May 1975

The building for SL_n over a field with discrete valuation.

Let R be a discrete valuation ring with quotient field F and residue field $R/\pi R$. Let V be a vector space over F of dimension n. By an R-lattice in V we shall mean an R-submodule which is free of rank n.

Using lattices we shall define two simplicial complexes X, X' associated to V. X' is the simplicial complex whose vertices are the R-lattices in V and in which a simplex is a chain of lattices $L_0 < L_1 < \cdots < L_q$ such that $\pi L_q \subset L_0$. Note that if a lattice L is fixed, then the lattices L' such that $\pi L \subset L' \subset L$ are in 1-1 correspondence with subspaces of the n-dimensional vector space $L/\pi L$ over the residue field $R/\pi R$. Consequently $L_0 < L_1 < \cdots < L_q$ is a simplex of maximum dimension in X' iff $L_0 = \pi L_q$, L_i/L_{i-1} is one-dimensional over $R/\pi R$, and $q = n$. Thus $\dim(X') = n$ and every simplex is contained in a n-simplex.

By the homothety class of a lattice L we mean the chain $cl(L) = \{\pi^j L, j \in \mathbb{Z}\}$. X is the simplicial complex whose vertices are the homothety classes of lattices, and in which a set of vertices is a simplex iff the union of these homothety classes is a chain. Thus a simplex of X may be viewed as a chain of lattices closed under multiplying by π, π^{-1}. If the chain is enumerated: $\sigma = \{L_j, j \in \mathbb{Z}\}$, starting with some lattice L_0 in σ, and putting L_j = least member of σ containing L_{j-1}, then $\pi L_j = L_{j-k}$ for some k, and σ is a $(k-1)$-simplex with the vertices $cl(L_0), \dots, cl(L_{k-1})$. A simplex of maximal dimension is one such that $\dim(L_i/L_{i-1}) = 1$, ~~such simplices will be called chambers~~ and the dimension of such a simplex is $n-1$, hence $\dim(X) = n-1$. The $(n-1)$-simplices of X will be called <u>chambers</u>. Every simplex is the face of some chambres.

There is an evident simplicial map from X' to X. We can lift X back to a subcomplex of X' as follows. Given $0 \neq w \in V$, it is clear that each homothety class of lattices contains a unique lattice L such that $L \cap Fw = Rw$. The subcomplex of X' made up of such lattices L maps isomorphically onto X. Another method is to fix a lattice $R\omega$ in $\Lambda^n V$ and define the volume of any lattice L to be the integer $v(L)$ such that $\Lambda^n L = \pi^{v(L)} R\omega$. Any homothety class of lattices contains a unique L with $0 < v(L) \leq n$, and the subcomplex of X' made up of these lattices maps isomorphically onto X.

The complex X is the building for SL_n over F in the sense of Bruhat + Tits. X' is in some sense the building for GL_n over F (compare ...)

Apartments:

Suppose now that $V = F^n$, the vector space of column vectors of length n over F, and let $e_1, \ldots, e_n$ be the standard basis. Call a lattice special if it of the form $L = \sum_{i=1}^{n} R\pi^{d_i} e_i$ with $d_i \in \mathbb{Z}$. The subcomplexes of X' and X made up of special lattices will be denoted A' and A and called the fundamental apartments of X' and X respectively.

~~We propose to determine the realizations of A' and A. A' may be identified with the complex whose vertices are elements of $\mathbb{Z}^n$ and whose simplices are chains $v_0 < \ldots < v_q$ for the product ordering in $\mathbb{Z}^n$ such that We identify a vertex L of A' with a sequence of integers $(m_1, \ldots, m_n)$ in $\mathbb{Z}^n$ by associate~~

~~We propose to determine the realizations of A' and A. Let D denote the simplicial complex of dimension 1 having integers for vertices and consecutive integers for 1-simplices. Then D is a triangulation of $\mathbb{R}$.~~

We propose to determine the realizations of A' and A. Recall that an ordered simplicial complex is one equipped with ~~an order~~ a partial ordering on its vertex such that each simplex is linearly ordered. The product $K_1 \times K_2$ of two ordered simplicial complexes is naturally an ordered simplicial complex such that $|K_1 \times K_2| = |K_1| \times |K_2|$ for the compactly generated topology. Let D be the ordered simplicial complex ~~[illegible]~~ ~~[illegible]~~ of dimension one having integers with the usual ordering for vertices and pairs of consecutive integers for 1-simplices. Then $|D| = \mathbb{R}$. The product D^n has $\mathbb{Z}^n$ for vertices and a simplex is a subset of $\mathbb{Z}^n$ of the form $\{v_0, \ldots, v_q\}$ where

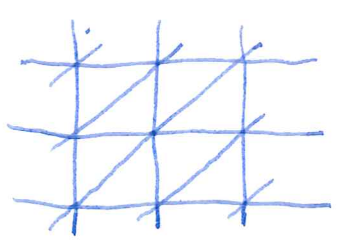
$$v_{0i} \le v_{1i} \le \cdots \le v_{qi} \le v_{0i} + 1$$

for $i = 1, \ldots, n$.

By associating to $(d_1, \ldots, d_n) \in \mathbb{Z}^n$ the lattice $\sum R\pi^{-d_i} e_i$ it is easily seen that we obtain an isom. of D^n with A'. Thus

$$|A'| = |D^n| = \mathbb{R}^n.$$

Picture of A' when $n = 2$:



Note that each vertex in A, i.e. homothety class of special lattices, contains a unique lattice of the form $\sum R\pi^{-d_i}e_i$ where $d_n = 0$, so that A is isomorphic to the subcomplex of A' made up out of lattices with $L \cap Fe_n = Re_n$. Thus we get an isom. of D^{n-1} with A by associating to $(d_1, \dots, d_{n-1})$ ~~the lattice~~ $cl(\sum R\pi^{-d_i}e_i)$, where $d_n = 0$. So

$$|A| \cong \mathbb{R}^{n-1}.$$

It is customary to identify $|A|$ with the space $E = \{x \in \mathbb{R}^n \mid \sum x_i = 0\}$ as follows. One sends the vertex $cl(\sum R\pi^{-d_i}e_i)$ to the point $(d_1, \dots, d_n) - (\frac{1}{n}\sum d_i)(1, \dots, 1)$ of E and extends linearly over simplices to get a map $|A| \longrightarrow E$. This map is ~~an isom.~~ a homeom. because its composition with the homeo above is the ~~map~~ isom: $(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, x_{n-1}, 0) - (\frac{1}{n}\sum x_i)(1, \dots, 1)$ of $\mathbb{R}^{n-1}$ with E. This picture has the advantage of being compatible with the action of Σ_n permuting coordinates.

Note: L special $\Leftrightarrow$ $L = \sum L \cap Fe_i$. In general given a splitting $V = W_1 \oplus \dots \oplus W_n$ of V into 1-dimensional spaces, we get apartments in X', X made up of lattices $L \ni L = \sum L \cap W_i$.

B-orbits

Suppose $V = F^n$ with basis ~~[illegible]~~ e_i, $1 \leq i \leq n$. We identify automorphisms g of V with invertible matrices $(g_{ij}) \in GL_n F$ by the formula

$$g e_i = \sum g_{ji} e_j$$

~~Define lattices V_p for $p \in \mathbb{Z}$ by putting~~

~~$V_i = Re_1 + \cdots + Re_{i-1} + R\pi e_i + \cdots + R\pi e_n$~~

~~for $i = 1, \ldots, n$ and setting $V_p = \pi^t$~~

Let $V_i = Re_1 + \cdots + Re_{i-1} + R\pi e_i + \cdots + R\pi e_n$ for $i = 1, \ldots, n$ and ~~[illegible]~~ set $e_p = \pi^{-t} e_i$, $V_p = \pi^{-t} V_i$ if $p = i + tn$. Then $\{V_p\}$ is a chambre of X which we call the fundamental chambre; V_p / V_{p-1} is one-dimensional space over $R/\pi R$ with basis e_p. The subgroup of $SL_n(F)$ preserving the chain $\{V_p\}$ will be denoted $\mathcal{B}$ and called the fundamental Iwahori subgroup of $SL_n(F)$. We have

$$\mathcal{B} = \{ g \in SL_n(R) \mid g_{ij} \in \pi R,\ i > j \}.$$

We propose now ~~[illegible]~~ to compute the $\mathcal{B}$-orbits on X' and X.

~~[illegible]~~

Let $\{L_j, j \in \mathbb{Z}\}$ be an increasing sequence of lattices such that $\pi L_j = L_{j-k}$ for all j. Define a map $\phi : \mathbb{Z} \to \mathbb{Z}$ by

$$(*) \qquad V_p \cap L_j + V_{p-1} = \begin{cases} V_{p-1} & j < \phi(p) \\ V_p & j \geq \phi(p). \end{cases}$$

Then $\phi(p+n) = \phi(p+k)$ for all p, and the function ϕ is an invariant of the B-orbit of the sequence L_j.

<u>Lemma</u>: Let Z be a set of representatives for the πR-cosets in R. Then there exists a unique element $x_p \in L_{\phi(p)}$ of the form

$$(**) \qquad x_p = e_p + \sum_{\substack{a<p \\ \phi(a) > \phi(p)}} z_{ap} e_a$$

with $z_{ap} \in Z$.

<u>Proof</u>: By $(*)$ above can find $y_p \in V_p \cap L_{\alpha(p)}$ such that $y_p - e_p \in V_{p-1}$. By induction on N we can find $z_{ap} \in Z$ for $-N < a < p$ such that

$$y_p - e_p - \sum_{-N<a<p} z_{ap} e_a \in V_{-N}.$$

Choose N large enough so that $V_{-N} \subset L_{\phi(p)}$; and let a be largest such that $\phi(a) \leq \phi(p)$ and the coefficient $z_{ap} \neq 0$. ~~Replacing~~ Then $y_p - z_{ap} y_a$ is another choice for y_p ~~with the same coefficients~~ having same coefficients in the places above a, but 0 coefficient at the place a. Continuing we get an element $x_p \in L_{\phi(p)}$ of the form (**). If x_p' is another element with the coefficients Z'_{ap}, let a be largest such that $Z'_{ap} \neq Z_{ap}$. Then $x_p' - x_p \in V_a$ but not V_{a-1}, so

$$V_a \cap L_{\phi(p)} + V_{a-1} = V_a$$

contradicting the fact that $\phi(a) > \phi(p)$. This proves the uniqueness of the elements x_p.

In virtue of the uniqueness we have:

$$\pi x_p = x_{p-k}$$

and hence if g is the endo. of V with $ge_i = x_i$ for $i=1,\dots,n$, then $g(e_p) = x_p$ for all p.

Now $g(e_i) = \sum g_{ji} e_j$, where from (**) we have:

$$g_{ji} = \delta_{ji} + \sum_t Z_{j-tn,i}\,\pi^t$$

where t runs over integers such that $j - tn < i$ and $\alpha(j-tn) = \alpha(j) - tk > \alpha(i)$, that is

$$\frac{j-i}{n} < t < \frac{\alpha(j)-\alpha(i)}{k}$$

By virtue of the canonical isomorphisms

$$\frac{V_p\cap L_j + V_{p-1}}{V_p\cap L_{j-1}+V_{p-1}} \xleftarrow{\sim} \frac{V_p\cap L_j}{V_p\cap L_{j-1}+V_{p-1}\cap L_j} \xrightarrow{\sim} \frac{V_p\cap L_j + L_{j-1}}{V_{p-1}\cap L_j + L_{j-1}}$$

and (*), the filtration $V_p\cap L_j + L_{j-1}$ of the layer $L_{j-1}\subset L_j$ has 1-dimensional jumps generated by x_p for those p such that $\phi(p)=j$. Thus L_j/L_{j-1} has a basis over $R/\pi R$ given by the images of the x_p with $\phi(p)=j$. Similarly, $L_j/\pi L_j = L_j/L_{j-k}$ has the basis given by the x_p with $j-k<\phi(p)\le j$. As L_j is a free R-module, these x_p form a basis for L_j over R:

(1) $$L_j = \bigoplus_{j-k<\phi(p)\le j} R x_p .$$

By the uniqueness assertion in the lemma one has $\pi x_p = x_{p-k}$, hence if g is the endomorphism of V with $ge_i = x_i$ for $1\le i\le n$, then $ge_p = x_p$ for all p in $\mathbb{Z}$. Let $g(e_i)=\sum g_{ji}e_j$, whence from (**) we have

(2) $$g_{ji} = \delta_{ji} + \sum_t z_{j-tn,i}\,\pi^t$$

where t runs over integers such that $j-tn<i$ and $\phi(j-tn)=\phi(j)-tk>\phi(i)$, that is,

$$\frac{j-i}{n} < t < \frac{\phi(j)-\phi(i)}{k} .$$

Clearly $g_{ji} = \delta_{ji}$ if $\phi(j) \leq \phi(i)$, hence $\det(g) = 1$, so $g \in \mathcal{B}$.

We are now in a position to classify $\mathcal{B}$-orbits on the set of increasing sequences of lattices L_j, $j \in \mathbb{Z}$, such that $\pi L_j = L_{j-k}$ for all j. Call this set $\mathcal{S}_k$. To such a sequence we have associated a function $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $\phi(p+n) = \phi(p) + k$ for all p, which depends only on the $\mathcal{B}$-orbit of the sequence. On the other hand, to each such ϕ we can associate the sequence of special lattices:

$$(3) \qquad L_j^{\phi} = \bigoplus_{j-k < \phi(p) \leq j} R e_p .$$

It is easily checked that the function assoc. to (L_j^{ϕ}) is ϕ, hence we obtain a 1-1 correspondence between sequences in $\mathcal{S}_k$ made up of special lattices and the functions ϕ. From the above formulas (1)-(3) we see that (L_j) is in the $\mathcal{B}$-orbit of (L_j^{ϕ}) if ϕ is associated to (L_j). Thus we have:

$$\boxed{\mathcal{S}_k^{sp} \xrightarrow{\sim} \mathcal{B} \backslash \mathcal{S}_k \xrightarrow{\sim} \{\phi : \mathbb{Z} \to \mathbb{Z} \mid \phi(p+n) = \phi(p) + k\}}$$

where $\mathcal{S}_k^{sp} \subset \mathcal{S}_k$ is the subset of "special" sequences.

Let's now consider the $\mathcal{B}$-orbit of (L_j^ϕ). From the lemma we have associated to a sequence (L_j) in this orbit a set of $z_{ap} \in Z$ for $a<p$ and $\alpha(a) > \alpha(p)$ satisfying $z_{a+n,p+n} = z_{ap}$.

From the z_{ap} we can construct a $g \in \mathcal{B}$ (see (2)) such that $g(L_j^\phi) = (L_j)$. So I get a map

$$\mathcal{B}(L_j^\phi) \longrightarrow (Z)^{\ell(\phi)}, \quad \ell(\phi) = \operatorname{card}\{(a,p) \mid \substack{1 \le p \le n \\ a<p}, \phi(a) > \phi(p)\}$$

$$(L_j) \longmapsto (z_{ap})$$

; and also a map the other way sending (z_{ap}) to $g(L_j^\phi)$, where g is given by (2). It is clear from the preceding that these two maps are inverses of each other, so we get a bijection:

$$\boxed{\mathcal{B}(L_j^\phi) \xrightarrow{\ \sim\ } (R/\pi R)^{\ell(\phi)}}$$

Geometric interpretation of $\ell(\phi)$: According to the calculation made at the bottom of page 9, $\ell(\phi)$ is the sum over pairs (i,j) such that $1 \le i,j \le n$ of the number of integers t such that

$$\frac{j-i}{n} < t < \frac{\phi(j)-\phi(i)}{k}$$

This is the same as the sum over $1 \le i < j \le n$ of the number of integers strictly between $\frac{j-i}{n}$ and $\frac{\phi(j)-\phi(i)}{k}$, so we have the following

__Assertion:__ $\ell(\phi)$ is the number of hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = t$ with $1 \le i < j \le n$ and $t \in \mathbb{Z}$ which are crossed in going along the straight line joining $(-\frac{1}{n}, \ldots, -\frac{n}{n})$ and $(-\frac{\phi(1)}{k}, \ldots, -\frac{\phi(n)}{k})$.

If $p = i + tn$, $1 \le i \le n$, satisfies $j-k < \phi(p) \le j$, then $(j-\phi(i))/k - 1 < t \le (j-\phi(i))/k$, so

$$t = [j-\phi(i)/k]$$

where $[x]$ = greatest integer $\le x$. Hence (3) page 10 can be written

$$L_j^\phi = \sum_{i=1}^{n} R\,\pi^{-[j-\phi(i)/k]} e_i \,.$$

Under ~~[illegible]~~ the identification of special lattices with integral points of $\mathbb{R}^n$, the lattice L_j^ϕ corresponds to the sequence $i \mapsto [j-\phi(i)/k]$. Now we have the

identity
$$\sum_{j=0}^{k-1} [j - \phi(i)/k] = -\phi(i)$$

hence the point $\left(-\frac{\phi(1)}{k}, \ldots, -\frac{\phi(n)}{k}\right)$ is the average of the integral points of $\mathbb{R}^n$ corresponding to the lattices $L_0^\phi, \ldots, L_{k-1}^\phi$. This is an interior point of the simplex in $\mathbb{R}^n$ (for the triangulation $\mathbb{R}^n = |D^n| = |A'|$) having these lattices for vertices.

Next note that the function $x_i - x_j$ from $\mathbb{R}^n$ to $\mathbb{R}$ maps any simplex to a simplex, so ~~the number of hyperplanes crossed in going along the line joining two points $\xi, \eta \in \mathbb{R}^n$ is the same~~ the set of hyperplanes of the form $x_i - x_j = t \in \mathbb{Z}$ crossed in going along the line joining ~~the points~~ ~~[illegible]~~ two points $\xi, \eta \in \mathbb{R}^n$ is the same as for ξ', η where ξ' is another point with the same support at ξ (meaning ξ, ξ' are interior points of the same simplex). Consequently the point $\left(-\frac{\phi(1)}{k}, \ldots, -\frac{\phi(n)}{k}\right)$ can be replaced by any point with the same support in so far as computing $l(\phi)$.

Next note that the functions $x_i - x_j$ are defined on $\mathbb{R}^n/\mathbb{R}\delta$ where $\delta = (1, \ldots, 1)$, where I recall $|A| \cong \mathbb{R}^n/\mathbb{R}\delta$, this homeom. being induced by sending a special lattice $L = \sum R\pi^{-d_i} e_i$ to the class of $(d_1,\dots,d_n)$ mod $\mathbb{R}\delta$. So we get the following:

<u>Assertion</u>: Given the sequence (L_j^ϕ) of special lattices assoc. to $\phi: \mathbb{Z}\to\mathbb{Z}$ such that $\phi(p+n)=\phi(p)+k$. Then the dimension $\ell(\phi)$ of the B-orbit of this sequence can be computed as follows. ~~Take~~ Choose any interior point ξ in the simplex in $|A| = \mathbb{R}^n/\mathbb{R}\delta$ with vertices $cl(L_j^\phi)$. ~~...~~ Then $\ell(\phi)$ is the number of hyperplanes of the form $x_i - x_j = t$, $1\le i<j\le n$, $t\in\mathbb{Z}$ crossed in going along the straight line joining ξ to an interior point of <u>the</u> fundamental chambre.

~~...~~

Now we apply what precedes to determine the B-orbits on X' and X. Given $\sigma: L_0 < \dots < L_{k-1}$ a $(k-1)$-simplex of X', we can extend it to a sequence L_j by putting $L_{i+tk} = \pi^{-t} L_i$. In this way $(k-1)$-simplices of X' become identified with increasing sequences of lattices (L_j) satisfying $\pi L_j = L_{j-k}$ and such that $L_0 < L_1 < \dots < L_{k-1}$ (note that L_{k-1} can be equal to $\pi^{-1}L_0 = L_k$).

According to what has been shown, the B-orbit of (L_j) contains exactly one sequence made up of special lattices, hence the orbit $\mathcal{B}\sigma$ contains exactly one simplex ~~..~~ in the apartment A'. Therefore we have an isomorphism

$$\boxed{A' \xrightarrow{\sim} \mathcal{B}\backslash X'}$$

and we get a retraction $\rho : X' \to A'$ by associating to σ the unique simplex in $\mathcal{B}\sigma$ which is in A'. This map ρ is essentially the one sending σ to the function ϕ associated to (L_j).

Given a $(k-1)$-simplex σ of X, we have $\sigma = \{L_j, j \in \mathbb{Z}\}$, where (L_j) is an increasing sequence of lattices such that $\pi L_j = L_{j-k}$ for all j and $L_0 < L_1 < \cdots < L_k = \pi^{-1} L_0$. If (L_j') is another sequence giving σ, then $L_j' = L_{j+c}$ for some integer c. Thus $(k-1)$-simplices of X can be identified with equivalence classes of sequences (L_j) in $\mathcal{S}_k$ which are strictly increasing $(L_j < L_{j+1})$, and where sequences are considered equivalent when they coincide after a translation of the indices.

According to what's been shown, the $\mathcal{B}$ orbit of σ contains a unique simplex in the apartment A. Thus

$$\boxed{A \xrightarrow{\sim} \mathcal{B}\backslash X}$$

and we have a retraction $\rho : X \to A$ whose fibres are the $\mathcal{B}$-orbits. ~~The dimension of the~~

Note that these two results in boxes imply:

$$|A'| \xrightarrow{\sim} \mathcal{B}\backslash|X'| \qquad \text{and} \qquad |A| \xrightarrow{\sim} \mathcal{B}\backslash|X| .$$

In effect the map $|A| \longrightarrow \mathcal{B}\backslash|X|$ is onto, and ρ induces a map $\mathcal{B}\backslash|X| \longrightarrow |A|$ which is a retraction for the preceding map.

Given $x \in |X|$ let σ be the support of x so that $x = \sum_{v \in \sigma} \lambda_v v$ where $\lambda_v > 0$ and $\sum \lambda_v = 1$. Then the stabilizers of x and of σ in $\mathcal{B}$ are the same, because any $b \in \mathcal{B}$ such that $b\sigma = \sigma$ satisfies $bv = v$ for all v in σ. (To see this I can apply the retraction $\rho : X \to A$; or I can argue that if b preserves $\{L_j\}$, then $bL_j = L_j$ because $\det(b) = 1$ and either $bL_j \subset$ or $\supset L_j$.)

From this + the assertion on page 14, the dimension of the $\mathcal{B}$-orbit of a point x is the number of hyperplanes in the apartment crossed in going along the line joining ρx to an interior point of the fundamental chamber.

Chambres in A, the Weyl group, and the Bruhat decomposition:

Let $\mathcal{N}' = \Sigma_n \ltimes (F^*)^n$, $\mathcal{H}' = (R^*)^n$ be considered as subgroups in $GL_n(F)$ in the standard way. Put $\mathcal{W}' = \mathcal{N}'/\mathcal{H}' \simeq \Sigma_n \ltimes \mathbb{Z}^n$. $\mathcal{N}'$ is the subgroup of $GL_n(F)$ preserving the splitting $V = Fe_1 \oplus \cdots \oplus Fe_n$ into lines. Hence $\mathcal{N}'$ acts on special lattices. Since $\mathcal{H}'$ carries a special lattice into itself, we get an action of $\mathcal{W}'$ on special lattices, hence an action of $\mathcal{W}'$ on A' and A.

Recall we have identified increasing sequences (L_j) of special lattices such that $\pi L_j = L_{j-k}$ with functions $\phi : \mathbb{Z} \to \mathbb{Z}$ satisfying $\phi(p+n) = \phi(p) + k$. Hence $\mathcal{W}'$ acts on such functions ϕ.

Let $\alpha \in \mathcal{N}'$. There is a permutation $\sigma \in \Sigma_n$ and a sequence of integers $t_1, \dots, t_n$ such that

$$\alpha(e_i) = u_i \pi^{-t_i} e_{\sigma i} \qquad u_i \in R^*.$$

Hence α determines a map $\nu : \mathbb{Z} \to \mathbb{Z}$ by $\alpha(e_p) \in R^* e_{\nu(p)}$. The map ν is an isomorphism satisfying $\nu(p+n) = \nu(p) + n$. It depends only on the image of α in $\mathcal{W}'$.

On one hand we have identified $\mathcal{W}'$ with $\Sigma_n \ltimes \mathbb{Z}^n$ as follows: w corresponds to (σ, t), $\sigma \in \Sigma_n$, $t \in \mathbb{Z}^n$, iff for any $\alpha \mapsto w$ we have

$$\alpha(e_i) \in R^* \pi^{-t_i} e_{\sigma i}$$

On the other hand we have identified $\mathcal{W}'$ with permutations of $\mathbb{Z}$ commuting with translation by n as follows: w corresponds to ν iff for any $\alpha \mapsto w$ we have

$$\alpha(e_p) \in R^* e_{\nu(p)}.$$

Thus ν corresponds to (σ, t) iff

$$\nu(i) = \sigma i + t_i n$$

Suppose now (L_j) is an increasing sequence of special lattices corresp. to ϕ; this means $\phi(p) = j$ iff e_p is in the natural basis of L_j / L_{j-1}. Apply α to get a new sequence (αL_j), and let ψ be the corresponding function. Then $\phi(p) = j$ $\Longleftrightarrow$ e_p is a basis elt of L_j / L_{j-1} $\Longleftrightarrow$ $e_{\nu p}$ is a basis elt of $\alpha L_j / \alpha L_{j-1}$ $\Longleftrightarrow$ $\psi(\nu p) = j$. Thus $\phi = \psi \nu$ or $\psi = \phi \nu^{-1}$. Therefore we have:

<u>Assertion</u>: If we identify $\mathcal{W}'$ with perms. ν of $\mathbb{Z}$ commuting with translation, then the action of $\mathcal{W}'$ on functions $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $\phi(p+n) = \phi(p) + k$ is

given by $\quad \nu(\phi) = \phi \circ \nu^{-1}$.

Consider the set of chambres in A'. These are classified by functions ϕ which are isos. such that $\phi(p+n) = \phi(p)+n$. Consequently $\mathcal{W}'$ acts simply-transitively on the set of chambres in A'.

Let $\mathcal{B}' = \{g \in GL_nR \mid g_{ij} \in \pi R \text{ if } i>j\}$. $\mathcal{B}'$ is the stabilizer in GL_nF of the fundamental chambre $V_0 < \cdots < V_n$ of A'. Thus $GL_nF/\mathcal{B}'$ is essentially the set of chambres of X'. Since each $\mathcal{B}'$-orbit on X' meets A' at exactly one point, $\mathcal{B}'\backslash GL_nF/\mathcal{B}'$ may be identified with the set of chambres in A'. Thus we get the Bruhat decomposition

$$\boxed{\mathcal{W}' \xrightarrow{\sim} \mathcal{B}' \backslash GL_nF/\mathcal{B}'}$$

Next let's discuss the modifications for SL_nF. Put $\mathcal{N} = \mathcal{N}' \cap SL_nF$, $\mathcal{H} = \mathcal{H}' \cap SL_nF$, and $\mathcal{W} = \mathcal{N}/\mathcal{H}$. Then $\mathcal{W} \subset \mathcal{W}'$. Let $\nu \in \mathcal{W}$ be interpreted as an auto of $\mathbb{Z}$ commuting with translation by n, and also as a pair (σ, t), so that $\nu(i) = \sigma i + t_i n$. If $\alpha \mapsto \nu$ with $\alpha \in \mathcal{N}$, then the lattice

$$\alpha R^n = \sum R\,\pi^{-t_i} e_{\sigma i}$$

must have same volume as R^n, hence $\sum t_i = 0$.

Conversely given $\nu \leftrightarrow (\sigma, t)$ with $\sum t_i = 0$, the monomial matrix sending e_i to $\pi^{-t_i} e_{\sigma i}$ has determinant sgn(σ), and we can mulifly by a diagonal matrix: $e_i \mapsto \pm e_i$ to get an element of N mapping to ν. Thus

$$\mathcal{W} = \Sigma_n \ltimes \{t \in \mathbb{Z}^n \mid \textstyle\sum t_i = 0\}.$$

I can think of elements of $\mathcal{W}$ as being perms. of $\mathbb{Z}$ commuting with $p \mapsto p+n$ which preserve volume:

$$p + N = \operatorname{card}\{p \mid -N < a,\ \nu(a) \leq p\}$$

for N large enough so that $a \leq -N \Rightarrow \nu(a) \leq p,\ a \leq p$.

Consider chambres in X. These are chains $\gamma = \{L_j\}$, $\pi L_j = L_{j-n}$, $\dim(L_j/L_{j-1}) = 1$. We can normalize the indexing of the chain so that $\Lambda^n L_j = \Lambda^n V_j$ for each j. In this case the function ϕ describing the B-orbit of γ is a volume-preserving auto of $\mathbb{Z}$. As before the $\mathcal{W}$-action on ϕ will be given by $\nu(\phi) = \phi \circ \nu^{-1}$, hence $\mathcal{W}$ acts simply-transitively on the chambres in A. As in the case of GL_n we get the Bruhat decomposition:

$$\boxed{\mathcal{W} \simeq B \backslash SL_n F / B.}$$

(this map is induced by inclusion of N in $SL_n F$)

Link of a simplex in X.

Let σ be a $(k-1)$-simplex of X, say $\sigma = \{L_j\}$ where (L_j) is an increasing sequence $\ni$ $\pi L_j = L_{j-k}$ and $L_0 < L_1 < \cdots < L_k = \pi^{-1} L_0$. A vertex in $\mathrm{Link}(\sigma)$ is of the form $cl(L)$ where $\exists\, j$ such that $L_{j-1} < L < L_j$, and we can suppose $1 \le j \le k$. A simplex τ in $\mathrm{Link}(\sigma)$ is a simplex of X such that each vertex is in $\mathrm{Link}(\sigma)$, because any two lattices in $\sigma \cup \tau$ are comparable.

If W is a vector space over $R/\pi R$, let $T(W)$ denote the simplicial complex associated to the poset of proper subspaces of W. Any simplex τ of $\mathrm{Link}(\sigma)$ is a join $\tau = \tau_0 * \tau_1 * \cdots * \tau_k$, where τ_j is either empty or a simplex of $T(L_j/L_{j-1})$, and not all τ_j are empty. Thus

$$\mathrm{Link}(\sigma) = T(L_1/L_0) * \cdots * T(L_k/L_{k-1}).$$

($T(L_j/L_{j-1})$ is a bouquet of $(\dim(L_j/L_j)-2)$-spheres, hence this join is a bouquet of spheres of dimension

$$(\dim(L_1/L_0)-2)+1+(\dim(L_2/L_1)-2)+1+\cdots+1+(\dim L_k/L_{k-1}-2)$$

$$= n-1-k.$$

The local homology $H_*(X, X-\{x\})$ is therefore concentrated in degree $n-1$.)

Lemmas on simplices and chambres of X.

Suppose σ is a $(k-1)$-simplex of X given by ~~$\cdots$~~ (L_j), and let $\gamma = \{M_q\}$ be a chambre containing σ. Let ϕ, ψ be the functions assoc. to σ and γ respectively. Thus $\phi(p) = j$ means that the quotient V_p/V_{p-1} of the composition series $\{V_p\}$ appears in L_j/L_{j-1} in the sense of the Jordan-Holder-Schreier theorem. ~~$\cdots$~~ Let $R_\phi = \{(a,p) \mid a<p,\ \phi(a) > \phi(p)\}$. ~~$\cdots$~~ Then $\mathbb{Z}$ acts on R_ϕ by $(a,p) \mapsto (a+t_j^n, p+t_n)$ and we have

$$\ell(\phi) = \text{card } \mathbb{Z}\backslash R_\phi$$

$$= \text{card } \{(a,p) \mid a<p,\ 1\le p\le n,\ \phi(a) > \phi(p)\}$$

$$= \text{card } \{(a,p) \mid a<p,\ \phi(a)>\phi(p),\ 1\le \phi(p) \le k\}$$

We put $\ell(\gamma) = \ell(\phi)$; $\ell(\gamma)$ is well-defined since ϕ is determined up to translation by γ.

Now since $\{M_q\}$ refines $\{L_j\}$, it is clear that $\phi(a) > \phi(p) \implies \psi(a) > \psi(b)$. Thus $R_\phi \subset R_\psi$ so $\ell(\sigma) \le \ell(\gamma)$. If $\psi(a) > \psi(p)$, then $\phi(a) \ge \phi(p)$, so $R_\psi - R_\phi$ consists of pairs in R_ψ with $\phi(a) = \phi(p)$. ~~$\cdots$~~ Thus

$$\ell(\gamma) - \ell(\sigma) = \sum_{j=1}^{k} \text{card } \{(a,p) \mid \phi(a) = \phi(p) = j,\ a<p,\ \psi(a)>\psi(p)\}.$$

Let us consider the j-th term of this sum, denote it S_j. This is the number of pairs in $\phi^{-1}(j)$ whose order is reversed by ψ. Translating the indexing of the sequence (M_j), I can suppose that $M_0 = L_{j-1}$ whence I get a composition series:

(1) $$L_{j-1} = M_0 < M_1 < \cdots < M_s = L_j$$

and $\phi^{-1}(j) = \{\psi^{-1}(1), \dots, \psi^{-1}(s)\}$. Then $S_j = \operatorname{card}\{(a,b) \mid 1 \leq a < b \leq s \text{ and } \psi^{-1}(a) > \psi^{-1}(b)\}$. Note that the fundamental chambre $\{V_p\}$ induces a comp. series:

(2) $$V_p \cap L_j + L_{j-1} \qquad p \in \phi^{-1}(j)$$

in the layer $L_{j-1} \subset L_j$. When the quotients of the two composition series (1) + (2) are compared via Jordan-Holder, the p-th quotient of (2) is isomorphic to the $\psi(p)$-th quotient of (1).

Suppose $\ell(\gamma) = \ell(\sigma)$, that is, $a < b \Leftrightarrow \psi^{-1}(a) < \psi^{-1}(b)$. One sees easily then that the series (1) and (2) coincide, so we obtain:

Lemma 1: Given a simplex $\sigma = \{L_j\}$, there is a unique chambre γ containing σ such that $\ell(\gamma) = \ell(\sigma)$. γ is the chambre obtained by taking the filtration in each layer L_j / L_{j-1} induced by $\{V_p\}$.

~~[crossed out]~~

Suppose next that $\ell(\sigma) < \ell(\gamma)$. Then for some j, the ψ-ordering on $\phi^{-1}(j)$ is different from the usual ordering. ~~supposing without loss of generality that we then (1) holds, we see there exists~~ Supposing without loss of generality that (1) hold, we see there exists an integer i with $1 \le i, i+1 \le s$ such that $\psi^{-1}(i) > \psi^{-1}(i+1)$. It is easily seen that if ρ is the panel (codim 1 face) of γ obtained by deleting $cl(M_i)$ from γ, then $\ell(\rho) < \ell(\gamma)$, in fact $\ell(\rho) = \ell(\gamma) - 1$. So we obtain

<u>Lemma 2</u>: If σ is contained in a chambre γ with $\ell(\sigma) < \ell(\gamma)$, then there exists a panel ρ of γ such that $\rho \supset \sigma$ and $\ell(\rho) < \ell(\gamma)$.

Corollary of lemma 1: Let $\sigma \subset \gamma$ be as in lemma 1. If τ is any simplex containing σ, then $\ell(\sigma) \le \ell(\tau)$ with equality iff $\tau < \gamma$.

For if γ' is the unique chambre containing τ with $\ell(\gamma') = \ell(\tau)$, then $\sigma < \gamma'$ so $\ell(\sigma) \le \ell(\gamma') = \ell(\tau)$. If $\ell(\sigma) = \ell(\tau)$, then $\ell(\gamma') = \ell(\sigma) \Rightarrow \gamma' = \gamma$.

Suppose now γ is a chambre $\gamma = \{M_j\}$ associated to the isomorphism ϕ. Let ρ_i be the panel of γ obtained by deleting $cl(M_i)$ from γ

for $i = 0, \ldots, n-1$. Then $\ell(\rho_i) = \ell(\gamma)$ iff $\psi^{-1}(i) < \psi^{-1}(i+1)$, otherwise $\ell(\rho_i) = \ell(\gamma) - 1$. If $\ell(\rho_i) = \ell(\gamma)$ for all panels of γ, then

$$\psi^{-1}(0) < \psi^{-1}(1) < \cdots < \psi^{-1}(n) = \psi^{-1}(0) + n$$

which is possible only if $\psi^{-1}(j) = j + \psi^{-1}(0)$. In this case ψ preserves ordering on $\mathbb{Z}$, so $\ell(\gamma) = 0$ which means $\gamma \in A$ since its B-orbit is trivial, hence γ = fundamental chambre. Finally if $\ell(\rho_i) < \ell(\gamma)$ for all panels of γ, then we would have

$$\psi^{-1}(0) > \cdots > \psi^{-1}(n) = \psi^{-1}(0) + n$$

which is impossible. Thus we have

Lemma 3: Given a chambre γ there exist panels of the same length. There exist panels of smaller length iff $\gamma \neq$ fundamental chambre.

Contractibility of X:

Let F_nX be the subcomplex consisting of simplices of length $\leq n$; it is the union of the closures of all the chambres of length $\leq n$. We show by induction that F_nX is contractible; as $X = \bigcup F_nX$ this will show X is contractible. F_0X is the closure of the fundamental chambre which is an $(n-1)$-simplex hence is contractible.

$F_n X$ is the union of $F_{n-1} X$ and $|\gamma|$ each chambre γ of length n. By lemma 1 any two chambres of length n intersect inside $F_{n-1} X$. It suffices therefore to show that for any chambre $\gamma \neq$ fixed chambre that $|\gamma|$ has $|\gamma| \cap |F_{n-1} X| = |\{\sigma < \gamma \mid \ell(\sigma) \leq n-1\}|$ as a strong deformation retract. But by lemmas 2 & 3 this subcomplex of $|\gamma|$ is a non-empty union of panels, but not all the panels of γ, so this point is OKAY.

Further topics to be explored.

1) Case of a sfield F with maximal order R and $R\pi$ = unique maximal ideal in R

2) $\mathcal{W}$ generated by reflections, $\mathcal{W}$ is a Coxeter group exchange condition

3) $\mathcal{B} \backslash \mathcal{G} / \mathcal{P}_I \simeq \mathcal{W} / \mathcal{W}_I \quad ; \quad I \subset S$

May 1975

Scattering Theory

Scattering theory is concerned with the following situation. A Hilbert space $\mathcal{H}$ is given with two 1-parameter unitary groups $U_0(t)$ and $U(t)$. $U_0(t)$ is understood, and $U(t)$ is a perturbation of $U_0(t)$ which one wants to understand. The scattering operator S arises as follows: One starts with $x \in \mathcal{H}$ and lets $J_- x$ be the element of $\mathcal{H}$ such that $U_0(t)x$ and $U(t)J_-x$ are asymptotic as $t \to -\infty$ (assuming this exists). Then $U(t)J_-x$ will be asymptotic to $U_0(t)(J_+^{-1}J_-x)$ as $t \to +\infty$. One sets $S = J_+^{-1}J_-$. The problems of the theory consist in proving these operators are well-defined, etc. (see Kato's address at NICE congress.)

I propose to discuss a simplified example of the theory in order to understand the basic phenomena. First I consider the discrete case: where one is given two unitary operators U_0, U on $\mathcal{H}$. I suppose U_0 given explicitly as multiplication by z on $\mathcal{H} = L^2(S^1)^n$. Put $W = \mathbb{C}e_1 + \cdots + \mathbb{C}e_n \subset \mathcal{H}$. The unitary operator U will be required to coincide with U_0 on $z^i W$ for $|i|$ sufficiently large. Thus $U = U_0 \Theta$ where Θ is a unitary operator on the finite dimensional space

$$z^{-N}W + \cdots + z^{N}W$$

extended by the identity on the orthogonal complement.

Let $\Delta = \mathbb{C}[z, z^{-1}]$, $\Delta_\pm = \mathbb{C}[z^{\pm 1}]$; let $D_\pm$ be the closure of $\Delta_\pm$ in $L^2(S^1)$. Note that U carries the dense subspace $\Delta^n \subset \mathcal{H}$ into itself. (Thus the situation at hand is very algebraic).

For large N, $U = U_0$ on $z^{-N}\Delta^n_-$, so $U(z^{-N}\Delta^n_-) = z^{-N+1}\Delta^n_-$. Taking perpendicular spaces (in Δ^n), one gets $U^{-1}(z^{-N+1}\Delta^n_+) = z^{-N}\Delta^n_+$ or $U(z^{-N}\Delta^n_+) = z^{-N+1}\Delta^n_+$. Put $L_2 = z^{-N}\Delta^n_+$ and $L_1 = z^{N+1}\Delta^n_+$; these are stable under U.

Let P be the orthogonal projection of L_2 onto $Y = L_2 \ominus L_1 = \bigoplus_{-N \le i \le N} z^i W$. The operator PU is a contraction operator on Y.

Let $v \in Y$ be an eigenvector for PU with eigenvalue ξ of absolute value 1. Then as P is a projection, for $PUv = \xi v$ to have the same norm as Uv means that $PUv = Uv$, hence v is an eigenvector for U.

~~Let v be an eigenvector for U in $\mathcal{H}$, and suppose it projects non-trivially into $z^i W$ for some i with $i < -N$ or $i \geq N$.~~

Let v be an eigenvector for $\mathcal{U}$ in $\mathcal{H}$ with eigenvalue $\lambda \in S^1$. If $v = \sum v_i$ with $v_i \in z^i W$, then for $i > N$, $\mathcal{U}v$ has the projection $z v_i$ in $z^{i+1} W$, hence $z v_i = \lambda v_{i+1}$. This implies $\|v_i\| = \|v_{i+1}\| = \cdots\cdots$, which is impossible unless $v_i = 0$ for $i > N$. Similarly $v_i = 0$ for $i < -N$. Thus any eigenvector v of $\mathcal{U}$ in $\mathcal{H}$ is contained in $Y = \bigoplus_{-N \le i \le N} z^i W$. And $P\mathcal{U}v = \zeta P v = \zeta v$.

Let T = subspace of $\mathcal{H}$ spanned by the eigenvectors of $\mathcal{U}$. It is the subspace of Y consisting of the eigenvectors of Z whose eigenvalues are on S^1. Put $\mathcal{H}' = \mathcal{H} \ominus T$, $L_2' = L_2 \ominus T$. Then on $L_2' \ominus L_1$, Z has all eigenvalues inside S^1.

Let p be the minimal polynomial of Z acting on $L_2' \ominus L_1$. The roots of p are inside of S^1. Since $p(Z) = 0$, we have

$$p(\mathcal{U})\, L_2' \subset L_1 .$$

Enlarging N by 1 if necessary, I can suppose $z^{-N} W$ is perpendicular to T, hence is in L_2'. Then for each $w \in W$, we have

1) $$p(\mathcal{U})\left(z^{-N} w\right) = \sum_{0 \le i \le m} z^{N+i} s_i(w)$$

where the $S_i \in \mathrm{End}(W)$.

Now let us compute the scattering operator S. First we note that for any $x \in \Delta^n$, $\mathcal{U}^{-t}\mathcal{U}_0^t x$ is contant for $t \ll 0$ and $t \gg 0$, so we get operators on Δ^n defined by

$$(2) \qquad J_\pm x = \lim_{t \to \pm\infty} \mathcal{U}^{-t}\mathcal{U}_0^t x$$

such that $\mathcal{U} J_\pm = J_\pm \mathcal{U}_0$. ~~The scattering~~

Since $\|J_\pm x\| = \|x\|$ and Δ^n is dense in $\mathcal{H}$, these operators [illegible] extend to isometries on $\mathcal{H}$. and the formula (2) holds for any $x \in \mathcal{H}$. (Proof: Given $\varepsilon > 0$, let $y \in \Delta^n$ be such that $\|x-y\| < \varepsilon$. Then

$$\|\mathcal{U}^{-t}\mathcal{U}_0^t x - \mathcal{U}^{-s}\mathcal{U}_0^s x\| \le \|\mathcal{U}^{-t}\mathcal{U}_0^t(x-y)\| + \|\mathcal{U}^{-s}\mathcal{U}_0^s(x-y)\| + \|\mathcal{U}^{-t}\mathcal{U}_0^t y - \mathcal{U}^{-s}\mathcal{U}_0^s y\|$$

$$< 2\varepsilon$$

for s, t sufficiently close to $\pm\infty$).

~~I now wish to show there is an operator S such that $J_- x = J_+ S x$ for all x in $\mathcal{H}$.~~ ~~If $x \in W$, then [illegible] applying [illegible] we get~~ ~~S is unique~~

Given $w \in W$, let's try to compute Sw. Then ~~[illegible]~~

$$J_- w = \mathcal{U}^N z^{-N} w$$

so applying 1) we get

$$p(U)\, J_- w = U^N \sum_{0 \le i \le m}' z^{N+i} s_i(w)$$

$$= J_+\Big(\sum' z^{2N+i} s_i(w)\Big)$$

But because p has no roots on S^1, $p(U)$ is an isomorphism of $\mathcal{H}$, so

$$J_- w = J_+\left(\frac{\sum_i z^{2N+i} s_i}{p(z)}\, w \right).$$

~~Actually it might be clearer to argue that we have~~

$$\cancel{J_-\big(p(z)w\big) = J_+\Big(\sum z^{2N+i} s_i(w)\Big)}$$

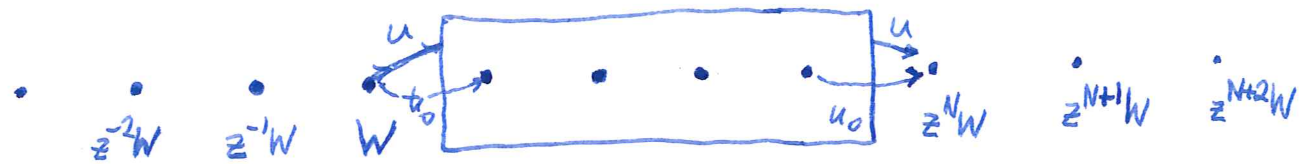
Applying U^k to both sides of this formula we see it holds with w replaced by $z^k w$. Hence one has for any $x \in \mathcal{H}$ that

$$J_- x = J_+(Sx)$$

where S is multiplication by the matrix function $\frac{1}{p(z)} \sum' z^{2N+i} s_i$. Thus we see the scattering operator exists. Because things are symmetric with respect to U, U_0 versus U^{-1}, U_0^{-1} the preceding argument can be modified so as to show that $\exists$ an S' with $J_- S' = J_+$. Thus we have that

$J_+ \mathcal{H} = J_- \mathcal{H}$ and that S is unitary. Since $J_+ = id$ on L_1, $J_+ \mathcal{H} \supset L_1 \supset p(U) L_2'$, so $J_+ \mathcal{H} \supset L_2' \oplus L_2^\perp$. It follows that $J_+ \mathcal{H} = J_- \mathcal{H} = \mathcal{H}' =$ orthogonal complement of T.

Here is a simple way to think of the scattering operator S. Suppose that the perturbation is supported in $W \oplus \cdots \oplus z^N W$ $\quad N \geq 1$ in the following sense:



Thus I want $\quad U_0^{-1} = U^{-1}$ on $z^i W$, $i \leq 0$

$\qquad U_0 = U$ on $z^i W$, $i \geq N$

whence $\quad J_+ = id$ on $z^N D_+^n$

$\qquad J_- = id$ on D_-^n

~~Let us take a U-path $U^t w$ beginning with $z^i w$ for $i \leq 0$. Thus~~

Let us now consider the U-path $U^t w$ as it moves through the ~~scattering~~ scatterer. As soon as $U^t w$ has a component in $z^N W$, this component translates along without change. Thus

$$U^0 w = w$$

$$U^1 w = \text{elt in box} + z^N \varphi_{N-1}(w), \quad \text{box} = zW \oplus \cdot \oplus z^{N-1} W$$

$$U^2 w = \text{elt in box} + z^N \varphi_{N-2}(w) + z^{N+1} \varphi_{N-1}(w)$$

$$U^3 w = \quad '' \quad + z^N \varphi_{N-3}(w) + z^{N+1} \varphi_{N-2}(w) + z^{N+2} \varphi_{N-1}(w)$$

etc. So

$$Sw = J_+^* J_- w = \lim_{t \to +\infty} U_0^{-t} U^t w$$

$$= \sum_{i \leq N-1} z^i \varphi_i(w)$$

Notation from Schweber: The generalized eigenvectors for U_0 are of the form

$$\delta_a(z) w = \sum_{i \in \mathbb{Z}} a^{-i} z^i w$$

where $a \in S^1$, $w \in W$, and δ_a is δ-function at the point a. One has

$$U_0(\delta_a(z) w) = a \cdot \delta_a(z) w.$$

~~Denote this eigenvector by $\varphi_{a,w}$. The operator J_- transforms it into a generalized eigenvector for U:~~

$$\cancel{\varphi^+_{a,w} = \sum a^{-i} U^i w}$$

The operators $J_\pm$ transforms this eigenvector for U_0 into ones for U with the same eigenvalue a

$$J_-(\delta_a w) = \sum a^{-i} U^i w$$

$$J_+(\delta_a w) = \sum a^{-i} U^i J_+ w.$$

The scattering operator S applied to $\delta_a w$ is

$$S(\delta_a w) = \sum_i z^i \, \delta_a(z) \, \varphi_i(w)$$

$$= \delta_a \cdot S(a) w$$

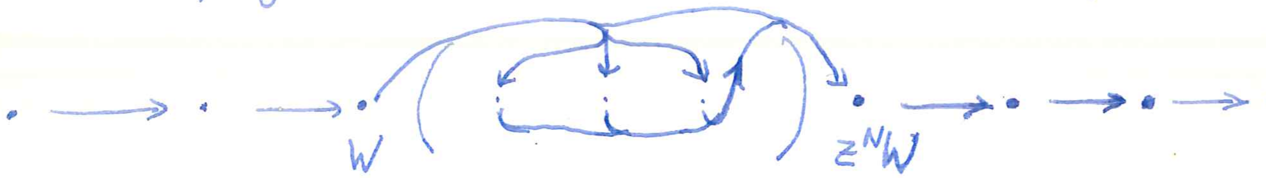
where $S(a) = \sum a^i \varphi_i \in \operatorname{End}(W)$. We have

$$J_-(\delta_a w) = J_+(S\,\delta_a w) = J_+(\delta_a \, S(a) w)$$

so we get the following interpretation for the scattering operator:

Take the generalized eigenfunction of U of the form $J_-(\delta_a w) = \sum a^{-i} U^i w$ which has the component $a^{-i} z^i w$ in degrees ≤ 0. Then its components in degrees $\geq N$ are $a^{-i} z^i (S(a) w)$, where $S(a)$ is the value of the scattering matrix at $z=a$.

Let's reconcile the two expressions for S given on pages 5 and 7. Recall the picture



It is clear from this picture that

$$T \subset zW \oplus \cdots \oplus z^{N-1}W.$$

We take $L_1 = z^N \Delta_+^n$, $L_2 = \Delta_+^n$, and let $p(t)$ be the minimal polynomial of $\mathbb{Z}$ on $L_2 \ominus (L_1 \oplus T)$. As $W \subset L_2 \ominus (L_1 \oplus T)$, we have $p(U) W \subset L_1$, hence

$$(*) \qquad p(U) e_i = \sum g_{ji}(U)\, z^N e_j \qquad g_{ji} \in \mathbb{C}[U].$$

By an analogous argument there is a poly $q(t)$ with roots inside S^1 such that

$$q(U^{-1})(z^N e_i) = \sum h_{ji}(U^{-1})\, e_j \qquad h_{ji} \in \mathbb{C}[U^{-1}].$$

Applying J_+^* to $(*)$ we get

$$p(z) J_+^* e_i = \sum g_{ji}(z)\, z^N e_j$$

or

$$S(z) e_i = \sum \alpha_{ji}(z)\, z^N e_j$$

where $\alpha_{ji}(z) = \dfrac{g_{ji}(z)}{p(z)}$ is a rational function with poles inside S^1. The inverse of (α_{ji}) is the matrix $\dfrac{h_{ji}(z^{-1})}{q(z^{-1})}$ which has poles outside of S^1. Thus the scattering matrix $S(z)$ is a rational function of z, holomorphic on S^1 and its exterior while $S(z)^{-1}$ is holomorphic on S^1 and the interior. (0 and ∞ are excluded.)

The above argument shows that the scattering matrix $S(z)$ is such that the matrix $U^{-N}S(U) = \alpha(U)$ transforms the sequence $\{z^N e_j\}$ into the sequence e_i. In fact looking at $\mathcal{H}\ominus T$ as isomorphic to $L^2(S^1)^n$ with $z \longleftrightarrow U$ and $e_i \longleftrightarrow z^N e_i$, the matrix α is a scattering matrix carrying the outgoing space $\overline{L}_1 = z^N D_+^n$ into the outgoing space $\overline{L}_2 \ominus T$. Thus I know that

$$
\begin{aligned}
-\text{degree}(\det\alpha : S^1 \to S^1) &= \text{index of } \overline{L}_2\ominus T \text{ wrt } \overline{L}_1 \\
&= \dim\left(\overline{L}_2 \ominus T \ominus \overline{L}_1\right) \\
&= nN - \dim T.
\end{aligned}
$$

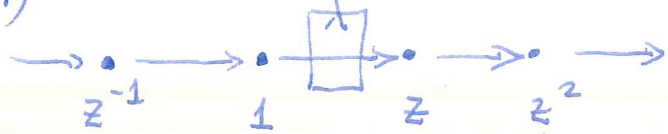
So

$$
\boxed{\text{degree}(\det S : S^1 \to S^1) = \dim T}
$$

(This degree is the number of zeros minus poles for $\det(S)$ in the unit circle.)

Examples: Take $n = 1$.

1)

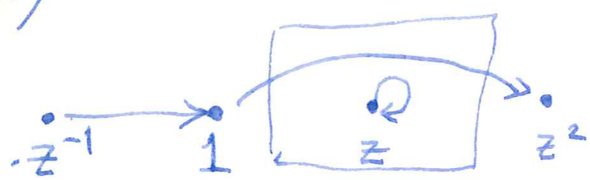


Thus $U1 = \lambda z$

where $|\lambda| = 1$. Then we have

$$U1 = \lambda z$$
$$U^2 1 = \lambda z^2$$
$$U^k 1 = \lambda z^k \qquad \therefore \quad S(z) = \lambda.$$

2)



$$Uz = z$$
$$U1 = z^2$$

$\boxed{\therefore\ z \text{ is a bound state}}$

Then
$$U1 = z^2$$
$$U^2 1 = z^3$$
$$\cdots$$
$$U^k 1 = z^{k+1} \qquad \therefore \quad S(z) = z.$$

3) Let
$$\Theta z = (\cos\alpha) z - (\sin\alpha) z^2$$
$$\Theta z^2 = (\sin\alpha) z + (\cos\alpha) z^2$$
$$\Theta z^m = z^m \quad \text{if} \quad m \neq 1, 2.$$

and put $U = \Theta U_0$.

$$U1 = \Theta z = (\cos\alpha) z - (\sin\alpha) z^2$$
$$Uz = \Theta z^2 = (\sin\alpha) z + (\cos\alpha) z^2$$
$$Uz^2 = \Theta z^3 = z^3$$

$$U^2 1 = (\cos\alpha)\left[(\sin\alpha) z + (\cos\alpha) z^2\right] - \sin\alpha\, z^3$$
$$= \cos\alpha \sin\alpha\, z + \cos^2\alpha\, z^2 - \sin\alpha\, z^3$$

$$\cancel{U^3 1 = \cos^2\alpha \sin\alpha\, z - \cos\alpha \sin^2\alpha\, z^2 + \cos^2\alpha\, z^3 - \sin\alpha\, z^4}$$

$$\cancel{U^4 1 = \cos^3\alpha \sin\alpha - \cos^2\alpha \sin^2\alpha\, z - \cos\alpha \sin^2\alpha\, z^3}$$

$$U^3 1 = (\cos \sin)\left[\sin z + \cos z^2\right] + \cos^2 z^3 - \sin z^4$$

$$= \cos \sin^2 z + \cos^2 \sin z^2 + \cos^2 z^3 - \sin z^4$$

$$U^4 1 = \cos \sin^3 z + \cos^2 \sin^2 z^2 + \cos^2 \sin z^3 + \cos^2 z^4 - \sin z^5$$

$$U^5 1 = \cos \sin^4 z + \cos^2 \sin^3 z^2$$

Thus

$$S = (-\sin\alpha) z + \cos^2\alpha \left(1 + z^{-1}\sin\alpha + z^{-2}\sin^2\alpha + \cdots\right)$$

$$= (-\sin\alpha) z + \frac{\cos^2\alpha}{1 - z^{-1}\sin\alpha}$$

$$= \frac{\sin^2\alpha - (\sin\alpha) z + \cos^2\alpha}{1 - z^{-1}\sin\alpha}$$

$$\boxed{S = \frac{1 - z\sin\alpha}{1 - z^{-1}\sin\alpha}}$$