August 8, 1974.

Let me try to understand the following example from the Dold-Thom theory of quasi-fibrations. Suppose I have a map $\pi : E \longrightarrow X$ and a filtration

$$Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_n = X$$

of X by closed subsets such that π is trivial over each stratum $Y_k - Y_{k-1}$ with fibre F_k. In addition I want there to be maps $F_k \longrightarrow F_{k-1}$ of specialization eventually. Assuming all these spec. maps are heq's, I want to show that the homotopy-fibres and fibres are heq's.

In this special case, we have a global map

$$E \longrightarrow X \times F_0$$

which we want to show is an equivalence. So we show by induction on k that

$$\pi^{-1}(Y_k) \longrightarrow Y_k \times F_0$$

is an equivalence. Reduction to the case:

$$\begin{tikzcd}
\pi^{-1} Y_{k-1} \arrow[r, hook] \arrow[d] & \pi^{-1} Y_k \arrow[d] & (Y_k - Y_{k-1}) \times F_k \arrow[l, hook'] \arrow[d] \\
Y_{k-1} \arrow[r, hook] & Y_k & Y_k - Y_{k-1} \arrow[l, hook']
\end{tikzcd}$$

So how should I proceed – Dold method
Suppose first I have a sequence of strata

$$Y_0 \subset Y_1 \subset Y_2 \subset \cdots$$

Then around Y_k I want an open set W_k which deforms down to Y_k. My examples – symmetric product $SP(X) = \bigcup_{n \geq 0} SP^n(X)$ $Y_n = SP^n(X)$. Then by using a function λ like barycen. coord of basepoint one gets to each $y \in SP^n(X)$ a sequence $y = \{x_1, \dots, x_n, *, *, \dots\}$

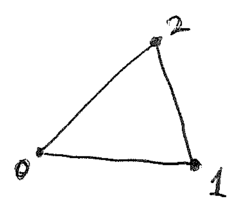
$$\lambda(x_1) \leq \cdots \leq \lambda(x_n) \leq 1$$

and we define Y_n where $s_n(y) = 1$ and W_n where $s_{n-1}(y) < s_n(y)$? No. It seems that if

$$V_n \text{ is defined by } s_n < s_{n+1}$$

then V_n is an open tube around $X_n = SP^n(X - *)$.
Now one really tends in this example to be interested in the open covering V_n, which corresponds to the barye. coordinate covering of the simplex.

Y_k is where $s_{k+1} = 1$.
W_k is where maybe $0 < s_{k+1}$ $\qquad W_0 = U_0$
$W_1 = U_0 \cup U_1$



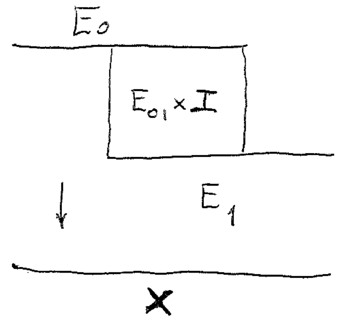
so one wants to divide the vertices into $0, 1, \dots, k$ and the rest and then to know that at least one of $t_0, t_1, \dots, t_k$ is non-zero i.e. that $s_{k+1} = t_0 + t_1 + \dots + t_k > 0$.

<u>August 9, 1974</u>

Suppose $X = U_0 \cup U_1$ where U_i is open. Put $U_{01} = U_0 \cap U_1$. Suppose given spaces + maps such that

$$\begin{array}{ccccc} E_0 & \longleftarrow & E_{01} & \longrightarrow & E_1 \\ \downarrow & & \downarrow & & \downarrow \\ U_0 & \hookleftarrow & U_{01} & \hookrightarrow & U_1 \end{array}$$

commutes. Then I can form the space



$$Cyl\,(E_0 \leftarrow E_{01} \rightarrow E_1)$$

which maps to X.

Given $f: X' \rightarrow X$ we can pull-back U_i, E_i via f to get a similar system over X'. Claim

$$f^* Cyl\,(E_0 \leftarrow E_{01} \rightarrow E_1) \simeq Cyl(f^*E_0 \leftarrow f^*E_{01} \rightarrow f^*E_1)$$

provided I give the cylinder the coarse topology. To prove this, suppose I determine the set

$$A = \underset{Sp/X}{Hom}(T, Cyl\,(E_0 \leftarrow E_{01} \rightarrow E_1)).$$

~~that of $Cyl(E_0 \leftarrow E_{01} \rightarrow E_1)$ as being~~ First of all there is a canonical map

$\xi : Cyl(E_0 \leftarrow E_{01} \rightarrow E_1) \longrightarrow [0,1]$, ~~and maps~~

such that

$\xi^{-1}[0,1)$ sits over U_0

$\xi^{-1}(0,1]$ —— U_1.

Secondly one has canonical maps

$\xi^{-1}[0,1) \xrightarrow{p_0} E_0$ over U_0

$\xi^{-1}(0,1] \xrightarrow{p_1} E_1$ over U_1

$$\xi^{-1}(0,1) \xrightarrow{p_{01}} E_{01} \quad \text{over } U_{01}$$

such that

$$\begin{tikzcd}
\xi^{-1}[0,1) \arrow[d, "p_0"'] & \xi^{-1}(0,1) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "p_{01}"] & \xi^{-1}(0,1] \arrow[d, "p_1"] \\
E_0 & E_{01} \arrow[l] \arrow[r] & E_1
\end{tikzcd}$$

commutes. Too complicated.

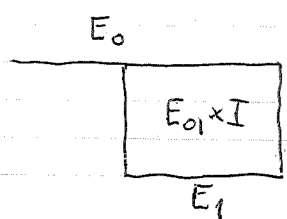
(Recall that a map $T \longrightarrow Cyl(E_0 \leftarrow E_{01} \rightarrow E_1)$ "consists" of a function $\lambda : T \longrightarrow [0,1]$ and maps p_0, p_1, p_{01} such that

$$\begin{tikzcd}
\lambda^{-1}[0,1) \arrow[d, "p_0"] & \lambda^{-1}(0,1) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "p_{01}"] & \lambda^{-1}(0,1] \arrow[d, "p_1"] \\
E_0 & E_{01} \arrow[l] \arrow[r] & E_1
\end{tikzcd}$$

commutes. In the present case T, and E_σ are all over X, so one has only to ask that p_σ be a map over X.)

Now suppose I have a class of maps which are to be thought of as equivalences (e.g. homology isomorphism). Then I would like to understand when the map $C = Cyl(E_0 \leftarrow E_{01} \rightarrow E_1) \longrightarrow X$, has the property that given an heq $X' \longrightarrow X''$ of spaces over X, then $X' \times_X C \longrightarrow X'' \times_X C$ is an equivalence.

Old analysis showed that for $C \to X$ to have this good property is local wrt numerable coverings of X, which reduces one to the case where $X = U_0$ whence C looks like



Assuming now that $E_{01} \longrightarrow E_1$ is a <u>universal equivalence</u> over $U_{01}=U_1$, I should be able to ~~[illegible]~~ replace ~~up to~~ ~~equivalence~~ $Cyl(E_0 \leftarrow E_{01} \to E_1)$ by $Cyl(E_0 \leftarrow E_{01} \to E_{0\,1})$ up to universal equivalence over X. Then we get a universal equivalence $\overbrace{\text{of}}^{\text{over } X}$ $Cyl(E_0 \leftarrow E_{01} \to E_1)$ with E_0.

So what it seems we get is the following:

<u>Theorem</u>: Let $X = U_0 \cup U_1$ be a numerable open covering and suppose given spaces E_σ and maps $\ni$

$$\begin{array}{ccccc} E_0 & \longleftarrow & E_{01} & \longrightarrow & E_1 \\ \downarrow & & \downarrow & & \downarrow \\ U_0 & \longleftarrow & U_{01} & \longrightarrow & U_1 \end{array}$$

commutes. Assume $E_0 \longrightarrow U_0$, $E_1 \longrightarrow U_1$ are ~~[illegible]~~ ~~[illegible]~~ "good" in the sense that base change by these maps carries heq's into equivalences (of the type considered). Assume also that

$$E_{01} \longrightarrow U_{01} \times_{U_0} E_0 \qquad E_{01} \longrightarrow U_{01} \times_{U_1} E_1$$

are universal ~~[illegible]~~ equivalences over U_{01} (meaning that after base change by an arb. space over U_{01} one has an equivalence). Then $Cyl(E_0 \longleftarrow E_{01} \longrightarrow E_1) \longrightarrow X$ is "good".

Proof involves following steps the essential one being:

<u>Thm</u>: Let $\mathcal{U}$ be a numerable covering of X. Then $C \to X$ is "good" iff $U \times_X C \longrightarrow U$ is "good" for each $U \in \mathcal{U}$.

Proof: We want to show that given maps $Y \xrightarrow{\alpha} Z \xrightarrow{u} X$ with α a h.eq., then $Y \times_X C \longrightarrow Z \times_X C$ is an equivalence.

Lemma 1: Enough to consider the case where $Z = Y \times [0,1]$ and $\alpha(y) = (y, 0)$.

First suppose I have $Y \underset{\alpha_1}{\overset{\alpha_0}{\rightrightarrows}} Z \xrightarrow{u} X$ with α_0 and α_1 homotopic. Then

$$\begin{tikzcd}
Y \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[rr, bend left, "\alpha_0"] \arrow[rr, bend right, "\alpha_1"'] & Y \times I \arrow[r, "h"] & Z \arrow[r, "u"] & X
\end{tikzcd}$$

so one has

$$\begin{tikzcd}
\alpha_0^* u^* C \arrow[r] & h^* u^* C \arrow[r] & u^* C \\
\alpha_1^* u^* C \arrow[ur] & &
\end{tikzcd}$$

where the first two maps are equivalences by hypothesis (clear for i_0 and to get i_1, reverse $[0,1]$). Thus $\alpha_0^* u^* C \to u^* C$ is an equivalence iff $\alpha_1^* u^* C \to u^* C$ is. This means that for $\alpha^* u^* C \longrightarrow u^* C$ to be an equiv. depends only on the homotopy class of α.

But if α is a h.eq., then we can find $\beta : Z \to Y$ with $\beta\alpha, \alpha\beta \sim id$. Thus one gets

$$Z \xrightarrow{\beta} Y \xrightarrow{\alpha} Z \xrightarrow{u} X$$

$\alpha\beta \sim id_Z \implies$ composite $\beta^* \alpha^* u^* C \longrightarrow \alpha^* u^* C \longrightarrow u^* C$ is an equivalence. Analogously one has that the composite

$$\alpha^*\beta^*\alpha^*u^*C \longrightarrow \beta^*\alpha^*u^*C \longrightarrow \alpha^*u^*C$$

is an equivalence. Hence we need the following fact about equivalences — if one has arrows $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet$

and ba, cb are equivalences, then a, b, c are equivalences. (This is always the case if there exists a functor inverting precisely the equivalences.)

<u>Lemma 2</u>: Given a numerable covering $\mathcal{U}$ of $Y \times I$, there exists a numerable covering $\mathcal{V}$ of Y such that for each V there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $V \times [t_{k-1}, t_k]$ is contained in a member of $\mathcal{U}$ for each $k = 1, \ldots, n$.

Proof for $\mathcal{U}$ finite, i.e. defined by a map $Y \times I \longrightarrow \Delta(n)$. The universal situation is where $Y = \Delta(n)^I$ which is a metric space, hence paracompact, whence every covering has a numerable refinement.

Still need a proof for the above lemma when $\mathcal{U}$ is infinite.

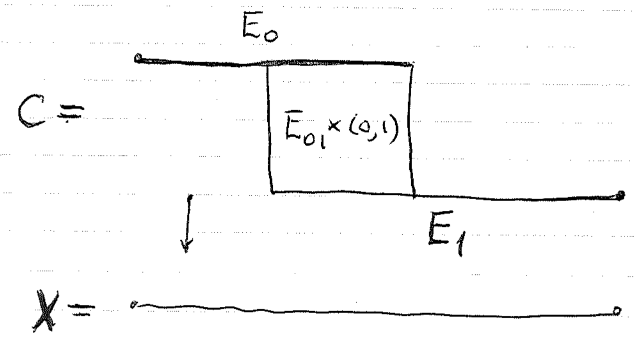
Now given V in $\mathcal{V}$, one can see inductively that

<u>August 10, 1974</u>

Suppose $X = U_0 \cup U_1$ with U_i open; put $U_{01} = U_0 \cap U_1$ and suppose given maps $\ni$

$$\begin{tikzcd} E_0 \arrow[d] & E_{01} \arrow[l] \arrow[r] \arrow[d] & E_1 \arrow[d] \\ U_0 & U_{01} \arrow[l, hook'] \arrow[r, hook] & U_1 \end{tikzcd}$$

commutes. Let $C = \mathrm{Cyl}(E_0 \leftarrow E_{01} \rightarrow E_1)$ regarded as a space over X in the obvious way:



A map of a space T over X into C may be identified with a map $\lambda : T \longrightarrow [0,1]$ and maps p_σ $\ni$

$$\begin{tikzcd} \lambda^{-1}[0,1) \arrow[d, "p_0"'] & \lambda^{-1}(0,1) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "p_{01}"] & \lambda^{-1}(0,1] \arrow[d, "p_1"] \\ E_0 & E_{01} \arrow[l] \arrow[r] & E_1 \end{tikzcd}$$

commutes, and where p_σ is a map over X. (Note this implies that $\lambda^{-1}[0,1) \subset U_0$, $\lambda^{-1}(0,1] \subset U_1$, so that λ is a partition of 1 subordinate to the covering - roughly.)

From this description of maps, one has that for $f : Y \longrightarrow X$

$$f^* \mathrm{Cyl}(E_0 \leftarrow E_{01} \rightarrow E_1) = \mathrm{Cyl}(f^* E_0 \leftarrow f^* E_{01} \rightarrow f^* E_1).$$

Suppose given a class of maps called <u>ε-equivalences</u>. These will be subjected to various conditions to be explained.

$\varepsilon 0$) Any homotopy equivalence is an ε-equivalence. ε-equivalences are closed under composition.

I will call a map $C \to X$ "<u>ε-good</u>" if base change by this map carries homotopy-equivalences into ε-equivalences – i.e. given $Y \xrightarrow{\alpha} Z \to X$ with α a heq, then $Y \times_X C \to Z \times_X C$ is an ε-equivalence. (example: Any Hurewicz fibration is ε-good, because base change by a Hurewicz fibration preserves homotopy equivalences.) Clearly a base change of an ε-good map is ε-good.

(Alternative formulation: Given $Y \to X$ it can be factored $Y \xrightarrow{heq} Z \xrightarrow{fib} X$, hence if $C \to X$ is ε-good one has $Y \times_X C \to Z \times_X C$ is an ε-equiv., i.e. the actual and h-theoretic pull-back of C to Y are ε-equivalent. Observe this is ind. of Z:

$$\begin{array}{ccc} Y \times_X C & \longrightarrow & Z \times_X C \\ \downarrow & & \downarrow{\scriptstyle heq} \\ Y \times_X E & \xrightarrow{heq} & Z \times_X E \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{heq} & E \\ & \searrow & \downarrow{\scriptstyle fib} \\ & & X \end{array}$$

$$Y \xrightarrow{heq} Z \xrightarrow{fib} X$$

Conversely, if $Y \times_X C \to Y \times_X E$ is an ε-equiv. for any Y, then $Y \to Z$ a heq $\Longrightarrow$ $Y \times_X C \to Z \times_X C$ is a heq. Thus one sees that $C \to X$ is "ε-good" if on factoring it $C \xrightarrow{heq} E \xrightarrow{fibn} X$, then $C \to E$ is an <u>universal ε-equivalence</u> over X.)

In the course of the preceding argument I used

~~(ε1) If fg is an ε-equivalence and f is a heq, then g is an ε-equivalence.~~
~~fg is an ε-equivalence and f is an heq~~

(ε1) If fg, f are ε-equivalences, then so is g.

(In effect, given $Y \xrightarrow{\sim} Z \longrightarrow X$ and knowing $C \to E$ is a universal ε-equiv. gives

$$\begin{tikzcd} Y \times_X C \arrow[r] \arrow[d, "\varepsilon"'] & Z \times_X C \arrow[d, "\varepsilon"] \\ Y \times_X E \arrow[r, "heq"'] & Z \times_X E \end{tikzcd}$$
).

<u>Assertion</u>: Assume $E_0 \to U_0$, $E_1 \to U_1$ are ε-good and that the maps

$$E_{01} \longrightarrow U_{01} \times_{U_0} E_0, \quad E_{01} \longrightarrow U_{01} \times_{U_1} E_1$$

are <u>universal ε-equivalences</u>. Then $C = Cyl(E_0 \leftarrow E_{01} \to E_1) \to X$ is ε-good when restricted to U_0 or U_1.

Proof. $U_0 \times_X C =$

$$\begin{tikzcd} Cyl(E_0 \leftarrow E_{01} \longrightarrow U_{01} \times_{U_1} E_1) & \\ Cyl(E_0 \leftarrow E_{01} \longrightarrow E_{01}) \arrow[u] \arrow[r, "\text{fiber heq over } U_0"] & E_0 \end{tikzcd}$$

These arrows will be ε-equivalences even after any base change. Better proof – let T be a space over U_0. Then

$$T \times_X C = \mathrm{Cyl}\,(T \times_X E_0 \longleftarrow T \times_X E_{01} \longrightarrow T \times_X E_1)$$

$$= \mathrm{Cyl}\,\big(T \times_{U_0} E_0 \longleftarrow (T \times_{U_0} U_{01}) \times_{U_{01}} E_{01} \longrightarrow (T \times_{U_0} U_{01}) \times_{U_{01}} (U_{01} \times_{U_1} E_1)\big)$$

ε-equivalence by hyp.

so one has an ε-equiv.

$$T \times_{U_0} E_0 \longrightarrow T \times_C X = T \times_{U_0} (U_0 \times_X C)$$

for all T over U_0, showing that C is universally ε-equiv. to E_0 over U_0. Since E_0 is ε-good over U_0, the same is true for $\underline{U_0 \times_X C}$.

Have used:

(ε2) [redacted] $A \longrightarrow \mathrm{Cyl}(A \longleftarrow B \longrightarrow C)$ is an ε-equivalence if $B \longrightarrow C$ is.

(ε1$'$) If fg, g are ε-equivalences, then so is f.

$$\begin{tikzcd}
Y \times_X E_0 \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"'] & Y \times_X C \arrow[d] \\
Z \times_X E_0 \arrow[r, "\varepsilon"'] & Z \times_X C
\end{tikzcd}$$

Remark: It seems useful to try to get ε-equivalences to include n-equivalences, i.e. maps whose homotopy fibres begin in dimension $n+1$. Then one has ($\varepsilon 1'$) but not ($\varepsilon 1$). This means that ε-good $\Rightarrow$ pull-back ε-equivalent to homotopy-pull-back but not conversely. (ε-equivalent may be bad terminology).

Generalization: Suppose now I start with an open covering $\{U_i, i \in I\}$ of X, and for each non-empty finite set $\sigma \subset I$, I give a space E_σ over $U_\sigma = \bigcap_{i\in\sigma} U_i$ varying contravariantly in σ. I assume that $E_\sigma \to U_\sigma$ is ε-good for each σ and that for $\sigma \subset \tau$

$$E_\tau \longrightarrow U_\tau \times_{U_\sigma} E_\sigma$$

is a universal ε-equivalence over U_τ. (Assuming ε-1, this last would imply $E_\sigma \to U_\sigma$ is ε-good, once one knows it for some vertex of σ). Now form

$$C = Cyl(\sigma \mapsto E_\sigma)$$

which will be a space over X. A map from T to C over X will consist of a partition of unity $\sum \rho_i = 1$ on T ~~[illegible]~~ together with a natural transf of functors

$$\bigcap_{i\in\sigma} \rho_i^{-1}(0,1] \longrightarrow E_\sigma$$

to spaces over X. (This implies that $\rho_i^{-1}(0,1]$ sits over U_i.)

Useful to think of $\overline{U_i}$ as being locally-finite, for then

f_i will be an "honest" partition of unity.

So now to show that C is locally ε-good over X. ~~[illegible]~~ Let T be a space over U_{i_0}. Then

$$T \times_X \mathrm{Cyl}(\sigma \mapsto E_\sigma) = \mathrm{Cyl}(\sigma \mapsto T \times_X E_\sigma)$$

$$\begin{aligned} T \times_X E_\sigma &= T \times_{U_{i_0}} U_{i_0\sigma} \times_{U_\sigma} E_\sigma \\ &= (T \times_{U_{i_0}} U_{i_0\sigma}) \times_{U_{i_0\sigma}} (U_{i_0\sigma} \times_{U_\sigma} E_\sigma) \end{aligned}$$

$$\uparrow \ \varepsilon\text{-equiv.}$$

$$T \times_{U_{i_0}} E_{i_0\sigma} = (T \times_{U_{i_0}} U_{i_0\sigma}) \times_{U_{i_0\sigma}} E_{i_0\sigma}$$

So what you want to show is

(i) If $E_\sigma = E_{i_0\sigma}$, then $\mathrm{Cyl}(\sigma \mapsto E_\sigma)$ fibre-wise deforms ~~to~~ E_{i_0} (because one has a natural transf. $E_\sigma = E_{i_0\sigma} \longrightarrow E_{i_0}$).

(ii) If $E'_\sigma \longrightarrow E_\sigma$ is an ε-equiv. for each σ, then so is $\mathrm{Cyl}(\sigma \mapsto E'_\sigma) \longrightarrow \mathrm{Cyl}(\sigma \mapsto E_\sigma)$.

For (ii) one would cover the cylinder using open stars and identify the open stars of σ with E_σ via the deformation encountered in (i).

Observe that we then get

$$\begin{array}{ccc} T \times_{U_{i_0}} E_{i_0} & \xrightarrow{\text{heq (i)}} & \mathrm{Cyl}(\sigma \mapsto T \times_{U_{i_0}} E_{i_0\sigma}) \\ & & \downarrow \ \varepsilon \text{ equiv. (ii)} \\ & & T \times_X C = \mathrm{Cyl}(\sigma \mapsto (T \times_{U_{i_0}} U_{i_0\sigma}) \times_{U_{i_0\sigma}} (U_{i_0\sigma} \times_{U_\sigma} E_\sigma)) \end{array}$$

hence we have a universal ε-equivalence

$E_{i_0} \longrightarrow U_{i_0} \times_X C$ over U_{i_0}.