June 9, 1975. Roots for symmetric spaces.

Let $U$ be a compact connected Lie group with an involution $\sigma$, put $K = U^\sigma$ and [redacted] $X = U/K$. $X$ is a compact Riemannian manifold with $U$ acting as [redacted] isometries. $X$ is symmetric because there is an isometry [redacted] $\varphi_x$ for any $x$ which fixes $x$ and acts as $-1$ on the tangent spaces [redacted] at $x$. [redacted] (It is known that $U$ maps onto the connected component of the group of isometries of $X$. Thus [redacted] when $U$ acts faithfully on $X$, i.e. $\bigcap_{g\in G} gKg^{-1} = 1$, $U$ can be recovered from $X$.)

Suppose $X$ is simply-connected. This forces $\pi_0 K = 0$, $\pi_1 K \twoheadrightarrow \pi_1 U$. Hence $\sigma$ acts trivially on $\pi_1 U$. Let $C$ be the connected component of the center of $U$. Then

$$\begin{array}{c} \pi_2(U/C) \longrightarrow \pi_1 C \longrightarrow \pi_1 U \\ \| \\ 0 \end{array}$$

so $\sigma$ acts trivially on $\pi_1 C$, hence trivially on $C$. Thus $C \subset K$ and so $C = 0$ if we assume $U$ acts faithfully except for a finite group. Replacing $U$ by $\tilde{U}$ which is compact, we can assume $U$ is compact and 1-connected. Conversely it is known for $U$ compact + 1-connected that $\pi_0 K = 0$, hence $U/K = X$ is

1-connected. This is the good situation to keep in mind.

Next denote by $S$ a torus of $U$ on which $\sigma$ acts as $-1$ and maximal with this property. $S$ is the same thing as a maximal abelian subspace $\mathfrak{s}$ of $\mathfrak{u}^-$. Let $T$ be a maximal torus of $U$ containing $S$. If $X \in \mathfrak{t}$, $Y \in \mathfrak{s}$ then $[X-\sigma X, Y] = [X,Y] - \sigma[X,\sigma Y] = [X,Y] + \sigma[X,Y] = 0$ and as $X - \sigma X \in \mathfrak{u}^-$, it follows $X - \sigma X \in \mathfrak{s}$ by maximality. Thus $\sigma\mathfrak{t} \subset \mathfrak{t}$ and $\sigma T \subset T$.

One proves all such tori $S$ are conjugate under the $K$-action as follows. First let $H \in \mathfrak{s}$ generate a 1-parameter subgroup dense in $S$, whence the centralizer in $\mathfrak{u}^-$ of $H$ is $\mathfrak{s}$. Next let $X \in \mathfrak{u}^-$ and choose $k \in K$ so that the distance from $Ad(k)X$ to $H$ is minimum, which means that $(H, Y)$ is minimum for $Y = Ad(k)X$. Thus for $A \in \mathfrak{k}$

$$\frac{d}{dt}\left(H, Ad(\exp tA)Y\right)\Big|_{t=0} = 0$$

or $\quad \left(H, [A,Y]\right) = 0$

or $\quad \left([H,Y], A\right) = 0 \quad$ for all $A \in \mathfrak{k}$.

This means $[H,Y]=0$ since $[H,Y] \in \underset{\sim}{k}$ and $Y \in \underset{\sim}{s}$. There we have shown any $X$ in $\mathfrak{u}_-$ is $K$-conjugate to an element of $\underset{\sim}{s}$, hence that any $\underset{\sim}{s}'$ is $K$-conjugate to $\underset{\sim}{s}$.

Put $M =$ centralizer of $S$ in $K$ whence $K/M$ is the space of such tori $S$.

Next we wish to discuss roots. First review roots for $(U,T)$. The Lie algebra $\mathfrak{u}$ splits as $T$-module:

$$\mathfrak{u} = t + \sum e_i$$

where the $e_i$ are distinct 2-dimensional non-trivial representations of $T$. In each $e_i$ one has a unique up to scalar inner product invariant under $T$, hence if one orients $e_i$ one gets a linear function $\alpha : \mathrm{Lie}(T) \longrightarrow \mathbb{R}$ such that for $u \in \mathrm{Lie}(T)$

$$\mathrm{Ad}(\exp u) \text{ on } e_i = \text{rotation thru } 2\pi\,\alpha(u).$$

Changing the orientation changes $\alpha$ to $-\alpha$. Note $\alpha$ takes integral values on lattice points in $T$. The different hyperplanes $\alpha = 0$ partition $\mathrm{Lie}(T)$ into chambres. Choosing a chamber $C$ gives a consistent family of orientations of the $e_i$, namely ~~one~~ from each pair $\pm\alpha$ one choose the one positive on the interior of $C$. $\Phi$ = roots, $\Phi^+$ = positive roots and

$$\mathfrak{u} = \mathfrak{t} + \sum_{\alpha\in\Phi^+} e_\alpha$$

$\exists$ a unique element $\tau_\alpha \in \Lambda^2 e_\alpha \subset \mathfrak{t}$ such that $\alpha(\tau_\alpha) = 2$. The $\tau_\alpha$ generate the lattice points of $T$ when $U$ is simply connected.

$N$ = Normalizer of $T$; $W = N/T$. For each $\alpha$ one has a reflection $s_\alpha$ in $W$. $W$ acts simply transitively on the Weyl chambres of Lie$(T)$ and is generated by the reflections thru panels of $C$.

Next we consider the symmetric space situation.

Question: ~~Prop~~ Let $W_0$ = Weyl group of $S$ = Normalizer of $S$ in $K/M$. Then is

$$W_0 \backslash S \xrightarrow{\sim} K\backslash U/K ?$$

~~Proof.~~ We know that any $K$-orbit on $U/K = X$ meets $S\cdot K/K$. (This is because any $X$ in $\mathfrak{r}$ is $K$-conjugate to an element of $\mathfrak{s}$.) Consequently the above map is surjective. It remains to show that if $x, y \in S$ are such that $kyK = xK$ for some $k \in K$, then $k$ can be chosen to normalize $S$.

So we consider $S \cdot K = S \cdot xK$ and $kSK$ both of which are maximal flat submanifolds of $X$ passing thru $K, xK$ and hence ought to be conjugate under the subgroup of $K$ fixing $xK$. ~~Let $J$ = Conn comp of $K \cap xKx^{-1}$.~~ ~~Any element fixing $K$ and $xK$ must fix the symmetric point $x^{1/2}K$, so $J$ is stable under $\sigma$.~~

Let $J$ be the identity component of the centralizer of $x$ in $U$. Since $\sigma x = x^{-1}$, $\sigma J = J$. ~~Then $S$, $kSk^{-1}$ are two tori~~ If it were the case that $ky = xk$ then $kSk^{-1}$ would be a torus containing $kyk^{-1}x$, hence $kSk^{-1}$, $S$ would be two tori in $J$ reversed under $\sigma$, hence conjugate under $J^\sigma$. This means $kSk^{-1} = zSz^{-1}$ where $z \in J \cap K$, hence $z^{-1}k$ normalizes $z$ and yet $(z^{-1}k)yK = z^{-1}xK = xK$.

So we seem to again run into the problem of the non-injectivity of the map

$$\{x \in U \mid \sigma x = x^{-1}\} \longrightarrow U/K.$$

For example $S \cap K$ = elements of order 2 in $S$.

I propose to determine $K\backslash U/K$.

Denote $U/K$ by $X$ and the basepoint by $o$. The stabilizer of $o$ in $S$ is $S\cap K = \{s\in S \mid s^2=1\}$. ~~So~~

$$\begin{array}{ccc} S/S\cap K & \xrightarrow{\sim} & S\cdot o \\ \big\downarrow{\scriptstyle s} & & \\ S & & \end{array}$$

where the vertical map is induced by $s \longmapsto s^2$. Put another way we can identify $S$ with the maximal flat submanifold $So$ of $X$ by the map

$$s \longmapsto s^{1/2} o.$$

A point of $s$ is of the form $e^Y$ with $Y\in \mathrm{Lie}(S)$, thus gets sent to $e^{\frac{1}{2}Y}\cdot o$. Note that this identification of $S$ with $So$ is compatible with the action of $N$ (= Norm. of $S$ in $K$), since

$$k s^{\frac{1}{2}} o = k s^{\frac{1}{2}} k^{-1} o = (ksk^{-1})^{1/2} o.$$

Suppose now that two points $x_1, x_2$ of $So$ are $K$-conjugate: $k$[redacted]$\cdot x_1 = x_2$. Put $x_i = s_i o$, whence $k$[redacted]$s_1 = s_2 k_0$ for some $k_0 \in K$. Then applying the involution we get

$$k s_1^{-1} = s_2^{-1} k_0 \quad \text{or} \quad s_2 k = k_0 s_1$$

hence

$$k s_1^2 = k s_1 s_1 = s_2 k_0 s_1 = s_2 s_2 k$$

or

$$k s_1^2 k^{-1} = s_2^2$$

Therefore let $J$ be the connected component of the centralizer in $U$ of $s_2^2$. Because $\sigma(s_2^2) = (s_2^2)^{-1}$, $\sigma J \subset J$. Then $S \ni s_2^2$ so $S \subset J$. Also $kSk^{-1} \ni s_2^2$ so $kSk^{-1} \subset J$. Thus $S$, $kSk^{-1}$ are two tori of $J$ reversed by $\sigma$ and maximal with this property, so $\exists\, z \in J$ with $zkSk^{-1}z^{-1} = S$. $zk \in N$ and

$$zk\, s_1^2 (zk)^{-1} = s_2^2$$

hence by what's been shown $zk x_1 = x_2$.

This means we have proved:

<u>Proposition:</u> Let $W_0 = N/M$ be the group of autos of $S$ produced by elements of $K$. (These are the symmetries of $X$ fixing $o$ and carrying $So$ into itself.) Then

$$W_0 \backslash S \xrightarrow{\sim} K \backslash X$$

the map associating to $W_0 s$ the element $Ks^{1/2}o$ of $X$.

Suppose next we determine the isotropy group in $K$ of a point $s\mathcal{O}$ of $S\mathcal{O}$. If $ks\mathcal{O} = s\mathcal{O}$ then

$$ks = sk' \xrightarrow{\text{apply } \overline{\ \ } \text{ then } ^{-1}} sk^{-1} = k'^{-1}s$$

$$\Longrightarrow \quad ks^2k^{-1} = s^2.$$

Conversely assume that $ks_1^2k^{-1} = s_2^2$, and write $ks_1k^{-1} = s_2t$, $t \in U$. Then apply $\overline{\ \ }$-inverse

$$ks_1k^{-1} = \bar{t}^{-1}s_2$$

so $t\bar{t}^{-1} = s_2^{-1}ks_1k^{-1}ks_1k^{-1}s_2^{-1} = s_2^{-1}s_2^2s_2^{-1} = 1$ and $t \in K$. Thus at least we get that the $K$-isotropy group of $s\mathcal{O}$ is the centralizer of $s^2$, for $\bar{s} = s^{-1}$.

Consider the map

$$(*) \qquad U/K \longrightarrow \{\omega \in U \mid \bar{\omega} = \omega^{-1}\}$$

$$uK \longmapsto u\bar{u}^{-1}$$

Claim injective: $u\bar{u}^{-1} = v\bar{v}^{-1} \Longrightarrow$

$$\overline{(v^{-1}u)} = \bar{v}^{-1}\bar{u} = v^{-1}u, \text{ so } v^{-1}u \in K$$

and $uK = vK$.

Note that $(*)$ is $K$-equivariant

It is clear then that $X$ is just the component of the identity in the set of $\omega$ with $\sigma(\omega) = \omega^{-1}$. The $\mathcal{U}$ action is given by

$$u.\,\omega = u\omega\bar{u}^{-1}$$

$$\left(\bar{u}\omega^{-1}u^{-1} = (u\omega\bar{u}^{-1})^{-1}\right)$$

So from now on $X$ is the identity component of $\{\omega \in \mathcal{U} \mid \bar{\omega} = \omega^{-1}\}$, with $\mathcal{U}$ action given by

$$u * x = u x \bar{u}^{-1}.$$

An added virtue of this is that our previous endpt. map $\mathcal{X} \longrightarrow X$ sending $h(t)$ to $h(\tfrac{1}{2})^{-1}\sigma$ is now given by

$$h(\tfrac{1}{2})^{-1}\overline{h(\tfrac{1}{2})} = h(\tfrac{1}{2})^{-1}h(-\tfrac{1}{2})$$

if $h(t) = f e^{tY}$

$$= \left(h(-\tfrac{1}{2})\,h(1)\right)^{-1}h(-\tfrac{1}{2}) = h(1)^{-1}$$

$$= e^{-Y}.$$

So changing the endpt. map by sign, we get the desired Galois invariance:

$$\begin{array}{ccccc}
\mathcal{K}' & \longrightarrow & \mathcal{X} & \xrightarrow{\ \phi\ } & \mathcal{U} \\
\cup & & \cup & & \cup \\
{}_0\mathcal{K}' & \longrightarrow & {}_0\mathcal{X} & \xrightarrow{\ \phi\ } & X
\end{array}$$

$\phi$ is given by $\phi(h) = h(1)$.

Change back to old notation: $K$ compact group, connected, $G$ is its complexification, $\sigma$ is an involution on $K$ extended anti-linearly to $G$. $\mathcal{X}$ is the set of special paths in $K$. I make $\sigma$ act on $\mathcal{X}$ via

$$(\sigma h)(t) = \overline{h(-t)}.$$

$S$ is a ~~maximal~~ torus in $K$ maximal with respect to being reversed by $\sigma$. $T$ is a maximal torus of $K$ containing $S$; $T$ is stable under $\sigma$. $H$ is the complexification of $T$. $E$ is $\frac{1}{2\pi i} L(T)$; $E^\sigma$ ~~is maximal~~ is the Lie algebra of a maximal split torus in $G^\sigma$.

To each element $x \in E^\sigma$ I get a special path $e^{2\pi i t x}$ in $\mathcal{X}^\sigma$ which I denote $\tilde{x}$. $\tilde{x}(1) = e^{2\pi i x}$ will be denoted $\bar{x}$. Because of the formulas for the action of $\mathcal{K}^\sigma$ on $\mathcal{X}^\sigma$:

$$\cancel{(f\cdot h)(t) = f(e^{2\pi i t})\, h(t)\, f(1)^{-1}}$$

Because every element of $X$ is $K$-conjugate to an element of $S$ which in turn is of the form $\bar{x}$, we get a surjection

$$\mathcal{K}^{\sigma} \times E^{\sigma} \twoheadrightarrow \mathcal{X}^{\sigma}$$

$$(g\ ,\ x) \longmapsto g(e^{2\pi i t})\, e^{2\pi i t x}\, g(1)^{-1}$$

and we wish to describe the equivalence relation on $\mathcal{K}^{\sigma} \times E^{\sigma}$ thus defined. So we let $(g_1, x_1)$, $(g_2, x_2)$ have the same image. It will be enough to replace these by $(g, x_1)$, $(1, x_2)$, $g = g_2^{-1} g_1$.

So we have $g \cdot \tilde{x}_1 = \tilde{x}_2$. Taking endpts. we have

$$g(1)\, \bar{x}_1\, g(1)^{-1} = \bar{x}_2$$

where $g(1) \in K^{\sigma}$. I know there exists $n \in N$ such that $n \bar{x}_1 n^{-1} = \bar{x}_2$, so if I also denote by $n$ the constant element of $\mathcal{K}^{\sigma}$, then certainly

$$g n^{-1} \cdot (n \cdot \tilde{x}_1) = \tilde{x}_2$$

$$(n \cdot \tilde{x}_1)(t) = n e^{2\pi i t x_1} n^{-1} = \widetilde{Ad(n) x_1}.$$

So $(g, x_1)$ is equivalent to $(g n^{-1}, \widetilde{Ad(n) x_1})$ which reduces me to describing equivalent pairs

$(g, x_1)$ $(1, x_2)$ such that $\bar{x}_1 = \bar{x}_2$.

In this case $x_1 - x_2$ is a lattice point in $E^\sigma$ i.e. it exponentiates to 1. This means that $e^{2\pi i t(x_1 - x_2)} = f(z)$ where $f : S^1 \longrightarrow S$ is a 1-parameter subgroup. So $(g, x_1)$ is equivalent to $(gf, -x_1 + x_2 + x_1)$ which reduces one to describing equivalent pairs $(g, x_1)$ $(1, x_2)$ with $x_1 = x_2$. This means $g$ is in the centralizer of $\tilde{x}_1$, which I denote $\mathcal{K}^\sigma_{x_1}$. I know

$$
\begin{array}{ccc}
\mathcal{K}^\sigma_{x_1} & \xrightarrow{\sim} & K_{x_1} \\
g & \longmapsto & g(1) \\
(e^{2\pi i t x_1} \xi e^{-2\pi i t x_1}) & \longleftarrow & \xi
\end{array}
\qquad
\begin{array}{l}
K_{x_1} = \text{centr. of} \\
\bar{x}_1 = e^{2\pi i x_1}
\end{array}
$$

Let $\mathcal{N}$ be the subgroup of $\mathcal{K}^\sigma$ which is the semi-direct product of $\mathrm{Hom}_{gp}(S^1, S)$ and $N$. Thus

$$\mathcal{K}^\sigma = K^\sigma \ltimes \mathcal{K}^{\sigma'}$$

$$\mathcal{N} = N \rtimes \mathrm{Hom}_{gp}(S^1, S)$$

June 12, 1975. Buildings (continued)

I have seen before that the (spherical) Tits building associated to $SL_n(\mathbb{C})$ has geometric realization the unit sphere in self-adjoint matrices of trace 0.

Generalization: Let $K$ be a compact connected Lie group, $G$ its complexification. Then the building of parabolics of $G$ can be naturally identified with the unit sphere in $\mathrm{Lie}(K)$.

Recall Tits' description of the building which I will denote $I$. Fix some invariant metric on $K$ (or else we work with rays in $\mathrm{Lie}(K)$ instead of unit vectors) to define $S(\mathfrak{k})$. ~~[illegible]~~

Let $E = \frac{1}{2\pi i}\cdot \mathrm{Lie}(T)$ where $T$ is a maximal torus in $K$. Tits describes $I$ as a quotient of $G \times S(E)$ as follows. To each element $x \in E$ we can associate the parabolic group $P_x \ni$

$$\mathrm{Lie}(P_x) = \mathfrak{h} + \sum_{\alpha(x)\geq 0} \mathbb{C}X_\alpha$$

The equivalence relation says that $(g_1,x_1)\sim(g_2,x_2)$ iff there exists $n \in N$ such that $w(n)x_1 = x_2$

(where $\nu(n) = Ad(n)$) and such that $g_2^{-1} g_1 n^{-1} \in P_{x_2}$. (Note that $\nu(n)x = x \iff n \in P_x$ so it is irrelevant which $n$ is choosen.)

(Proof that $\nu(n)x = x \iff n \in P_x$. ($\Leftarrow$): The reductive part of $P_x$ has those roots $\alpha$ such that $\alpha(x) = 0$, hence the Weyl group of $P_x$ is generated by the reflections thru these hyperplanes $\Rightarrow \nu(n)x = x$. ($\Rightarrow$) It is known that the stabilizer of a point of $E$ in $W$ is generated by the reflections through the hyperplanes fixing that point, hence $n$ will be congruent mod $H$ to something in $P_x$, hence $n \in P_x$.)

Let $C \subset \$E$ be a fundamental domain for the $W$-action. Then $\underline{I}$ can be described as the quotient of $G \times C$ by the equivalence relation $(g_1, x_1) \sim (g_2, x_2) \iff x_1 = x_2$ and $g_2^{-1} g_1 \in P_{x_1}$.

Next consider $\$\underset{\sim}{k}$. We know that every element of $\$\underset{\sim}{k}$ is $K$-conjugate to an element of $\$E$ and that two elements of $\$E$ are $K$-conjugate iff they are $W$-conjugate. This means $\$\underset{\sim}{k}$ is the quotient of $K \times \$E$ by the relation $(k_1, x_1) \sim (k_2, x_2)$ if $\exists\, n \in N \ni \nu(n) x_1 = x_2$

and $k_2^{-1}k_1 n^{-1} \in K_{x_2}$. Using the fundamental domain $C$ we see that $S_{\underline{k}}$ is the quotient of $K \times C$ by the relation $(k_1, x_1) \sim (k_2, x_2) \iff x_1 = x_2$ and $k_2^{-1}k_1 \in K_{x_1}$.

But $K_x$ = stabilizer in $K$ of $x$ is the subgroup of $K$ having the roots $\alpha \ni \alpha(x) = 0$. Thus $K_x$ is a maximal compact subgroup of $P_x$ so

$$G/P_x = K/K_x$$

by analogy with what I have done in the loop space case.

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Suppose now we have an involution $\sigma$ on $K$ in which case I expect to be able to identify the building of parabolic subgroups of $G^\sigma$ with the sphere in $k_-$. Principle is to take $\sigma$-invariants.

$I^\sigma$ is the building of parabolics in $G^\sigma$ (this is essentially because parabolics in $G^\sigma$ are the same as parabolics in $G$ stable under $\sigma$.)

One selects $T$ to be invariant under $\sigma$.

~~[illegible]~~

Somehow $I$ is the directions to $\infty$ in $G$. Specifically ~~[illegible]~~ one can identify a point in $I$ with a ray $e^{tA}$ $t \geq 0$ where $A \in \mathfrak{p}$ is a length 1. Changing the maximal compact subgroup doesn't alter the effective asymptotic behavior of these paths to $\infty$.

Anyhow $I$ is something invariantly attached to $G$, and is independent of the choice of the maximal compact subgroup $K$.

When we want the directions to $\infty$ in $G^\sigma$ we want to consider those paths $e^{tA}$ fixed under $\sigma$: $\sigma A = A$. Under the identification of $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$ with $\mathfrak{k}$ given by $x \longmapsto 2\pi i x$ from $\mathfrak{p}$ to $\mathfrak{k}$, we want to look at $B \in \mathfrak{k}$ such that $\sigma(B) = -B$.

Now suppose I have a maximal torus $T$ in $K$ which is stable under $\sigma$, but do not suppose until necessary that $T$ contains an $S$. Suppose I take an element of

Therefore $I^{\sigma}$ is identified with the unit sphere in $\underset{\sim}{k}$; this identification is compatible with the action of $K^{\sigma}$. But we understand very well the $K^{\sigma}$-orbit structure of $\mathbb{S}(\underset{\sim}{k})$. We know that every $K^{\sigma}$-orbit meets $\mathbb{S}\,\mathrm{Lie}(S) \xleftarrow{\sim} \mathbb{S}(E_0)$ in a $W_0$-orbit.

Also we know that $W_0$ is a reflection group on $E_0$. Thus if $\mathbb{S}C_0$ is a chambre for $W_0$ acting on $\mathbb{S}(E_0)$, $\mathbb{S}C_0$ is a fundamental domain for $K^{\sigma}$ on $\mathbb{S}(k_-)$.

So let's examine roots. Let $\Phi$ be the roots of $K$ wrt $T$; $\Phi$ consists of linear maps $\alpha : E \to \mathbb{R}$ carrying lattice points to $\mathbb{Z}$.

$\Phi_0$ = linear functions $\alpha : E_0 \to \mathbb{R}$ which are $\neq 0$ and restrictions of $\alpha \in \Phi$. Recall $\sigma$ is the identity on $E_0$. $\Phi_1 = \{\alpha \in \Phi \mid \alpha | E_0 = 0\}$ $= \{\alpha \in \Phi \mid \alpha\sigma = -\alpha\}$.

Choose chambre $C_0$ for $W_0$ and then choose $C$ for $W$ so that $C_0 \subset C$. These chambres determine $\Phi_0^+$, $\Phi^+$. If $\alpha \in \Phi$ and $\alpha | E_0 \in \Phi_0^+$, then $\alpha \in \Phi^+$.

$\sigma$ acts on $E$, $\Phi$, $W = N/T$.

First suppose inside of $T$ we have a torus $S$ with centralizer $Z$. I know $K/Z$ has a cell decomposition indexed by its $T$-fixpts. Now if $kZ$ is a $T$-fixpt,

$$TkZ = kZ \iff k^{-1}Tk \subset Z;$$

then $T$, $k^{-1}Tk$ are two maximal tori of $Z$, so $\exists$ $z$ with $T = z^{-1}k^{-1}Tkz$, or $kz \in N$. Thus

$$(K/Z)^T = NZ/Z$$

is isomorphic to $N/N \cap Z \simeq W/W'$ where $W' = N \cap Z/T$ is the subgroup of $W$ acting trivially on $S$.

A basic fact is that for any $x \in E$, its stabilizer $W_x$ is generated by $s_\alpha$ where $\alpha(x) = 0$. Moreover if $x \in C$, then it suffices to take $\alpha$ a "simple" root, i.e. $\alpha = 0$ a panel of $C$. Thus $W'$ will be the subgroup of $W$ generated by the reflections through the roots vanishing on $E_0 = \mathrm{Lie}(S)$

~~The cells of $K/Z$ are $B$-orbits. The cell indexed by $w$ is parameterized by (I mean acted on simply-transitively) by the subgp $B \cap wBw^{-1}$.~~

Next consider the Weyl group of $S$. Let $g \in \mathrm{Norm}_K(S)$, i.e. $gSg^{-1} = S$. Then $T$ and $gTg^{-1}$ are both tori containing $S$, hence $\exists\, z$ in $Z$ such that $zg \in N$. Thus

$$\frac{\mathrm{Norm}_K(S)}{\mathrm{Cent}_K(S)} = \frac{N \cap \mathrm{Norm}_K(S)/T}{N \cap \mathrm{Cent}_K(S)/T} = W''/W'$$

where $W'' = \{w \mid wE_0 = E_0\}$.

Go back to the case where $S$ arises from $\sigma$, in which case $Z = MS$. For $w \in W$ to belong to $W''$ means that $w(E_0) \subset E_0$ i.e.

$$\sigma x = -x \implies \sigma w x = -wx \implies \sigma w x = w \sigma x \quad \forall x \in E_0.$$

But then $w$ will also preserves the orthogonal complement to $E_0$ i.e.

$$\sigma x = x \implies \sigma w x = wx \implies \sigma w x = w\sigma x \quad \forall x \in E_0^{\perp}.$$

Thus $W'' = \{w \in W \mid \sigma w = w \sigma\}$.

June 13, 1975

I am still try to relate $W$ and $W_0$.

$(K/Z)^\sigma = K^\sigma/M$ ?  Let $\xi \in E_0$ be such that the corresponding 1-parameter subgroup in $S$ is dense. Then $K/Z \xrightarrow{\sim} K\cdot\xi$; moreover $(K/Z)^\sigma \xrightarrow{\sim} K\xi \cap k_-$. Let $k\xi \in k_-$. $\exists\, k_0 \in K^\sigma$ such that $k_0 k\xi \in E_0$, so if I am to show $K^\sigma\xi \ni k\xi$, I can suppose $k\xi \in E_0$. It follows that $kSk^{-1} = S$. $\exists\, z \in Z$ such that $zTz^{-1} = kTk^{-1}$; factoring $z = mt$ we can assume $z = m$. Thus I can suppose $kTk^{-1} = T$, whence $k \in N$ preserves $S$. The image of $k$ in $W$ is in the subgroup $W''$ centralizing $\sigma$. So we have reached:

<u>Problem</u>: Given $w \in W$ such that $wE_0 = E_0$ show that there exists $w_0 \in W_0$ such that $w = w_0$ on $E_0$.

I am going to try to give a geometric proof. Fix a chambre $C_0$ in $E_0$ whence $wC_0$ is also a chambre of $E_0$. Now take a line joining generic points in $wC_0$ and $C_0$. As we go along this line we cross various hyperplanes, call them:

$$P_{\alpha_1} \cap E_0, \ldots\ldots, P_{\alpha_k} \cap E_0 \qquad \alpha_i \in \Phi_0^+$$

where $P_\alpha = \{x \in E \mid \alpha(x) = 0\}$. Denote by $\tau_\alpha$ the reflection in $E_0$ thru the hyperplane $P_\alpha \cap E_0$. Suppose I know that $\tau_\alpha \in W_0$.*) Then it is clear that we get a gallery in $E_0$

$$wC_0, \ \tau_{\alpha_1} w C_0, \ \ldots, \ \tau_{\alpha_k} \cdots \tau_{\alpha_1} w C_0 = C_0$$

Since $w' = \tau_{\alpha_1} \cdots \tau_{\alpha_1} w$ satisfies $w' C_0 = C_0$, it follows that $w' =$ identity on $E_0$, whence $w|E_0 \in W_0$.

*(To be more precise, I want to know $\exists$ elements $\gamma_\alpha$ in $W$ induced by elements of $N_0 = \mathrm{Norm}_{K^\sigma}(S)$, whose restrictions to $E_0$ are the reflections $\tau_\alpha$).

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More generally suppose two elements $s_1, s_2$ of $S$ become conjugate in $K$: $ks_1k^{-1} = s_2$. In $Z_K(s_2)$, which is stable under $\sigma$, we have the max. reversed toris $kSk^{-1}$, $S$; hence can suppose $kSk^{-1} = S$. Similarly we can suppose $kTk^{-1} = T$. Then $k$ gives us an element of $W$

not nec. reversed

~~which preserves $E_0$, hence I know that $\exists\ n_0 \in \mathrm{Norm}_{K_0}(S)$ such that $n_0$ acts the same as $k$ on $S$. $\therefore\ n_0 s_1 n_0^{-1} = s_2$ which shows that $s_1, s_2$ are $W_0$ conjugate. This proves~~

$$W_0 \backslash S \hookrightarrow K \backslash K$$

Suppose we have a conjugacy class in $K$ anti- stable under $\sigma$. Let $C'$ be the fundamental simplex in $E$; the is the fine chambre containing $0$ and contained in the cone $C$. Every $K$-conj. class has a unique representative of the form $e^{\gamma}$ with $\gamma \in C'$. Let $w_0$ be the unique element of $W$ such that

$$-\sigma C = w_0 C.$$

Then if $cl(e^{\gamma})$ is (anti-) $\sigma$-stable, we have

$$\sigma e^{\gamma} = k e^{-\gamma} k^{-1}$$

$$\text{or} \quad w_0^{-1} e^{-\sigma\gamma} w_0 = w_0^{-1} k e^{\gamma} k^{-1} w_0$$

$$\begin{array}{c} \| \\ e^{-w_0^{-1}\cdot\sigma\gamma} \end{array}$$

Now both $-w_0^{-1}\sigma\gamma$ and $\gamma$ are in $C$, hence we conclude they are equal:

$$\sigma\gamma = -w_0\gamma$$

In particular this holds for $\gamma \in C_0'$ showing that $w_0 = \mathrm{id}$ on $E_0$, hence $w_0$ commutes with $\sigma$, so

$$C = (-\sigma)^2 C = -\sigma w_0 C = w_0 w_0 C$$

so $w_0^2 = 1$.

This calculation should be done for the Lie algebras first. So consider the map

$$\begin{tikzcd}
K^\sigma \backslash k_- \arrow[r] & K \backslash k \\
W_0 \backslash E_0 \arrow[r] \arrow[u, "s"] & W \backslash E \arrow[u, "s"'] \\
C_0 \arrow[u, "s"] & C \arrow[u, "s"']
\end{tikzcd}$$

As $C_0 \hookrightarrow C$ this map is injective. (Direct proof: Given $x_1, x_2 \in E_0$ with $k \cdot x_1 = x_2$; work in the centralizer of $x_2$, can suppose $k$ normalizes $E_0$; then work in the centralizer of $E_0$, can suppose $k$ normalizes $T$. So we have $w \in W$ preserving $E_0$ such that $w x_1 = x_2$ and I have seen that there exists a $w_0 \in W_0$ with same effect on $E_0$, $\therefore$ $w_0 x_1 = x_2$).

Next let $w_\sigma$ be the unique elt. of $W$ with $-\sigma C = w_\sigma C$. Then $\gamma \mapsto w_\sigma(-\sigma\gamma)$

is a map from $C$ to itself which agrees with the map on $W|E$ induced by $-\sigma$. In particular, assuming $C_0 \subset C$ as we have been, we find that $\omega_\sigma(-\sigma\gamma) = \gamma$ for $\gamma \in C_0$ i.e. $\omega_\sigma \gamma = \gamma$ for $\gamma$ in $C_0$. This implies $\omega_\sigma = \mathrm{id}$ on $E_0$, hence that $\omega_\sigma \in W'$ and that it commutes with $\sigma$. Then

$$C = (-\sigma)^2 C = -\sigma\omega_\sigma C = \omega_\sigma(-\sigma C) = \omega_\sigma^2 C$$

so $\omega_\sigma^2 = 1$.

Suppose next that $\gamma \in C$ is an element whose image in $W|E$ is invariant under $-\sigma$. Then $\gamma = \omega_\sigma(-\sigma\gamma)$ or $\sigma\gamma = -\omega_\sigma\gamma$. Conversely also, so

$$(W|E)^{-\sigma} = \{\gamma \in C \mid \sigma\gamma = -\omega_\sigma\gamma\}.$$

Now

$$E' = \{x \in E \mid -\sigma x = \omega_\sigma x\}$$

is the linear space containing $E_0$ and the part of $E \ominus E_0$ where $\omega_\sigma$ is $-1$. There are examples where $\omega_\sigma \neq 1$, so $E'$ can be bigger that $E_0$. Thus $W_0|E_0$ is not in general the $-\sigma$ invariants in $W/E$.

June 19, 1975:

Let $\xi$ be an element of $E_0$, and let $Z_\xi$ be its centralizer in $K$. Then $K/Z_\xi$ is a projective variety. Claim

$$K^\sigma/Z_\xi^\sigma \xrightarrow{\sim} (K/Z_\xi)^\sigma$$

In effect suppose $kZ_\xi = \bar{k}Z_\xi$ whence $k\cdot\xi = \bar{k}\cdot\xi = -\sigma(k\cdot\xi)$. Thus $k\cdot\xi \in \mathfrak{k}_-$. But I have seen this implies* there exists $k_0 \in K^\sigma$ with $k_0\cdot\xi = k\cdot\xi$ or $k_0^{-1}k \in Z_\xi$. Hence $kZ_\xi = k_0Z_\xi$ is a point of $K^\sigma/Z_\xi^\sigma$.

(* Recall proof: Given $k\xi_1 = \xi_2$ with $\xi_1, \xi_2 \in \mathfrak{k}_-$ to show $\exists\, k_0 \ni k_0\xi_1 = \xi_2$. Can suppose $\xi_1, \xi_2 \in E_0$. Next in $\mathrm{Cent}_K(\xi_2)$, which is $\sigma$-stable, we have the two tori $kTk^{-1}$, $T$ hence I can suppose $k \in N$. Thus I have ~~an element~~ two elements of $E_0$ which are in the same $W$-orbit. [redacted] (acting) on $\xi_1, \xi_2$ by elements of [redacted] $W_0$ we can suppose that $\xi_1, \xi_2 \in C_0$. However $C_0 \subset C$ and any two $K$-conjugate elements of $C$ are equal. QED.)

I can identify the $\sigma$ fixpts of the spherical building for $G$ with the spherical building for $G^\sigma$. ~~Any point of the G building is of the form~~ ~~$K/K_x$~~

$$\text{G-building} = \coprod_{x \in \$C} G/P_x \times \{x\} = \coprod_{x \in \$C} K/K_x \times \{x\}$$
$$= \$k$$

where $K_x$ is the centralizer of $x$. When is this point $\sigma$-invariant? ~~On the other hand an element~~

$$I = G \times \$E/\text{rel.} = K \times \$E/\text{relations}.$$

$\sigma$ acts on both sides, on $E$ by $-$ what I've been calling $\sigma$. Thus an invariant point $cl(k,x)$ is one such that

$$(*) \qquad (\bar{k}, -\sigma x) \sim (k, x)$$

i.e. such that $\exists\, w \in W$ with $wx = -\sigma x$ and $k^{-1}\bar{k}w^{-1} \in K_x$. Certainly it is not obvious that $x$ is in the $W$-orbit of a point of $E_0$. Yes it is, because the condition $(*)$ says that $\sigma(k \cdot x) = -k \cdot x$, hence $kx \in \mathfrak{k}_-$ so we know $\exists\, k_0$ with $k_0 k x \in E_0$, and we know also $\exists\, w \ni wx = k_0 k x$.

Let me next consider the question of whether the map $K_0\backslash X \longrightarrow K\backslash K$ is injective. If I wish to proceed in analogy with the analysis of $K_0\backslash \mathfrak{k} \longrightarrow K\backslash \mathfrak{k}$, then I really must to understand $K$-orbits in $K$, i.e. conjugacy classes.

<u>Questions:</u> Let $s \in T$, and let $K_s$ be its centralizer. This is known to be connected if $K$ is simply-connected. Is $K/K_s$ a projective variety? Is $K_s$ the centralizer of a torus? ~~Under these circumstances~~ When $s \in S$, then is

$$(K/K_s)^\sigma = K^\sigma/K_s^\sigma \ ?$$

These things are true if $s$ generates a torus, but now we want to understand what happens in general.

Question: Is $K_s = K_{T'}$ where $T'$ is a torus containing $s$?

Let $Z =$ center of $K_s$, and $Z^{(0)}$ its identity component. Clearly $T'$ if it exists is contained in $Z^{(0)}$, and one can take $T' = Z^{(0)}$.

June 15, 1975

Let $\theta$ be an auto. of $K$ compact and connected. Look at the action of $\theta$ on $\mathfrak{k}$. We try to find ~~[illegible]~~ a non-trivial abelian subspace of $\mathfrak{k}$ stable under $\theta$. Suppose none were to exist. Then $\mathfrak{k}$ has zero center. Also ~~[illegible]~~ there are no ~~[illegible]~~ eigenvectors for $\theta$. Let $W$ be a minimal ~~[illegible]~~ dimension $\theta$-invariant subspace; ~~[illegible]~~ then $\dim W = 2$. ~~[illegible]~~ The bracket defines $u: \Lambda^2 W \longrightarrow \mathfrak{k}$. Hence either $u = 0$ and $W$ is abelian, or $u \neq 0$ and $\operatorname{Im} u$ contains an eigenvector. So $\mathfrak{k}$ has a non-trivial abelian subspace invariant under $\theta$.

<u>Proposition</u>: If $\theta$ is an auto. of a compact Lie group $K$, there exists a maximal torus $T$ of $K$ which is ~~[illegible]~~ normalized by $\theta$.

<u>Proof</u>. We have to find a $\theta$-invariant abelian subspace $\mathfrak{a}$ of $\mathfrak{k}$ which is its own centralizer. ~~[illegible]~~ Let $\mathfrak{z}$ be the center of $\mathfrak{k}$ and $[\mathfrak{k},\mathfrak{k}]$ the derived subalgebra, so that $\mathfrak{k} = \mathfrak{z} \oplus [\mathfrak{k},\mathfrak{k}]$. We argue by induction on $\dim K$. If $\mathfrak{z} \neq 0$, then $\dim [\mathfrak{k},\mathfrak{k}] < \dim \mathfrak{k}$ so there is self-centralizing $\theta$-subspace $\mathfrak{a}'$ of $[\mathfrak{k},\mathfrak{k}]$, hence

we can take $\mathfrak{a} = \mathfrak{z} \oplus \mathfrak{a}'$. [redacted]
[redacted] Suppose then $\mathfrak{z} = 0$. We know $\mathfrak{k}$ has an abelian $\theta$-subspace $W$. The centralizer $Z(W)$ is then of smaller dimension, so it contains a self-centralizing $\theta$-subspace $\mathfrak{a}$. Clearly $W \subset \mathfrak{a}$, so $\mathfrak{a}$ is self-centralizing in $\mathfrak{k}$. QED.

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[redacted] Centralizer case: Here we suppose $\theta$ is conjugation by an element $s$ of $T$, whence $K^{\theta} = Z(s)$. Let $k \in K^{\theta}$. Consider the two tori $T$ and $kTk^{-1}$ containing $s$. They have to be conjugate by an element $k_0$ of the identity component of $K^{\theta}$, hence replacing $k$ by $k_0^{-1}k$, one sees each element of $\pi_0(K^{\theta})$ is represented by an element of $W$ fixing $s$.

Therefore [redacted] to see if $Z(s)$ is connected we have to show that [redacted] $W_s = \{w \mid ws = s\}$ is comes from $Z(s)^{(0)}$, or that $N_s = \{n \in N \mid ns = s\}$ is contained in $Z(s)^{(0)}$. Claim [redacted] that if $\alpha$ is a [redacted] root such that the reflection $s_\alpha$ centralizes $s$, [redacted] then $s_\alpha$ [redacted] comes from $Z(s)^{(0)}$. [redacted] Let $s = \exp(2\pi i v)$. Then $s_\alpha(v) = v - \alpha(v) H_\alpha$ and thus $s_\alpha(s) = s$ means $\alpha(v) H_\alpha$ is a lattice point. One can always arrange that $\alpha$ is a simple root (choose fundamental chambre to have $\alpha=0$ as a wall), in which case $H_\alpha$ is a generator for the group of lattice pts.* This means $\alpha(v)\in\mathbb{Z}$, whence $Z(s)^{(o)}$ will contain the root $\alpha$, hence $s_\alpha$. Here I use that $K$ is simply-connected.

So one has only to prove that $W_s$ is generated by the $s_\alpha$ such that $\alpha(v)\in\mathbb{Z}$. Let $w\in W_s$ and let $v$ be chosen so that $\exp(2\pi i v)=s$ and $|v|$ is least. Then $w(v)-v$ is a lattice point so it is a sum of roots vector $H_\alpha$. Thus we get a sequence $\alpha_1,\ldots,\alpha_n$ $\ni$

$$(*)\qquad w(v)-v=\sum_{i=1}^{n}H_{\alpha_i}$$

and I will suppose the sequence chosen so that $n$ is least. Since

$$s_\alpha(H_\beta)=H_\beta-(\dot\alpha,H_\beta)H_\alpha$$

and one knows that the roots of the form $\beta+k\alpha$ (when $\beta$ not prop. to $\alpha$) form a segment $-p\le k\le q$ of $\mathbb{Z}$, it follows that

$$(H_\alpha,H_\beta)<0 \implies \text{either } \alpha+\beta=0 \text{ or } \alpha+\beta \text{ is a root.}$$

Thus in $(*)$ all the $H_{\alpha_i}$ have inner product $\geq 0$ with each other by the minimality. So

$$n|v|^2 \le \sum_{i=1}^{n} |v + H_{\alpha_i}|^2 \qquad \text{by min. of } |v|$$

$$\le \sum |v+H_{\alpha_i}|^2 + 2\sum_{i<j} (H_{\alpha_i}, H_{\alpha_j})$$

$$= n|v|^2 + \underbrace{2\sum (v, H_{\alpha_i})}_{2(v, \sum H_{\alpha_i})} + \underbrace{\sum_i |H_{\alpha_i}|^2 + 2\sum_{i<j} (H_{\alpha_i}, H_{\alpha_j})}_{|\sum H_{\alpha_i}|^2}$$

$$= (n-1)|v|^2 + |v + \sum H_{\alpha_i}|^2 = (n-1)|v|^2 + |wv|^2$$

$$= n|v|^2$$

∴ All $\le$ are equal, so $(H_{\alpha_i}, H_{\alpha_j}) = 0 \quad i<j$ and $|v + H_{\alpha_i}| = |v|$. $\Rightarrow$ $w_{\alpha_i}$ commute and also $w_{\alpha_i}(v) = v + H_{\alpha_i}$, so

$$\left(\prod w_{\alpha_i}\right)(v) = v + \sum H_\alpha = wv.$$

Thus as $w_{\alpha_i}$ are reflections, we see that $w$ can be moved into $W_v$, whence it is a product of the reflections thru hyperplanes containing $v$. Done!

Morse theory approach toward $K/K_\eta$ where $\eta \in L(K)$. Let $\xi$ be an interior point of the fundamental chambre and consider the function

$$k \cdot K_\eta \mapsto \operatorname{dist}(k\eta, \xi)^2$$

$$= |\eta|^2 + |\xi|^2 - 2(k\eta, \xi)$$

on $K/K_\eta$. We compute its critical points. Call this function $f(k\eta)$. Then

$$f(\exp(X)\, k\eta) = \text{const} - 2(\exp(\operatorname{ad} X)\, k\eta, \xi)$$

$$= \text{const} - 2([X, k\eta], \xi)$$

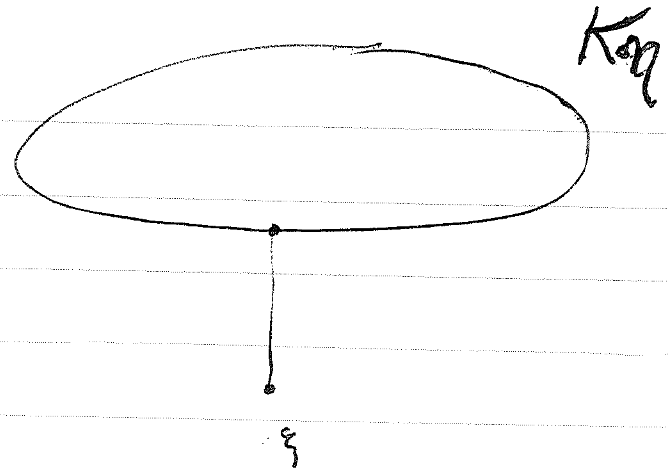
$$- 2((\operatorname{ad} X)^2 k\eta, \xi) \quad + \text{higher}$$

For $k\eta$ to be a critical point means $\forall X$

$$0 = ([X, k\eta], \xi) = +(X, [k\eta, \xi])$$

i.e. $[k\eta, \xi] = 0$, whence $k\eta \in E = L(T)$.

Geometric meaning: $K$ orbits are perpendicular to $E$, because $([X, \xi_1], \xi_2) = (X, [\xi_1, \xi_2]) = 0$ for $\xi_i \in E$. Those points on an orbit where the distance to $\xi$ becomes critical are where the straight line from $\xi$ is $\perp$ to the orbit:




Note that conjugation by $T$ must preserve these geodesics for $\xi$ sufficiently generic. Thus the critical points of the distance function are where $K\eta$ meets $E$.

Hessian: $\quad -2\left((\mathrm{ad}X)^2 k\eta, \xi\right)$ where $k\eta, \xi \in E$.

$$\begin{aligned} &\| \\ &+2\left((\mathrm{ad}X)k\eta, (\mathrm{ad}X)\xi\right) \\ &\| \\ &2\left([k\eta, X], [\xi, X]\right) \end{aligned}$$

Now use $\mathfrak{g} = \mathfrak{h} + \sum' \mathbb{C}X_\alpha$ as usual and write

$$X = H + \sum{}' \lambda_\alpha X_\alpha$$

where $\lambda_\alpha \in \mathbb{C}$ satisfy $\lambda_{-\alpha} = -\overline{\lambda_\alpha}$. Then Hessian becomes

$$2\left(\sum \lambda_\alpha \alpha(k\eta) X_\alpha, \sum \lambda_\alpha \alpha(\xi) X_\alpha\right)$$

and since the basis $\{X_\alpha\}\{X_{-\alpha}\}$ are ~~orthonormal~~ dual we get

$$2\sum_{\alpha\in\Phi} \lambda_\alpha \lambda_{-\alpha}\, \alpha(k\eta)\left(-\alpha(\xi)\right)$$

or $\quad 2\sum_{\alpha\in\Phi} |\lambda_\alpha|^2 \alpha(k\eta)\alpha(\xi) = 4 \sum_{\alpha\in\Phi^+} |\lambda_\alpha|^2 \alpha(k\eta)\alpha(\xi).$

Since $\alpha(\xi) > 0$ for $\alpha \in \Phi^+$, this shows that the number of negative eigenvalues is the number of $\alpha \in \Phi^+ \ni \alpha(k\eta) < 0$. Note this form is non degenerate on $\mathfrak{k}/\mathfrak{k}_{k\eta}$ (has roots $\alpha \ni \alpha(k\eta) \neq 0$).

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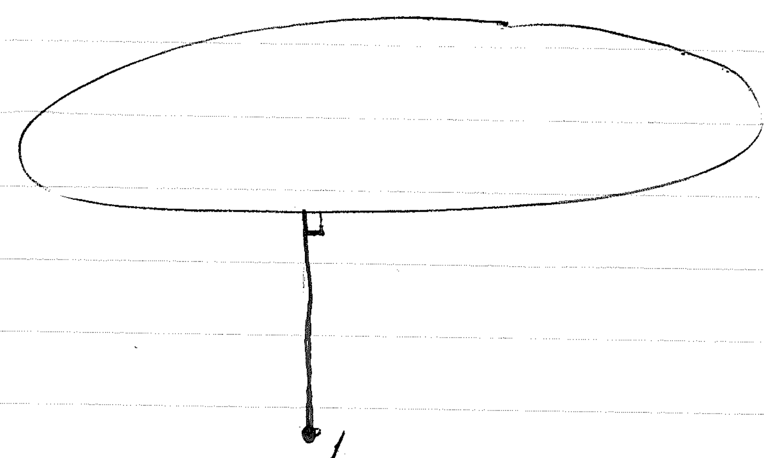
Next I want to generalize this argument to the group case. I have $K$ acting on itself by conjugation, and an orbit $Ks$. One can choose a generic point which is not a focal point for the geodesics issuing $\perp$ly from $Ks$ and then conjugate it to a point $t_0 \in T$. Then all geodesics from $t_0$ to $Ks$ have to be invariant under $T$-conjugation, hence must lie in $T$.



$t_0$

Now I want to compute the index of one of these geodesics following Bott–Samelson. ~~...~~ This means we count the conjugate points along the geodesics and look at dimensions of the stabilizers (these are the multiplicities).

So we end up with the following setup. We have a $W$-orbit inside $T$ which we lift back to a $\tilde{W}$ orbit in $E$. Each point $\zeta$ in this $\tilde{W}$ orbit will give us a geodesic namely the straight line joining $\xi$ to $\zeta$ where $\xi$ is a generic point in the fundamental small chambre. To get the index of the geodesic one computes the hyperplanes crossed counted according to their multiplicities. In the case of $K$ acting by conjugation on itself, these multiplicities are always 2.

Return to page 53. We have seen that $K\eta$ meets $E$ perpendicularly and transversally with zero-dimensional intersection. The points of this intersection are critical points for the function $k\eta \longmapsto |k\eta - \xi|^2 = |\eta|^2 + |\xi|^2 - 2(k\eta, \xi)$ for any $\xi$ in $E$. If no root vanishes on $\xi$, then these are exactly the critical points of the function, the critical points are non-degenerate, and the index at $k\eta \in E$ is twice the number of roots $\alpha$ such that $\alpha(\xi) > 0$ and $\alpha(k\eta) < 0$. Consequences

1) $K\eta \cap E \neq \phi$ for the function has a minimum.

2) $K\eta$ is a CW complex with even dimensional cells, one for each point $k\eta \in E$, of dimension the no. of hyperplanes crossed in going from $\xi$ to $k\eta$ in $E$.

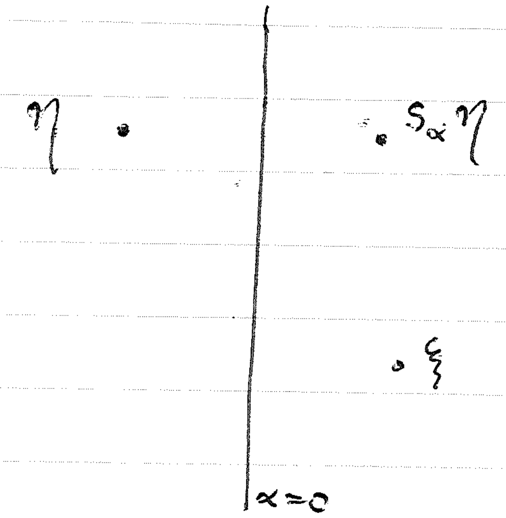
3) There is a unique point of $K\eta \cap E$ in the region $\{\xi \in E \mid \alpha(\xi) \geq 0$ if $\alpha(\xi) > 0\}$. Because the cells, being even-dimensional, corresp. to a basis of the homology, and $K\eta$ is connected. This unique point is where the Morse function is minimum (in fact, the unique point where the Morse function has a local minimum).

Here is a translation of 3):

<u>Prop</u>: Let $\eta, \xi \in E$ and suppose $\xi$ regular: $\alpha(\xi) \neq 0$ for all $\alpha$. Let $C_\xi$ be the Weyl cone containing $\xi$: $C_\xi = \{x \mid \alpha(x) \geq 0$ if $\alpha(\xi) > 0\}$. Then

$$\eta \in C \iff |\eta - \xi| < |w\eta - \xi| \quad \text{for all } w\eta \neq \eta,\ w \in W.$$

Direct proof of the proposition goes as follows. Assume that $\eta$ is on the wrong side of $\alpha=0$ with respect to $\xi$:




Then $s_\alpha \eta$ is closer to $\xi$. Therefore if $\eta$ is a member of its W-orbit with minimum distance from $\xi$, one has $\eta \in C_\xi$. Now you have to argue that no two points of $C_\xi$ are W-conjugate. ( $wx = x'$ with $x, x' \in C_\xi$, then $wC_\xi$ and $C_\xi$ are two cones containing $x'$, hence related by reflections thru hyperplanes containing $x'$. So in $W_{x'}$ we get a $w' \ni w'C_\xi = wC_\xi \Rightarrow w'^{-1}wx = x'$ and $w'^{-1}wC_\xi = C_\xi \Rightarrow w'^{-1}w = id$. This last step is because the finite subgroup of W preserving $C_\xi$ fixes some point of Int $C_\xi$, and there can be no fixpt $\xi$ in the interior because $\xi$ generates T whose centralizer is T itself.)

Now lets apply Morse theory to paths starting on $t \in T$ (regular point) and ending on conjugacy class $K\cdot s$. Lift $t$ to $\xi$ not lying on any of the hyperplanes $\alpha \in \mathbb{Z}$, say $0<\alpha(\xi)<1$ for all $\alpha \in \Phi^+$. Then we want geodesics in $E$ starting from $\xi$ ending at the points of $E$ over points of $W\cdot s$.

Take $s = 1$ whence geodesics end at lattice points. The index of a geodesic ending at $\eta$ is twice the no. of hyperplanes $\alpha^{-1}(n)$, $n \in \mathbb{Z}$ crossed in going from $\xi$ to $\eta$. Let $\tilde{W}$ be the reflection group generated by reflections thru: $\alpha(x)=n$; it is the semi-direct product of $W$ and the lattice $\Lambda$ generated by the $H_\alpha$. Let $\Lambda_1$ be the lattice points.

Because the space of paths from $t$ to $1$ has the homotopy type described by Morse theory, I know the cells described by geodesics form a basis for the homology. Thus $\pi_1 K$ has its elements in 1-1 correspondence with the geodesics of index 0. These will be those lattices points contained in the fundamental chambre

$$C' = \{x \mid 0 \leq \alpha(x) \leq 1 \quad \text{all } \alpha \in \Phi^+\}.$$

Therefore if $K$ is simply-connected I know that the only lattice point in $C'$ is $0$.
~~prove that given any [illegible] K-orbit~~

Let us now suppose $K$ is simply-connected, and consider the space of paths joining $t_0$ to the orbit $K\cdot s$. $K\cdot s \cap T = W\cdot s$ if $s \in T$ say. So we have to consider paths joining $\xi$ to the different points of $p^{-1}(W\cdot s)$, $p: E \to T$ the canonical map. Now we know the path space in question $\Omega(K; t_0, K\cdot s)$ is the fibre of the inclusion $K\cdot s \subset K$, hence we have a fibration

$$\Omega(K; t_0, K\cdot s) \to K\cdot s \longrightarrow K .$$

Thus if $\Omega(K; t_0, K\cdot s)$ is simply-connected, so is $K\cdot s$, implying the stabilizer $K_s$ is connected.

Because $K\cdot s$ is connected and $K$ is 1-conn. it follows that $\Omega(K; t_0, K\cdot s)$ is connected, hence $p^{-1}(W\cdot s)$ contains a unique point in $C'$. This shows $C'$ is a fundamental domain for $W$ on $T$.

But also we know that this path space $\Omega(K; t_0, K\cdot s)$ has even-dimensional cells, hence it is 1-connected, which implies $K.s$ is 1-connected.