August 11, 1975. Compactifying the [redacted] symmetric space

$X = G/K$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $S\mathfrak{p}$ be the set of rays in $\mathfrak{p}$: $S\mathfrak{p} = \mathbb{R}^+ \backslash (\mathfrak{p} - 0)$. $\mathfrak{p} \cup S\mathfrak{p}$ can be topologized so as to be a disk. We have an isom.

$$X \xleftarrow{\sim} \mathfrak{p}$$ ? ? ?

so we can define $\bar{X} = X \cup S\mathfrak{p}$. The problem is to make $G$ act on this compactification $\bar{X}$, or rather to [redacted] show the set-theoretic action I have is continuous. ? ? This doesn't seem to work. The compactification is not $X \cup S\mathfrak{p}$.

Example: $G = GL_n(\mathbb{C})$. $X$ can be identified with the set of inner products on $\mathbb{C}^n$. Specifically a point of $x$ is a function $Q : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that $Q(\lambda x) = |\lambda|^2 Q(x)$, $Q$ is smooth at $0$, $Q(x) \geq 0$ with $= \iff x = 0$. On polarization one gets from $Q$ a hermitian form $\tilde{Q}(x,y)$. If I fix an inner product on $\mathbb{C}^n$ e.g. the usual one, then $\tilde{Q}$ may be identified with a positive definite hermitian matrix $A$: $\tilde{Q}(x,y) = \langle Ax, y \rangle$ and $A = e^{\xi}$ for a unique $\xi \in \mathfrak{p}$ = hermitian matrices.

The obvious limit points to add to $X$ are the following, which one might call forms allowed to take on the value $\infty$. Specifically I mean a pair consisting of a subspace $W$ of $\mathbb{C}^n$ and hermitian form $Q : W \longrightarrow \mathbb{R}_{\geq 0}$. In the presence of a fixed inner product such a thing can be written

$$Q(x) = \int_0^\infty \lambda \langle E_\lambda x, x\rangle d\lambda$$

where $E_\lambda$ is a projection valued measure on $[0,\infty]$. In other words $Q(x) = \langle Ax, x\rangle$ where $A$ is an unbounded self-adjoint operator $\geq 0$.

So its clear now what $\overline{X}$ has to be. Normally $X$ consists of pos. def $A$ with $g$ acting: $g \cdot A = gAg^*$. So I extend $A$ to include possibly unbounded operators $\geq 0$, but with the same $G$-action.

To describe $\overline{X}$ as a space, recall that one defines unbounded operators using their graphs. Hence we associate to $A$ the subspace

$$\Gamma_A = \{(x,y) \in V \times V \mid Ax = y\}$$

which has the same dimension as $V$.

$$\Gamma_A^\perp = \{(u,v) \mid \langle x,u\rangle + \langle Ax, v\rangle = 0\}$$

$$= \{ (u,v) \mid u + A^* v = 0 \} = \{ (-A^* u, u) \}.$$

So if $J$ is the operator $J(x,y) = (-y, x)$ on $V \times V$, we have

$$(\Gamma_A)^\perp = J \Gamma_{A^*}$$

so

$$A \text{ hermitian} \iff (J\Gamma_A, \Gamma_A) = 0.$$

On $V \times V$ we have the sesqui-linear form

$$\langle J(x,y), (u,v) \rangle = \langle (-y,x), (u,v) \rangle$$

$$= -\langle y, u \rangle + \langle x, v \rangle$$

which is skew-hermitian:

$$\overline{\langle J(x,y), (u,v) \rangle} = -\langle u, y \rangle + \langle v, x \rangle$$

$$\langle J(u,v), (x,y) \rangle = -\langle v, x \rangle + \langle u, y \rangle$$

Hence not nec. bounded hermitian operators correspond to isotropic subspaces for this skew-hermitian form.

I need the $G$-action. $g \cdot A = g A g^*$

$$\Gamma_{g \cdot A} = \{ (x, g A g^* x) \mid x \in V \} = \{ (g^{*-1} y, g A y) \mid y \in V \}.$$

Hence $\Gamma_{g\cdot A} = (g^{*-1}, g)\,\Gamma_A$. So I make $G$ act on $V\times V$ by $g\cdot(x,y) = (g^{*-1}x, gy)$. Since

$$\langle J(g\cdot(x,y)), g\cdot(u,v)\rangle = \langle (-gy, g^{*-1}x), (g^{*-1}u, gv)\rangle$$
$$= \langle -gy, g^{*-1}u\rangle + \langle g^{*-1}x, gv\rangle$$
$$= -\langle y,u\rangle + \langle x,v\rangle$$

the skew-hermitian form is preserved. Hence $G$ acts on maximal isotropic subspaces.

Now $X$ is an invariant open set in the space of maximal isotropic subspaces of $V\times V$ for the $J$-form. Hence the closure $\bar{X}$ will be compact and $G$ will act continuously on it.

So the idea seems to be this. $\bar{X}$ consists of hermitian positive (possibly unbounded) forms on $V$.

Somehow here is the picture: Let $X$ be the symmetric space. If I give a point $x$ of $X$ I can identify $\bar{X}$ NO with the spherical compactification of the space $p_x$. So I have the disk bundle of the tangent bundle to $X$

and an exponential map of this disk bundle to $\bar{X}$. Thus I should be able to describe $\bar{X}$ as some sort of quotient of D. So each point of $\bar{X}$ occurs once for each element of X. In some way

$$D \cong X \times \bar{X}$$

$$\begin{tikzcd} T_X \arrow[r, "\simeq"] \arrow[d, phantom, "\wr\wr"] & X \times X \\ G \times^K \mathfrak{p} & \end{tikzcd}$$

Precisely: $T = G \times^K \mathfrak{p}$

$D = G \times^K \bar{\mathfrak{p}}$

Now one has
$$\begin{tikzcd} G \times^K \mathfrak{p} \arrow[r, "\text{endpoint}"] \arrow[d, "\text{initial pt.}"] & X \\ X & \end{tikzcd}$$

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Natural stratification of $\mathfrak{p} \cong \mathfrak{k}^-$: Given elements $\xi, \eta$ in $\mathfrak{k}^-$ let's say they belong to the same stratum if $K_\xi = K_{\xi + t(\eta - \xi)}$ for $0 \le t \le 1$

This condition forces $\xi$ and $\eta$ to commute ($\mathfrak{k}_\xi = \mathfrak{k}_\eta \Rightarrow \eta \in \mathfrak{k}_\xi$). Hence we can find a maximal abelian subspace $E^-$ of $\mathfrak{k}^-$ containing both. Let

$$\mathfrak{k} = \mathfrak{z} + \sum_{\pm\alpha\in\Phi} \mathfrak{k}^{\pm\alpha} \qquad \mathfrak{z} = \mathfrak{k}_{E^-}$$

be the root space decomposition of $\mathfrak{k}$ with respect to $E^-$. One has

$$\mathfrak{k}_\xi = \mathfrak{z} + \sum_{\substack{\pm\alpha\in\Phi \\ \alpha(\xi)=0}}' \mathfrak{k}^{\pm\alpha}$$

hence $\mathfrak{k}_\xi = \mathfrak{k}_\eta$ means $\alpha(\xi)=0 \iff \alpha(\eta)=0$. The condition $\mathfrak{k}_\xi = \mathfrak{k}_{\xi+t(\eta-\xi)}$ for $0\le t\le 1$ means that for each $\alpha\in\Phi$, $\alpha(\xi)$ has the same sign $(+,-,\text{or } 0)$ as $\alpha(\eta)$.

One sees from the above description that a stratum in $\mathfrak{k}^-$ is the same thing as a stratum in a maximal abelian subspace of $\mathfrak{k}^-$.

Another description: As $\xi$ is in the center of $\mathfrak{k}_\xi$, $\mathfrak{k}_\xi \subseteq \mathfrak{k}_\eta$ implies $\eta$ is in the center of $\mathfrak{k}_\xi$. Take $\xi$ and let $\mathfrak{a}_\xi$ denote (center of $\mathfrak{k}_\xi$) $\cap\, \mathfrak{k}^-$. Let $E^-$ be a maximal abelian subspace of $\mathfrak{k}^-$ containing $\xi$. Then $\mathfrak{a}_\xi \subset E^-$, so using the formula $\mathfrak{k}_\xi = \mathfrak{z} + \sum_{\alpha(\xi)=0} \mathfrak{k}^{\pm\alpha}$, one sees that

$\mathfrak{a}_\xi$ is the subspace of $E^-$ killed by all $\alpha\in\Phi$ with $\alpha(\xi)=0$.

So it is clear now that the stratum of $\xi$ is the chambre inside $\mathfrak{a}_\xi$ containing $\xi$. So

<u>Proposition:</u> Given $\xi \in \mathfrak{p}^-$, let $\mathfrak{a}_\xi$ be the center of $\mathfrak{k}_\xi$ intersected with $\mathfrak{k}^-$. (If $\xi \in E^-$, then $\mathfrak{a}_\xi$ is the intersection of the root hyperplanes containing $\xi$). Let $\sigma_\xi$ be the stratum in $\mathfrak{a}_\xi$ containing $\xi$ (for the roots of $\mathfrak{k}$ with respect to $\mathfrak{a}_\xi$ say, or if one wants, for the roots of $\mathfrak{k}$ with respect to $E^-$). Then $\sigma_\xi$ is the stratum of $\xi$ in the sense of the bottom of page 5.

Recall $B_\xi = G_\xi \ltimes B_\xi^u$, where $G_\xi$ is the centralizer of $\xi$. [Here $G$ is a real reductive group and now $\xi \in \mathfrak{p}$] but first discuss the case where $G$ is the complexification of $K$. Then $G_\xi$ has Lie algebra $\mathfrak{g}_\xi = \mathfrak{k}_\xi \otimes \mathbb{C}$, and the center of $\mathfrak{g}_\xi$ is the center of $\mathfrak{k}_\xi \otimes \mathbb{C}$.

$$\mathfrak{b}_\xi = \mathfrak{h} + \sum_{\substack{\alpha \in \Phi \\ \alpha(\xi) \geq 0}} \mathfrak{g}^\alpha$$

Thus $\xi$ and $\eta$ are in the same stratum $\Longrightarrow$ $B_\xi = B_\eta$. Conversely if $B_\xi = B_\eta$, then intersecting with

$K$ we have $K_\xi = K_\eta$, so $\xi$ and $\eta$ commute and hence can be put in a maximal abelian space $E$ of $\mathfrak{p}$. Then we have $\alpha(\xi) \geq 0 \iff \alpha(\eta) \geq 0$, showing that $\alpha(\xi)$ has the same sign as $\alpha(\eta)$.

Let $A_\xi = \exp(\mathfrak{a}_\xi) \subset G_\xi$. If $W$ is the connected center of $G_\xi$, then $W$ is a torus, so $W = (W \cap K) \times A_\xi$. Let $M_\xi$ be the conn subgroup of $G_\xi$ with Lie algebra $[\mathfrak{g}_\xi, \mathfrak{g}_\xi] \oplus i\mathfrak{a}_\xi$. Then one has $G_\xi = M_\xi^\times A_\xi$ and

$$B_\xi = M_\xi \times A_\xi \times B_\xi^u .$$

This decomposition is called the Langlands decomposition of $B_\xi$. The philosophy here is that in arithmetic questions the fundamental invariant is the degree. It is the point of major interest.

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Assume $K$ is not connected. Still we have an action of $K$ on $\mathfrak{k}$ preserving the root stratification. Let $E$ be a maximal abelian subspace of $\mathfrak{k}$ and $C$ a chambre of $E$. Each $K$ orbit on $\mathfrak{k}$ intersects $E$ in a $W$ orbit. Let $W^{(0)}$ be the Weyl group of $E$ with respect to $K^{(0)}$ = identity comp, and let $W' \subset W$ be the stabilizer of $C$. Then because

$W^{(0)}$ acts simply-transitively on the chambres

$$W^{(0)} \xrightarrow{\sim} W/W'.$$

~~If $N$ is the normalizer of $E$ in $K$ and $Z$ is the centralizer, then $Z \cap K^{(0)} = T = \exp(E)$~~

~~$W^{(0)} = \dfrac{N \cap K^{(0)}}{Z \cap K^{(0)}} \hookrightarrow N/Z \times K/K^{(0)}$.~~

Since $W^{(0)}$ is generated by reflections thru root hyperplanes, and these hyperplanes are preserved by $W$, it follows $W^{(0)}$ is normal in $W$. Thus

$$W \cong W' \ltimes W^{(0)}$$

Let $N$ be the normalizer of $E$ in $K$, and $Z$ the centralizer so that $W = N/Z$. Given $k \in K$, $k \cdot E$ is another max. ab. subspace of $\mathfrak{k}$ hence $\exists$ $x \in K^{(0)}$ with $x^{-1}k \in N$. This shows we have ontoness in

$$1 \longrightarrow N \cap K^{(0)} \longrightarrow N \longrightarrow \pi_0 K \to 1$$

Now $Z \cap K^{(0)} = T = \exp(E)$ so we get

$$1 \longrightarrow W^{(0)} \longrightarrow N/T \longrightarrow \pi_0 K \longrightarrow 1$$

$$1 \longrightarrow \underset{\pi_0 Z}{\overset{\shortparallel}{Z/T}} \longrightarrow N/T \longrightarrow W \longrightarrow 1$$

So $\quad 1 \longrightarrow \pi_0 Z \longrightarrow \pi_0 K \longrightarrow W' \longrightarrow 1$ 

Consider the $G$-action on $\mathfrak{k}$. I claim it preserves strata. Recall $\xi, \eta$ are in the same stratum if $B_\xi \cap G^{(0)} = B_\eta \cap G^{(0)}$, so this is evidently invariant.

The finite group $W'$ leaves fixed an interior point $\xi$ of $C$. The stabilizer of $\xi$ in $G$, namely $B_\xi$, contains $Z$ the centralizer of $E$ in $K$ and $W'$ so it should meet every component of $K$.

Better we know that each $G$-orbit on $\mathfrak{k}$ meets $E$ in a $W$-orbit and meets $C$ in a $W'$-orbit. Moreover by the Morse theory, the $W'$-orbit is identifiable with the set of components of the $G$-orbit. So if $\xi$ is an interior point of $C$ invariant under $W'$, then $G/B_\xi \simeq G\cdot\xi$ is connected, so it is clear that

$$G^{(0)}/B_\xi^{(0)} \simeq G/B_\xi .$$

<u>August 18, 1975.</u>

Review construction on pg. 1. Let $P = GL_n\mathbb{C}/U_n$ pos. def. hermitian $n \times n$ matrices. $\overline{P}$ = unbounded hermitian operators $A \geqslant 0$. The problem is to generalize the construction $\overline{P}$ to other $G$.

<u>Example</u>: $SL_n \subset GL_n$. Let $SP_n \subset P_n$ denote <u>those</u> <u>pos.</u> def. matrices of determinant 1. Then $\overline{SP_n} \subset \overline{P_n}$ consists of unbounded $A \geqslant 0$ such that i) $0 < A < \infty$ i.e. $A \in SP_n$ or ii) both $0, \infty$ occur as eigenvalues of $A$.

This example indicates to me that $\overline{SP_n}$ is not the gadget I seek for $SL_n$. Given an eigenvalue sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \infty$ I want all the simple roots $\lambda_{i+1}/\lambda_i$ to be defined i.e. to be a definite number in $[1, \infty]$. Recall the intuition.

Now $GL_n$ acts on $P_n$ by $g(A) = gAg^*$ which means that if we interpret $A$ as the form $b(x,y) = \langle Ax, y \rangle$, then $(g(b))(x,y) = \langle gAg^*x, y \rangle = \langle Ag^*x, g^*y \rangle = b(g^*x, g^*y)$. Now we want to compactify the space $P_n$ by adding in points which correspond to limits of forms. The basic invariants of a form are the eigenvalues and eigenspaces. Let $b_\nu(x,y) = \langle A_\nu x, y\rangle$ be a sequence of positive definite forms. For each $A_\nu$ we consider its eigenvalue sequence

$$0 < \lambda_1^\nu \leq \lambda_2^\nu \leq \cdots \leq \lambda_n^\nu < \infty,$$

or better we should arrange them:

$$\infty > \lambda_1^\nu \geq \lambda_2^\nu \geq \cdots \geq \lambda_n^\nu > 0$$

so that $\lambda_i^\nu / \lambda_{i+1}^\nu \geq 1$. Replacing $A_\nu$ by a subsequence we can assume convergence

$$\lambda_i^\nu / \lambda_{i+1}^\nu \longrightarrow a_i \in [1, \infty].$$

for each $i = 1, \ldots, n-1$. Then it follows that $\lambda_i^\nu / \lambda_j^\nu$ converges for each $i < j$.

Suppose $a_i > 1$ so that $\lambda_i^\nu > \lambda_{i+1}^\nu$ for large $\nu$; we can suppose for all $\nu$ if we want. Then it makes sense to talk of the subspace $W_i^\nu$ of $V$ invariant under $A^\nu$ having eigenvalues $\lambda_1^\nu, \ldots, \lambda_i^\nu$. By extracting a subsequence again we can suppose $W_i^\nu$ converges to $W_i$ for each $i$ such that $a_i > 1$.

This gives one the picture of the compactification for the $SL_n$ case. What one wants is a

G-space whose K-orbits are described by $[1,\infty]^{n-1}$. Thus you have

$$\{\lambda_1 \geq \cdots \geq \lambda_n \mid \Pi \lambda_i = 1\} \xrightarrow{\sim} \prod_1^{n-1} \{a \geq 1\}$$

$$\lambda_1, \ldots, \lambda_n \longmapsto (\lambda_1/\lambda_2, \ldots, \lambda_{n-1}/\lambda_n)$$

and you wish to compactify it by allowing $a_i$ to be $+\infty$. It's clear I ought to be able to write down a K-space of the sort, in fact you could pull $[1,\infty]$ down into a finite interval.

But I want a G-space, so what I want to do is to embed P in a suitable compact G space so as to get these roots. Adjoint representation of $SL_n$ will carry a positive matrix with eigenvalues $\lambda_i$ ~~to~~ a matrix with eigenvalues $\lambda_i/\lambda_j$.

General construction. Let $\rho : G \longrightarrow Aut(V)$ be a representation. Fix an inner product on $V \ni \rho(K)$ is in the unitary group. Then $\rho\Theta = \Theta\rho$, hence $\rho$ carries $P = \{g(\Theta g)^{-1}\}$ into $P_V =$ pos. def. hermitian operators on V. Now we can consider the closure of $\rho(P)$ in the unbounded operators. If $\rho$ is faithful this gives us a compactification of $P$.

Take the case where $G$ is a torus, whence $P = \exp \mathfrak{p}$ where $\mathfrak{p} = \mathrm{Hom}(\mathbb{G}_m, G) \otimes \mathbb{R}$. The representation is a sum of characters

$$V = \bigoplus L_i \qquad \text{orthogonal direct sum}$$

where

$$g = \text{mult. by } \chi_i(g) \text{ on } L_i$$

$$\chi_i : G \longrightarrow \mathbb{G}_m$$

Then $\rho(P)$ consists of operators of the form

$$A = \sum \chi_i(p)\, E_i$$

where $E_i$ = proj. on $L_i$. Thus $\overline{\rho(P)}$ can be identified with the closure of the map

$$\mathfrak{p} \xrightarrow{\dot{\chi}_i} \prod_i \mathbb{R} \cup \{\pm\infty\}$$

In the general case $P$ is the union:

$$\mathfrak{p} = \bigcup_{k \in K} k \cdot \mathfrak{a}$$

$$P = \bigcup_{k \in K} kAk^{-1}$$

so $$\rho(P) = \bigcup \rho(k)\,\rho(A)\,\rho(k)^{-1}$$

But $\bigcup_{k\in K} \rho(k)\,\overline{\rho(A)}\,\rho(k)^{-1}$ is closed in $\overline{P_V}$

as $K$ is compact, therefore

$$\overline{\rho(P)} = \bigcup_{k\in K} \rho(k)\,\overline{\rho(A)}\,\rho(k)^{-1}.$$

Notice in this argument since $p = \bigcup kC$ I can replace $A$ by $\exp C$.

This argument gives me a compact $G$-space having the correct $K$-orbit structure. Taking the adjoint representation with $G$ semi-simple $C$ is described by $\alpha_1, \dots, \alpha_\ell \geq 0$. Since any root $\alpha$ is a sum $\sum n_i \alpha_i$ with $n_i \geq 0$ once we specify the values of $\alpha_1, \dots, \alpha_\ell$ in $[0, \infty]$, then the values of all the roots are known. This shows

$$\overline{\rho(\exp C)} \simeq \prod_{i=1}^{\ell} [1, \infty]$$

as I want.

Therefore I get a $G$-space compactifying the symmetric space whose $K$-fundamental domain is $\overline{C} \simeq \prod^{\ell} [0, \infty]$.

I should carefully check that this procedure works.

for $SL_n$ and that it gives me the compactification I want.

August 14, 1975. Representations of $U_m$.

As a consequence of Peter-Weyl thm. + Weierstrass theorems, etc. I know

$$\mathbb{C}[X_{ij}, (\det X)^{-1}] \xrightarrow{\sim} A(U_m)$$

Review the steps in the proof:

1) Injectivity: If $f$ is a holom. function on $GL_m$ vanishing on $U_m$, then $f \circ \exp$ is a holom. fn. on $gl_m = \mathfrak{u}_m \otimes \mathbb{C}$ vanishing on $\mathfrak{u}_m$, $\Rightarrow f \circ \exp = 0$.

2) Because $U_m \subset \mathbb{C}^{m^2}$ via functions $X_{ij}$, one knows (Weierstrass) the alg. of functions on $U_m$ generated by $X_{ij}$ and $\bar{X}_{ij}$ is dense in $C(U_m)$. But for a unitary matrix $\bar{A} = (A^t)^{-1} = (\det A)^{-1} \operatorname{cof}(A^t)$. Thus as functions on $U_m$, we have

$$\bar{X} = (\det X)^{-1} \operatorname{cof}(X^t)$$

so now we know $\mathbb{C}[X_{ij}, (\det X)^{-1}]$ has dense image inside $C(U_m)$.

3) Let $A' \subset A(U_m)$ be the image of $\mathbb{C}[X_{ij}, (\det X)^{-1}]$. $A'$ is a $U_m \times U_m$-module, so from the formula

$$A(K) = \bigoplus V_i^* \otimes V_i$$

if $A' < A$, then there is an $f \in A$ orthogonal to all of $A'$.

Impossible by density.

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Preceding shows simply that representations of $U_m$ are the same as alg. repns. of $GL_m$. So from now on take algebraic viewpoint.

$$A(GL_m) = \mathbb{C}[X_{ij}, d^{-1}] \qquad d = \det(X).$$

Let $V = \mathbb{C}^m$ with standard action of $GL_m$:

$$g e_i = \sum_j g_{ji} e_j$$

i.e. action is given by comodule structure map

$$V \longrightarrow V \otimes A(GL_m)$$

$$e_i \longrightarrow \sum_j e_j \otimes X_{ji}$$

Thus the map

$$(*) \qquad V^* \otimes V \longrightarrow A(GL_m)$$

$$\lambda \otimes v \longmapsto (g \mapsto \lambda(gv))$$

is the map sending

$$e_j^* \otimes e_i \longmapsto (g \mapsto \underset{\substack{\parallel \\ g_{ji}}}{e_j^*(ge_i)}) = X_{ji}$$

Recall the operators $R_g$ and $L_g$ on $A(G)$ giving right and left regular representaton

$$(R_g f)(x) = f(xg)$$
$$(L_g f)(x) = f(g^{-1}x)$$

Thus

$$(R_g X_{ij})(x) = (xg)_{ij} = \sum_k x_{ik} g_{kj} = \Big(\sum_k X_{ik}\, g_{kj}\Big)(x)$$

$$(L_g X_{ij})(x) = (g^{-1}x)_{ij} = \sum_k (g^{-1})_{ik}\, x_{kj} = \sum_k (g^{-1})_{ik} X_{kj}(x)$$

So

$$\boxed{\begin{aligned} R_g X_{ij} &= \sum_k X_{ik}\, g_{kj} \\ L_g X_{ij} &= \sum_k (g^{-1})_{ik} X_{kj} \end{aligned}}$$

Return to

$$(*) \qquad V^* \otimes V \longrightarrow A(GL_m)$$

$$e_i^* \otimes e_j \longmapsto X_{ij}$$

Then the image of $V^* \otimes V$ is simply the [redacted] linear polynomials in the $X_{ij}$. I can think of these as linear functions of an [redacted] invertible matrix. So from our formula for $A(GL_m)$ we see that $A(GL_m)$ is the localization with respect to $d$ of

$$\bigoplus_{n \geq 0} S^n(V^* \otimes V)$$

which I [redacted] can think of as polynomial functions of a matrix.

What is $S^n(V^* \otimes V)$? We have isos.

$$S^n(V^* \otimes V) = \Sigma_n \backslash (V^* \otimes V)^{\otimes n} = \Sigma_n \backslash (V^*)^{\otimes n} \otimes V^{\otimes n}$$

$$= \Sigma_n \backslash (V^{\otimes n})^* \otimes V^{\otimes n}$$

We are thinking of [redacted] $V^* \otimes V$ as the dual of $\mathrm{End}(V)$, hence I ought to think of $(V^{\otimes n})^* \otimes V^{\otimes n}$ as the dual of $\mathrm{End}(V^{\otimes n})$, and hence

[redacted] $S^n(V^* \otimes V) =$ dual of $\mathrm{End}_{\Sigma_n}(V^{\otimes n})$.

This identification proceeds as follows: For each $A \in \mathrm{End}(V)$ we have $A^{\otimes n} \in \mathrm{End}_{\Sigma_n}(V^{\otimes n})$, in fact this map

$$\mathrm{End}(V) \longrightarrow \mathrm{End}(V^{\otimes n}) = (\mathrm{End}(V))^{\otimes n}$$

is just the "diagonal"$_n$ embedding: $A \longmapsto A^{\otimes n}$. Thus an element of $\mathrm{End}_{\Sigma_n}(V^{\otimes n})^*$ is a polynomial function on $\mathrm{End}(V)$, of degree $n$, i.e. in $S^n(V^* \otimes V)$.

<u>Assertion</u>: If we make the identification $\mathrm{End}(V)^{\otimes n} \xrightarrow{\sim} \mathrm{End}(V)^{\otimes n}$, then the map

$$\mathrm{End}(V) \longrightarrow \mathrm{End}_{\Sigma_n}(V^{\otimes n}) \qquad A \longmapsto A^{\otimes n}$$

can be identified with the "diagonal" map $\delta_n : W \longrightarrow F_n'(W)$ for $W = \mathrm{End}(V)$. Consequently

$$\left(\mathrm{End}_{\Sigma_n}(V^{\otimes n})\right)^* \xrightarrow{\sim} \text{poly functions of degree } n \text{ on } \mathrm{End}\, V. \quad = S^n((\mathrm{End}\, V)^*) = S^n(V^* \otimes V).$$

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Next we want the structure of $S^n(V^* \otimes V)$ as a $GL_m \times GL_m$-module. Maybe there is a point to introducing an isomorphic copy $W$ of $V$. Thus I am now interested in isomorphisms of $V$ with $W$, so I replace

$GL_m \subset \mathrm{End}(V)$ with $\mathrm{Isom}(V,W) \subset \mathrm{Hom}(V,W)$, and I denote by $W^* \otimes V$ the linear functions on $\mathrm{Hom}(V,W)$, whence I will have as before:

$$\begin{cases} \mathrm{Hom}_{\Sigma_n}(V^{\otimes n}, W^{\otimes n}) = \Gamma_n \, \mathrm{Hom}(V,W) \\ \mathrm{Hom}_{\Sigma_n}(V^{\otimes n}, W^{\otimes n})^* = \text{poly functions degree } n \text{ on } \mathrm{Hom}(V,W) \end{cases}$$

$$= S^n(W^* \otimes V)$$

Moreover these formulas do not require $W, V$ to be isomorphic.

I am interested in $S^n(W^* \otimes V)$ as a $\mathrm{Aut}(W) \times \mathrm{Aut}(V)$-module. Essentially this is equivalent to the structure of the dual module $\mathrm{Hom}_{\Sigma_n}(V^{\otimes n}, W^{\otimes n})$. Now suppose we let $\pi_1, \ldots, \pi_\ell$ be the different irreducible repns. of $\Sigma_n$. Then

$$V^{\otimes n} = \bigoplus_{i=1}^{\ell} \mathrm{Hom}_{\Sigma_n}(\pi_i, V^{\otimes n}) \otimes \pi_i$$

so

$$\mathrm{Hom}_{\Sigma_n}(V^{\otimes n}, W^{\otimes n}) = \bigoplus_{i=1}^{\ell} \mathrm{Hom}\left(\mathrm{Hom}_{\Sigma_n}(\pi_i, V^{\otimes n}), \mathrm{Hom}_{\Sigma_n}(\pi_i, W^{\otimes n})\right)$$

~~Hence if V = W, this is a sum of matrix rings~~

We know that in $A(K)$ each irreducible repn. of $K \times K$ occurs at most once. Thus if $V = W$, we know that $S^n(V^* \otimes V)$ contains each irreducible representation of $GL_m \times GL_m$ at most once, and more precisely we have

$$S^n(V^* \otimes V) = \bigoplus_{i=1}^{r} V_i^* \otimes V_i$$

where $V_1, \ldots, V_r$ are distinct irreducible repns. of $GL_m$. ~~The same will be true for the dual~~

Combine the formulas above to get:

$$\boxed{S^n(W^* \otimes V) = \bigoplus_{i=1}^{\ell} \operatorname{Hom}_{\Sigma_n}(\pi_i, W^{\otimes n})^* \otimes \operatorname{Hom}_{\Sigma_n}(\pi_i, V^{\otimes n})}$$

$$\boxed{\operatorname{Hom}_{\Sigma_n}(V^{\otimes n}, W^{\otimes n}) = \bigoplus_{i=1}^{\ell} \operatorname{Hom}\left(\operatorname{Hom}_{\Sigma_n}(\pi_i, V^{\otimes n}), \operatorname{Hom}_{\Sigma_n}(\pi_i, W^{\otimes n})\right)}$$

Now apply result on structure of $S^n(V^* \otimes V)$ and you get

<u>Theorem</u>: If $\pi_i$ is an irreducible representation of $\Sigma_n$ which occurs in $V^{\otimes n}$, then $\mathrm{Hom}_{\Sigma_n}(\pi_i, V^{\otimes n})$ is an irreducible representation of $GL_m$. In this way we get a 1-1 correspondence between irreducible representations of $GL_m$ and $\Sigma_n$ which occur in $V^{\otimes n}$.

<u>Corollary</u>: Every irreducible representation of $\Sigma_n \times GL_m$ has multiplicity $\leq 1$ in $V^{\otimes n}$.

<u>Remark</u>: If $W$ is a representation of $G$, then $\mathrm{End}_G(W) = \bigoplus_i \mathrm{End}(\mathrm{Hom}_G(\pi_i, W))$ as $\pi_i$ runs over the irreducible repns. of $G$. Thus $\mathrm{End}_G(W)$ is commutative $\Longleftrightarrow$ multiplicities are $\leq 1$.

Weyl's argument: ~~[illegible]~~

Inside $\mathrm{End}(V^{\otimes n}) = \mathrm{End}(V)^{\otimes n}$ the centralizer of $\Sigma_n$ is $\Gamma_n \mathrm{End}(V)$ which is spanned by $A^{\otimes n}$ $A \in \mathrm{End}\, V$, and by density, $A \in GL_m$. Thus if $R$ is the subring generated by $A^{\otimes n}$, one has $R =$ centralizer of $\mathbb{C}[\Sigma_n]$, whence because repns. of $\mathbb{C}[\Sigma_n]$ are completely reducible, the density theorem says $\mathbb{C}[\Sigma_n]$ is the centralizer of $R$.

But the centralizer$^{R}$ of $\mathbb{C}[\Sigma_n]$ is a sum of matrix rings, one for each irred. rep. of $\Sigma_n$ occurring in $V^{\otimes n}$. Thus the irreducible reps. of $R$ are in 1-1 correspondence with those of $\Sigma_n$ occuring in $V^{\otimes n}$ etc. But irred. repns. of $R$ and $GL_m$ inside $V^{\otimes n}$ are the same.

Essentially this is the same argument as I have given except this one seems simpler.

---

Suppose I try to reconstruct Atiyah's analysis of $\bigoplus_n R(\Sigma_n)^*$ which is to end up as a ring of operations on K-theory. It is made into a ring using the restriction homos.

$$R(\Sigma_{p+q}) \longrightarrow R(\Sigma_p \times \Sigma_q) = R(\Sigma_p) \otimes R(\Sigma_q)$$

Next one takes the torus $T$ with generic element $t_1, \dots, t_m$ inside $GL_m$ and looks at $V^{\otimes n}$.

~~If $e_1, \dots, e_m$ is the standard basis of $V$, then $V^{\otimes n}$ has basis $e_{i_1} \otimes \cdots \otimes e_{i_n}$ $1 \le i_1, \dots, i_n \le m$, on which $T$ acts via the character $t_{i_1} \cdots t_{i_n}$. The~~

~~possible $\Sigma_m$ orbits are classified by partitions $\alpha$ of $n$, into $\leq m$ pieces.~~

~~$[V^n] \in R(GL_m \times \Sigma_n)$~~

This procedure gives a homomorphism

$$(*) \qquad \bigoplus_{n\geq 0} R(\Sigma_n)^\vee \longrightarrow \mathbb{Z}[t_1, \ldots, t_m]^{\Sigma_m} = \mathbb{Z}[\lambda_1, \ldots, \lambda_m]$$

where $\lambda_i$ = $i$th elementary symmetric function of $t_1, \ldots, t_m$. This homomorphism preserves the grading with $\deg(t_i) = 1$, $\deg(\sigma_i) = i$. The claim is $(*)$ is an isomorphism in degrees $\leq m$. Onto: the image is a subring, hence you only have to show it contains $\sigma_i$ for $1 \leq i \leq m$. But take the element of $R(\Sigma_i)^\vee$ given by inner product with the sign representation of $\Sigma_i$; this gives the exterior power $\Lambda^i V$ which has char. $\lambda^i$ on $T$. Injective in degrees $\leq m$. Rank of $R(\Sigma_n)$ is number of conjugacy classes in $\Sigma_n$ = number of partitions $\alpha \vdash n$ with $\leq m$ pieces if $n \leq m$ = rank of the degree $n$ part of $\mathbb{Z}[\lambda_1, \ldots, \lambda_m]$.

A corollary is that $R(\Sigma_n)$ has $\mathbb{Z}$-basis formed of the repres. $\mathrm{ind}_{\Sigma_\alpha \to \Sigma_n}(1)$ where $\alpha$ is a partition of $n$ say $\alpha_1 \geq \cdots \geq \alpha_r > 0$, $\sum \alpha_i = n$. and

$\Sigma_\alpha = \Sigma_{\alpha_1} \times \cdots \times \Sigma_{\alpha_r}$. This is because $V^{\otimes n}$ has basis $e_{i_1} \otimes \cdots \otimes e_{i_r}$ and so splits according to the $\Sigma_n$ orbits of these monomials. $\mathbb{Z}[\lambda_1, \ldots, \lambda_m]_{\deg n}$ has as basis the symmetrization $S_\alpha$ of the monomials $t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ as $\alpha$ ranges over partitions. Taking the coefficient of $S_\alpha$ is the map $R(\Sigma_n)' \longrightarrow \mathbb{Z}$ corresponding to the representation $\mathrm{ind}_{\Sigma_\alpha \to \Sigma_n}(1)$.

In more details. Let $V$ has basis $e_1, \ldots, e_m$ where $t \cdot e_i = t_i e_i$. Then $V^{\otimes n}$ has the basis $e_{i_1} \otimes \cdots \otimes e_{i_n}$ with $1 \le i_1, \ldots, i_n \le m$, and

$$t(e_{i_1} \otimes \cdots \otimes e_{i_n}) = t_{i_1} \cdots t_{i_n} \; e_{i_1} \otimes \cdots \otimes e_{i_n}$$

Let me fix a monomial $t_1^{a_1} \cdots t_m^{a_m}$ with $\sum_{i=1}^m a_i = n$. The corresponding eigenspace ~~consists of the~~ comprises the monomials $e_{i_1} \otimes \cdots \otimes e_{i_n}$ where $a_1$ of the $i$'s are 1, $a_2$ are 2. These monomials form a orbit under $\Sigma_n$ with isotropy group $\Sigma_{a_1} \times \cdots \times \Sigma_{a_m}$. Thus

$$V^{\otimes n} = \sum_{\substack{a_1, \ldots, a_m \ge 0 \\ \sum a_i = n}} \mathrm{ind}_{\Sigma_{a_1} \times \cdots \times \Sigma_{a_m} \to \Sigma_n}(1) \otimes \begin{array}{l}\text{1 diml repn. with} \\ \text{character} \\ t_1^{a_1} \cdots t_m^{a_m}\end{array}$$

$$= \sum_{\substack{a_1 \ge \cdots \ge a_m \ge 0 \\ \sum a_i = n}} \mathrm{ind}_{\Sigma_{a_1} \times \cdots \times \Sigma_{a_m} \to \Sigma_n}(1) \otimes S_{a_1 \cdots a_m}(t)$$

where $s_{a_1 \cdots a_m}(t)$ is the symmetrization of $t_1^{a_1} \cdots t_m^{a_m}$
i.e.

$$s_{a_1 \cdots a_m}(t) = \sum_{\sigma \in \Sigma_m / \text{stabilizer of } (a_1, \ldots, a_m)} t_1^{a_{\sigma 1}} \cdots t_m^{a_{\sigma m}}$$

i.e. $s_{a_1 \cdots a_m}$ is the sum of the monomials conjugate to $t_1^{a_1} \cdots t_m^{a_m}$ under $\Sigma_m$.

Now it is clear from the formula at the bottom of the preceding page that the bases

$$\mathrm{ind}_{\Sigma_\alpha \to \Sigma_n}(1) \quad \text{of } R(\Sigma_n)$$

$$s_\alpha(t) \quad \text{of} \quad \mathbb{Z}[\lambda_1, \ldots, \lambda_m]_{\deg = n},$$

where $\alpha$ ranges over partitions of $n$, are dual.

---

Can I now calculate the representation ring of $GL_m$? Better: let $M_m$ be the monoid $\mathrm{End}(V)$. We then have maps

$$\bigoplus_{n \geq 0} R(\Sigma_n)^\vee \xrightarrow{\alpha} R(M_m) \xrightarrow{\beta} \mathbb{Z}[\lambda_1, \ldots, \lambda_m]$$

where the former $\alpha$ is the map we get by decomposing $V^{\otimes n}$, and the latter by restricting to $T$.
Because the irreducible repns for $\Sigma_n$ and $M_m$ occurring

in $V^{\otimes n}$ are in 1-1 correspondence, I know that $\alpha$ is onto. Here I use that all the representations of $M_m$ are obtained by decomposing $V^{\otimes n}$ for each $n$.

On page 10, I showed Ker $\beta\alpha$ is the ideal generated by $\lambda^n$ for $n > m$. Since these $\lambda^n$ becomes 0 on $GL_m$ we win. So we get

<u>Theorem</u>: The representation ring of the alg. monoid $\mathrm{End}(\mathbb{C}^m) = M_m$ is $\mathbb{Z}[\lambda^1, \ldots, \lambda^m]$, where $\lambda^i$ is the $i$-th exterior power of the standard representation. Moreover

$$\begin{aligned} R(GL_m) &= \mathbb{Z}[\lambda^1, \ldots, \lambda^m][(\lambda^m)^{-1}] \\ &= \mathbb{Z}[t_1, \ldots, t_m][t_1^{-1}, \ldots, t_m^{-1}]^{\Sigma_m} \\ &= R(T)^W. \end{aligned}$$

August 15, 1975.

I want to work out the details of Deligne's $\lambda$-operation theory. He looks at the category of algebraic functors from finite dimensional $k$-vector spaces to itself. These gadgets in my opinion ought to be like representations of $GL$, as they will be built out of representations of $GL_n$ for various $n$ by a stabilization process.

$T$ is an algebraic functor if

$$\mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(T(V),T(W))$$

is algebraic. When $k$ is finite one has to enrich things by giving the defining maps:

$$T(V) \longrightarrow T(W) \otimes A(\mathrm{Hom}(V,W))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad S(\mathrm{Hom}(V,W)^*).$$

Any algebraic functor decomposes into homogeneous algebraic functors: $T = T_0 + \cdots$

Suppose then we consider functors of a fixed degree **d**. Thus the defining maps are:

$$T(V) \longrightarrow T(W) \otimes S_d(\mathrm{Hom}(V,W)^*).$$

The dual of $S_d(\mathrm{Hom}(V,W)^*)$ is $\mathrm{Hom}_{\Sigma_d}(V^{\otimes d}, W^{\otimes d})$.

Let $\mathcal{A}$ be the abelian category of finite type $k[\Sigma_d]$-modules, and $\mathcal{A}'$ the subcategory comprising the modules $V^{\otimes d}$. The algebraic functor $T$ provides us with homs

$$\mathrm{Hom}_{\Sigma_d}(V^{\otimes d}, W^{\otimes d}) \longrightarrow \mathrm{Hom}(T(V), T(W))$$

compatible with composition. Thus $T$ can be interpreted as a $k$-linear functor from $\mathcal{A}'$ to $\mathrm{Modf}(k)$.

One should extend $T$ to the full subcategory $\mathcal{A}''$ consisting of direct summands of direct sums of objects in $\mathcal{A}'$. Then $\mathcal{A}''$ is a Karoubian additive category.

If char. $k = 0$, then $\dim V \geq d \Longrightarrow$ $V^{\otimes d}$ contains the regular representation of $\Sigma_d$ as a direct summand. Thus $\mathcal{A}'' = \mathcal{A} = \mathrm{Modf}(k)$, and so one has an equivalence of categories between the category of algebraic functors of degree d and representations of $\Sigma_d$. The formula giving the correspondence has to be

$$T(V) = \mathrm{Hom}_{\Sigma_d}(\pi, V^{\otimes d})$$

What is the universal functor of degree d? Thus I seek ~~an additive category~~ a k-linear additive category $\mathcal{P}$ with a degree d functor $T : \mathrm{Modf}(k) \longrightarrow \mathcal{P}$, which is universal. Answer: $\mathcal{P}$ is the full subcategory of $\mathrm{Modf}(k[\Sigma_d])$ consisting of finite direct sums of $V^{\otimes d}$. No harm in making $\mathcal{P}$ Karoubian.

Is it possible to showing that $V^{\otimes d}$ for $\dim V \geqslant d$ is a generator for $\mathcal{P}$? If this is so then $\mathcal{P}$ becomes equivalent to finitely generated projective modules over $R = \mathrm{End}_{\Sigma_d}(V^{\otimes d})$.

But look: If $V = ke_1 \oplus \cdots \oplus ke_m$, then as a $\Sigma_d$-module, we know that $V^{\otimes d}$ is a direct sum of induced modules from the subgroups $\Sigma_{a_1} \times \cdots \times \Sigma_{a_m}$ $\quad \sum a_i = d$

Review: We have identified algebraic functors of degree $d$, with $k$-linear functors on the full sub-category of $\Sigma_d$-modules comprising ones of the form $V^{\otimes d}$.

Now we can decompose $V^{\otimes d}$ as a $\Sigma_d$-module into modules induced from the subgroups $\Sigma_\alpha$ where $\alpha$ ranges over the partitions of $d$.

So I should look at the preadditive category consisting of the modules over $k[\Sigma_d]$

$$\mathrm{ind}_{\Sigma_\alpha \to \Sigma_d} 1$$

as $\alpha$ ranges over partitions of $d$. Now this module is defined over $\mathbb{Z}$.

$$\operatorname{Hom}_{k[\Sigma_d]}\left(\mathrm{ind}_{\Sigma_\alpha \to \Sigma_d} 1,\ \mathrm{ind}_{\Sigma_\beta \to \Sigma_d} 1\right)$$

$$= \operatorname{Hom}_{k[\Sigma_\alpha]}\left(1,\ \mathrm{ind}_{\Sigma_\beta \to \Sigma_d} 1\right)$$

But we know something about

$$\mathrm{res}_{\Sigma_\alpha \to \Sigma_d}\left(\mathrm{ind}_{\Sigma_\beta \to \Sigma_d} 1\right)$$

in terms of the orbits of $\Sigma_\alpha$ on $\Sigma_d / \Sigma_\beta$.

These formulas should work over $\mathbb{Z}$. In fact we get

$$\mathrm{res}_{\Sigma_\alpha \to \Sigma_d}\left(\mathrm{ind}_{\Sigma_\beta \to \Sigma_d} 1\right) = \sum_{i \in \Sigma'_\alpha \backslash \Sigma_d / \Sigma_\beta} \mathrm{ind}_{\Sigma_{\gamma_i} \to \Sigma_\alpha} 1$$

where $\gamma_i$ is a finer partition than $\alpha$. It should be true that

$$\mathrm{Hom}_{k[\Sigma_\alpha]}\left(1, \mathrm{ind}_{\Sigma_\beta \to \Sigma_d} 1\right)$$

$$\cong k[\Sigma_\alpha \backslash \Sigma_d / \Sigma_\beta]$$

In fact I bet there is a Hecke algebra one can construct out of these double cosets which gives the <u>game in question</u>.

Suppose we go back to a degree $d$ repn. $T$ of $\mathrm{End}(V)$:

$$T \longrightarrow T \otimes S_d(\mathrm{End}(V)^*)$$

or as I have seen $T$ is just an $R$-module where

$$R = \mathrm{End}_{\Sigma_d}(V^{\otimes d}).$$

Thus it appears that the category of representations of End(V) of degree d is just the category of R-modules. But what exactly is the structure of R? As a $\Sigma_d$-module

$$V^{\otimes d} = k[X^d]$$

where X is a basis for V. Hence one wants the structure of $\mathrm{End}_G(k[S])$ where S is a G-set.

$$\mathrm{Hom}_G(k[S], k[S']) = \mathrm{Map}_G(S, k[S'])$$

free over $k$.

---

Analogy: To get the notion of algebraic functor of degree d you take vector spaces but with different morphisms, namely $\Gamma_d(\mathrm{Hom}(V,W)) = \mathrm{Hom}_{\Sigma_d}(V^{\otimes d}, W^{\otimes d})$, and look at k-linear functors on this category.

On the other hand given a group G we can start with G-sets S and define a map from S to S' to be a G-map $k[S] \longrightarrow k[S']$, and then we study k-linear functors on this category. I have a hunch I've seen this category before.

Suppose I look for "Frobenius" functors on G-sets. Such a functor consists of giving for every $S$ an abelian gp $F(S)$ and for every map $f: S \to S'$ two maps

$$f^*: F(S') \longrightarrow F(S)$$
$$f_*: F(S) \longrightarrow F(S')$$

satisfying some formal properties. What is the universal gadget of this type?

As usual morphisms consist of correspondences with some type of equivalence relation depending on the axioms. To find the axioms which give $\mathrm{Hom}_G(\mathbb{Z}[S], \mathbb{Z}[T])$ for the set of maps.

$$\mathrm{Hom}_G(\mathbb{Z}[S], \mathbb{Z}[T]) = \mathrm{Map}_G(S, \mathbb{Z}[T])$$

Suppose $S = G/H \qquad T = G/K.$

$$\mathrm{Map}_G(G/H, \mathbb{Z}[T]) = \mathbb{Z}[T]^H \cong \mathbb{Z}[H\backslash T]$$

The last isomorphism is given by associating to an $H$-orbit $\mathcal{O}$ in $T$ the corresponding cycle in $\mathbb{Z}[T]$. How can I interpret such a cycle? If $\mathcal{O} = Ht_0$, then we have

$$\begin{tikzcd} G \times^H Ht_0 \arrow[r] \arrow[d] & T \\ G/H & \end{tikzcd}$$

Thus I see that the type of map I want from $G/H$ to $T$ is a $\mathbb{Z}$-linear combination of $G$ orbits in $G/H \times T$.

Claim: $\operatorname{Hom}_G(\mathbb{Z}[S], \mathbb{Z}[T])$ is canonically the free abelian group generated by $G$ orbits in $S \times T$.

Remaining axiom is that $f_* f^* =$ mult. by degree $f$, where degree $f$ is a function on the base.

---

See if we can make sense out of the following generalization: Up to now I have considered algebraic functors $T$ which I can view as being given by maps:

$$T(V) \longrightarrow T(W) \otimes A(\operatorname{Hom}(V,W))$$

where $A(\operatorname{Hom}(V,W)) = S(\operatorname{Hom}(V,W)^\vee)$ is the ring of algebraic functions on $\operatorname{Hom}(V,W)$. The idea will be to generalize to non-algebraic functions. Here $T(V)$ will now be some kind of good topological vector space, and $A$ will be replaced by some kind of functions say $C^\infty$ functions, or maybe distributions.

First ingredient is that $T$ is some sort of topological functor, i.e. given $x \in T(V)$ and $A$ in $\mathrm{Hom}(V, W)$

$$T(A)x$$

is some sort of map from $\mathrm{Hom}(V, W)$ into $T(W)$. So we have a function

$$\mathrm{Hom}(V, W) \longrightarrow T(W)$$

and the first thing to ask about is its degree, or rather to decompose it into homogeneous functions.

Note that we have a nice action $\mathbb{C}$ on $T$. Since we have a good harmonic analysis for abelian groups $\mathbb{C}$ we must be able to decompose $T$ according to the characters of $\mathbb{C}^*$. But a character $\rho : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ extends smoothly to $\rho : \mathbb{C} \rightarrow \mathbb{C}$ iff it is of the form $\rho(z) = z^n$ $n \geq 0$. Thus the functor $T$ will be a sum of homogeneous functors. Similar argument shows that if we want $T(A)x$ to be smooth at the origin then $T(A)$ is a polynomial in $A$. Thus we get nothing new this way.

August 31, 1975.

What is the relation between the two vector fields on $\mathcal{O} = K\eta$ $\eta \in k$ given by the gradient of the function $(?, \xi)$ and the vector field tangent to the flow $e^{it\xi} * (?)$.

Compute the gradient of the function $f(x) = (x, \xi)$, $x \in \mathcal{O}$ at a point $\eta$. The tangent space to $\mathcal{O}$ at $\eta$ may be identified with $[k, \eta]$. $df$ applied to a tangent vector $[X, \eta]$ is

$$df([X,\eta]) = ([X,\eta], \xi)$$

hence the gradient of this function is the projection of $\xi$ onto $[k, \eta]$. Now

$$k = k_\eta \underset{\substack{\uparrow \\ \text{orthogonal}}}{\oplus} [k, \eta] = k_\eta \oplus \sum_{\alpha(\eta) > 0} k^{\pm\alpha}$$

so if $\xi = \xi_0 + \sum \xi_\alpha$ is the decomposition of $\xi$, the gradient is $\sum \xi_\alpha$.

Now $e^{it\xi} * \eta$ is isomorphic to the image of $e^{it\xi} \cdot B_\eta$ in $G/B_\eta \xleftarrow{\sim} K/K_\eta$. It derivative is $i\xi \bmod b_\eta$ in $\mathfrak{g}/b_\eta \xleftarrow{\sim} k/k_\eta \xrightarrow{\sim} [k, \eta]$. Thus I want to take

~~[illegible]~~ $[\xi, \eta]$ and take its image under multiplication by $i$ for the natural complex structure on $\mathfrak{g}/\mathfrak{b}_\eta$.

$$[\xi, \eta] = -[\eta, \xi] = -\sum_\alpha i\alpha(\eta)\, \xi_\alpha$$

so up to sign the vector field obtained has the value

$$\sum_{\alpha(\eta)>0} \alpha(\eta)\, \xi_\alpha$$

at $\eta$, whereas the gradient has the values

$$\sum_{\alpha(\eta)>0} \xi_\alpha$$

Thus these two are not the same.

However $\left( \sum_{\alpha(\eta)>0} \alpha(\eta)\, \xi_\alpha \,,\ \xi \right) = \sum_{\alpha(\eta)>0} \alpha(\eta)\, |\xi_\alpha|^2 > 0$ provided $[\xi, \eta] \neq 0$. Thus the vector field assoc. to $e^{it\xi}{}_*$ is always pointing so as to increase the function $f$ except at critical points.

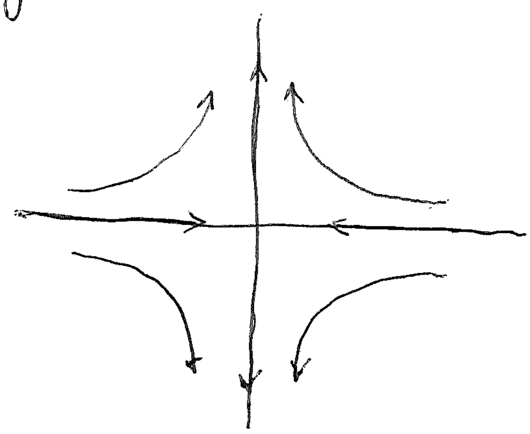
Consider next the set $S$ of limit points of $e^{t\xi}{}_* \eta$ where $\eta$ is not a critical point: $[\xi, \eta] \neq 0$. As $f(e^{t\xi}{}_* \eta)$ is monotone strictly increasing and bounded, $A = \lim_{t\to+\infty} f(e^{t\xi}{}_*\eta)$ exists. Clearly $f(S) = A$. Also $S$ is stable under $e^{t\xi}$, thus $S \subset \mathfrak{p}_\xi$.

Now let $\eta \in \mathfrak{p}_\xi$. I suppose known that $B^u_{-\eta} \xrightarrow{\sim} B^u_{-\eta} * \eta$ is an open nbd of $\eta$ in $G\eta$.

If this is known then we have a nbd of $\eta$ in $G\eta$ invariant under $e^{t\xi}$ and moreover this nbd is isomorphic to $\mathfrak{b}^u_{-\eta}$ with $e^{t\xi}$ acting via the adjoint action. Break up $\mathfrak{b}^u_{-\eta}$ according to the eigenspaces of $\xi$ and you immediately see what points have $\eta$ as limit point, namely $\mathfrak{b}^u_{-\eta} \cap \mathfrak{b}^u_{\xi}$.

September 1, 1975.

Let $M$ be a compact manifold, $f$ a Morse function on $M$, and let $X$ be a gradient-like vector field with respect to $f$ (this means $Xf > 0$ away from the critical points of $f$ and maybe also that $X$ vanishes at the critical points). Using $X$ we get through each critical point an incoming and outgoing manifold which meet transversally. Picture:




arrows point toward increasing $f$. $f$ is like $y^2 - x^2$

For each $x \in C =$ set of critical points, let $W_x$ be the incoming manifold. We have $\dim W_x =$ index of the critical point $x$.

Question: If the critical points are arranged in order of increasing index, say $x_1, \ldots, x_n$ then is the closure of $W_{x_i}$ contained in the union of the $W_{x_j}$ with $j<i$? Better: Is $\bigcup_{i_x \leq m} W_x$ closed?

Take a point $y$ in $M$ and follow its path $e^{tX}y$ as $t \to +\infty$. If $y \notin C$, then $f(e^{tX}y)$ is strictly increasing and bounded so it has a limit $L$. If $S$ is the set of limit points of $e^{tX}y$ as $t \to +\infty$, then $S$ is stable under the flow and $f(S) = L$, hence $S \subset C$. Again local analysis at a critical point $y$ shows that $S$ consists of a single point $x$ and that $y$ is in $W_x$.

I can consider all closed subspaces $Z$ of $M$ invariant under $e^{tX}$ for $t \leq 0$, i.e. such that $e^{tX} Z \subset Z$ for $t \leq 0$. For example, $\overline{W_x}$ is such a $Z$. Better, consider the closed sets $\overline{W_x}$ as $x$ ranges over critical points. These sets are stable under $e^{tX}$ for all $t$ because $W_x$ is.

Suppose the situation is like the Schubert cell decomposition of a flag manifold. Then the $W_x$ form the cells of a CW decomposition, and so we get a chain complex by filtering by dimension. Maybe it is always possible to construct a gradient-like vector field which produces a CW decomposition from a Morse function.