September 20, 1975

Green's algebra for Σ_n. This is $\bigoplus_{n\geq 0} R(\Sigma_n)$ with the product

$$\alpha\cdot\beta = \mathrm{Ind}_{\Sigma_i\times\Sigma_j\to\Sigma_{i+j}}(\alpha\otimes\beta)$$

Since one has a duality $R(G)^\vee \cong R(G)$ such that restriction and induction are ~~adjoints~~ transposes, the Green algebra is $\bigoplus_{n\geq 0} R(\Sigma_n)^\vee$ with product

$$R(\Sigma_i)^\vee\otimes R(\Sigma_j)^\vee \longrightarrow R(\Sigma_{i+j})^\vee$$

the transpose of restriction from Σ_{i+j} to $\Sigma_i\times\Sigma_j$. Atiyah analyzed this last algebra, so we will go over his results.

Let $V = F^d$ be regarded as a repn. of M_d = $d\times d$-matrices over F. Then $V^{\otimes n}$ is a representation of $\Sigma_n\times M_d$, hence a rep. of $\Sigma_n\times$ diag. Representations of diag. are characters $(t_1,\dots,t_d)\longmapsto t_1^{\alpha_1}\cdots t_d^{\alpha_d}$. Thus we get a distinguished element of

$$[V^{\otimes n}]\in R(\Sigma_n\times \mathrm{diag}) = R(\Sigma_n)\otimes\mathbb{Z}[t_1,\dots,t_d].$$

Specifically one has

$$V^{\otimes n} = \sum \mathrm{Hom}_{\Sigma_n}(\pi, V^{\otimes n})\otimes\pi$$

where π runs over the irreducibles of Σ_n, and

$[V^{\otimes n}] = \sum \pi\, \varphi_\pi(t_1,\dots,t_n)$ where $\varphi_\pi(t_1,\dots,t_n)$ is the trace of $(t_1,\dots,t_n)$ acting on $\mathrm{Hom}_{\Sigma_n}(\pi, V^{\otimes n})$.

Contracting with these polynomials we get a map

$$\bigoplus_{n\geq 0} R(\Sigma_n)^\vee \longrightarrow \mathbb{Z}[t_1,\dots,t_d]^{\Sigma_d} = \mathbb{Z}[\sigma_1,\dots,\sigma_d]$$

which can be described as follows. Given $f \in R(\Sigma_n)^\vee$ one sends it to the polynomial

$$\sum f(\pi)\cdot \text{trace of } t \text{ on } \mathrm{Hom}_{\Sigma_n}(\pi, V^{\otimes d}).$$

~~But for example, that~~ I can be more precise

$$V = Fe_1 + \dots + Fe_d \quad \text{where} \quad t.e_i = t_i e_i$$

so $$V^{\otimes n} = \bigoplus_{i_1,\dots,i_n \in \{1,\dots,d\}} F e_{i_1} \otimes \dots \otimes e_{i_n} \quad \text{where}$$

$$t\cdot e_{i_1} \otimes \cdot \otimes e_{i_n} = t_{i_1} \cdots t_{i_n}\, e_{i_1} \otimes \cdot \otimes e_{i_n}$$

~~We can break this up according to the orbits of Σ_n on $\{1,\dots,d\}^n$; an orbit is simply a partition $n = d_1 + \dots + d_n$ with $d_1 \geq \dots \geq d_n$. Thus~~

$$\cancel{[V^{\otimes n}] = \sum_{\substack{d_1 \geq \dots \geq d_n \\ d_1 + \dots + d_n = d}} \mathrm{Ind}_{\Sigma_{d_1} \times \dots \times \Sigma_{d_n}}^{\Sigma_d}}$$

We can split $V^{\otimes n}$ according to the orbits of Σ_n on $\{1,\dots,d\}^n$. An orbit is given by an ordered partition $n = n_1 + \dots + n_d$, and is represented by the basis element $e_1^{\otimes n_1} \otimes e_2^{\otimes n_2} \otimes \cdots \otimes e_d^{\otimes n_d}$, whose stabilizer is $\Sigma_{n_1} \times \dots \times \Sigma_{n_d}$. Thus

$$[V^{\otimes n}] = \sum_{n_1 + \dots + n_d = n} \mathrm{Ind}_{\Sigma_{n_1} \times \dots \times \Sigma_{n_d} \to \Sigma_n} 1 \cdot t_1^{n_1} \cdots t_d^{n_d}$$

$$= \sum_{\substack{n_1 + \dots + n_d = n \\ n_1 \geq n_2 \geq \dots \geq n_d}} \left(\mathrm{Ind}_{\Sigma_{n_1} \times \dots \times \Sigma_{n_d} \to \Sigma_n} 1 \right) \cdot s_{n_1 \dots n_d}(t)$$

where $s_{n_1 \dots n_d}(t)$ = the symmetrization of $t_1^{n_1} \cdots t_d^{n_d}$

Consider in $R(\Sigma_n)^\vee$ the element which gives the multiplicity of the trivial repn. In $V^{\otimes n}$ the trivial part for Σ_n is $S^n V$ which has the basis $e_{i_1} \cdots e_{i_n}$ $\quad 1 \leq i_1 \leq \dots \leq i_n \leq d$ hence the character

$$\sum_{1 \leq i_1 \leq \dots \leq i_n \leq d} t_{i_1} \cdots t_{i_n}$$

I see now that I want to look at instead the element of

$R(\Sigma_n)^\vee$ which gives the multiplicity of the sign representation, call this element $\lambda_n \in R(\Sigma_n)^\vee$. Applied to $V^{\otimes n}$ it gives $\Lambda^n V$, which has the basis $e_{i_1}\wedge\cdots\wedge e_{i_n}$ where $1 \le i_1 < \cdots < i_n \le d$, and hence the character $\sum_{1\le i_1<\cdots<i_n\le d} t_{i_1}\cdots t_{i_n} = \sigma_n(t_1,\dots,t_d)$.

Atiyah argues that $\bigoplus R(\Sigma_n)^\vee \longrightarrow \mathbb{Z}[t_1,\dots,t_d]^{\Sigma_d}$ is a ring homom. whose image contains $\sigma_1,\dots,\sigma_d$, hence it's onto. But rank $R(\Sigma_n)^\vee$ = no. of conj. classes in Σ_n = no. of partitions of n = rank $\mathbb{Z}[t_1,\dots,t_d]^{\Sigma_d}$ in degree n if $n \le d$.

So by Atiyah one knows that

$$\bigoplus_{n\ge 0} R(\Sigma_n)^\vee \xleftarrow{\ \sim\ } \mathbb{Z}[\lambda_1,\dots\dots]$$

where $\lambda_i \in R(\Sigma_n)^\vee$ is inner product with the sign representation. It follows that

$$\mathbb{Z}[\lambda_1,\dots,] \xrightarrow{\ \sim\ } \bigoplus_{n\ge 0} R(\Sigma_n) \quad \text{with Green product}$$

where $\lambda_n \longmapsto$ sign repn. of Σ_n.

But on $\bigoplus_{n\ge 0} R(\Sigma_n)$ we also have the coproduct given by restriction. I want to show this makes $\bigoplus_{n \geq 0} R(\Sigma_n)$ into a Hopf algebra. This means I have to prove commutativity of

$$\begin{tikzcd} R(\Sigma_i) \otimes R(\Sigma_j) \arrow[r, "\text{ind}"] \arrow[d, "\text{res}"] & R(\Sigma_{i+j}) \arrow[d, "\text{res}"] \\ \bigoplus_{\substack{a'+b'=i \\ a''+b''=j}} R(\Sigma_{a'}) \otimes R(\Sigma_{b'}) \otimes R(\Sigma_{a''}) \otimes R(\Sigma_{b''}) \arrow[r, "(\text{ind} \otimes \text{ind} \circ 1 \otimes T \otimes 1)"'] & \bigoplus_{a+b=i+j} R(\Sigma_a) \otimes R(\Sigma_b) \end{tikzcd}$$

which hopefully results from the Mackey formula.

$$\Sigma_a \times \Sigma_b \backslash \Sigma_n / \Sigma_i \times \Sigma_j$$

is the set of $\Sigma_a \times \Sigma_b$ orbits on splittings of $\{1, \ldots, n\}$ into an i and $j = n - i$ subset.

The sign representation λ_n of Σ_n restricted to $\Sigma_i \times \Sigma_j$ is $\lambda_i \otimes \lambda_j$, so one has

$$\Delta \lambda_n = \sum_{i+j=n} \lambda_i \otimes \lambda_j$$

for the coalgebra structure. Thus the Hopf alg $\bigoplus_{n \geq 0} R(\Sigma_n)$ corresponds to the algebraic group of series: $1 + a_1 t + a_2 t^2 + \cdots$.

Next we consider the case of the Green algebra $\bigoplus_{n\geq 0} R(G_n)$ where $G_n = GL_n(\mathbb{F}_q)$. Here the product is defined by

$$\alpha \cdot \beta = \mathrm{Ind}_{G_{i,j} \to G_{i+j}} \mathrm{Res}_{G_{i,j} \to G_i \times G_j} \alpha \otimes \beta$$

Is is true that this algebra is a Hopf algebra?

To prove this we have to use the Mackey formula. [redacted]

$$\mathrm{Res}_{Q \to G} \mathrm{Ind}_{P \to G} W = \bigoplus_{Q \times P} \mathrm{Ind}_{Q \cap {}^xP \to Q} \mathrm{Res}_{Q \cap {}^xP \to P} {}^xW$$

So I want to calculate

$$\mathrm{Ind}_{Q \twoheadrightarrow Q/Q^u} \mathrm{Res}_{Q \hookrightarrow G} \mathrm{Ind}_{P \hookrightarrow G} \mathrm{Res}_{P \twoheadrightarrow P/P^u} W$$

$$= \bigoplus_{Q \times P} \mathrm{Ind}_{Q \cap {}^xP \to Q/Q^u} \mathrm{Res}_{Q \cap {}^xP \to {}^x(P/P^u)} {}^xW$$

$$= \bigoplus_{Q \times P} \mathrm{Ind}_{Q \cap {}^xP / Q^u \cap {}^x(P/P^u) \to Q/Q^u} \mathrm{Res}_{Q \cap {}^xP / Q^u \cap {}^x(P/P^u) \to {}^xP/P^u} {}^xW$$

So now I want to apply this when $G = GL_n$, $P = G_{i,j}$, $Q = G_{a,b}$. $G/P = \mathrm{Grass}_i(F^n)$, so

$Q\backslash G/P =$ pairs (L^a, M^i) in F^n mod G action

Think of a representation of G_n as a functor on the groupoid of n-dim'l vector spaces. So if $F' \in \mathrm{Rep}(G_i)$, $F'' \in \mathrm{Rep}(G_j)$, then $F' \cdot F''$ Ind Res $F' \otimes F''$ is the functor

$$(F' \cdot F'')(V) = \sum_{L \in \mathrm{Grass}_i(V)} F'(L)\, F''(V/L)$$

I then want to restrict this to the category of vector spaces with a given a-dimensional subspace M, and to take that part which is invariant under Q^u.

So begin by dividing up according to the orbits of $G_{a,b} = \mathrm{Aut}(M \subset V)$ on $\mathrm{Grass}_i(V)$.

$$(F' \cdot F'')(V) = \sum_{\substack{\dim(L \cap M) = a' \\ 0 \le a' \le \min(a,i)}}' F'(L)\, F''(V/L).$$

?

I think the point is the following. Take the double coset QP, and consider the diagonal

$$\begin{tikzcd}
Q\cap P/Q^u\cap P^u \arrow[r, two heads] \arrow[d] & Q\cap P/Q\cap P^u \arrow[r, hook] \arrow[d] & P/P^u \\
Q\cap P/Q^u\cap P \arrow[r, two heads] \arrow[d, hook] & Q\cap P/(Q\cap P)^u & \\
Q/Q^u & &
\end{tikzcd}$$

Check this. $\mathrm{Lie}\left(B_\xi \cap B_\eta\right) = h + \sum\limits_{\substack{\alpha(\xi)\geq 0\\ \alpha(\eta)\geq 0}} g^\alpha$

$$\mathrm{Lie}\left(B_\xi \cap B_\eta / B_\xi \cap B_\eta^u\right) = h + \sum_{\substack{\alpha(\xi)\geq 0\\ \alpha(\eta)=0}} g^\alpha$$

$$\mathrm{Lie}\left(\left(B_\xi\cap B_\eta\right)^u\right) = \sum_{\substack{\alpha(\xi)\geq 0\\ \alpha(\eta)\geq 0}\ \text{and}\ \alpha(\xi)+\alpha(\eta)>0} g^\alpha$$

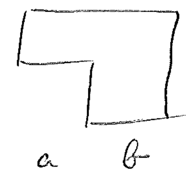
$$\mathrm{Lie}\left(\left(B_\xi\cap B_\eta / B_\xi \cap B_\eta^u\right)_{red}\right) = h + \sum_{\substack{\alpha(\xi)=0\\ \alpha(\eta)=0}} g^\alpha$$

Seems to be OKAY. Thus we get $(*)$ is equal to

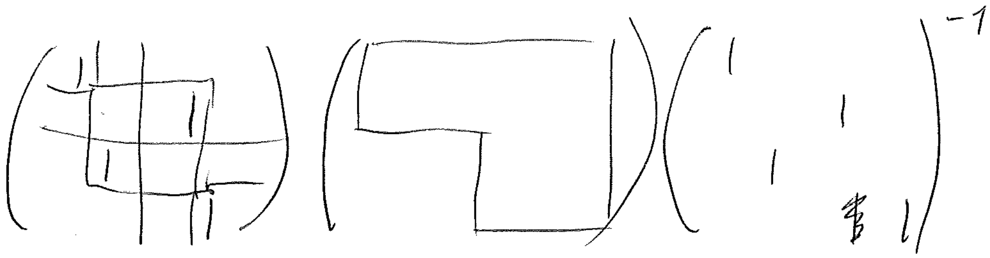
$$\bigoplus_{Q\times P} \mathrm{Ind}_{Q\cap {}^xP/Q^u\cap {}^xP \hookrightarrow Q/Q^u}\ \mathrm{Res}_{Q\cap {}^xP/Q^u\cap {}^xP \twoheadrightarrow Q\cap {}^xP/(Q\cap {}^xP)^u}$$

$$\mathrm{Ind} \qquad \twoheadrightarrow \qquad \mathrm{Res} \qquad \hookrightarrow$$

etc.

So I want to take $Q =$ 

and P is to be



$$= \begin{pmatrix} * & * & * & * \\ & & * & * \\ * & * & * & * \\ & & * & * \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

$$P = \begin{array}{c} \begin{pmatrix} * & * & * & * \\ & * & & * \\ * & * & * & * \\ & * & & * \end{pmatrix} \\ \;\; a' \;\; a'' \;\; b' \;\; b'' \end{array} \qquad Q = \begin{array}{c} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \\ \;\; a' \;\; a'' \;\; b' \;\; b'' \end{array}$$

$$\begin{cases} a'+b' = i & a''+b'' = j \\ a'+a'' = a, & b'+b'' = b. \end{cases}$$

$$Q \cap P = \begin{pmatrix} * & * & * & * \\ & * & & * \\ & & * & * \\ & & & * \end{pmatrix} \qquad (Q \cap P)^u = \begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

$$Q^u = \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad P^u = \begin{pmatrix} 1 & * & & * \\ & 1 & & \\ & * & 1 & * \\ & & & 1 \end{pmatrix}$$

Diagram on pg 8.

$$
\begin{tikzcd}
\begin{pmatrix} * & * & * & 0 \\ & * & & * \\ & & * & * \\ & & & * \end{pmatrix} \arrow[r, two heads] \arrow[d] & \begin{pmatrix} * & 0 & * & 0 \\ & * & & * \\ & & * & 0 \\ & & & * \end{pmatrix} \arrow[r, hook] \arrow[d, two heads] & \begin{pmatrix} * & 0 & * & 0 \\ & * & & * \\ * & 0 & * & 0 \\ & * & & * \end{pmatrix} \\
\begin{pmatrix} * & * & 0 & 0 \\ & * & & 0 \\ & & * & * \\ & & & * \end{pmatrix} \arrow[r, two heads] \arrow[d, hook] & \begin{pmatrix} * & 0 & 0 & 0 \\ & * & & 0 \\ & & * & 0 \\ & & & * \end{pmatrix} & \\
\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ & & * & * \\ & & * & * \end{pmatrix} & &
\end{tikzcd}
$$

Everything seems to be OK although a convincing proof would [redacted] be messy by this method. So lets assume that we can show $\bigoplus_{n \geq 0} R(G_n)$ is a Hopf algebra (bicommutative).

[redacted]

September 21, 1975

In addition to the [redacted] char. 0 representations, it should be possible to work into the theory the modular representations of $GL_n(\mathbb{F}_q)$. So I suppose $k \leftarrow A \rightarrow K$ is a d.v.r., A Henselian with $K \subset \mathbb{C}$ and $k = \bar{\mathbb{F}}_q$. Then I have for finite groups G maps

$$\begin{tikzcd} R_k(G) \arrow[r, "i_*"] & R_A(G) \arrow[r, "j^*"] \arrow[d, "i^*"', "\text{reduction}"] & R_K(G) \arrow[r] & 0 \\ & R_k(G) & & \end{tikzcd}$$

and $i_* = 0$ by a theorem of Swan. So I get a homomorphism

$$d : R_K(G) \longrightarrow R_k(G)$$

compatible with multiplication, induction, and restriction. d is even surjective with splitting given by the Brauer lifting. (Recall [redacted] that in the [redacted] context of non-semi-simple reps, induction is defined only for injections, and that [redacted] Frobenius reciprocity doesn't necessarily hold, so one can't extend it to surjections.)

So from this it is clear that the map

$$d : \bigoplus_n R_K(G_n) \longrightarrow \bigoplus_n R_k(G_n)$$

is a surjection of rings when both are given the Green product.

<u>Question</u>: Is $\bigoplus_n R_k(G_n)$ a Hopf algebra with coproduct defined via restricting to $G_a \times G_b \subset G_n$, $a+b=n$?

First notice that if $G' = G/N$ where N is a p-group, then any irreducible k-representation V of G comes from G'. In effect $V^N \neq 0$ and it is G invariant, so $V^N = V$. Thus $R_k(G') \xrightarrow{\sim} R_k(G)$. In particular

$$R_k(G_a \times G_b) \xrightarrow{\sim} R_k(G_{a,b})$$

Now I want to prove commutativity of

$$\begin{tikzcd}
R_k(G_i)\otimes R_k(G_j) = R_k(G_{i,j}) \arrow[r, "\mathrm{ind}"] \arrow[d, "\mathrm{res}\otimes\mathrm{res}"] & R_k(G_n) \arrow[d, "\mathrm{res.}"] \\
\bigoplus_{\substack{a'+b'=i \\ a''+b''=j}} R_k(G_{a'})\otimes R_k(G_{b'})\otimes R_k(G_{a''})\otimes R_k(G_{b''}) \arrow[r, "(\mathrm{ind}\otimes\mathrm{ind})(1\otimes T\otimes 1)"'] & R_k(G_a)\otimes R_k(G_b)
\end{tikzcd}$$

Put $R_k = R'$

By the Mackey formula the composite

$$R'(G_{i,j}) \xrightarrow{\text{ind}} R'(G_n) \xrightarrow{\text{res}} R'(G_{a,b})$$

is the sum of maps for each (a', a'', b', b'') such that $a'+a''=a$, $b'+b''=b$, $a'+b'=i$, $a''+b''=j$. Recall

$$P = G_{i,j} \qquad Q = G_{a,b} \qquad x = \begin{array}{c} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \\ \;\; a' \;\; a'' \;\; b' \;\; b'' \end{array}$$

$$Q \cap {}^xP = \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ & & \ast & \ast \\ & & \ast & \ast \end{pmatrix} \cap \begin{pmatrix} \ast & \ast & \ast & \ast \\ & \ast & & \ast \\ \ast & \ast & \ast & \ast \\ & \ast & & \ast \end{pmatrix} = \begin{pmatrix} \ast & \ast & \ast & \ast \\ & \ast & & \ast \\ & & \ast & \ast \\ & & & \ast \end{pmatrix}$$

So the term corresponding to (a', a'', b', b'') is the composite

$$\begin{tikzcd}
R'(P) \arrow[r, phantom, "\cong"] & R'({}^xP) \arrow[r, "\text{res}"] & R'(Q \cap {}^xP) \arrow[r, "\text{ind}"] \arrow[d, "\boxed{\text{ind}}"] & R'(Q). \\
R'\begin{pmatrix} \ast & \ast & & \\ \ast & \ast & & \\ & & \ast & \ast \\ & & \ast & \ast \end{pmatrix} \arrow[u, "\wr"'] \arrow[r, phantom, "\cong"] \arrow[d, "\text{res}"] & R'\begin{pmatrix} \ast & & \ast & \\ & \ast & & \ast \\ \ast & & \ast & \\ & \ast & & \ast \end{pmatrix} \arrow[u, "\wr"'] \arrow[d, "\text{res}"] & R'\begin{pmatrix} \ast & \ast & \ast & \ast \\ & \ast & \ast & \ast \\ & & \ast & \ast \\ & & & \ast \end{pmatrix} \arrow[r, "\text{ind}"] & R'\begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ & & \ast & \ast \\ & & \ast & \ast \end{pmatrix} \arrow[u, equal] \\
R'\begin{pmatrix} \ast & & & \\ & \ast & & \\ & & \ast & \\ & & & \ast \end{pmatrix} \arrow[r, "1 \otimes T \otimes 1", "\sim"'] & R'\begin{pmatrix} \ast & & & \\ & \ast & & \\ & & \ast & \\ & & & \ast \end{pmatrix} \arrow[r, "\sim"] \arrow[uur, leftrightarrow] \arrow[ur, "\sim"] & R'\begin{pmatrix} \ast & \ast & & \\ & \ast & & \\ & & \ast & \ast \\ & & & \ast \end{pmatrix} \arrow[r, "\text{ind}"] \arrow[u, "\wr"'] & R'\begin{pmatrix} \ast & \ast & & \\ \ast & \ast & & \\ & & \ast & \ast \\ & & \ast & \ast \end{pmatrix} \arrow[u, "\wr"']
\end{tikzcd}$$

Seems not to work because of the induction in the box $\diamond$. Off by $q^{b'a''}$.

It seems to me that in the arguments using the Mackey formula there must be something worth axiomatizing.

[redacted] It seems one has a functor $R(G)$ contravariant in G, with induction maps. [redacted]
I need the Mackey formula

(1) $$\mathrm{res}_{Q\to G}\ \mathrm{ind}_{P\to G} W = \sum_{QxP} \mathrm{ind}_{Q\cap {}^xP\to Q}\ \mathrm{res}_{Q\cap {}^xP\to {}^xP}\ {}^xW$$

I will want $R(G)$ to be independent of [redacted] the unipotent radical, i.e. it depends only on G/G^u. The key appears to be the following: I have to be able [redacted] to replace the above formula by

(2) $$\mathrm{res}_{Q/Q^u \dashrightarrow G}\circ \mathrm{ind}_{P/P^u \dashrightarrow G} W = \sum_{QxP} \mathrm{ind}_{(Q\cap {}^xP)/(Q\cap {}^xP)^u \dashrightarrow Q/Q^u}\circ \mathrm{res}_{O \dashrightarrow P/P^u}\ {}^xW$$

because this is the effective way to use the Bruhat decomposition

~~Suppose next that one has a functor~~

<u>Example:</u> Take $R(G)$ = mod l cohomology of G.

How can we derive (2) from (1)? Let me try to define $\mathrm{ind}_{P/P^u \dashrightarrow G} W$ as $\mathrm{ind}_{P \to G}\, \mathrm{res}_{P \to P/P^u} W$.

$$\begin{tikzcd}
Q \cap P \arrow[rrr, hook] \arrow[dr] \arrow[ddd, hook] & & & P \arrow[d, two heads] \\
& Q\cap P/Q^u \cap P^u \arrow[r, two heads] \arrow[d, two heads] & Q \cap P/Q \cap P^u \arrow[r, hook] \arrow[d, two heads] & P/P^u \\
& Q \cap P/Q^u \cap P \arrow[r, two heads] \arrow[dd, hook] & (Q\cap P)/(Q \cap P)^u & \\
& & & \\
Q \arrow[r, two heads] & Q/Q^u & &
\end{tikzcd}$$

Now, how do I define $\mathrm{res}_{Q/Q^u \dashrightarrow G}$? First try is $\mathrm{ind}_{Q \to Q/Q^u}\, \mathrm{res}_{Q \hookrightarrow G}$, however ind is only defined for embeddings. In this example, I should try a scalar (e.g. order of Q^u) times the isomorphism $R(Q) \xleftarrow{\sim} R(Q/Q^u)$. In fact, because $Q \simeq M \ltimes Q^u$ one has

$$\begin{tikzcd}
R(M) \arrow[r, bend left, "\mathrm{ind}"] \arrow[r, bend right, "\sim"', "\mathrm{res}"'] & R(Q)
\end{tikzcd}
\qquad
\begin{tikzcd}
BM \arrow[d, "f"] \\
BQ
\end{tikzcd}$$

and $\mathrm{ind} \circ \mathrm{res} = f_* f^* = [Q:M] = |Q^u|$. Thus $\mathrm{ind}_{Q \to Q/Q^u}$ should be division by $|Q^u|$. Consequently $\mathrm{res}_{Q/Q^u \dashrightarrow G}$ should be $\mathrm{res}_{Q \to G}$ followed by division by $|Q^u|$.

So I want to start with $\alpha \in R(P/P^u)$ and compare

$$\frac{1}{|Q^u|} \, \mathrm{ind}_{Q\cap P \to Q} \, \mathrm{res}_{Q\cap P \to P/P^u} (\alpha)$$

with

$$\mathrm{ind}_{(Q\cap P)_r \to Q_r} \, \mathrm{res}_{(Q\cap P)_r \to P_r} \, \alpha \qquad G_r = G/G^u$$

Let $\alpha' =$ Im of α in $R(Q\cap P/Q^u \cap P^u)$.
We then want to see if

$$\frac{1}{|Q^u|} \, \mathrm{Ind}_{Q\cap P \hookrightarrow P} \, \mathrm{Res}_{Q\cap P \to Q\cap P/Q^u\cap P} \, \alpha'$$

$$= \frac{|Q^u \cap P^u|}{|Q^u \cap P|} \, \mathrm{Res}_{Q \to Q/Q^u} \, \mathrm{Ind}_{Q\cap P/Q^u \cap P \to Q/Q^u} \, \alpha'$$

This doesn't work.

I was unable to find a variant definition which would make the formulas work.

Sept. 23, 1975

Amazing fact: If $f: G \to G'$ is a homomorphism of finite groups, then all of the functors

$$\mathrm{Modf}(\mathbb{C}[G]) \rightleftarrows \mathrm{Modf}(\mathbb{C}[G'])$$

induce maps on the Grothendieck groups. In fact I know that $f_!$ and f_* have the same effect. Thus we get maps $f_*: R(G) \to R(G')$ defined for all f with the usual functorial properties such that $f_* f^* = 1$ for f surjective.

With modular representations however $f_!, f_*$ will not be exact functors, unless the kernel of f is prime to the characteristic p. Thus we do not get f_* maps except for injections, or such f.

Let's look at $H^*(G) = H^*(G, \mathbb{F}_\ell)$, where ℓ is a prime number dividing $q-1$. In this case it should be so that $\bigoplus_n H^*(G_n)$, $G_n = GL_n(\mathbb{F}_q)$ is a Hopf algebra in a new way.

Recall that $\bigoplus_n H_*(G_n)$ is an algebra with product given by the maps

$$H_*(G_a) \otimes H_*(G_b) \longrightarrow H_*(G_a \times G_b) \longrightarrow H_*(G_{a+b})$$

So this gives me a coproduct on $\oplus H^*(G)$ with $\Delta\alpha = \sum_{a+b=n} \mathrm{res}_{G_a \times G_b \longrightarrow G_n} \alpha$.

In addition $\oplus H_*(G_n)$ has a coproduct given by $\Delta : G_n \longrightarrow G_n \times G_n$.

I can characterize the ring structure on $\bigoplus_{n\geq 0} H_*(G_n)$ by identifying gr. ring homs.

$$\bigoplus_{n\geq 0} H_*(G_n) \longrightarrow R = \bigoplus_{n\geq 0} R_n$$

with ~~character~~ exponential characteristic classes

$$\theta : \left\{\begin{array}{l}\text{iso. classes of reps.}\\ \text{of } G \text{ over } \mathbb{F}_q\end{array}\right\} \longrightarrow \prod_i H^i(G, R_i)$$

Here "exponential" means $\theta(W \oplus V) = \theta(W) \cdot \theta(V)$ and that $\theta(0) = 1$. The coproduct on $\oplus H_*(G_n)$ corresponds to the operation of pointwise multiplying two exponential classes.

I want to define another coproduct on $\oplus H_*(G_n)$ this time using induction (transfer map). Dually, it is the map

$$H^*(G_a) \otimes H^*(G_b) \xrightarrow{\sim} H^*(G_a \times G_b) \xrightarrow{\sim} H^*(G_{a,b}) \xrightarrow{\mathrm{ind}} H^*(G_{a+b})$$

so in homology it is the composition

$$H_*(G_{a+b}) \xrightarrow{\mathrm{ind}} H_*(G_{a,b}) \simeq H_*(G_a) \otimes H_*(G_b).$$

Now I wish to check carefully that this coproduct operation is compatible with product:

$$
\begin{tikzcd}
H_*(G_i)\otimes H_*(G_j) \arrow[d] & H_*(G_{i,j}) \arrow[l, "\sim"'] \arrow[r] & H_*(G_n) \arrow[d] \\
\bigoplus\limits_{\substack{a'+b'=i\\ a''+b''=j}} H_*(G_{a',b'})\otimes H_*(G_{a'',b''}) \arrow[d, "\wr"] & & \bigoplus\limits_{a+b=n} H_*(G_{a,b}) \arrow[d, "\wr"] \\
\text{"}\quad H_*(G_{a'})\otimes H_*(G_{b'})\otimes H_*(G_{a''})\otimes H_*(G_{b''}) \arrow[dr, "\wr"] & & \bigoplus\limits_{a+b=n} H_*(G_a)\otimes H_*(G_b) \\
& H_*(G_{a'})\otimes H_*(G_{a''})\otimes H_*(G_{b'})\otimes H_*(G_{b''}) \arrow[ur] &
\end{tikzcd}
$$

By the Mackey formula $H_*(G_{i,j}) \longrightarrow H_*(G_n) \longrightarrow H_*(G_{a,b})$ is a sum of terms for each a', a'', b', b''. The term corresponding to such indices is

$$
G_{i,j} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}
\xrightarrow[\sim]{x}
\underset{P}{\begin{pmatrix} * & * & * & * \\ & * & & * \\ * & * & * & * \\ & * & & * \end{pmatrix}}
\xrightarrow[\text{tr}]{}
\underset{P\cap Q}{\begin{pmatrix} * & * & * & * \\ & * & & * \\ & & * & * \\ & & & * \end{pmatrix}}
\xrightarrow[\text{in}]{}
\underset{Q}{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}}
$$

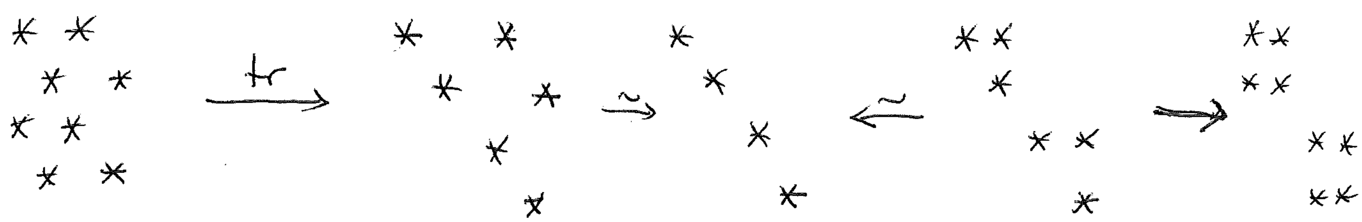
I want to show it is the same as

$$
\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}
\xrightarrow{\text{qu.}}
\begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix}
\xrightarrow{\text{tr}}
\begin{pmatrix} * & * & & \\ & * & & \\ & & * & * \\ & & & * \end{pmatrix}
\xrightarrow{\text{Gal}}
$$

$$P/P^u \xleftarrow{\ \simeq\ } P \xrightarrow{tr} P \cap Q \longrightarrow Q \longrightarrow Q/Q^u$$

$$\begin{array}{l} P/P^u \xrightarrow{tr} P\cap Q \cdot P^u/P^u \xrightarrow{\simeq} P\cap Q/(P\cap Q)^u \xleftarrow{\simeq} P \cap Q \cdot Q^u/Q^u \longrightarrow Q/Q^u \\ \parallel \end{array}$$

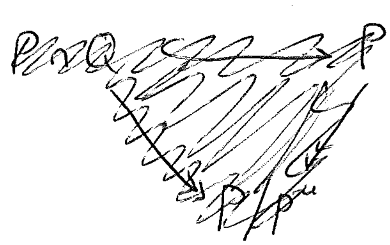
[redacted]



So therefore I am trying to show the comm. of ~~the~~ traipsing thru the diagram.

$$\begin{tikzcd}
P\cap Q \arrow[rrr, hook, "\overleftarrow{tr}"] \arrow[dr, "s"] \arrow[ddd, hook] & & & P \arrow[d, "s"] \\
& P\cap Q/P^u \cap Q^u \arrow[r, "\cong"] \arrow[d, "\cong", "\text{commutes}"'] & P\cap Q \cdot P^u/P^u \arrow[r, hook, "\overleftarrow{tr}"] \arrow[d, "\simeq"] & P/P^u \\
& P\cap Q \cdot Q^u/Q^u \arrow[r, "\cong"] \arrow[dd, hook] & P \cap Q/(P\cap Q)^u & \\
Q \arrow[dr, "s"] & & & \\
& Q/Q^u & &
\end{tikzcd}$$

So I have to establish commutativity of the top square



$$\begin{tikzcd}
P\cap Q \arrow[r, hook, "i"] \arrow[dr, two heads, "p"'] & P\cap Q\cdot P^u \arrow[r, hook, "g'"] \arrow[d, "f'"] & P \arrow[d, "f"] \\
 & P\cap Q\cdot P^u/P^u \arrow[r, hook, "g"] & P/P^u
\end{tikzcd}$$

$$p_* i^! g'^! = f'_* i_* i^! g'^! = [P\cap Q : P\cap Q\cdot P^u]\, f'_* g'^!$$
$$= [P\cap Q : P\cap Q\cdot P^u]\, g^! f_*$$

where $f^!$ denotes the transfer. Since

$$[P\cap Q : P\cap Q\cdot P^u] = \frac{|P\cap Q|\cdot|P^u\cap Q|}{|P\cap Q|\cdot|P^u|} = \frac{|P^u\cap Q|}{|P^u|}$$

is a power of q and $q\equiv 1 \mod l$, we see $p_* i^! g'^! = g^! f_*$. QED.

Now I recall that

$$\bigoplus H_*(G_n) = \mathbb{F}_l[\xi_0, \xi_1, \ldots] \otimes \Lambda[\eta_1, \ldots]$$

where ξ_i is the good basis element of $H_{2i}(G_1)$ and η_i is the ________________ of $H_{2i-1}(G_1)$. So to determine the coproduct ψ defined above, it suffices to give $\psi(\xi_i)$, $\psi(\eta_i)$. These are clearly primitive

$$G_1 \supset G_{0,1} = G_0 \times G_1$$

A homomorphism $\bigoplus H_*(G_n) \xrightarrow{\alpha} R_*$ is the same thing as a power series

$$\sum_{n\geq 0} \alpha(\xi_n)\, x^n + \alpha(\eta_n)\, x^{n-1} y$$

i.e. element of $H^0(G_1, R)$. We've seen that the "additive" coproduct defined using transfer corresponds to addition of power series. The "multiplicative" product which is defined using diagonal maps $G_n \to G_n \times G_n$ corresponds to the pointwise multiplication of power series, where x and y refer ~~the~~ to the distinguished elements of $H^2(G_1)$ and $H^1(G_1)$ resp.

Try next $\bigoplus_{n\geq 0} R(\Sigma_n)^\vee$, where $R(\Sigma_n)$ is the complex representation ring of Σ_n. We've seen that

$$\bigoplus_{n\geq 0} R(\Sigma_n)^\vee = \mathbb{Z}[\lambda_1, \lambda_2, \cdots]$$

where $\lambda_i \in R(\Sigma_i)^\vee$ gives the inner product with the sign representation. Take $\Delta^+(\lambda_i)$ where $+$ refers to the coproduct defined using transfer from [redacted] $\Sigma_{a,b}$ to Σ_{a+b}. So it is necessary first of all to understand what representation of Σ_{a+b} we get by inducing the sign representation on $\Sigma_{a,b}$.

$$R(\Sigma_n)^\vee \xrightarrow[t_{ind}]{} R(\Sigma_a \times \Sigma_b)^\vee$$

Since

$$\begin{array}{ccc} R(\Sigma_n)^\vee & \xrightarrow{\ \mathrm{ind}^\vee\ } & R(\Sigma_a \times \Sigma_b)^\vee \\ \uparrow s & & \uparrow s \\ R(\Sigma_n) & \xrightarrow{\ \mathrm{res}\ } & R(\Sigma_a \times \Sigma_b) \end{array}$$

commutes and $\mathrm{sgn}_n \longmapsto \lambda_n$ one has

$$\mathrm{ind}^\vee_{\Sigma_a \times \Sigma_b \to \Sigma_n} \lambda_n = \lambda_a \otimes \lambda_b$$

Thus $$\Delta^+(\lambda_n) = \sum_{a+b=n} \mathrm{ind}^\vee_{\Sigma_a \times \Sigma_b \to \Sigma_n} \lambda_n = \sum_{a+b=n} \lambda_a \otimes \lambda_b$$

as it should be.

Next we have $\Delta^\times$ to calculate. The point to keep in mind here is that $\oplus R(\Sigma_n)^\vee$ has a natural interpretation as operations on K-theory as follows. Given a bundle E and $\varphi \in R(\Sigma_n)^\vee$, then $\varphi(E)$ is what you get by decomposing $E^{\otimes n}$ according to the irred. repns of Σ_n and then capping with φ.

$$[E^{\otimes n}] \in K(X) \otimes R(\Sigma_n) \longrightarrow K(X)$$

September 24, 1975

I am trying to calculate the ring $\bigoplus R(\Sigma_n)^\vee$ with its two coproducts. This ring is to be interpreted as operations in K-theory.

My previous approach was to take the power operations maps

$$K(X) \longrightarrow K_{\Sigma_n}(X^n) \qquad\qquad E \longmapsto E^{\otimes n}$$

and to fit them all together. I formed

$$\prod^* K_{\Sigma_n}(X^n) = \left\{ (\alpha_n) \in \prod_n K_{\Sigma_n}(X^n) \,\middle|\, \mathrm{res}_{\Sigma_i \times \Sigma_j \to \Sigma_n} \alpha_n = \alpha_i \otimes \alpha_j \right\}$$

~~[illegible]~~ Restrict to diagonal:

$$\begin{aligned} K_{\Sigma_n}(X) &= R(\Sigma_n) \otimes K(X) \\ &= \mathrm{Hom}(R(\Sigma_n)^\vee, K(X)) \end{aligned}$$

Thus

$$\prod^* K_{\Sigma_n}(X) = \left\{ (\alpha_n) \,\middle|\, \alpha_n : R(\Sigma_n)^\vee \longrightarrow K(X) \text{ s.t.} \right.$$

$$\begin{tikzcd} R(\Sigma_i)^\vee \otimes R(\Sigma_j)^\vee \arrow[r, "\alpha_i \otimes \alpha_j"] \arrow[d, "\mathrm{res}^\vee"] & K(X) \otimes K(X) \arrow[d] \\ R(\Sigma_{i+j}) \arrow[r, "\alpha_{i+j}"] & K(X) \end{tikzcd}$$

$$= \mathrm{Hom}_{\text{alg.}}\left(\bigoplus_{n \geq 0} R(\Sigma_n)^\vee, K(X) \right)$$

On the other hand on $\Pi^* K_{\Sigma_n}(X^n)$ I put a ring structure by defining

$$(\alpha+\beta)_n = \sum_{i+j=n} \mathrm{ind}_{\Sigma_i \times \Sigma_j \to \Sigma_n} \alpha_i \otimes \beta_j$$

$$(\alpha\cdot\beta)_n = \alpha_n \beta_n$$

Note that these formulas work for $\alpha_n = E^{\otimes n}$ because

$$(E\oplus F)^{\otimes n} = \sum_{i+j} \mathrm{ind}_{\Sigma_i\times\Sigma_j \to \Sigma_n} E^{\otimes i} \otimes F^{\otimes j}$$

Let me check the distributive law

$$((\alpha+\beta)\cdot\gamma)_n = (\alpha+\beta)_n \gamma_n = \Big(\sum_{i+j=n} \mathrm{ind}_{\Sigma_i\times\Sigma_j\to\Sigma_n} \alpha_i\otimes\beta_j\Big)\gamma_n$$

Look at the (i,j)-th term and let $f : \Sigma_i\times\Sigma_j \longrightarrow \Sigma_n$ be the inclusion.

$$\begin{aligned} f_*(\alpha_i\otimes\beta_j)\cdot\gamma_n &= f_*\big((\alpha_i\otimes\beta_j) f^*\gamma_n\big) \\ &= f_*\big((\alpha_i\otimes\beta_j)(\gamma_i\otimes\gamma_j)\big) \\ &= f_*(\alpha_i\gamma_i\otimes\beta_j\gamma_j) \end{aligned}$$

$$\therefore\quad ((\alpha+\beta)\cdot\gamma)_n = \sum_{i+j=n} \mathrm{ind}_{\Sigma_i\times\Sigma_j\to\Sigma_n} \alpha_i\gamma_i\otimes\beta_j\gamma_j$$

$$= (\alpha\cdot\gamma+\beta\cdot\gamma)_n .$$

Segal remark: Consider functors from finite sets + autos to vector spaces, i.e. vector bundles over the category of finite sets + autos; denote this category by Σ_*. The Grothendieck group of such bundles is $\bigoplus_n R(\Sigma_n)$; denote this $R(\Sigma_*)$. The operations of union and product give functors

$$\Sigma_* \times \Sigma_* \longrightarrow \Sigma_*$$

hence to maps $R(\Sigma_*) \longrightarrow R(\Sigma_* \times \Sigma_*)$. In the case of the union, this map comprises the maps

$$R(\Sigma_n) \xrightarrow{\text{res}} \bigoplus_{i+j=n} R(\Sigma_i \times \Sigma_j)$$

and for direct product it comprises

$$R(\Sigma_n) \longrightarrow \bigoplus_{ij=n} R(\Sigma_i \times \Sigma_j).$$

Therefore it is clear at least that union gives a map $\Delta^+ : R(\Sigma_*) \longrightarrow R(\Sigma_*) \otimes R(\Sigma_*)$

Actually since
$$R(\Sigma_* \times \Sigma_*) = \bigoplus_{i,j} R(\Sigma_i \times \Sigma_j)$$
$$= \bigoplus_{i,j} R(\Sigma_i) \otimes R(\Sigma_j)$$
we also get $\Delta^\times$.

Question: Is $\bigoplus R(\Sigma_n)$ equipped with its usual product (cup product in each $R(\Sigma_n)$) and with $\Delta^+, \Delta^\times$ ring object?

Answer is probably that Δ^+ can't have an antipode until one stabilizes.

Go back to $\bigoplus R(G_n)^\vee$ with $G_n = GL_n(\mathbb{F}_q)$. We have made this into a Hopf algebra with product defined by

$$R(G_a)^\vee \otimes R(G_b)^\vee \xrightarrow{ind^\vee} R(G_{a,b})^\vee \xrightarrow{res^\vee} R(G_{a+b})^\vee$$

and with coproduct defined by

$$\Delta^+ : R(G_n)^\vee \xrightarrow{ind^\vee} \bigoplus R(G_{ab})^\vee \xrightarrow{res^\vee} \bigoplus R(G_a)^\vee \otimes R(G_b)^\vee$$

This is a self-dual Hopf algebra. One also has

$$\Delta^\times : R(G_n)^\vee \longrightarrow R(G_n \times G_n)^\vee = R(G_n)^\vee \otimes R(G_n)^\vee$$

induced by the diagonal of G_n; the corresponds to the product of representations.

To show $\bigoplus R(G_n)^\vee$ is a ring object, suppose we have homomorphisms $\alpha = (\alpha_n) : \bigoplus R(G_n)^\vee \longrightarrow R$. and $\beta = (\beta_n)$, $\gamma = (\gamma_n)$. I can identify $\alpha_n : R(G_n)^\vee \to R$ with an elt. α_n of $R(G_n) \otimes R$. Then the condition that α is a ring homom. is that

$$\begin{tikzcd}
R(G_a)^\vee \otimes R(G_b)^\vee \arrow[r, "\alpha_a \otimes \alpha_b"] \arrow[d, "\mathrm{ind}^\vee"] & R \otimes R \arrow[dd, "\mathrm{prod.}"] \\
R(G_{ab})^\vee \arrow[d, "\mathrm{res}^\vee"] & \\
R(G_{a+b})^\vee \arrow[r, "\alpha_{a+\beta}"] & R
\end{tikzcd}$$

i.e.

$$\mathrm{ind}_{G_{a,b} \to G_a \times G_b} \, \mathrm{res}_{G_{a,b} \to G_{a+b}} \, \alpha_{a+b} = \alpha_a \otimes \alpha_b$$

Addition $\alpha + \beta$ is defined by

$$(\alpha+\beta)_n = \sum_{a+b=n} \mathrm{ind}_{G_{a,b} \to G_{a+b}} \, \mathrm{res}_{G_{a,b} \to G_a \times G_b} \, \alpha_a \otimes \beta_b .$$

So

$$\big((\alpha+\beta)\cdot\gamma\big)_n = \Big(\sum_{a+b=n} \mathrm{ind}_{G_{a,b} \hookrightarrow G_n} \, \mathrm{res}_{G_{a,b} \to G_a \times G_b} \, \alpha_a \times \beta_b\Big) \cdot \gamma_n$$

Calculate the a,b-th term; lets label arrows

$$G_a \times G_b \xleftarrow{j} G_{a,b} \xhookrightarrow{i} G_n$$

The term in question is

$$i_* j^*(\alpha_a \otimes \beta_b) \cdot \gamma = i_*\big(j^*(\alpha_a \otimes \beta_b) \cdot i^*\gamma\big)$$

?

Doesn't seem to work. Maybe this means I ought to go back to $\oplus R(G_n)$.

Recall that $\mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{C})$ has a basis given by the different conjugacy classes in G. What is a conjugacy class in G_n? These are represented by the different Jordan forms of invertible matrices. Look at unipotents first; there is one Jordan form for each partition of n. Next look at a semi-simple matrix. It is determined up to conjugacy by its characteristic polynomial. The building blocks would consist of all irreducible monic polys, i.e. orbits of Frobenius $x \mapsto x^q$ on $\overline{\mathbb{F}}_q^*$. So a general conjugacy class in G_n would be given by a map from irreducible monic polys. to partitions (a decreasing sequence of nat. nos. $a_1 \geq \cdots \geq a_i \geq \cdots$), denote it $f \longmapsto \pi_f$, such that $\sum_f |\pi_f| = d$.

Therefore under direct sum what I find is something like a set of generators corresponding to irreducible modules tensored with the standard unipotent classes of various degrees.

Recall the formula for the product on the algebra $\oplus R(G_n)$. Let $\alpha \in R(G_p)$, $\beta \in R(G_q)$ then for c a conjugacy class in G_n we have

$$(\alpha . \beta)(c) = \sum_{c', c''} g^{c}_{c'c''} \, \alpha(c') \beta(c'')$$

where the sum is taken over conjugacy classes c' in G_p and c'' in G_q. If $\theta \in c$, then $g^{c}_{c';c''}$ is the number of subspaces $0 < L^p < \mathbb{F}_q^{p+q}$ stable under θ such that $\theta | L^p \in c'$, $\theta | \mathbb{F}^{p+q}/L \in c''$. Therefore the coalgebra $\bigoplus_{n\geq 0} R(G_n)^\vee \otimes \mathbb{C}$ has as basis the conjugacy classes c with the coproduct

$$\Delta^+ c = \sum g^{c}_{c'c''} \; c' \otimes c''$$

The algebra structure is probably the following:

$$c' \cdot c'' = \sum_{c} g^{c}_{c'c''} \; c$$

In effect associate to c the central function δ_c on G which is the char. function of c. Then $c \mapsto \delta_c$ gives the duality between $R(G)^\vee \otimes \mathbb{C}$ and $R(G) \otimes \mathbb{C}$ and it transforms mult. into Δ^+.

$$(\delta_{c'} \cdot \delta_{c''})(c) = \sum g^{c}_{a'a''} \; \delta_{c'}(a')\,\delta_{c''}(a'') = g^{c}_{c'c''}$$

$$\therefore \; \delta_{c'} \cdot \delta_{c''} = \sum g^{c}_{c'c''} \, \delta_c$$

Consequences of this formula. Denote by F the set of irreducible monic polynomials over $\mathbb{F}_q$. Suppose we let C_f be the set of conjugacy classes of matrices with characteristic polynomial a power of f. Then $\mathbb{C}[C_f] \subset \mathbb{C}[C] \cong \bigoplus R(G_n)^\vee \otimes \mathbb{C}$.

Sept 26, 1976

It will be more convenient from now ~~on~~ to consider the algebra $\oplus R(G_n) \otimes \mathbb{C} \cong \mathbb{C}[\mathcal{C}]$ with basis δ_c, $c \in \mathcal{C}$. I claim $\mathbb{C}[\mathcal{C}_f]$ is a subalgebra. In effect if $c', c'' \in \mathcal{C}_f$ and if $g^{c}_{c'c''} \neq 0$ then $c \in \mathcal{C}_f$. This is because if I have an auto θ of $W \subset V$ such that $\theta / W \in c'$ and θ on V/W is in c'', then the char. poly of θ is the product of the char. polys. of θ_W and $\theta_{V/W}$.

Next point is to consider an auto θ of V and let its char. poly factor

$$\det(X - \theta) = f_1^{r_1} \cdots f_s^{r_s}.$$

Then V is canonically a direct sum

$$V = V_1 \oplus \cdots \oplus V_s$$

with $\det(X - \theta_{V_i}) = f_i^{r_i}$. There is a unique flag $0 < W_1 < W_2 < \cdots < W_s = V$ stable under θ such that $\theta_{W_i / W_{i-1}}$ has char. poly $f_i^{r_i}$, namely $W_i = V_1 \oplus \cdots \oplus V_i$. Let c_i be the iso. class of θ_{V_i}. Then one has

$$\delta_{c_1} \cdots \delta_{c_s} = \delta_c$$

where c is the class of θ.

I should check this carefully. The claim is that any conjugacy class c has a primary decomp.

We have the idea of a primary conjugacy class (char poly is primary). Any conjugacy class c has primary components c_f, $f \in \mathcal{F}$. The map

$$\mathcal{C} \xrightarrow{\sim} \prod{}' \mathcal{C}_f$$

($'$ denotes restricted direct product) is an isom. If $c_i \in \mathcal{C}_{f_i}$ with f_i distinct and $g^{c}_{c_1 \cdots c_s} \neq 0$, then c is a conjugacy class with primary components c_i; it is the unique class with these components and $g^{c}_{c_1 \cdots c_s} = 1$.

It is clear that we get:

<u>Proposition</u>: Let $\mathcal{C}$ be the set of iso classes of pairs (V, θ) consisting of an autom. of a $\mathbb{F}_q$-vector space of finite dimensional; for each irreducible monic poly f over $\mathbb{F}_q$ let $\mathcal{C}_f$ be those iso. classes of (V, θ) such that θ has characteristic polynomial a power of f. Form an algebra $\mathbb{Z}[\mathcal{C}]$ by putting

$$\delta_{c'} \cdot \delta_{c''} = \sum_{c} g^{c}_{c', c''} \, \delta_c$$

Then $\mathbb{Z}[\mathcal{C}]$ is the tensor product of the algebras $\mathbb{Z}[\mathcal{C}_f]$ as f runs over $\mathcal{F}$.

$$\mathbb{Z}[\mathcal{C}] = \bigotimes_{f \in \mathcal{F}} \mathbb{Z}[\mathcal{C}_f]$$

The next point should be that the algebras $\mathbb{Z}[C_f]$ should be isomorphic to $\mathbb{Z}[C_1]$ for the extension $\mathbb{F}_q[T]/(f(T))$.

Let's start with an element of C_f given by $c = cl(V, \theta)$. Take the element θ and factor it $\theta_s \theta_u$ with θ_s semi-simple and θ_u unipotent. Because θ has characteristic polynomial f^N, so does θ_s, hence θ_s has minimal polynomial f. This means that V becomes a $\mathbb{F}_q[T]/(f(T))$ module in a canonical way with T acting as θ_s. In fact it should be clear that any (V, θ) is the same thing as a fd vector space over $\mathbb{F}_q[T]/f(T) = F'$ together with a unipotent operator in this vector space. Similarly subspaces of V invariant under θ are the same as F'-subspaces invariant under the unipotent operator. So we have:

<u>Prop.</u> One has a (canonical) algebra isomorphism

$$\mathbb{Z}[C_f^F] \simeq \mathbb{Z}[C_1^{F'}]$$

where $F' = F[T]/(f(T))$.

Next problem: Structure of the algebra $\mathbb{Z}[\mathcal{C}_1]$. This algebra has as basis the conjugacy classes of unipotents, which by Jordan canonical forms can be identified with partitions. This is called the Hall algebra $\mathcal{H}$ in Springer's talk. Notation: λ is a partition, c_λ is the corresponding unipotent, $g^{\nu}_{\lambda\mu} = g^{c_\nu}_{c_\lambda c_\mu}$ and $\mathcal{H}$ has basis e_λ indexed by partitions such that

$$e_\lambda e_\mu = \sum g^{\nu}_{\lambda\mu}\, e_\nu$$

Let $x_n = e_{\{1^n\}}$ where $\{1^n\}$ denotes the partition $\{1,\dots,1\}$ n-times.

$\mathcal{C}_1 \longrightarrow \mathbb{N}$ sends a partition λ to $|\lambda|$. Let $P_n = \{$partitions of $n\}$.

P_0	contains	ϕ	empty seq.
P_1		(1)	
P_2		$(2), (1,1)$	
P_3		$(3), (2,1), (1,1,1)$	

$$e_{(1)} \cdot e_{(1)} = \alpha e_{(2)} + \beta e_{(1,1)}$$

To compute α we take a unipotent operator with Jordan form corresp. to partition (2), and count invars.

subspaces of dimension 1. There is exactly 1 so $\alpha = 1$. Clearly β is the number of lines in F^2 which is $q+1$. !

$$e_{(1)}^2 = e_{(2)} + (q+1)\, e_{(1,1)}.$$

So we need the generators $e_{(1)}, e_{(1,1)}$ so far. So we want to use the generators $x_n = e_{(1,\dots,1)}$ n-times. Check that $x_1^3, x_1 x_2, x_3$ form a basis for the degree 3 space.

Notice that there is a basic ordering on conjugacy classes, namely, $c \geq c'$ if c an associated graded of c' for some filtration.

Transitivity: Suppose given $c = cl(V, \Theta)$ and a filtration on V invariant under Θ and $c' = cl(\operatorname{gr} V, \operatorname{gr}\Theta)$. Suppose also we have a filtration on $\operatorname{gr} V = V'$ invariant under $\operatorname{gr}\Theta$. ?

Better description of this ordering might be in terms of the closure.

So what I am going to look at is the set of unipotent elements in GL_n. These fall into finitely many conjugacy classes which are described by Jordan canonical forms, i.e. by partitions of n. We get a partial ordering on these classes by saying $c' \leq c$ if the class c' is contained in the closure of c. By algebraic geometry c is open and dense in $\bar{c}$, so $\bar{c} = \bar{c'} \Rightarrow c \cap c' \neq \phi \Rightarrow c = c'$.

Example: Suppose $g \in c$ and W is a subspace invariant under g; let c' be the iso-class of g_W and c'' the class of $g_{V/W}$. Then $c' \oplus c'' \leq c$. This is because

$$e^{-t\xi} g e^{t\xi} \longrightarrow \bar{g} \in G_\xi \qquad \text{as } t \to \infty$$

for any $g \in B_\xi$.

As a special case of this one sees that for any partition λ one has $c_\mu \leq c_\lambda$ if μ is a refinement of λ, that is, μ is obtained by partitioning the parts of λ.

So observe: If $g^\nu_{\lambda\mu} \neq 0$, then c_ν has $c_\lambda \oplus c_\mu$ for associated graded, hence $c_\nu \geq c_\lambda \oplus c_\mu$.

Dimension of the conjugacy class belonging to a partition λ?