K compact 1-conn Lie group, G its complexification, R a discrete valuation ring over $\mathbb{C}$ with max ideal πR and $\mathbb{C} \simeq R/\pi R$, F its quotient field. Put $\mathcal{G} = G(F)$. According to Bruhat-Tits there is a building $\mathcal{I}$ attached to $\mathcal{G}$. Our aim in this section is to identify $\mathcal{I}$ with the set $\mathcal{X} = \mathcal{X}(K)$ of special paths in K.

~~(Nature of the identification: It will be a continuous bijection $\mathcal{I} \longrightarrow \mathcal{X}$, not a homeomorphism. For example, if K is simple, $\mathcal{I}$ is a simplicial complex with compactly generated topology, and $\mathcal{X}$ has a topology roughly a product of the topology in a simplex + topology on the vertices which comes from the topology on $\mathbb{C}$.)~~

T = max. torus of K, H its complexification, B a Borel subgroup containing H, Φ the set of roots of G wrt H, Φ^+ roots of B, $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}, \underline{\mathfrak{b}}, \underline{\mathfrak{k}}, \underline{\mathfrak{t}}$ Lie algebras.

$$\underline{\mathfrak{g}} = \underline{\mathfrak{h}} + \sum_{\alpha \in \Phi} \mathbb{C} X_\alpha$$

where X_α is a Chevalley basis: $\alpha(H_\alpha) = 2$ where $H_\alpha = [X_\alpha, X_{-\alpha}]$. E = $\mathbb{R}$-subspace of $\underline{\mathfrak{h}}$ spanned by H_α. Then $\underline{\mathfrak{t}} = iE$, and the map $x \longmapsto 2\pi i x$ is an isom of $E \xrightarrow{\sim} \underline{\mathfrak{t}}$ such that $\exp(2\pi i x) = 1$ in $T \iff x \in \mathbb{Z}$-lattice gen. by H_α. This is because K is 1-connected.

$$\underset{\sim}{k} = iE + \sum_{\alpha \in \Phi^+} \mathbb{R}(X_\alpha - X_{-\alpha}) + \mathbb{R}(iX_\alpha + iX_{-\alpha})$$

$C = \{ x \in E \mid 0 \le \alpha(x) \le 1 \text{ all } \alpha \in \Phi^+ \}$ fundamental chambre. It is known that every conjugacy class in K contains $\bar{x} = \exp(2\pi i x)$ for a unique $x \in C$.

$\mathcal{K}$ = group of special loops $S^1 \to K$, acts on $\mathcal{X}$. $g \in \mathcal{K}$, $h \in \mathcal{X}$, say $h(w) = f(e^{2\pi i w}) \exp(wX)$, $X \in \underset{\sim}{k}$ then

$$g(e^{2\pi i w})\, h(w)\, g(1)^{-1} = g(e^{2\pi i w}) f(e^{2\pi i w}) g(1)^{-1} \exp(w \operatorname{Ad}(g(1))X)$$

is again a space path. Thus we have an action of $\mathcal{K}$ on $\mathcal{X}$ given by the formula

$$(g * h)(w) = g(w)\, h(w)\, g(1)^{-1}.$$

Given $x \in C$, let $\tilde{x} \in \mathcal{X}$ be the element $\tilde{x}(w) = \exp(2\pi i w x)$. We now show every $\mathcal{K}$-orbit in $\mathcal{X}$ contains $\tilde{x}$ for a unique $x \in C$. ~~Then $\exists!\, x \in C \ni h(1)$ is conjugate in K to $\bar{x}$.~~ If $g * \tilde{x} = h$, then $g(1)\bar{x}g(1)^{-1} = h(1)$ so x is uniquely determined by $h(1)$. Now given h, choose $k \in K$, and $x \in C \ni h(1) = k\bar{x}k^{-1}$. Then $k^{-1}hk$ and $\tilde{x}$ are two elements of $\mathcal{X}$ with same proj. in K, hence $\exists!\, f \in \mathcal{K}' \ni$

$$k^{-1}h(w)k = f(e^{2\pi i w})\, \tilde{x}(w)$$

$\therefore\ h = g * \tilde{x}$ where $g(w) = kf(e^{2\pi i w})$.

~~Let $\mathcal{K}_x = \{g \in \mathcal{K} \mid g * \tilde{x} = \tilde{x}\}$, $K_x = \{k \in K \mid k\bar{x}k^{-1} = \bar{x}\}$. Then we have an isom~~

$$\cancel{\mathcal{K}_x \xrightarrow{\sim} K_x \qquad g \longmapsto g(1)}$$

~~with inverse sending $\xi \in K_x$ to the loop $w \longmapsto \tilde{x}(w)\,\xi\,\tilde{x}(w)^{-1}$.~~

So we have established that X is the quotient of $\mathcal{K} \times C$ by the equivalence relation $(g, x) \sim (g', x') \iff x = x'$ and $g^{-1}g' \in \mathcal{K}_x$. where $\mathcal{K}_x = \{g \in \mathcal{K},\ g\tilde{x} = \tilde{x}\}$.
The building $\mathcal{I}$ is defined to be the quotient of $\mathcal{G} \times C$ by the equivalence relation $(g, x) \sim (g', x') \iff x = x'$ and $g'g^{-1} \in P_x$ where P_x is a certain subgroup of $\mathcal{G}$ to be defined below. ~~To establish the identification of $\mathcal{I}$ and X~~ In order to identify $\mathcal{I}$ and X we only have to prove that the inclusion of $\mathcal{K}$ in $\mathcal{G}$ induces isos. $\mathcal{K}/\mathcal{K}_x \xrightarrow{\sim} \mathcal{G}/P_x$.

Let K_x = centralizer of $\bar{x}$ in K. Then we have isom

$$\mathcal{K}_x \xrightarrow{\sim} K_x \qquad g \longmapsto g(1)$$

with inverse sending $\xi \in K_x$ to the loop $w \longmapsto \tilde{x}(w)\,\xi\,\tilde{x}(w)^{-1}$.

Let G_x = centralizer of $\bar{x}$ in G. It is known that K_x, G_x are connected, that G_x is the reductive subgroup of G with roots $\Phi(x) = \{\alpha \in \Phi \mid \alpha(x) \in \mathbb{Z}\}$. G_x is generated by H and the 1-parameter subgroups $x_\alpha(t) = \exp(t X_\alpha)$, $t \in \mathbb{C}$, for $\alpha \in \Phi(x)$. Note

$$\tilde{x}(\omega)\, \xi\, \tilde{x}(\omega)^{-1} = \xi \qquad \text{if } \xi \in H$$

$$\begin{aligned}\tilde{x}(\omega)\, x_\alpha(t)\, \tilde{x}(\omega)^{-1} &= \exp(t \operatorname{Ad} \tilde{x}(\omega) X_\alpha) \\ &= x_\alpha(t e^{2\pi i \omega\, \alpha(x)}) \\ &= x_\alpha(t z^{\alpha(x)})\end{aligned}$$

if $z = e^{2\pi i \omega}$ and $\alpha(x) \in \mathbb{Z}$. Thus if we denote by $\mathcal{G}_x$ the subgp of $G(\mathbb{C}[z, z^{-1}])$ generated by H and the 1-param. subgps $X_\alpha(t z^{\alpha(x)})$ for $\alpha \in \Phi(x)$, then we have an isom

$$\mathcal{G}_x \xrightarrow{\sim} G_x \qquad g \longmapsto g(1)$$

with inverse sending ξ into the g $\ni$ $g(e^{2\pi i \omega}) = \tilde{x}(\omega)\, \xi\, \tilde{x}(\omega)^{-1}$. Of course K_x is a maximal compact subgroup of $\mathcal{G}_x$.

The stabilizer of the image of $x \in C$ in the building $\mathcal{I}$ is the subgroup P_x of G generated by the subgroups $H(R)$ and $x_\alpha(\pi^n R)$ for $\alpha \in \Phi$, $n \in \mathbb{Z}$ such that $\alpha(x) + n \geq 0$.

~~P_x has the structure of a pro algebraic group over $\mathbb{C}$. It contains the normal subgroup P_x^u generated by $\mathrm{Ker}\{H(R) \to H(R/\pi R)\}$ and $x_\alpha(\pi^n R)$ for~~ The group P_x^u generated by $\mathrm{Ker}\{H(R) \longrightarrow H(R/\pi R)\}$ and $x_\alpha(\pi^n R)$ for $\alpha(x) + n > 0$ is a normal subgroup of P_x. P_x has the structure of pro-algebraic group over $\mathbb{C}$ such that P_x^u is an inverse limit of unipotent groups and P_x / P_x^u is reductive. Clearly P_x is generated by P_x^u, H, and $x_\alpha(\pi^{-\alpha(x)}\mathbb{C})$ for $\alpha \in \Phi(x)$, hence $P_x = G_x P_x^u$. Also $G_x \cap P_x^u = 1$ as G_x is reductive and hence has no non-trivial normal unipotent subgroups. Thus we have a semi-direct product decomposition

(1) $$P_x = G_x \times P_x^u .$$

Let P be the Iwahori subgroup of G generated by $H(R)$ and $x_\alpha(R)$, $x_{-\alpha}(\pi R)$ for $\alpha \in \Phi^+$, ($P = P_y$ for y an interior point of C). One knows P is a Borel subgroup of P_x, hence we get a semi-direct prod. decomp.

(2) $\mathcal{P} = \mathcal{B}_x \times \mathcal{P}_x^u$

where $\mathcal{B}_x = \mathcal{G}_x \cap \mathcal{P}$ is a Borel subgp of $\mathcal{G}_x$ containing H.

As $\mathcal{K}_x$ is a maximal compact subgroup of $\mathcal{G}_x$, we have the Iwasawa decomp.

(3) $\mathcal{G}_x = \mathcal{K}_x \times^T \mathcal{B}_x$.

Combining (1), (2), (3) we get

$$\mathcal{K} \times^T \mathcal{P} = \mathcal{K} \times^{\mathcal{K}_x} \mathcal{K}_x \times^T \mathcal{B}_x \times \mathcal{P}_x^u$$

(4)
$$= \mathcal{K} \times^{\mathcal{K}_x} \mathcal{G}_x \times \mathcal{P}_x^u$$
$$= \mathcal{K} \times^{\mathcal{K}_x} \mathcal{P}_x.$$

for any x in C. Taking $x = 0$, we have $\mathcal{K}_x = K$, $\mathcal{P}_x = G(R)$ and we showed in the preceding section that

$$\mathcal{G} = \mathcal{K} \times^K G(R).$$

Hence we ~~conclude~~ have proved the following

<u>Theorem</u>: $\mathcal{G} = \mathcal{K} \times^{\mathcal{K}_x} \mathcal{P}_x$ for any $x \in C$, in particular $\mathcal{G} = \mathcal{K} \times^T \mathcal{P}$.

Consequently we have $\mathcal{G}/\mathcal{P}_x = \mathcal{K}/\mathcal{K}_x$ which is what we ~~had to~~ needed to identify $\mathcal{X}$ and $\mathcal{I}$.

Loop space of a symmetric space.

K ~~compact~~ compact connected Lie gp,
σ involution of K.
K/K^σ is the symmetric space determined by (K,σ)

Other interpretation: Let K act on itself via $k*(x) = kx(\sigma k)^{-1}$. Then K/K^σ is the orbit of 1 for this action. Put $Y = \{k \in K / \sigma k = k^{-1}\}$. Then $\mathrm{Im}(K/K^\sigma) \subset Y$. Using

Y stable under K action

$\exp : k \longrightarrow K$ we see $\exp : k_- \longrightarrow Y$ is a diffeomorphism at the identity. ~~Thus K/K^σ is the identity component of Y.~~

But if $k = \exp(X)$ $\quad X \in k^-$ then
$$k(\sigma k)^{-1} = e^{2X}$$

Hence $\exp(k_-) \subset K*1 \subset Y$. Thus $K*1 =$ identity component of Y.

Now using compactness of K, one knows that $\exp(k_-) = K*1$, because geodesics are $e^{tX}*1$ with $X \in k_-$.

So from now on introduce the symmetric space $H = \exp(k_-) =$ identity component of $\{k \mid \sigma k = k^{-1}\} \simeq K/K^\sigma$.

Let $h(t) = f(e^{2\pi i t})\, e^{tX}$, $f \in \mathcal{K}'$, $X \in \mathfrak{k}$
be a special path in K such that

$$h(t) = \overline{h(-t)} \qquad t \in \mathbb{R}$$

$$f(e^{2\pi i t})\, e^{tX} = \overline{f(e^{-2\pi i t})}\, e^{-t\bar{X}}$$

Put $t = \frac{1}{2}$ $\qquad f(-1)\, e^{\frac{1}{2}X} = \overline{f(-1)}\, e^{-\frac{1}{2}\bar{X}}$ or
if $k = f(-1)e^{-\frac{1}{2}X}$ then

$$k e^{X} = \bar{k}$$

or $e^X = k^{-1}\bar{k}$. But $k^{-1}\bar{k} = k^{-1} * 1$ is
of the form e^Y with $Y \in \mathfrak{k}_-$ (above discussion)
so $e^X = e^Y$. Thus we have

<u>Lemma</u>: Any $h \in \mathcal{I}(K)$ such that
$\overline{h(-t)} = h(t)$ is of the form

$$h(t) = f(e^{2\pi i t})\, e^{tY}$$

with $Y \in \mathfrak{k}_-$ and $f \in \mathcal{K}' \ni f(z) = \overline{f(\bar{z})}$.

Let σ acts on $\mathcal{I}(K)$ by $(\sigma h)(t) = \overline{h(-t)}$
and on $\mathcal{K}$ by $(\sigma f)(z) = \overline{f(\bar{z})}$. Then we see
from the above that $\mathcal{I}(K)^\sigma$ is a principal bundle:

$$\mathcal{K}'^{\sigma} \longrightarrow \mathcal{I}(K)^{\sigma} \longrightarrow H$$

<u>Conclusion:</u> $\mathcal{I}(K)^\sigma$ is the set of Laurent paths h in K satisfying $h(t) = \overline{h(-t)}$ topologized so as to be a compactly generated space whose compact sets are limited subsets closed under uniform convergence. So I see that $\mathcal{I}(K)^\sigma$ is a principal bundle over H with fibre $\mathcal{K}'^\sigma$.

Next step is to identify $\mathcal{I}(K)^\sigma$ with the Tits building of G^σ over $F^\sigma = \mathbb{R}[[x]][x^{-1}]$. Evident because σ acts on G via $(\sigma g)(z) = \overline{g(\bar{z})}$. and an invariant function can be viewed as a holom. map $g : \{x \mid 0 < x < \varepsilon\} \longrightarrow G^\sigma$ meromorphic at 0. $\mathcal{K}^\sigma$ consists of g such that

$$\theta\, g(\bar{z}^{-1}) = g(z)$$

(this is the condition that $z \in S^1 \Longrightarrow g(z) \in K$) which means

$$\theta\, g(x^{-1}) = g(x)$$

where θ is the Cartan involution of G^σ wrt K^σ. So $\mathcal{I}(K)^\sigma$ can be identified with maps

$$h(t) = f(e^{2\pi i t})\, e^{tY} \qquad Y \in k_- \quad f \in \mathcal{K}'^\sigma$$

$$= f(e^{+a})\, e^{a(2\pi i)^{-1} Y} \qquad (2\pi i^{-1}) Y \in p.$$

$$h(a) = f(e^a)\, e^{aX}$$

where $X \in \mathfrak{p}$ and $f \in G^\sigma(\mathbb{R}[x,x^{-1}])$ satisfies the symmetry condition

$$\theta f(x^{-1}) = f(x)$$

and $f(1) = 1$. (Thus $\theta h(a) = h(-a)$)

Let's review the structure of the building. I will now replace $\mathcal{G}^\sigma, \mathcal{K}^\sigma, \mathcal{I}$ by simply $\mathcal{G}, \mathcal{K}, \mathcal{I}$ in the following. Thus G will be the Lie group of rational points of a reductif connected algebraic group, K will be a maximal compact subgroup, θ the Cartan involution of G w.r.t. K. I define $\mathcal{I}$ to be paths in G of the form

$$h(a) = f(e^a)\, e^{aX}$$

where $f \in G(\mathbb{R}[x,x^{-1}])$ satisfies (i) $f(1) = 1$ (ii) $\theta f(x) = f(x^{-1})$, and where $X \in \mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}$.

Given $h \in \mathcal{I}$, define $P_h \subset \mathcal{G}$ to be the group of g such that $h(a)^{-1} g(e^a) h(a)$ converges as $a \longrightarrow -\infty$. P_h^u is the subgroup such that it converges to 1. Now $h(a)^{-1} g(e^a) h(a)$ is a matrix ~~exponential poly~~ whose entries are linear combinations of functions $e^{a\lambda}$ with

$\lambda \in \mathbb{R}$. Thus if it converges as $a \to -\infty$ all such exponentials appearing have $\lambda \geq 0$, so the limit

$$l = \lim_{a \to -\infty} h(a+ib)^{-1} g(e^{a+ib}) h(a+ib)$$

will exist in the complex group G_c (= old G), and will be the same for all b. So taking $b = 2\pi$ we get

$$e^{-2\pi i X} \, l \, e^{2\pi i X} = l$$

Conversely given such an l the function

$$h(a) \, l \, h(a)^{-1}$$

$$= f(e^a) e^{aX} l e^{-aX} f(e^a)^{-1}$$

will be in $\mathcal{G}$. The point is that $e^{aX} l e^{-aX}$ $= e^{aX} l e^{-aX} l^{-1} \cdot l$ will be a Laurent polynomial in e^a.

Thus if $\quad \mathcal{G}_h = \{ h l h^{-1} \mid l \in G \text{ commutes with } h(2\pi i) \}$

we have $\quad \mathcal{P}_h = \mathcal{G}_h \ltimes \mathcal{P}_h^u$

Assuming the Iwasawa decomposition: $\mathcal{G} = \mathcal{K}\mathcal{P}_h$

we therefore get a $\mathcal{G}$-action on $\mathcal{I}$ such that $\mathcal{K}$ acts transitively on each $\mathcal{G}$-orbit.

What are the $\mathcal{G}$-orbits: $\mathcal{G}\backslash\mathcal{I} = \mathcal{K}\backslash\mathcal{I}$ $= K\backslash H = W\backslash S$.

Go back to the old notation with σ. Let $\xi = e^{tX}$ with $X \in \mathrm{Lie}(T)$. Calculate P^u_ξ. I do this first inside GL_n. Assume X diagonal with entries $2\pi i\lambda_j$ $j=1,\dots,n$ where $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$(\xi^{-1} g\, \xi)_{ij} = e^{2\pi i(\lambda_i-\lambda_j)t}\, g_{ij}(e^{2\pi i t})$$

and we want this to converge (resp. converge to δ_{ij}) as $e^{2\pi i t} \to 0$, i.e. $\mathrm{Im}\, t \to +\infty$. Let $g_{ij}(z) = c_{ij} z^{n_{ij}}$ + higher degree with $c \neq 0$. Then

$$e^{2\pi i(\lambda_i-\lambda_j)t}\, e^{2\pi i(n_{ij})t}\, c_{ij}$$ converges

iff $\lambda_i - \lambda_j + n_{ij} \geq 0$

and it converges to δ_{ij} iff $\lambda_i - \lambda_j + n_{ij} > 0$ for $i \neq j$

~~and $c_{ii}=1$, $n_{ii}=0$.~~

~~So in this case we see that P^u_ξ is the subgroup:~~

$$\cancel{\begin{pmatrix} 1+zR & zR & \cdots & zR \\ & & & \\ zR & zR & \cdots & 1+zR \end{pmatrix}}$$

Goal: You want to make explicit the subgroups involved.

$$P^u_\xi \qquad \xi(t) = e^{tX} \qquad X \in \mathrm{Lie}(T)$$

Look at $\xi^{-1}\ \xi$ on Lie algebra

$$\mathrm{Ad}(\xi^{-1})X_\alpha = \left(\mathrm{Ad}\, e^{-tX}\right)X_\alpha$$
$$= e^{-t\alpha(X)}X_\alpha$$

This holds for any X in h. But $X = 2\pi i \mathcal{X}$ where $\mathcal{X} \in E$. So

$$e^{-2\pi i t\alpha(\mathcal{X})}X_\alpha$$

Thus $z^m X_\alpha \in \mathrm{Lie}\, P^u_\xi \iff e^{-2\pi i t\alpha(x)}z^m$ converges to 0 as $t \to \infty$.

$\iff -\alpha(x) + m > 0$

$$\mathrm{Lie}\ P^{u}_{\tilde{x}} = zR\otimes h + \sum_{\alpha\in\Phi} z^{\langle t+\alpha(x)\rangle} R X_\alpha$$

$\xi(t) = e^{2\pi i t x}$ $\quad x \in E.$

where $\langle\alpha(x)\rangle$ is the least integer $> \alpha(x)$.

~~$\mathrm{Lie}(P^{u}_{\tilde{x}}) = zR\otimes h$~~

Still I want to understand what goes on. ~~So I have a sheaf of sorts and what one unders~~

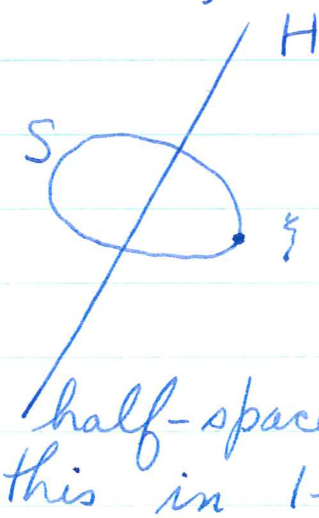
So I suppose I have to go back to the beginning. ~~I start with a~~

I have defined the building I and I have produced

<u>Setup</u> -
K comp. conn. with involution σ.
E max. abelian subspace of k_-.
ξ is a regular element of E
C the corresponding chambre
$\alpha = 0$ a wall of C
η a generic element of this wall
($\beta(\eta) = 0 \iff \beta$ prop. to α).

Claim: $K_\eta^\sigma / K_\xi^\sigma$ is a sphere.

<u>Proof</u>: Replace K by K_η; this doesn't change E and ξ still is regular; so we can suppose η is in the center of K. We want to show K^σ_ξ is a sphere. Let S be the sphere obtained by rotating ξ around the subspace $H = \{x \in E \mid \alpha(x) = 0\}_\xi$ of k_-. Since H is stable under K^σ, S is stable under K^σ, hence it is a union of K^σ-orbits.



~~Because~~ Each K^σ-orbit ~~meets E in a W-orbit. But because H is of codim 1 in H, S meets E at ξ and the reflected point. Clear.~~ meets the half-space $\alpha \geq 0$ in a single point. S meets this in 1-point ξ, so
$$S = K^\sigma / K^\sigma_\xi$$

Summary: ~~Let K be compact + connected~~

Given a wall $\psi = m$ in E & form the connected group K' in K fixing $e^{2\pi i \cdot}$ wall; it is generated by $\mathfrak{z}$ and the $\mathfrak{k}^\alpha$ such that $\psi(x) = m \Longrightarrow \alpha(x) \in \mathbb{Z}$.

Specifically K' is the ~~conn. subgroup~~ conn. cent. of $\{e^{2\pi i x} \mid \psi(x) = m\}$. It's Lie alg. is $\mathfrak{z} + \sum \mathfrak{k}^\alpha$ where α such that

$$\psi(x) = m \longrightarrow \alpha(x) \in \mathbb{Z}$$

(this implies $\psi(y) = 0 \Longrightarrow \alpha(y) = 0$ so $\alpha \sim \psi$).

Then K'^σ / Z^σ $\quad$ Z = cent. of E is a sphere, in fact it is canonically the 1-point compactification of $\sum \mathfrak{k}^\alpha$ as above.

Reason (K', σ) is a rank 1 symm space situation.

<u>Setup:</u> K compact conn. with σ

$$C_0 = \{x \in E \mid 0 \leq \alpha(x) \leq 1 \quad \alpha \in \Phi^+\}$$

Consider a wall of C_0 of the form $\psi(x) = 1$ with $\psi \in \Phi^+$. Denote this wall $C_0 \cap H_{\psi,1}$. By assumption it contains a non-empty open subset of $H_{\psi,1} = \{x \in E \mid \psi(x) = 1\}$. Put

$$K' = \{k \in K \mid k \text{ centralizes } e^{2\pi i x},\ x \in C_0 \cap H_{\psi,1}\}$$

$$K'' = \{k \in K \mid \text{———} \quad H_{\psi,0}\}$$

$$Z = \{k \in K \mid k \text{ centralizes } E\}.$$

Let $x_0 \in \mathrm{Int}(C_0 \cap H_{\psi,1})$. Then for any small element y in $H_{\psi,0}$ we have $x_0 + y \in C_0 \cap H_{\psi,1}$ so it is clear that K' centralizes $e^{2\pi i y}$ for all small y in $H_{\psi,0}$, hence K' centralizes $H_{\psi,0}$ i.e.

$$K' \subset K''.$$

But K'' is connected and its Lie algebra is the sum of $\mathfrak{t}$ and the roots spaces with roots α vanishing on $H_{\psi,0}$ (i.e. α proportional to ψ). If $x \in H_{\psi,1}$, then $Ad(e^{2\pi i x})$ has eigenvalue

$$e^{2\pi i \alpha(x)} \quad \text{on} \quad \mathfrak{k}^\alpha.$$

~~[illegible]~~ First part will include a complete discussion of U_n, SU_n.

[Lattices + Scattering operators

$GL_n(F) = \mathcal{U} \times^{\mathcal{U}} GL_n(R)$.

[Special Paths in U_n, GL_n, identification with solution of D.E. with regular singular points building

[modification for SL_n.

[building for SL_n < B-orbits, length chambres and Weyl group lemmas on chambres.

[K, $\mathcal{K}$, $\mathcal{X}(K)$, statement of the theorem and reduction to 1-conn. case.

[1-connected simple case. Notation + identification of $\mathcal{X}$ and $\mathcal{J}$. Proof of the theorem. Theorems of Bott.

Buildings and the Loop Space of a Lie group

1. Algebraic Loops.

Let K be a compact, connected Lie group. The algebra $A(K)$ consisting of real-valued representative functions on K is the coordinate ring of an algebraic group over $\mathbb{R}$ of which K is the group of points rational over $\mathbb{R}$. In this way we can identify compact Lie groups with certain reductive algebraic groups over $\mathbb{R}$.

Let S^1 denote the group of complex numbers of absolute value 1. The ring of complex-valued representative functions on S^1 is the ring $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials over $\mathbb{C}$. $A(S^1)$ is the fixed subring for the involution

1) $$\left(\sum a_i z^i\right)^- = \left(\sum \bar{a}_i z^{i}\right).$$

It is isomorphic to $\mathbb{R}[X, Y]/(X^2 + Y^2)$ where $X = \cos\theta$ $Y = \sin\theta$, $z = e^{i\theta}$.

A closed path $f: S^1 \longrightarrow K$ will be called <u>algebraic</u> if composition with f carries $A(K)$ into $A(S^1)$. Such a path is the same thing as a morphism ~~[illegible]~~ from S^1 to K considered as algebraic varieties over $\mathbb{R}$. The set of these paths is a group which will be denoted $\mathcal{K}$.

For example, take K to be the group U_n of unitary $n \times n$ matrices, in which case we write $\mathcal{U}_n$ for $\mathcal{K}$. The ring $A(U_n) \otimes \mathbb{C}$ is known to be the coordinate ring of GL_n over $\mathbb{C}$, which is $\mathbb{C}[X_{ij},\ 1 \le i,j \le n][\det(X)^{-1}]$. Using this, it follows without trouble that an element of $\mathcal{U}_n$ is given by a Laurent polynomial matrix

$$2) \qquad f = \sum_{|i| \le N} a_i z^i$$

where each a_i is an $n \times n$ matrix over $\mathbb{C}$, such that $f(z)^* f(z) = I$ for z in S^1. Such an f is the same thing as a unitary matrix over the ring $\mathbb{C}[z, z^{-1}]$ equipped with the involution 1).

Since $\mathcal{K}$ is contained in the space of maps from S^1 to K, it inherits from the latter a metric space topology given by uniform convergence with respect to a metric on K. However, ~~for our purposes~~ it is more natural for our purposes to put a finer topology on $\mathcal{K}$ as follows.

~~Choose an embedding of K in a unitary group U_n. We call a subset S of $\mathcal{K}$ bounded if the Laurent polynomials in $\mathcal{U}_n$ associated to the elements of S have bounded degree.~~

A subset S of $\mathcal{K}$ will be called **bounded** if for any p in $A(K)$, the set of Laurent polynomials $p(f(z))$, $f \in S$, has bounded degree, where the degree of $\sum a_i z^i$ is the largest $|i|$ such that $a_i \neq 0$. ~~If K is embedded in U_n, then it is easily seen that S is bounded if and only if $S \subset F_N \mathcal{U}_n$, where $F_N \mathcal{U}_n$ is the subset of $\mathcal{U}_N$ consisting of polys. of degree $\leq N$. Let us fix an embedding of K in a unitary group U_n.~~ Suppose K embedded in U_n, whence elements of $\mathcal{K}$ may be viewed as Laurent polynomial matrices f such that $f(z) \in K$ for z in S^1. Then it is easily seen that S is bounded if and only if $S \subset F_N \mathcal{U}_n$ for some N, where $F_N \mathcal{U}_n$ is the subset of Laurent polynomials matrices of degree $\leq N$. Put $F_N \mathcal{K} = \mathcal{K} \cap F_N \mathcal{U}_n$.

From the Cauchy formula:

$$a_i = \frac{1}{2\pi\sqrt{-1}} \int_{S^1} f(z)\, z^{i-1}\, dz$$

(f as in 2)) one sees that each entry of the matrix a_i has abs. value ≤ 1. Further, for elements of a bounded subset, uniform convergence is the same as convergence of the coefficients. It follows that by associating to $f \in F_N\mathcal{K}$, the sequence of its coefficients, we obtain a homeomorphism of $F_N\mathcal{K}$ with a compact subset of a Euclidean space. ~~Hence we have proved~~

~~Proposition: A bounded subset of $\mathcal{K}$ which is closed for the uniform convergence topology is compact.~~

~~Now we define the fine topology on $\mathcal{K}$ to be the one such that a set is closed~~

Next we put on $\mathcal{K}$ the topology making it the inductive limit of the compact spaces $F_N\mathcal{K}$. In this way $\mathcal{K}$ becomes a compactly generated (Hausdorff) space whose compact sets are the bounded subsets closed for the uniform convergence topology. Note that the ~~and inverse~~ product and inverse for $\mathcal{K}$ are continuous when restricted to the sets $F_N\mathcal{K}$, hence $\mathcal{K}$ is a group object in the category of compactly generated spaces.

2. Special paths.

As is customary we identify the Lie algebra of $GL_n\mathbb{C}$ with the algebra of $n \times n$ matrices over $\mathbb{C}$. The exponential map is given by $\exp(A) = \sum \frac{1}{k!} A^k$.

<u>Lemma 1</u>: Let A, B be matrices such that $\exp A = \exp B$. Then there is an f in $GL_n(\mathbb{C}[z, z^{-1}])$ such that

$$1) \qquad \exp(tA)\exp(-tB) = f(e^{2\pi i t})$$

for all t in $\mathbb{C}$.

Note that because both sides of 1) are holomorphic in t, the ~~function~~ matrix f is uniquely determined by 1) for $0 \leq t \leq 1$. Moreover $f(1) = 1$.

<u>Proof</u>. Let $A = A_s + A_n$, $B = B_s + B_n$ be the Jordan decompositions into commuting semi-simple and nilpotent elements. Then

$$\exp(A_s)\exp(A_n) = \exp(B_s)\exp(B_n)$$

and by the uniqueness of the multiplicative Jordan decomposition, we have $\exp(A_s) = \exp(B_s)$ and $\exp(A_n) = \exp(B_n)$. Since exponential is bijective between nilpotent and unipotent matrices, we have

$A_n = B_n$, and

$$\exp(tA)\exp(-tB) = \exp(tA_s)\exp(-tB_s).$$

So we may suppose A, B are semi-simple.

Conjugating A, B by a suitable matrix, we can suppose A is diagonal, hence so is $\exp(A)$. Since A and B commute with $\exp(A)$, we can split $\mathbb{C}^n$ into the different eigenspaces of $\exp(A)$ and check the lemma in each eigenspace. Thus we can suppose $\exp(A) = \lambda I$.

Choose μ with $e^{\mu} = \lambda$. We can replace A, B by $A - \mu I$, $B - \mu I$ without changing the left side of 1). Hence we can suppose $\exp(A) = \exp(B) = I$, in which case we need only prove that if $\exp(A) = I$ then $\exp(tA) = f(e^{2\pi i t})$ with $f \in GL_n(\mathbb{C}[z, z^{-1}])$.

Supposing A to be a diagonal matrix with entries a_j, then $e^{a_j} = 1$, so $a_j = 2\pi i n_j$ with $n_j \in \mathbb{Z}$. Therefore f is the diagonal matrix with entries z^{n_j}. QED.

When the matrices A, B are skew-hermitian, i.e. in the Lie algebra of U_n, it is clear that $f \in \mathcal{U}_n$. Moreover inspection of the proof of the lemma shows that if the eigenvalues of A and B respectively $2\pi i a_j$, $2\pi i b_j$, then the degree of f is bounded by $\max|a_j| + \max|b_j| + 2$. This entails that the mapping $(A,B) \mapsto f$ from pairs of skew-hermitian matrices with $\exp(A)=\exp(B)$ to $\mathcal{U}_n$ is continuous.

Let $\mathcal{K}'$ be the subgroup of $\mathcal{K}$ consisting of f preserving basepoint: $f(1)=1$. Let $\mathrm{Lie}(K)$ denote the Lie algebra of K. If K is embedded in U_n, then $\mathcal{K}'$ is the subspace of $\mathcal{U}_n'$ consisting of paths contained in K. Thus the preceding discussion implies the following.

__Lemma 2.__ If X, Y are elts of $\mathrm{Lie}(K)$ such that $\exp(X)=\exp(Y)$, then there is a unique f in $\mathcal{K}'$ such that

$$\exp(tX)\exp(-tY) = f_{X,Y}(e^{2\pi i t})$$

for $0 \le t \le 1$. Furthermore, the map $(X,Y) \longmapsto f_{XY}$ is continuous from pairs (X,Y) such that $\exp(X)=\exp(Y)$ to $\mathcal{K}'$.

__Definition:__ A path $h: [0,1] \to K$ will be called a __special path__ if it is of the form

$$h(t) = f(e^{2\pi i t})\exp(tX) \tag{2}$$

for some f in $\mathcal{K}'$ and X in $\mathrm{Lie}(K)$.

Let $\mathcal{X}$ denote the set of special paths in K, and let $\phi: \mathcal{X} \to K$ be the map such that $\phi(h) = h(1)$. The group $\mathcal{K}'$ acts on $\mathcal{X}$ by left multiplication; the action is free and it preserves the fibres of ϕ. If $k \in K$, then because exp is onto for compact connected Lie groups, there is a Y in Lie(K) such that $k = \exp(Y)$; hence $\phi^{-1}(k)$ contains the special path $\exp(tY)$, so ϕ is surjective. If h' is another point of $\phi^{-1}(k)$, say h is the form 2), then

$$h(t) = \left[f(e^{2\pi i t})\exp(tX)\exp(-tY)\right]\exp(tY)$$
$$= f_1(e^{2\pi i t})\exp(tY)$$

where $f_1 \in \mathcal{K}'$ by Lemma 2. Thus $\mathcal{K}'$ acts simply-transitively on the fibres of ϕ. Therefore, at least on the level of sets, we have a principal $\mathcal{K}'$-bundle:

3) $$\mathcal{K}' \longrightarrow \mathcal{X} \xrightarrow{\phi} K .$$

~~However we can make $\mathcal{X}$ into a principal bundle topologically by equipping $\mathcal{X}$ with the quotient topology induced by the evident map $\mathcal{K}' \times \mathrm{Lie}(K) \to \mathcal{X}$. A cocycle describing this covering can be obtained by choosing open sets U_i' in Lie(K) such that the sets $U_i = \exp(U_i')$ form an open covering of K and such that $\exp: U_i' \to U_i$ is a diffeomorphism. The cocycle is the function which on $U_i \cap U_j$ sends k to f_{XY}, where $X \in U_i'$, $Y \in U_j'$, and $\exp(X) = \exp(Y) = k$. According to Lemma 2 this function is continuous~~

In order to make $\mathfrak{X}$ into a topological principal bundle it suffices to exhibit ~~[illegible]~~ ~~[illegible]~~ a suitable Čech cocycle on K with values in $\mathfrak{K}'$. Choose open sets U_i' in $Lie(K)$ such that the sets $U_i = \exp(U_i')$ form an open covering of K and such that $\exp : U_i \longrightarrow U_i'$ is a diffeomorphism. On $U_i \cap U_j$ the function sending k to f_{XY}, where $X \in U_i'$, $Y \in U_j'$, and $\exp(X) = \exp(Y) = k$, is continuous by Lemma 2. It is easily seen that this cocycle makes $\mathfrak{X}$ into a principal $\mathfrak{K}'$-bundle over K.

~~[illegible]~~

<u>Remark</u>: A matrix function of the form $h(t) = f(e^{2\pi i t}) \exp(tX)$, where $f \in GL_n(\mathbb{C}[z, z^{-1}])$ is such that $f(1) = 1$, is the ~~[illegible]~~ same thing as a solution matrix of an ordinary differential equation with regular singular points at 0 and ∞. The set of these matrices forms a principal bundle over $GL_n\mathbb{C}$ with structural group the subgroup of $GL_n(\mathbb{C}[z, z^{-1}])$ containing f such that $f(1) = 1$.

3. The main results.

Recall K is a compact connected Lie group, ~~$\mathcal{K}'$ is the group of "algebraic" loops in K equipped with a suitable topology the loop space ΩK, and $\mathcal{X}$ is the space of special paths in K~~ $\mathcal{K}'$ is the group of "algebraic" loops in K, and $\mathcal{X}$ is the space of special paths in K. There is an evident map of the fibration 3) page ⑧ to the path space fibration

$$\Omega K \longrightarrow EK \longrightarrow K$$

which is continuous for the topologies on $\mathcal{K}'$ and $\mathcal{X}$ and injective.

<u>Theorem 1</u>: The map $\mathcal{K}' \longrightarrow \Omega K$ is a homotopy equivalence. The space $\mathcal{X}$ is contractible.

Of course, these two assertions are equivalent by homotopy theory.

<u>Theorem 2</u>: $\mathcal{K}'$ has the structure of a CW complex with even-dimensional cells.

This means that $\mathcal{K}'$ is a model for the homotopy type of ΩK which is minimal in some sense, because the cells of $\mathcal{K}'$ are in one-one correspondence with elements of a basis for $H_*(\Omega K)$.

<u>Example</u>: $K = S^1 = U_1$. Because units in $\mathbb{C}[z, z^{-1}]$ are of the form az^m with a in $\mathbb{C}^*$, one has ~~[illegible]~~ $\mathcal{U}_1 = \{az^m \mid a \in S^1\}$, $\mathcal{U}_1' = \{z^m\} \cong \mathbb{Z}$. Special paths are of the form $e^{2\pi i t x}$ with x in $\mathbb{R}$, so the ~~[illegible]~~ bundle ( ) is the ~~[illegible]~~ exponential sequence

$$\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 .$$

Clearly both theorems are true in this case.

~~[illegible]~~

We now begin the proof of these theorems.

If K is a product: $K = K_1 \times K_2$, then it is not hard to verify that one has homeomorphisms $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ when the products are taken in the category of compactly generated spaces. Hence the above theorems hold for K if they do ~~[illegible]~~ for K_1 and K_2.

Now K has a finite covering K_1 which is a product of circle groups and simply-connected simple compact groups. We are now going to show the above theorems hold for K if they do for K_1. In the following sections we shall prove the theorems when K is simply-connected and simple, thereby completing the proof.

Suppose $p : K_1 \longrightarrow K$ is a finite covering, and put $A = \mathrm{Ker}(p)$. One has a diagram of groups

$$\begin{tikzcd} & \mathcal{K}_1 \arrow[r, "p_*"] \arrow[d, "i_1"'] & \mathcal{K} \arrow[r, "q"] \arrow[d, "i"] & A \arrow[d, equal] & \\ 1 \arrow[r] & \Omega K_1 \arrow[r, "p_*"] & \Omega K \arrow[r, "q"] & A \arrow[r] & 1 \end{tikzcd}$$

with the bottom row exact. The map q is obtained by lifting a loop in K to a path in K_1 starting at 1 and then taking the endpoint.

Let $a \in A$ and let $X \in \mathrm{Lie}(K_1)$ be such that $\exp(X) = a$. The path $p\exp(tX)$ in K is special with endpoint 1, hence it is of the form $f(e^{2\pi i t})$ with f in $\mathcal{K}$. Clearly $q(f) = a$, so we see q maps $\mathcal{K}$ onto A. ~~As A is finite, the kernel of q in $\mathcal{K}$ is the union of those components of $\mathcal{K}$ mapped to 1 in A.~~

<u>Lemma:</u> The map p_* is a homeomorphism of $\mathcal{K}_1$ onto the kernel of $q : \mathcal{K} \to A$.

Since A is finite, the kernel of q on ΩK (resp. on $\mathcal{K}$) is the union of those components of ΩK (resp. $\mathcal{K}$) mapped to 1 in A. It is thus clear that the map i_1 is a homotopy equivalence iff i is, and that the above theorems are true for K_1 iff they are true for K.

~~To prove the lemma we must show that if an algebraic loop f in K lifts to a loop f_1 in K_1, then f_1 is algebraic. Moreover we must show that if f~~

Proof of the lemma.