ act on the space $\mathcal{E}$ of functions $(S^{n-1} \times I, S^{n-1} \times 0, S^{n-1} \times 1) \longrightarrow (I, 0, 1)$ without critical points. The action is transitive and the stabilizer of the natural projection is the group of paths in $\mathrm{Diff}(S^{n-1})$ starting at the identity. Since this subgroup is contractible $\mathcal{G} \to \mathcal{E}$ is a homotopy equivalence.

Therefore if we want to understand the fibres of the functor which associates to any M/X the associated vector bundle E, we must understand the space $\mathcal{E}$ of normalized functions on $S^{n-1} \times I$ without critical points.

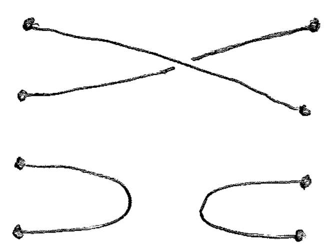
Cerf's pseudoisotopy theorem for S^{n-1} says that any M/X with $\dim X \leq 1$ is diffeomorphic to the linearized bundle E.

Go back to an old idea of yours which was to find a geometric proof that the K-theory of finite sets is stable homotopy. What I wanted to do was to define a suitable notion of signed set, so that I could see that a framed proper map $f: Y \to X$ is somehow equivalent to a bundle of signed sets over X.

My orignal approach went like this. Start with

$f: Y \longrightarrow X$ smooth and framed. Suppose X triangulated. Near each vertex of the triangulation, we can find a regular point for f, in fact, we ought to be able to jiggle the triangulation so that all simplices are transv. to f. Then we get an induced stratification of Y. Over vertices we get signed sets. Over a 1-simplex we get a 1-manifold

W:



with a function to I which I can jiggle to be a Morse function. Over a 2 simplex I get something difficult to analyze, and it is not clear what even the generic f should look like.

Maybe I should look at the special case of a differentiable fibre bundle $E \longrightarrow X$ and let Y be the zero submanifold of a generic section of the tangent bundle along the fibres

October 12, 1975 (Becky is 9!)

Problem: Let $M \to X$ be a differentiable fibre bundle and let Y be the zero submanifold of a generic section of the tangent bundle along the fibres of M/X. I want to describe Y/X as some sort of structure over a stratification of X.

This should be a local problem on X, hence I can suppose $M = X \times F$. Then $\Gamma(X \times F, T_{M/X}) = \mathrm{Map}(X, \Gamma(F, T_F))$, so I am trying to understand what is a generic map of X into the space of vector fields on the manifold F.

Change notation to $F = Z$, and replace T_Z by a vector bundle E over Z with $\mathrm{rank}(E) = \dim(Z)$. So now I want a natural stratification of $\Gamma(Z, E)$ so that I can define a "generic" map $X \longrightarrow \Gamma(Z, E)$.

Conjecture. Consider in $\Gamma(Z,E)$ ~~the~~ subset of sections s ~~such that~~ which vanish to finite order, that is such that the map

$$\Gamma(Z, E^\vee) \xrightarrow{s^\vee} \Gamma(Z, \mathcal{O}_Z)$$

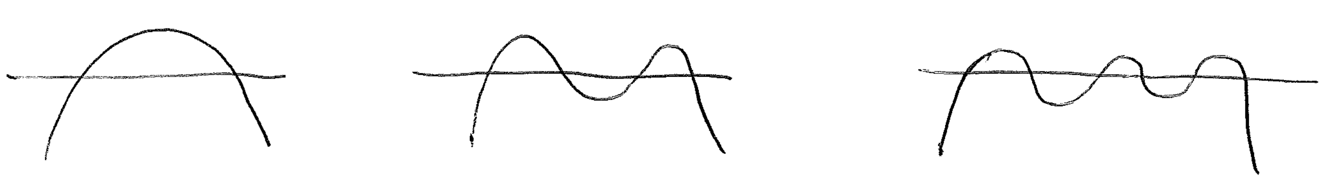
has finite dimensional cokernel. Then this subset is open and its complement is of infinite codimension. In particular we can always move a family $X \to \Gamma(Z,E)$ into a family in this open set.

Note: the last statement is probably true because any non-zero "analytic" section (suppose Z, E analytic) ought to vanish to finite order, and any family can be moved into a family of analytic sections. ??

If this conjecture is true, then we ought to know that the space of sections of finite order of vanishing is of the same homotopy type as $\Gamma(Z, E)$, i.e. contractible. Next possibility: We can count the singularities of a section vanishing to finite order. It has something to do with regular sequences.

Suppose Z is a 1-manifold, and E is the trivial bundle over Z. I am going to consider a stratification of the sections of E over Z which results from looking just at the zeroes of sections.

So suppose $Z = [0,1]$ and I want only functions f with $f(0), f(1) < 0$. Then the non-degenerate strata are represented by:



Each stratum is described by a sequence

$+1, -1$

$+1, -1, +1, -1$

etc.

Basic singularities are of the style $\pm x^k$ for some integer k. So the strata will be described by sequences $k_1, \ldots, k_m$ where k_m is $\pm$ a non-zero integer subject to certain rules.

October 14, 1975

Let E, F be vector spaces over a field k.
Problem: Determine the orbits of $Gr_r(E\oplus F)$ for the group $Aut(E) \times Aut(F)$.

Given $W \in Gr_r(E\oplus F)$, we can associate two subspaces: $K = W \cap F \subset F$, and $L = (W+F)\cap E \subset E$. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & W & \longrightarrow & L & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 0 & \longrightarrow & F & \longrightarrow & E\oplus F & \longrightarrow & E & \longrightarrow & 0 . \end{array}$$

so $\dim L = \dim W - \dim K$. Now put $e = \dim E$, $s = \dim K$. We have defined a map

$$Gr_r(E\oplus F) \xrightarrow{\varphi} \coprod_{0\le s\le r} Gr_s(F) \times Gr_{r-s}(E)$$

Next fix $K \in Gr_s(F)$ and $L \in Gr_{r-s}(E)$ and consider

$$\varphi^{-1}(K,L) = \left\{ W \in E\oplus F \;\middle|\; \begin{array}{l} W\cap F = K \\ (W+F) = L\oplus F \end{array} \right\}$$

$$\cong \left\{ \bar{W} \subset L\oplus F/K \;\middle|\; \begin{array}{l} \bar{W}\cap (F/K) = 0 \\ \bar{W} + (F/K) = L\oplus F/K \end{array} \right\}$$

(complements to $\bar{F}$ in $L\oplus\bar{F}$)

$$\cong Hom(L, F/K)$$

Now the stabilizer of (K,L) in the group $\mathrm{Aut}(F)\times\mathrm{Aut}(E)$ is $\mathrm{Aut}(F,K)\times\mathrm{Aut}(E,L)$ which acts on $\varphi^{-1}(K,L) \cong \mathrm{Hom}(L, F/K)$ thru the obvious epimorphism

$$\mathrm{Aut}(F,K)\times\mathrm{Aut}(E,L) \longrightarrow \mathrm{Aut}(F/K)\times\mathrm{Aut}(L).$$

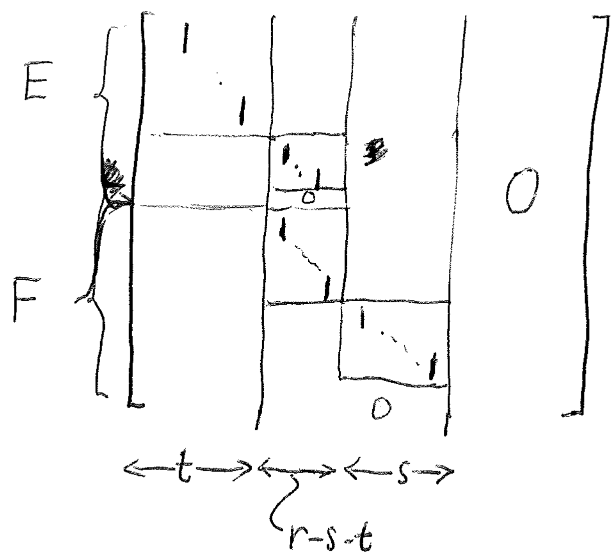
But we know the orbits of $\mathrm{Aut}(F/K)\times\mathrm{Aut}(L)$ on $\mathrm{Hom}(L, F/K)$ ~~correspond to~~ are classified by the nullity of the homomorphism. This nullity is essentially the dim of the intersection $L\cap W = E\cap W$. So I conclude that the orbits of $\mathrm{Aut}(E)\times\mathrm{Aut}(F)$ on $Gr_r(E\oplus F)$ are described by the pair of integers $\dim(W\cap F)$, $\dim(W\cap E)$.

<u>Check:</u> Let W be a subspace of dim r, let $s=\dim W\cap F$, $t=\dim W\cap E$. Let $f_1,\dots,f_s$ be a basis for $W\cap F$, let $e_1,\dots,e_t$ be a basis for $W\cap E$. Then $f_1,\dots,f_s, e_1,\dots,e_t$ is ind. inside W so we can extend it to a basis $w_1,\dots, w_{r-s-t}$ for W. Let $w_i = e_i' + f_i' \in E\oplus F$.

Claim e_i, e_i' are indep. For if $\sum c_ie_i + c_i'e_i' = 0$, then $\sum c_ie_i + c_i'w_i \in W\cap F \Rightarrow \sum c_ie_i + c_i'w_i = \sum c_jf_j$, impossible.

So I can extend $e_1,\dots,e_t, e_1',\dots,e_{r-s-t}'$ to a basis for E by adding $e_1'',\dots,e''_{\dim E - r+s}$. Analogously for F. Thus we get a canonical form depending only on s,t.

The canonical form looks like:



October 15, 1975.

Problem: Let T be a simplicial complex and let $\mathcal{M}$ be an exact category. To define [redacted] a category of chains on T with coefficients in $\mathcal{M}$.

Whatever a chain is, I ought to know that part of it in a given open set U. For example if $\sum a_x x$ is a chain on $|T|$ with coefficients in an abelian monoid A, then I should know $\sum_{x \in U} a_x$ [redacted]

So maybe I should consider sheaves F on T with coefficients in $\mathcal{M}$. (Say $\mathcal{M}$ is abelian to simplify). Then to each open set U in T I get an object of $\mathcal{M}$ denoted $F(U)$. If $V \subset U$, then we have a [redacted] map $F(U) \longrightarrow F(V)$. In the example, this map is [redacted] surjective, which suggests that I look only at flask sheaves.

Basic question: Let T be a finite simplicial complex. Consider the abelian category of sheaves on T with values in the abelian category $\mathcal{M}$; here I only consider open sets which are unions of "open" simplices, i.e. sets of "open" simplices closed under generalization, or comple[redacted]ments of subcomplexes. [redacted]

Is it true that the K-theory of such sheaves and the K-theory of the subcategory of flask sheaves is simply a direct sum of copies of the K-theory of $\mathcal{M}$, one for each simplex of T?

To prove this [redacted] first note that a sheaf is the same thing as a covariant functor on simplices. $\sigma \mapsto F_x$ any $x \in \mathrm{Int}(\sigma)$. If $\tau \subset \sigma$ then any nbd of τ contains a point of σ, so we get a map $F_\tau \to F_\sigma$. Then

$$F(\mathcal{U}) = \varprojlim_{\tau \in \mathcal{U}} F_\tau$$

$\mathcal{U}_\tau$ = smallest open containing $\tau = \bigcup_{\sigma \supset \tau} \sigma$. $F_\tau = F(\mathcal{U}_\tau)$.
$\sigma \subset \tau \Rightarrow \mathcal{U}_\sigma \supset \mathcal{U}_\tau \Rightarrow F(\mathcal{U}_\sigma) \to F(\mathcal{U}_\tau)$. If F is flask then $\sigma \subset \tau \Rightarrow F_\sigma \twoheadrightarrow F_\tau$. [redacted]

[redacted] The converse isn't true for if T:

then $F_1 \to F_{01} \times F_{12}$ isn't necessarily surjective, when $F_1 \twoheadrightarrow F_{01}$ and $F_1 \twoheadrightarrow F_{12}$.

<u>Example</u>:

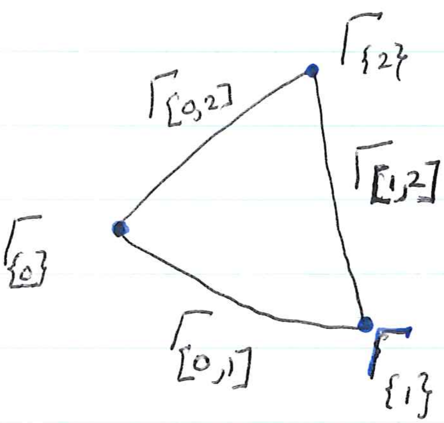
so a sheaf is a diagram $F_0 \to F_{01} \leftarrow F_1$. A flask sheaf consists of $F_{[0,1]} = F_0 \times_{F_{01}} F_1$) together with two submodule consisting of sections with support at 0 and 1.

$$0 \longrightarrow \underline{H}_{\{0\}}(F) \oplus \underline{H}_{\{1\}}(F) \longrightarrow F \longrightarrow F_{01} \longrightarrow 0$$

Example of a simplex: Let $T = \Delta(p)$ = simplex with vertices $\{0, 1, \dots, p\}$. Then a flask sheaf F over T ought to be an object F of $\mathcal{M}$ equipped with a filtration indexed by the simplices $\ni$

$$0 \longrightarrow \Gamma_{\overline{\sigma \cap \tau}}(F) \longrightarrow \Gamma_{\overline{\sigma}}(F) \oplus \Gamma_{\overline{\tau}}(F) \longrightarrow \Gamma_{\overline{\sigma \cup \tau}}(F)$$

is exact with admissible cokernel. e.g. if $p=2$, then one gets



with $\Gamma_{\{0\}} \oplus \Gamma_{\{1\}} \rightarrowtail \Gamma_{[0,1]}$ etc.

$$0 \to \Gamma_{[0]} + \Gamma_{[1]} + \Gamma_{[2]} \longrightarrow \Gamma_{[0,1]} + \Gamma_{[1,2]} + \Gamma_{[0,2]} \longrightarrow \Gamma_{\Delta(2)}$$

admissibly exact.

<u>Situation</u>: J a finite poset. We topologize J so that open sets are the subsets closed under generalization: $x \in U$, $x < y \Rightarrow y \in U$. A sheaf of sets over J is just a covariant functor F from J to Sets:

$$F_x = \Gamma(U_x, F) \qquad U_x = \{y \mid y \geq x\}.$$

$$x < y \Rightarrow U_x \supset U_y \Rightarrow F(U_x) \longrightarrow F(U_y).$$

I want to show any sheaf has a finite "flask" resolution. A sheaf is flask if $U \subseteq V \Rightarrow F(V) \twoheadrightarrow F(U)$. Example of a flask sheaf is

$$\Gamma(U, (i_y)_*(A)) = \begin{cases} 0 & y \notin U \\ A & y \in U. \end{cases}$$

$$\mathrm{Hom}(F, (i_y)_*(A)) = \mathrm{Hom}(F_y, A)$$

Standard resolution

$$0 \to F \longrightarrow \prod_x (i_x)_*(F_x) \longrightarrow \prod_{x \leq y} (i_x)_*(F_y) \longrightarrow \cdots$$

etc.

Note that

$$((i_y)_* A)_x = \Gamma(U_x, (i_y)_* A) = \begin{cases} 0 & y \notin U_x \\ A & y \in U_x \end{cases} = \begin{cases} A & x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Thus $(i_y)_* A$ is the constant sheaf A on $\overline{\{y\}}$ extended by 0.

<u>Question</u>: Let X be a simplicial set. Is

$$A[|X|] = |A[X]| \quad ?$$

Yes, because if M is a top abelian monoid, then

$$\operatorname{Hom}_{\substack{\text{top.ab.}\\ \text{mon.}}}(\mathbb{N}[|X|], M) = \operatorname{Hom}_{\text{sp.}}(|X|, M) = \operatorname{Hom}_{\text{s.sets.}}(X, \operatorname{Sing} M)$$

$$= \operatorname{Hom}_{\substack{\text{s. mon.}\\ \text{ab.}}}(\mathbb{N}[X], \operatorname{Sing} M)$$

$$= \operatorname{Hom}_{\substack{\text{top ab}\\ \text{mon}}}(|\mathbb{N}[X]|, M);$$

the general case should result because A has a presentation using $\mathbb{N}$.

Let us consider the family of ~~vector spaces~~ finite $\mathbb{C}[T]$-modules parameterized by $\operatorname{End}(V)$ such that the $\mathbb{C}[T]$-module at $A \in \operatorname{End}(V)$ is V with T acting via A. Let $V_A = V$ with T acting as A, i.e. $V_A = \mathbb{C}[T] \otimes V / f(T) \otimes v = 1 \otimes f(A)v$.

When A is semi-simple, V_A splits as the sum of eigen-spaces $V_A = \bigoplus V_{A,\lambda}$ where $V_{A,\lambda} = \operatorname{Ker}(A - \lambda)$. I can see how specialization occurs if I stay on semi-simple matrices: Two eigenvalues can coallesce and one adds the corresponding eigenspaces. Next I really want to understand what happens when one specializes to a non-semi-simple element A.

At a multiple eigenvalue λ the corresponding eigenspace is a filtered vector space. Suppose A nilpotent, and suppose $B_\nu \to A$ where B_ν is semi-simple with distinct eigenvalues. How can you describe what's happening?

Suppose that A is a regular nilpotent, hence exactly one flag in V is invariant under A. If L_ν is a line invariant under B_ν and $L_\nu \to L$, then $L_\nu = B_\nu L_\nu \longrightarrow AL$ so $AL = L$. This means that all the eigen-lines of B_ν have to converge to the unique eigen line of A. It seems that all the $n!$ flags fixed by B_ν converge to the unique flag fixed by A.