May 28, 1974 Karoubi's periodicity theorem.

A ring with involution: $a \mapsto \bar{a}$ from $A \xrightarrow{\sim} A^\circ$; $\varepsilon\bar{\varepsilon} = 1$; $P \in \mathcal{P}_A$. A ε-quadratic form on P is a bilinear form $f: P \times P \longrightarrow A$

$$f(ax, y) = a f(x,y)$$
$$f(x, ay) = \bar{a} f(x,y)$$
$$f(y,x) = \varepsilon \overline{f(x,y)}.$$

i.e. a map $P \longrightarrow P^*$ $x \mapsto (y \mapsto f(x,y))$ where $P^* = \mathrm{Hom}_A(P,A)$ (which is a right A-module) considered as a left module via $A \xrightarrow{\sim} A^\circ$. Denote the category of (non-degenerate) ε-quadratic modules by ${}_\varepsilon Q(A)$; morphisms are isometries.

Functors

$$\mathcal{P}_A^{iso} \underset{H}{\overset{F}{\rightleftarrows}} {}_\varepsilon Q(A)$$

forgetful

hyperbolic

Here $H(P) = P \oplus P^*$ with the duality

$$h: P \oplus P^* \longrightarrow (P \oplus P^*)^* = P^* \oplus P$$

given by the matrix

$$\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}.$$

Thus $h((x,\lambda),(x',\lambda')) = \langle x, \lambda' \rangle + \varepsilon \overline{\langle x', \lambda \rangle}$ where $\langle , \rangle$ denotes the canonical pairing $P \times P^* \to A$.

$$F(H(P)) = P \oplus P^*$$

$$H(F(Q)) = Q \oplus -Q \qquad (\text{assuming } \tfrac{1}{2} \in A)$$

Check the last formula — $H(F(Q)) = Q \oplus Q^* = Q \oplus Q$ with form $h((x,y),(x',y')) = f(x,y') + \varepsilon \overline{f(x',y)} = f(x,y') + f(y,x')$.

Thus if we map
$$\begin{aligned} Q &\longrightarrow Q \oplus Q \\ x &\longmapsto \alpha x + \beta x \end{aligned}$$
then
$$h(\alpha x + \beta x, \alpha y + \beta y) = f(\alpha x, \beta y) + f(\beta x, \alpha y)$$
$$= (\alpha\bar\beta + \beta\bar\alpha) f(x,y)$$
and we therefore find an isometry when $\alpha\bar\beta + \beta\bar\alpha = 1$.
Observe that if
$$\begin{aligned} Q &\longrightarrow Q \oplus Q \\ x &\longmapsto \alpha' x + \beta' x \end{aligned}$$
is to be orthogonal complementary to the preceding, then
$$\begin{vmatrix} \alpha' & \beta' \\ \alpha & \beta \end{vmatrix} = \text{unit} \qquad \alpha\bar{\beta'} + \beta\bar{\alpha'} = 0$$
Thus if I take
$$\alpha = \tfrac{1}{2} \quad \beta = 1$$
$$\alpha' = \tfrac{1}{2} \quad \beta' = -1$$
one gets $H(Q) \simeq Q \oplus (-Q)$.

Note that if we have a unit $u \in A$, then $x \mapsto ux$ carries f to $u\bar u f$. Thus when there is a u such that $u\bar u = -1$, then
$$H(F(Q)) \simeq Q \oplus Q.$$

For example, if $A = \mathbb{C}$ with usual involution, then $u\bar u$ is always positive. In fact we know any non-degenerate hermitian space splits into a positive def. $\oplus$ neg. definite one.

Over $A = \mathbb{F}_{q^2}$, one has $u\bar u = u^{1+q}$, and the set of these is $\mathbb{F}_q^*$, which contains -1.

Next: One defines ${}_{\varepsilon}L_n(A)$ $n \geq 1$ as the K-groups of ${}_{\varepsilon}Q(A)$ with $\oplus$. Put

$$ {}_{\varepsilon}O_{n,n}(A) = \text{Isometries of } H(A^n) $$

whence the basic connected H-space is

$$ B_{\varepsilon}O(A)^+ \qquad {}_{\varepsilon}O(A) = \varinjlim_n {}_{\varepsilon}O_{n,n}(A). $$

One has maps induced by F, H:

$$ F: \; B_{\varepsilon}O(A)^+ \underset{H}{\overset{}{\rightleftarrows}} BGL(A)^+ \qquad\qquad {}_{\varepsilon}L_n(A) \underset{H}{\overset{F}{\rightleftarrows}} K_n(A) $$

Forgotten to remark this: Changing f to αf changes

$$ (\alpha f)(y,x) = \alpha \varepsilon \overline{f(x,y)} $$
$$ = \alpha(\bar{\alpha}^{-1}) \varepsilon \overline{(\alpha f)(x,y)}. $$

Hence one has an equivalence of categories

$$ {}_{\varepsilon}Q(A) \sim {}_{\varepsilon'}Q(A) $$

if

$$ \varepsilon'(\varepsilon^{-1}) = \alpha / \bar{\alpha}. $$

Thus in $\mathbb{C}$ where ε can be anything $\ni \varepsilon\bar{\varepsilon} = 1$; one can solve always $\varepsilon = \alpha/\bar{\alpha}$, and so ε doesn't matter in this case.

Karoubi defines

${}_{\varepsilon}U$-theory = fibre theory of $H: K \longrightarrow {}_{\varepsilon}L$

${}_{\varepsilon}V$-theory = fibre theory of $F: {}_{\varepsilon}L \longrightarrow K$.

and his PERIODICITY THM is $\boxed{{}_{+\varepsilon}V = \Omega \, {}_{-\varepsilon}U}$

Examples:

1) Take $A = \mathbb{C}$ with its usual topology. One has that every quadratic module (ε doesn't matter here) is a direct sum of a positive and a negative one.

$$\therefore \quad BO^{top}(A) = BU \times BU$$

$$H = \Delta : BU \longrightarrow BU \times BU$$

observe $* = id$ on BU

$$F = \oplus : BU \times BU \longrightarrow BU$$

(Reason $H = \Delta$ is that given a $\mathbb{C}$-vector space V we can give it a pos. definite hermitian product, whence $H(V) = V \oplus V^*$ $\simeq V \oplus (-V)$.) Thus

$$U = \text{fibre of } H = U \times U / \Delta U = U$$

$$V = \text{fibre of } F = BU$$

and $V = \Omega U$ is the Bott periodicity

2) Take $A = B \times B^{\circ}$ with flipping involution. Clearly an $P(A)$-object is of the form (P, Q) where P is $P(B)$-module, Q is a $P(B^{\circ})$-module, and $(P, Q)^* = (Q^{\wedge}, P^{\wedge})$. A quadratic module is one such that $\alpha : P \xrightarrow{\sim} Q^{\wedge}$, the isom. of Q with $P^{\wedge}$ being ε times the transpose of α. ε doesn't matter.

$$BO(A)^+ = BGL(B)^+$$

$$BGL(A)^+ = BGL(B)^+ \times BGL(B^{\circ})^+ = BGL(B)^+ \times BGL(B)^+$$

where

$$H = \oplus : \left(BGL(B)^+\right)^2 \longrightarrow BGL(B)^+$$

$$F = \Delta \qquad \longleftarrow$$

Thus

$$U = BGL(B)^+$$

$$V = \Omega\, BGL(B)^+$$

so $V = \Omega U$ is trivial here

Question: Consider $\mathbb{C}$ as a discrete ring with the conjugation involution. Then I have have two endofunctors of $\mathcal{P}(\mathbb{C})$ namely conjugation and duality, the latter being contravariant. So I get ~~two~~ endos of $K_*(\mathbb{C})$ which are definitely different on $K_1\mathbb{C} = \mathbb{C}^*$. But do these functors agree on $K_*(\mathbb{C}, \mathbb{Z}/m\mathbb{Z})$ for any $\neq 0$ integer m?

Conjecture: On $K_*(\mathbb{C})$ I get two endos. - one corresponding to Ψ^k, the other the Galois auto. which coincides with Ψ^k on roots of unity. Then these two autos should agree on $K_*(\mathbb{C}, \mathbb{Z}/m\mathbb{Z})$.

Possible proof. One wants to show

$$\{\text{fibre of } F: BO_\varepsilon^+ \longrightarrow BGL^+\} = \Omega\{\text{fibre of } H: BGL^+ \to BO_\varepsilon^+\}$$

$$\underline{\underline{V}} \qquad\qquad\qquad \Omega\,\underline{\underline{U}}$$

or that $B\underline{\underline{V}} = \underline{\underline{U}}$. Observe if this is so then we get maps

$$BO^+ \xrightarrow{F} BGL^+ \longrightarrow B\underline{\underline{V}}$$

$$\qquad\qquad\qquad\qquad\quad \| $$

$$\qquad\qquad\qquad\qquad\quad \underline{\underline{U}} \longrightarrow BGL^+ \xrightarrow{H} BO^+$$

hence a map $BGL^+ \longrightarrow BGL^+$ which kills F coming in and H going out. Presumably, this maps is $id - *$.

(Note that F is compatible with products, since if one has an ε-quadratic module P, and an an η-quadratic module Q, then $P \otimes Q$ is naturally an $\varepsilon\eta$-quadratic module with

$$f(x\otimes y, x'\otimes y') = f(x,x') \otimes f(y,y')$$

Thus it would seem that U-theory is an "ideal" theory with respect to L.)

What I might try to do to prove this theorem is to establish a fibration of categories

$$B\underline{\underline{V}} \longrightarrow ? \longrightarrow B\underline{\underline{U}}$$

where ? is contractible. Note that I have nice models for the rest:

$B\,{}_{\varepsilon}\underline{\underline{V}}$: objects same as P_A, a map $V \to V'$ is a direct injection whose complement has an ε-quadratic structure.

$B\,{}_{-\varepsilon}\underline{\underline{U}}$: objects same as ${}_{\varepsilon}\underline{\underline{Q}}_A$, a map $Q \to Q'$ is a direct injection whose complement is hyperbolic.

Try to take ? as a category of paths in $B\underline{\underline{U}}$. So naturally we cut down $B\underline{\underline{U}}$ to its connected component so that it becomes connected, and we consider then $(-\varepsilon)$-quadratic spaces which are stably hyperbolic. Then we might consider paths starting from 0 to a given quadratic module Q. This roughly amounts to considering for a quadratic module Q the different paths from 0 to Q of the form

$$0 \longrightarrow H(L) \longleftarrow Q$$

i.e.

$$H(L_1) \oplus Q \simeq H(L_0).$$

In particular over $Q=0$, we find a hyperbolic module $H(L_0)$ with a Lagrangian L_1. This is I guess what Ranicki calls a formation. Anyway is it possible to relate this to $B\underline{\underline{V}}$?

Karoubi's basic map (which is due to Novikov) is
$$\tau_U : {}_{-\varepsilon}U(A) \longrightarrow {}_{\varepsilon}V(A[z,z^{-1}])$$
which is the analog of the map
$$K_0(A) \longrightarrow K_1(A[z,z^{-1}])$$
sending a projective A-module P into the auto mult by z on $P_z = A[z,z^{-1}]\otimes_A P$. In fact he shows that one has a commutative diagram
$$\begin{tikzcd}
{}_{-\varepsilon}U(A) \arrow[r] \arrow[d] & K(A) \arrow[d] \\
{}_{\varepsilon}V(A[z,z^{-1}]) \arrow[r, "\Delta"] & K_1(A[z,z^{-1}])
\end{tikzcd}$$
where I have to understand Δ.

His model for ${}_{\varepsilon}V(A)$: One takes the natural thing giving an exact sequence
$$ {}_{\varepsilon}V_0(A) \longrightarrow {}_{\varepsilon}L_0(A) \xrightarrow{F} K_0(A)$$
i.e. one considers triples $(Q, Q'; FQ \simeq FQ')$, or equivalently one considers a given projective modules E, with two ε-quadratic forms $g_i : E \longrightarrow E^*$ $g_i^* = \varepsilon g_i^{-1}$. Then this map Δ takes (E, g_1, g_2) into the automorphism $g_1^{-1}g_2$ of E.

Thus it seems what I have to understand is the map Δ which fits into a triangle
$$\begin{tikzcd}
{}_{\varepsilon}\underline{\underline{V}} \arrow[r, phantom, "\simeq"] \arrow[dr, "\Delta"'] & \Omega\underline{\underline{U}} \arrow[d] \\
 & \underline{\underline{GL}}
\end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd}
B\underline{\underline{V}} \arrow[dr, "\Delta"'] & \underline{\underline{U}} \arrow[d] \\
 & B\underline{\underline{GL}} \arrow[d, "H"] \\
 & B\underline{\underline{O}}
\end{tikzcd}$$

Idea: For $B_{\varepsilon}\underline{\underline{V}}$ we will take the category whose objects are $P(A)$ modules, whose arrows are direct injections with ε-quadratic cokernel. For $B\underline{\underline{GL}}$ I take the category of pairs (P,Q) in which the morphisms are direct injections with "diagonal" cokernel. I then have a functor

$$B_{\varepsilon}\underline{\underline{V}} \longrightarrow B\underline{\underline{GL}}$$
$$P \longmapsto (P, P^*)$$

which sends a morphism

$$P \simeq P' \oplus Q \qquad Q \in {}_{\varepsilon}Q(A)$$

into the $B\underline{\underline{GL}}$-morphism

$$(P,P^*) \simeq (P' \oplus Q, P'^* \oplus Q^*) \longleftarrow (P', P'^*)$$

where we use the given isom of $Q \xrightarrow{\sim} Q^*$. Now I have a hyperbolic functor

$$B\underline{\underline{GL}} \xrightarrow{H} B_{-\varepsilon}\underline{\underline{O}}$$
$$(P_1, P_2) \longmapsto (HP_1, HP_2)$$

and I notice that

$$H(P,P^*) = (HP, HP^*)$$

is ~~[illegible]~~ canonically contractible to zero. The only thing to be checked is that if I start with an ε-quadratic module Q, consider then the path

$$(0,0) \xrightarrow{f} (Q,Q^*) \qquad \text{in } B\underline{\underline{GL}}$$

~~[illegible]~~ then

$$\begin{tikzcd}
(H0, H0) \arrow[r, "Hf"] \arrow[d, phantom, "\parallel"] & (HQ, HQ^*) \\
(0,0) \arrow[ur] &
\end{tikzcd}$$

commutes. Thus I will have two isomorphisms of HQ with HQ^*. The first is H applied to the isom $Q \xrightarrow{f} Q^*$.

and the second is the natural isomorphism of HQ with HQ^*, namely

$$HQ = Q \oplus Q^*$$
$$HQ^* = Q^* \oplus (Q^*)^*$$

Formulas.

$$H(Q) = Q \oplus Q^* \quad \text{with}$$
$$\langle (x,\lambda), (y,\mu) \rangle = \langle x, \mu \rangle + \varepsilon \overline{\langle y, \lambda \rangle}$$

$H(Q^*) = Q^* \oplus Q^{**}$ with same formula.

Now identify Q^{**} with Q so that

$$\langle \lambda, \theta_\varepsilon x \rangle = \varepsilon \overline{\langle x, \lambda \rangle} \qquad \theta_\varepsilon : Q \to Q^{**}$$

then the map $(x,\lambda) \longmapsto (\lambda, \theta_\varepsilon x)$ satisfies

$$\begin{aligned}
\langle (\lambda, \theta_\varepsilon x), (\mu, \theta_\varepsilon y) \rangle &= \langle \lambda, \theta_\varepsilon y \rangle + \varepsilon \overline{\langle \mu, \theta_\varepsilon x \rangle} \\
&= \varepsilon \overline{\langle y, \lambda \rangle} + \varepsilon \overline{\varepsilon \langle x, \mu \rangle} \qquad \varepsilon \bar{\varepsilon} = 1 \\
&= \langle x, \mu \rangle + \varepsilon \langle y, \lambda \rangle \\
&= \langle (x,\lambda), (y,\mu) \rangle
\end{aligned}$$

and so we get an isometry of $HQ \cong HQ^*$.

But now if I have $f : Q \xrightarrow{\sim} Q^*$, then one has a map $\quad f^* : Q^* \leftarrow Q^{**}$

$$H(Q) \xrightarrow{\sim} H(Q^*)$$
$$(x,\lambda) \longmapsto (fx, (f^*)^{-1}\lambda)$$

$$\begin{aligned}
\langle (fx, f^{*-1}\lambda), (fy, f^{*-1}\mu) \rangle &= \langle fx, (f^*)^{-1}\mu \rangle + \varepsilon \overline{\langle fy, f^{*-1}\lambda \rangle} \\
&= \langle x, \mu \rangle + \varepsilon \langle y, \lambda \rangle \\
&= \langle (x,\lambda), (y,\mu) \rangle
\end{aligned}$$

which is an isometry. I was hoping this isometry would be the same as $(x,\lambda) \longmapsto (\lambda, \theta_\varepsilon x)$, i.e. that

$$(\lambda, \theta_\varepsilon x) = (fx, f^{*-1}\lambda)$$

for all λ, x. This is non-sense.

Problem: ~~[illegible]~~ I have the hyperbolic functor $H : P(A) \longrightarrow$ Quad$_\varepsilon(A)$ which comes with a canonical isomorphism

$$H(P) \simeq H(P^*)$$
$$(x,\lambda) \longmapsto (\lambda, \theta_\varepsilon x)$$

where $\theta_\varepsilon : P \xrightarrow{\sim} P^{**}$ is the iso $\ni$

$$\langle \lambda, \theta_\varepsilon x \rangle = \varepsilon \overline{\langle x, \lambda \rangle} \qquad \theta_\varepsilon = \bar{\varepsilon}\theta_1$$

On the other hand given an isom $f : P \xrightarrow{\sim} P'$ one has $H(f) : H(P) \xrightarrow{\sim} H(P')$, $(x,\lambda) \longmapsto (f(x), f^{*-1}(\lambda))$; this is an isometry.

So if we are given $f : P \xrightarrow{\sim} P^*$, then we get a unitary transformation of $H(P)$

$$H(P) \xrightarrow[\sim]{H(f)} H(P^*) \xleftarrow{\sim} H(P)$$

$$(x,\lambda) \longmapsto (f(x), f^{*-1}(\lambda)) \longmapsto (\underbrace{\theta_\varepsilon^{-1} f^{*-1}(\lambda)}, f(x))$$

$$P^* \xrightarrow{f^{*-1}} P^{**} \xrightarrow{\theta_\varepsilon^{-1}} P \qquad \theta_\varepsilon^{-1} = \varepsilon\theta_1^{-1}$$

So it appears that given $f : P \xrightarrow{\sim} P^*$ we get a unitary op

$$\boxed{\begin{array}{c} {}_\varepsilon H(P) \xrightarrow{\sim} {}_\varepsilon H(P) \\ (x,\lambda) \longmapsto (\varepsilon f^{*-1}(\lambda), f(x)) \end{array}}$$

Check: $\langle (\varepsilon f^{*-1}(\lambda), f(x)), (\varepsilon f^{*-1}(\mu), f(y)) \rangle = \varepsilon \langle f^{*-1}(\lambda), f(y) \rangle + \varepsilon \overline{\langle \varepsilon f^{*-1}(\mu), f(x) \rangle}$

$= \langle x, \mu \rangle + \varepsilon \langle y, \lambda \rangle$

Problem: Show that when f satisfies the appropriate symmetry conditions, I guess $f^* = -\varepsilon f$, then this auto. is canonically contractible in the K-theory of $\mathrm{Quad}_\varepsilon(A)$.

Reformulate:

$$ {}_{-1}\mathcal{Q}(A) \xrightarrow{F} \mathcal{P}(A) \underset{*}{\overset{id}{\rightrightarrows}} \mathcal{P}(A) \xrightarrow{H} \mathcal{Q}(A) $$

Here $H(P) = P \oplus P^*$ with $\langle (x,\lambda),(y,\mu)\rangle = \langle x,\mu\rangle + \overline{\langle y,\lambda\rangle}$ and I have a canonical isom.

$$ \begin{array}{ccc} H(P) & \overset{can}{=} & H(P^*) \\ \| & & \| \\ (P \oplus P^*) & \xrightarrow{\sim} & (P^* \oplus P^{**}) \end{array} $$

given by $(x,\lambda) \longmapsto (\lambda, \theta x)$ where $\theta : P \xrightarrow{\sim} P^{**}$ is the canonical iso. such that $\langle \lambda, \theta x\rangle = \overline{\langle x,\lambda\rangle}$.

Now given $f: P \xrightarrow{\sim} P^*$ I get a unitary auto

$$ H(P) \xrightarrow{H(f)} H(P^*) \overset{can}{=} H(P) $$

which in matrix form is

$$ P \oplus P^* \longrightarrow P^* \oplus P = P \oplus P^* $$

$$ \begin{pmatrix} f & 0 \\ 0 & f^{*-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f^{*-1} \\ f & 0 \end{pmatrix} $$

The question now is to show that (when f is skew-symmetric) this map I have defined from ${}_{-1}\mathcal{Q}(A)$ to $\Omega\mathcal{Q}(A)$ is trivial.

Andrew's interpretation: He has the following model for $\mathcal{U}(A)$. First recall that this is the Grothendieck group of $H: \mathcal{P}(A) \longrightarrow \mathcal{Q}(A)$, hence is generated by triples $(P, P', HP \simeq HP')$, or "equivalently" by a formation $(Q; F, G)$ where Q is a quadratic module and F, G are two lagrangians. Such a $(Q; F, G)$ is called a <u>formation</u>.

Now in forming a Grothendieck group out of these formations, one regards as trivial those formations of the form

$$(H(P),\ P,\ \Gamma_f)$$

where $\Gamma_f = \{(x, f(x))\}$ is the graph of f and f is skew-symmetric, so that

$$\begin{aligned}\langle (x, f(x)), (y, f(y)) \rangle &= \langle x, f(y) \rangle + \langle f(x), y \rangle \\ &= \langle (f^* + f)x, y \rangle = 0\end{aligned}$$

Next he defines the map

$$\begin{aligned} \mathcal{P}(A) &\xrightarrow{M} \mathcal{U} = (\text{cat made up of formations}) \\ P &\longmapsto (H(P), P, P^*) \end{aligned}$$

so that on composing with
$$\begin{aligned}\mathcal{U} &\longrightarrow \mathcal{P}(A)^{\wedge} \\ (Q; F, G) &\mapsto (F, G)\end{aligned}$$
one gets the map $P - P^*$ as I wanted. Now the point is to see why this map M is canonically trivializable when restricted to $\mathcal{Q}_{-1}(A) \xrightarrow{F} \mathcal{P}(A)$. But the point is that if $Q \in \mathcal{Q}_{-1}(A)$, then we have $f : Q \xrightarrow{\sim} Q^*$ $\ni$ $f^* = -f$. So we have the isom

$$\begin{aligned}(H(Q), Q, Q^*) &\xrightarrow{\ \sim\ } (H(Q), Q, \Gamma_f) \\ (x, \lambda) &\longmapsto (x + f^{-1}(\lambda), \lambda)\end{aligned}$$

$$\begin{aligned}\langle (x + f^{-1}(\lambda), \lambda), (y + f^{-1}(\mu), \mu) \rangle &= \langle x, \mu \rangle + \langle f^{-1}\lambda, \mu \rangle \\ &\quad \langle \lambda, y \rangle + \langle \lambda, f^{-1}\mu \rangle \\ &= \langle x, \mu \rangle + \langle \lambda, y \rangle \quad \text{for } (f^{-1})^* = (-f)^{-1}\end{aligned}$$

Now I can perhaps settle the problem of before. So I have for $f : P \xrightarrow{\sim} P^*$ an automorphism

$$H(P) \underset{\sim}{\xrightarrow{H(f)}} H(P^*) \xrightarrow{\text{can}} H(P)$$

which I want to show is trivial for a canonical reason, when f is skew-symmetric. What seems to go is this — given complementary Lagrangians we can transvect one keeping the other fixed. Thus we have an automorphism

$$\begin{array}{lccc}
 & (P \oplus P^*,\ P,\ P^*) & (x, 0) & (0, \lambda) \\
\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} & \downarrow & & \\
 & (P \oplus P^*,\ \Gamma_f,\ P^*) & (x, f(x)) & (0, \lambda) \\
\begin{pmatrix} 2 & -f^{-1} \\ f & 0 \end{pmatrix} & \downarrow & & \\
 & (P \oplus P^*,\ \Gamma_f,\ P) & (x, f(x)) & (-f^{-1}\lambda, 0) \\
\begin{pmatrix} 1 & -f^{-1} \\ 0 & 1 \end{pmatrix} & \downarrow & & \\
 & (P \oplus P^*,\ P^*,\ P) & (0, f(x)) & (-f^{-1}(\lambda), 0) \\
 & & & \shortparallel \\
 & & & (f^{*-1}(\lambda), 0)
\end{array}$$

This is a sequence of elementary unitary transfs. with product $H(f)$.

Andrew's suggestion:

Start with the groupoid consisting of formations (Q, F, G) and their isomorphisms. By a trivialized formation, mean a quadruple (Q, F, G, H) where F, G, H are Lagrangians in Q, and H is complementary to both F, G. Thus (Q, F, G, H) is canon. isom. to $(H(P), P, \Gamma_f, P^*)$ where $f: P \to P^*$ is $\ni f^* = -f$.

Now form the category $\langle \text{triv. form.}, \text{form.} \rangle$. According to Andrew this ought to have the homotopy type of $\underline{\underline{U}} =$ fibre of $H: \underline{\underline{K}} \to \underline{KQ}$.

Example: $A = B \times B^{\circ}$. A formation is of the form $((P, P^*); (M, M^{\perp}), (N, N^{\perp}))$ where M, N are two admissible subobjects of P. Thus formations $=$ triples (P, M, N), with $M, N \subset P$. A trivialized formation is a quadruple (P, M, N, C), where C is a complement for both M and N in P. (Check: $(P, P^*) = (M, M^{\perp}) \oplus (C, C^{\perp})$ means that $P = M \oplus C$.)

~~Now the model for $\underline{BV}$ was pairs (P, Q) modulo diagonal and the functor from $\underline{BV}$ to $\underline{U}$ is supposed ...~~

Some ideas of Graeme Segal:

Let G be a Lie group; suppose G connected. Let Γ be the underlying discrete group of G. Letting Γ act on G by left multiplication, one obtains a top. category (G, Γ) whose classifying space is $E\Gamma \times^{\Gamma} G$. I would like to prove $E\Gamma \times^{\Gamma} G$ has trivial homology with torsion coefficients.

Choose a right invariant metric on G, and let U be a small open ball around e, small enough so that it is convex. Then consider the covering γU of G. Any finite intersection of open sets in the covering if non-empty is contractible by convexity. Thus G as a space is homotopy equivalent to the nerve of this covering. N is the simplicial complex whose q-simplices are chains $\gamma_0, \gamma_1, \dots, \gamma_q$ such that

$$\gamma_0 U \cap \dots \cap \gamma_q U \neq \phi .$$

Consider N as a simplicial set; Γ acts freely so we can divide out: $\Gamma \backslash N$. A q-simplex of $\Gamma \backslash N$ is

$$(\gamma_0, \gamma_1, \dots, \gamma_q) \longmapsto (\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, \dots, \gamma_{q-1}^{-1}\gamma_q)$$

$$\gamma_0 U \cap \gamma_1 U \cap \dots \cap \gamma_q U = \gamma_0 U \cap (\gamma_0)(\gamma_0^{-1}\gamma_1) U \cap (\gamma_0)(\gamma_0^{-1}\gamma_1)(\gamma_1^{-1}\gamma_2) U$$

$\therefore$ a q-simplex is a $(g_1, \dots, g_q) \ni \quad U \cap g_1 U \cap g_1 g_2 U \cap \dots \cap g_1 \cdots g_q U \neq \phi$

Don't seem to get a partial monoid this way

June 1, 1974. K-homology

I want to construct for any space X (say compact) a space giving the K-homology of X.

Example 1. If X is a ~~[illegible]~~ simplicial set, ~~[illegible]~~ and P is a permutative category, one can form the simplicial space, which is $\bigoplus_{x \in X_0} P$ degree-wise.

From this example, one feels what I am after generalizes the idea of a chain on X with coefficients P. Thus I start with things of the form $\bigoplus_{x\in X} P_x$, finite sums indexed by the set X, and I want to topologize these when X is a space.

Example 2. If $P =$ finite sets, then what we want to look at is finite sets over X. Taking those ~~[illegible]~~ of card d, one gets an ~~[illegible]~~ equivalent category ~~[illegible]~~ whose objects are maps $\{1, \dots, d\} \longrightarrow X$ and in which the maps are permutations. Thus for card d we get the space

$$P\Sigma_d' \times^{\Sigma_d} X^d.$$

Idea for complex K-theory. I want some way of describing a $\mathbb{C}$ vector space V with a decomposition

$$V = \bigoplus_{x\in X} V_x$$

indexed by ~~[illegible]~~ the points of X. Such a decomposition is the same as a $\mathbb{C}^X$-module structure on V. In effect the maximal ideals in $\mathbb{C}^X$ are in one-one correspondence with the points of X; also $m_x^2 = m_x$ for continuous functions.

Thus it seems that to give a $\mathbb{C}^X$-module structure on V, supposed such that $f^* = \bar{f}$ for a metric on V, is the same as decomposing V into $V = \oplus V_x$

$$V_x = \{ v \mid f \cdot v = f(x) \cdot v \}.$$

But this brings to mind bundles with A structure, A a $\mathbb{C}$-algebra.

Question: Can we make a reasonable topological K-theory out of bundles with $\mathbb{C}^X$-structure?

Such a bundle is a vector bundle E over Y equipped with a continuous action of $\mathbb{C}^X$.

Example: Let $X =$ circle. Then to give a $\mathbb{C}^X$-action on a hermitian vector space V is the same as giving a unitary operator (action of the function $e^{i\theta}$).

Thus to give a bundle over a space Z with $\mathbb{C}^X$ structure, is the same as to give a vector bundle over Z together with a unitary isomorphism. Up to homotopy, this ought to be the same as a bundle over $Z \times S^1$

To describe the space of $\mathbb{C}^T$-structures on V. Here T is a compact space, V a finite dim $\mathbb{C}$ Hilbert space. A $\mathbb{C}^T$-structure is a $*$-homo. $\mathbb{C}^T \longrightarrow \mathrm{End}(V)$; or what amounts to the same thing an orthogonal decomposition

$$V = \bigoplus_{t \in T} V_t$$

indexed by points of T.

It is clear what one means by a converging sequence of decompositions: $y_n \longrightarrow y$. Let the support of y be the finite set $\{t_i\}$. Then for n large the support of y_n should split up into pieces tending to each of the t_i. More precisely suppose one chooses disjoint nbds. U_i of t_i. Then for n large the support of y_n is contained in $\bigcup_i U_i$, and then if $V_i(y_n) =$ sum of $V_t(y_n)$ for $t \in U_i$, then $\dim V_i(y_n) = \dim V_{t_i}$ and $V_i(y_n) \longrightarrow V_{t_i}$ in the Grassmannian, as $n \to \infty$.

Now denote by $D(V,T)$ the space of $\mathbb{C}^T$ structures on V. It should be possible to describe this as a stratified set.

There is a map

$$D(V,T) \longrightarrow T^d/\Sigma_d \qquad d = \dim(V).$$

given by sending $V = \oplus V_t$ into the divisor $\sum_t \dim(V_t)$. $D(V,T)$ is stratified according to

~~the stratification of T^d/Σ_d.~~

~~More precisely we have~~

$$\cancel{T^d/\Sigma_d = \coprod_{\substack{a_1 \geq \cdots \geq a_k > 0 \\ \sum a_i = d}} T^k - (\text{mult. diag})}$$

~~The union is taken over partitions of d. Better — let me~~ ~~[illegible]~~ ~~gives the sequence~~

$$\cancel{\alpha_1 = \text{no of } a_i = 1}$$

$$\cancel{\alpha_2 = \text{no. of } a_i = 2 \qquad \text{etc.}}$$

~~So that~~

$$\cancel{T^d/\Sigma_d = \coprod_{\substack{\alpha = (\alpha_1, \alpha_2, \ldots) \\ \sum i\alpha_i = d}} T^{\sum \alpha_i} - \text{mult. diag}}$$

~~Thus given a~~ ~~[illegible]~~ ~~divisor $\sum a_i t_i$ with $\sum a_i$~~

The stratification of T^d/Σ_d. Given $\sum a_i t_i$ with $\sum a_i = d$, all $a_i > 0$. We can arrange it according to the size of the a_i.

$$\sum_{a_i = 1} t_i + 2 \sum_{a_i = 2} t_i + \cdots$$

~~[illegible]~~ Let α_u = no. of $a_i = u$. Then

$$\sum u\alpha_u = d \qquad \alpha_u \geq 0$$

and it is clear that our point determines an element of

$$T^{\sum \alpha_u} - (\text{big diag}) \Big/ \Sigma_{\alpha_1} \times \cdots \times \Sigma_{\alpha_u} \times \cdots$$

Put $\sum \alpha_u = |\alpha|$, $\sum u\alpha_u = \|\alpha\| = d$, $\Sigma_\alpha = \Sigma_{\alpha_1} \times \cdots$.
Then we have the partition

$$T^d/\Sigma_d = \coprod_{\|\alpha\|=d} \left(T^{|\alpha|} - \text{big diag}\right)/\Sigma_\alpha$$

Now over a point $\sum a_i t_i$ with invariant $\alpha = (\alpha_1, \alpha_2, \cdots)$, one has all compatible decompositions of V. Thus given α, define a flag of type α to be a flag with jumps

$$\underbrace{1, 1, \cdots, 1}_{\alpha_1}, \underbrace{2, \cdots, 2}_{\alpha_2}, \cdots$$

etc and let $D_\alpha(V)$ be the space of these flags. Then we have an action of Σ_α on $D_\alpha(V)$. Then we have the partition

$$D(V, T) = \coprod_{\|\alpha\| = \dim V} \left(T^{|\alpha|} - \text{big diag}\right) \times^{\Sigma_\alpha} D_\alpha(V).$$

And if we put $\quad U^\alpha = (U_1)^{\alpha_1} \times (U_2)^{\alpha_2} \times \cdots$
one has the partition

$$BU(V) \times^{U(V)} D(V, T) = \coprod_{\|\alpha\| = \dim V} \left(T^{|\alpha|} - \text{big diag}\right) \times^{\Sigma_\alpha} \left(BU^\alpha\right)$$

I want now to make bundles with $\mathbb{C}^T$-structure into a K-theory.

Definition: If Y is a space, and if E is a unitary vector bundle over Y, then a T-structure on E will be a continuous $*$-homomorphism

$$\varphi : \mathbb{C}^T \longrightarrow \mathrm{End}(E) = \Gamma(Y, \underline{\mathrm{End}}(E)).$$

$*$-homomorphism means φ is a ring homo. such that $\varphi(\bar{f}) = \varphi(f)^*$. When Y is compact $\mathrm{End}(E)$ is a Banach algebra, and φ is continuous iff is it continuous as a map of Banach spaces. (It follows that φ decreases norms.)

Suppose Y compact from now on.

It is clear that the set of T-structures on E is a closed subspace of the Banach space of bounded linear maps from the Banach space $\mathbb{C}^T$ to the Banach space $\mathrm{End}(E)$.

Thus I can speak of two T-structures on E as being homotopic, i.e. the corresponding $*$-homs. $\mathbb{C}^T \longrightarrow \mathrm{End}(E)$ are joined by a path.

Now let $\mathrm{Vect}(Y;T)$ denote the set of equivalence classes of bundles over Y equipped with T-structure, where equivalence is generated by isomorphism and homotopy. More precisely given two vector bundles with T-structure (E,φ) and (E',φ'), call them equivalent if there exists an isomorphism $E \simeq E'$ such that φ is homotopic to the image of φ' by this isomorphism. This is an equivalence relation.

When Y is connected we can decompose according to rank

$$\mathrm{Vect}(Y;T) = \coprod_{n\geq 0} \mathrm{Vect}_n(Y;T)$$

and then extend $\mathrm{Vect}_n(Y;T)$.

To show $\mathrm{Vect}_n(Y;T)$ is representable as a functor of Y. It is obviously contravariant in Y. To show the homotopy axiom, it suffices to show that if (E,φ) is a bundle with T-structure on $Y\times I$, then the induced bundles with T-structure (T-bundles) on $Y\times 0$, $Y\times 1$ are equivalent. But E is of the form $E_0\times I$ where E_0 is a bundle on Y, and

$$\begin{aligned}\mathrm{End}(E_0\times I) &= \Gamma(Y\times I, \underline{\mathrm{End}}(E_0)\times I) = \Gamma(Y,\underline{\mathrm{End}}(E_0))^I \\ &= \mathrm{End}(E_0)^I\end{aligned}$$

Thus we get a cont. $*$-homo

$$\mathbb{C}^T \longrightarrow \mathrm{End}(E_0)^I$$

so we get a 1-parameter family of T-structures on E_0.

Suppose now that we take a Grassmannian G N-classifying for bundles of dim n. If P is the principal U_n bundle, let D be the space of T-structures on $\mathbb{C}^n$, and put

$$Z = P \times^{U_n} D.$$

Lifting the canonical bundle on G to Z, it has a canonical T-structure. (The point is that T-structures glue together). Denote this bundle with T-structure by ξ. Since $\mathrm{Vect}_n(?;T)$ is a homotopy functor, we get a map

$$\begin{aligned} [Y, Z] &\longrightarrow \mathrm{Vect}_n(?;T) \\ f &\longmapsto f^*(\xi) \end{aligned}$$

This map is onto if $\dim(Y) \le N$. In effect given $\eta = (E, \varphi)$ over Y there exists a map $Y \to G$ inducing E. Pulling by Z via this map we get the fibre bundle whose fibres are the different T-structures on the fibres of E. Since

$$\Gamma(U, \underline{\mathrm{End}}(V) \times U) = \mathrm{End}(V)^U$$

continuity of a T-action given on each fibre of E/Y is the same as continuity of the corresponding section of $Y \times_G Z$ over Y. So φ determines a section of $Y \times_G Z$ over Y, hence we get a map $Y \longrightarrow Z$ inducing η from ξ.

To prove injectivity suppose we have two maps $f, g : Y \longrightarrow Z$ inducing the same element of $\mathrm{Vect}_n(Y;T)$. Then the two maps $Y \to G$ induce isom. bundles, hence are homotopic if $\dim(Y) < N$. By the covering homotopy theorem, the homotopy lifts to a homot. starting with f, hence we reduce to the case where f, g induce the same map into G and we have a homotopy between the two T-structures on the bundle E. This homotopy is a homotopy between <u>the corresponding</u> sections of Z.

$\mathrm{Vect}(Y;T)$ is an abelian monoid; let $K(Y;T)$ denote the corresponding Grothendieck group. One wants to know if it is representable. This leads to the following problem.

Suppose T is connected with basepoint. One wants to be able to get $\mathrm{Coker}\{K(Y;pt) \longrightarrow K(Y;T)\}$ by stabilizing. Thus I want to know if every T-bundle E over Y is a direct summand of a T-bundle homotopic to a trivial one, i.e. where the T action is thru the homo $\mathbb{C}^T \longrightarrow \mathbb{C}$ given by evaluating at the basepoint.

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Suppose Y is connected. We can then stabilize $\mathrm{Vect}(Y,T)$ with respect to the monoid $\mathrm{Vect}(pt,T)$. We form

$\mathrm{Vect}(Y,T)/\mathrm{Vect}(pt,T)$ = equiv. classes of $\xi \in \mathrm{Vect}(Y,T)$ where $\xi \sim \xi' \iff \xi' + \eta = \xi + \eta'$, $\eta, \eta' \in \mathrm{Vect}(pt,T)$.

$$= \varinjlim \mathrm{Vect}_\alpha(Y,T)$$

where the limit is taken over the translation category of $\mathrm{Vect}(pt,T)$, and $\mathrm{Vect}_\alpha(Y,T)$ = those ξ which restrict to α over the basepoint.

Better: form the monoid

$$F(Y,T) = \mathrm{Vect}(pt,T)^{-1}\,\mathrm{Vect}(Y,T)$$

As a functor of Y it is a filtered limit of representable functors which are monoid-valued. If $Y = pt$, $F(pt,T)$ is a group. Thus I know that $F(Y,T)$ is always a group, hence we have the formula

$$\boxed{K(Y;T) = \mathrm{Vect}(pt;T)^{-1}\,\mathrm{Vect}(Y;T)}$$

I should also be able to prove this by showing directly that any T-bundle ξ on Y is a direct summand of one which is homotopic to one coming via $Y \to pt$. Take a point $y \in Y$, and a contractible nbd U. If I can show that $\xi|U$ extends to a bundle on Y homotopic to one coming from $Y \to pt$, then covering Y by such open sets, I win. Since U is contractible $\xi | U \sim U \times V$, V a T-bundle over a point. Thus I have the T-bundle $Y \times V$ over Y and a homotopy of its T-structure over U. Since the T-structure is simply a map of Y (or U) into the space of T-structures on V, it's clear we win by CHT.

Variant: I want to show that if T is a connected space with basepoint t_0, then $\mathrm{Vect}(pt, t_0)$ is cofinal in $\mathrm{Vect}(pt, T)$. (mean finite complex).

Suppose $T = A \cup B$ where $t_0 \in A \cap B$. Given a vector space with T-structure E I can split it into a direct sum of pieces coming from $\mathrm{Vect}(pt, A)$, and $\mathrm{Vect}(pt, B)$, i.e. I have an epi:

$$\mathrm{Vect}(pt; A) \times \mathrm{Vect}(pt; B) \twoheadrightarrow \mathrm{Vect}(pt; A \cup B).$$

Thus if $\mathrm{Vect}(pt; t_0)$ is cofinal in both $\mathrm{Vect}(pt; A)$ and $\mathrm{Vect}(pt; B)$ it is cofinal in $\mathrm{Vect}(pt; T)$.

Next note that if $h : T \times I \to T'$ is a homotopy then $h_{0*} = h_{1*} : \mathrm{Vect}(Y, T) \to \mathrm{Vect}(Y, T')$. In effect h induces

$$h^* : \mathbb{C}^{T'} \longrightarrow \mathbb{C}^{T \times I} = (\mathbb{C}^T)^I$$

which when composed with $\mathbb{C}^T \to \mathrm{End}(E)$, gives a map

$$\mathbb{C}^{T'} \longrightarrow (\mathrm{End}(E))^I$$

which is a homotopy between the two T'-structures on E.

~~So now it is clear that Vect(pt, t_0) = $\mathbb{N}$ is cofinal in Vect(pt; T) when T is connected. Because any element of Vect(pt; T) comes from a finite subset of T, and any finite subset~~

So it is now clear that

$$\mathrm{Vect}(pt;\, T) = \mathrm{Vect}(pt;\, t_0) = \mathbb{N}$$

when T is connected. Because any element of Vect(pt, T) comes from a finite subset of T, and any finite subset can be contracted to the basepoint.

Thus if we localize Vect(Y, T) with respect to Vect(pt, t_0) = $\mathbb{N}$, we get K(Y, T):

$$\boxed{K(Y;T) = \mathrm{Vect}(pt, t_0)^{-1}\,\mathrm{Vect}(Y;T)}$$

Now we should try to prove exactness of

$$K(Y;A) \longrightarrow K(Y;T) \longrightarrow K(Y;T/A, pt)$$

$$\mathrm{Vect}(Y;T) \qquad\qquad K(Y;T/A, pt) = K(Y;T/A)/K(Y;pt)$$

Suppose then we have $\alpha \in K(Y;T)$, $\beta \in K(Y;pt)$ with the same image $\bar\alpha = \bar\beta \in K(Y;T/A)$. ~~$K(Y;A) \to K(Y;pt)$~~ We want to show α comes from K(Y; A). Since $K(Y,A) \twoheadrightarrow K(Y,pt)$ I can assume β trivial; adding trivial elements to α I can suppose that α comes from a T-bundle $\xi = (E, \varphi)$ on Y. So what I get down to is a T-bundle $\xi = (E,\varphi)$ on Y such that as a T/A bundle it is homotopic to a trivial bundle. In other words, provided I lump together eigenspaces belonging to the points in A, I can deform all the eigenspaces into A.

<u>Situation:</u> E vector bundle on Y; φ is a T-structure on E such that the induced T/A structure is homotopic to the basepoint T/A-structure.

Let $\psi_t : \mathbb{C}^{T/A} \longrightarrow \mathrm{End}(E)$ be a one-parameter family such that $\psi_0 = \varphi$ rest. to functions constant on A, and $\psi_1 =$ evaluation on the basepoint.

Let U be a neighborhood of A. What I have to do is to deform the spectrum of φ which is outside of U into U.

Now ψ_t tells me what to do for spectral points away from A. Now what I want to is to modify ψ_t so that it will extend to T starting from φ, and pulling everything into U. Let θ_t be the homotopy I seek

~~Example: Let $\rho_t : \mathbb{C}^T \longrightarrow \mathrm{End}(E)$ pull everything to the basepoint. e.g.~~

$$\cancel{\rho_t(f) = f(tx) \qquad E = \mathbb{C}, \quad T = \mathbb{R}}$$

~~How to modify ρ_t so that it stops at $x = 1$.~~

$$\cancel{\theta_t(f) = f \max(}$$

~~?~~

Problem: Given a T-structure φ on E, and a deformation of the induced T/A structure to the basepoint, show that φ can be deformed into an A-structure.

Corresponding problem for the symmetric product. Given a divisor φ of degree d on T, and a deformation of the induced divisor on T/A to the basepoint, show that φ can be deformed into a divisor in A. To be very realistic think of φ as a map $Y \longrightarrow SP_d(T)$. One then wants to proceed by induction on the number of points outside of A.

$$\begin{tikzcd}
SP_d(A) \arrow[r, phantom, "\subset \cdots\cdots \subset \cdots\cdots \subset"] \arrow[d] & SP_d(T) \arrow[d, "f"] \\
SP_0(T/A) \arrow[r, phantom, "\subset \cdots\cdots \subset SP_{d-1}(T/A) \subset"] & SP_d(T/A)
\end{tikzcd}$$

Here one has that

$$SP_j(T/A) - SP_{j-1}(T/A) = SP_j(T-A)$$

and
$$f^{-1}SP_j(T/A) - f^{-1}SP_{j-1}(T/A) = SP_j(T-A) \times SP_{d-j}(A).$$

Thus f is a trivial bundle over the difference.

So it is clear now that the above problem is impossible, because a pair (φ, ψ) consisting of $Y \xrightarrow{\varphi} SP_d(T)$ and a null-homot. ψ of $Y \xrightarrow{f\varphi} SP_d(T/A)$ is an element of the homotopy-fibre of f. And this won't map to the honest fibre except in the limit.

Conjecture: One will be able to prove exactness of

$$K(Y; A) \longrightarrow K(Y; T) \longrightarrow K(Y; T/A, pt)$$

by Dold-Thom style arguments.

~~[illegible]~~

To be precise for T connected pointed finite complex put

$$K(Y; T_*) = \mathrm{Vect}(Y, T)/\mathrm{Vect}(Y, *)$$
$$= K(Y, T)/K(Y, pt).$$

Then every element here is represented by a T-structure on a trivial bundle over Y. In fact if we put

$$b(T_*) = \varinjlim_{n} \{T\text{-}\overset{\text{space of}}{\text{structures}} \text{ on } \mathbb{C}^n\}$$

then

$$K(Y, T_*) = [Y, b(T_*)].$$

Now by Dold-Thom argument, one ought to have a quasi-fibration

$$b(A_*) \longrightarrow b(T_*) \longrightarrow b(T/A_*).$$

Granted this, we get an exact sequence

$$\longrightarrow K^{-1}(Y; A_*) \longrightarrow K^{-1}(Y; T_*) \longrightarrow K^{-1}(Y; T, A) \longrightarrow$$
$$K^{0}(Y; A_*) \longrightarrow K^{0}(Y; T_*) \longrightarrow K^{0}(Y; T, A)$$

so that putting $T = C(A_*)$, one gets an isomorphism

$$K^{0}(SY; SA_*) = K^{0}(Y; A_*).$$

I hope to find a proof of the periodicity theorem along the following lines. Take $T = S^1$ in which case of T-structure on V is a unitary operators. Hence

$$b(S^1,*) = \varinjlim_n \{S^1 \text{ structures on } \mathbb{C}^n\} = U$$

where here $S^1 = \{|z|=1\}$ and $* = 1$. Now if I can prove

$$(*) \qquad \begin{array}{ccc} K(Y; S^0, *) & = & K(SY; S^1, *) \\ \| & & \| \\ K(Y) & & [SY, b(S^1,*)] \\ \| & & \| \\ [Y, \mathbb{Z} \times BU] & & [Y, \Omega U] \end{array}$$

then I have the periodicity thm. $\mathbb{Z} \times BU = \Omega U$.

Following the above ideas, one would like to establish $(*)$ by exhibiting a quasi-fibration

$$b(S^0,*) \longrightarrow b(I,*) \longrightarrow b(S^1,*).$$

However note this doesn't work, because

$$\left\{\begin{matrix} S^0 \text{ structures} \\ \text{on } V \end{matrix}\right\} = \coprod_{0 \le p \le \dim(V)} G_p(V) \qquad \text{(give 1-eigenspace)}$$

so

$$b(S^0,*) = \varinjlim_n \coprod_p G_p(\mathbb{C}^n) \qquad \text{(stabilizing by adding 0-eigenspace)}$$

$$= \coprod_p BU_p$$

which has the wrong homotopy type. This suggests however that we stabilize differently.

$I = [0,1]$. I-structure on V = self adjoint A $0 \le A \le 1$.

Map $I \longrightarrow S^1$
$t \longmapsto \exp(2\pi i t)$.

induces the exponential map

$$\begin{array}{ccc} \left\{\begin{array}{c} \text{self adj} \\ A,\ 0 \le A \le 1 \end{array}\right\} & \xrightarrow{\exp(2\pi i)} & \left\{\begin{array}{c} \text{unitary} \\ \text{operators} \end{array}\right\} \\ \text{on } \mathbb{C}^n & & \text{on } \mathbb{C}^n \\ \| \text{(notation)} & & \| \\ A(\mathbb{C}^n) & \xrightarrow{\quad f \quad} & U(\mathbb{C}^n) \end{array}$$

Now stabilize carefully. Consider the inductive system $n, m \longmapsto \mathbb{C}^n \times \mathbb{C}^m$ and form the inductive limit of $A(\mathbb{C}^n \times \mathbb{C}^m)$ where on increasing n adds to the zero eigenvalue and increasing m adds to the one eigenvalue. Then we get a limit map

$$\varinjlim A(\mathbb{C}^n \times \mathbb{C}^m) \xrightarrow{\quad f \quad} \varinjlim U(\mathbb{C}^n \times \mathbb{C}^m)$$

which might be a quasi-fibration. In effect fix a unitary matrix $\theta \in U(\mathbb{C}^n \times \mathbb{C}^n)$, and let K be the 1 eigenspace of θ. Then

$$f^{-1}(\theta) = \varinjlim_{m} \coprod G_p(K \times \mathbb{C}^m \times \mathbb{C}^m)$$

but where $G_p(K \times \mathbb{C}^m \times \mathbb{C}^m)$ goes into $G_{p+1}(K \times \mathbb{C}^{m+1} \times \mathbb{C}^{m+1})$.
Thus the limit is going to be $\mathbb{Z} \times BU$.

What does this approach to periodicity have to do with the Atiyah–Singer one which uses also the exponential map.

If one completes $\mathbb{C}^\infty \times \mathbb{C}^\infty$ into a Hilbert space and closes up $\varinjlim U(\mathbb{C}^n \times \mathbb{C}^n)$ in the uniform topology one gets a closed subgroup of unitary operators with essential spectrum 1. (Maybe the whole thing, but in any case one gets the same homotopy type U.)

Now I believe that the space of self-adjoint operators A with $\mu \leq A \leq \nu$ and essential spectrum $\{\mu, \nu\}$ is of the same homotopy type U.

Guess that the uniform closure of $\varinjlim A(\mathbb{C}^n \times \mathbb{C}^m)$ is the space of self-adjoint operators A with spectrum $[0,1]$, essential spectrum $\{0,1\}$ and which leave invariant up to compact operators the given splitting of $\mathbb{C}^\infty \times \mathbb{C}^\infty$, with the appropriate eigenvalues.

Thus if E is the projection operator assoc. to the $+1$ part of $\overline{\mathbb{C}^\infty \times \mathbb{C}^\infty}$, then I am considering all self adjoint operators A such that $A - E_0$ is compact. This is obviously contractible. Call this space $\mathcal{A}$.

The map f is now $\exp 2\pi i : \mathcal{A} \longrightarrow \mathcal{U}$. The fibre of f over 1 is the space of projectors E such that $E - E_0$ is compact. E is completely equivalent to its image which is a subspace of $V = H \oplus H$

"close" to $0 \oplus H$, hence $\operatorname{Im} E_0$ is the graph of a compact correspondence from H to H.

June 10, 1974. K-homology

If V is a unitary vector space and T is a space, then by a T-decomposition of V I will mean an orthogonal direct sum decomposition

$$V = \bigoplus_{t\in T} V_t$$

indexed by T. Let $D(V;T)$ be the set of T-decomp. of V. Then

$$D(V;T) = \varinjlim D(V;S)$$

where S runs over the cat. of finite sets over T.

~~Topology on $D(V,T)$:~~ ~~There is an evident notion of convergence for T-decompositions, at least for T-Hausdorff~~

For T-Hausdorff, there is an evident topology on $D(V,T)$:

~~[illegible] If $\sigma \in D(V,T)$ has support $t_1, \ldots, t_k$ [illegible] σ will.~~

Given $\sigma \in D(V,T)$ with support $t_1, \ldots, t_k$, a basic system of nbds of σ is obtained as follows. Take disj. nbds $U_1, \ldots, U_k$ of $t_1, \ldots, t_k$ resp., and a nbd of the decomposition $V = \bigoplus_{i=1}^{k} V_{t_i}$ in the space of splittings of V. etc.

<u>Topology on $D(V,T)$:</u> Three possibilities:

1) When T is Hausdorff one has an evident notion of convergence for T-decompositions

2) If S is a finite set, $D(V;S)$ is a disjoint union of flag manifolds of V. $S \mapsto D(V;S)$ is a functor

from the topological category of finite sets over T to spaces. One can topologize $D(V;T)$ as the inductive limit — give it the finest topology such that the maps

$$D(V;S) \times T^S \longrightarrow D(V;T)$$

are continuous. (Thus $D(V;T)$ is the contraction of $\begin{matrix} S \mapsto D(V;S) \\ S \mapsto T^S \end{matrix}$).

3) When T is compact, a T-decomp. of V is the same thing as a star-homomorphism

$$\mathbb{C}^T \longrightarrow \mathrm{End}(V).$$

This embeds $D(V;T)$ into the space of measures on T with values in $\mathrm{End}(V)$. One topologizes it using the weak topology on measures (so that $\delta_t \to \delta_{t_0}$ as $t \to t_0$)

For T compact at least, the above three possibilities ought to be equivalent. In this case $D(V;T)$ is <u>compact.</u>

<u>Unitary Vector bundles with T-decomposition.</u>

<u>Versions</u>: 1) A T-decomp. of a unitary bundle E over a space Y is a T-decomposition on each fibre which varies continuously, i.e. if $n = \mathrm{rank}(E)$, then the associated section of

$$\mathrm{Isom}(\mathbb{C}^n_Y, E) \times^{U_n} D(\mathbb{C}^n; T)$$

is continuous.

2) One has a topological groupoid whose objects are

unitary vector spaces with T-decomposition, where the T-decomp. is allowed to vary, and whose maps are unitary maps with topology. Precisely one takes the top. groupoid obtained by letting [redacted] U_n act on $D(\mathbb{C}^n;T)$, then disjoint union over $n \in \mathbb{N}$. A unitary bundle with T-decomp. then is a [redacted] torsor for this top. [redacted] groupoid.

The classifying space for this top. groupoid is

$$\coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T)$$

and the corresponding homotopy functor [redacted]

$$\mathrm{Vect}(Y;T) = \left[Y, \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T) \right]$$

assoc. to Y the homotopy classes of vector bundles with T-decomposition over Y. Here we call ξ, ξ' homotopic if there is an isom. of the underlying vector bundles [redacted] with respect to which the T-decompositions become homotopic (equivalently $\exists$ a η over $Y \times I$ restricting to ξ, ξ' at the ends).

[redacted] Now I want to form the K-theory out of $\mathrm{Vect}(Y;T)$. Suppose T connected with basepoint t_0. The problem as I see it is to represent the functor [redacted] $K(Y;T,t_0) = \mathrm{Vect}(Y;T)/\mathrm{Vect}(Y,t_0)$. [redacted]
I have the following idea: I will no longer be able to represent it by the classifying space of a top. groupoid, but rather a topological category.

[redacted] Other idea: Put $M(T) = \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n; T)$. Then the spaces I want is probably the quotient $M(T)/M(pt)$. Set-theoretically this should be $M(T-t_0)$. [redacted] It is reasonable to think that [redacted] $M(T;t_0)$ is stratified with $M(T-t_0)$ the disjoint of the strata. For example in the case of the symmetric product, it

$$SP(T,t_0) = \coprod_n SP_n(T) \Big/ \coprod_n pt = \varinjlim SP_n(T)$$

is stratified [redacted] by the closed sets

$$pt = SP_0(T) \subset SP_1(T) \subset \cdots\cdots\cdots$$

and

$$SP_n(T) - SP_{n-1}(T) = SP_n(T-t_0).$$

[redacted] Now $M(T-t_0)$ classifies bundles with $T-t_0$ decomposition. Thus maybe what I want is [redacted] $M(T;t_0)$ to be a topological category [redacted] with a filtration $F_n M(T;t_0)$ such that $F_n M(T;t_0) - F_{n-1} M(T;t_0)$ is the groupoid of $(T-t_0)$ decompositions of $\mathbb{C}^n$. Thus in some local sense a $M(T;t_0)$-bundle over a space Y will be family of $(T-t_0)$-bundles over a stratification. Suppose then I have a $F_n M(T;t_0)$-bundle over Y. ~~Presumably as necessary~~ Then I ought to able to write Y as a union $Y = A \cup B$, where on A I have a $F_{n-1}M(T;t_0)$-bundle, and on B I have a rank-n bundle with $T-t_0$ decomposition, and over $A \cap B$ I describe the specialization.

So given $\sigma \in D(V; T-t_0)$, $\dim(V) = n$, how do I specialize. Various possibilities. One should allow some of the support of σ to approach t_0. Think of being in the normal tube — one has the point where one is, its image on the lower strata, and the path joining the two.

Thus I should give a V of rank n with $(T-t_0)$-decomp. (where one is), a W of rank $m<n$ ———————. (where one ends), and a path between the two. This amounts to ....

Other possibility is to take as objects unitary vector spaces with T-action. The space of objects is then

$$\coprod_n D(\mathbb{C}^n;T).$$

and to define the spaces of maps as

$$\coprod_{m\le n}\left\{(\sigma,\tau,u)\;\middle|\;\begin{array}{l}\sigma\in D(\mathbb{C}^m;T),\quad \tau\in D(\mathbb{C}^n;T),\\ u \text{ unitary embeding of } \mathbb{C}^m \text{ in } \mathbb{C}^n\text{, compatible with } \mathbb{C}^T\text{-action}\\ \text{such that the orthogonal complement of the image has the}\\ \text{basepoint } \mathbb{C}^T\text{-action}\end{array}\right\}.$$

Thus

$$\mathrm{Hom}_{m,n} = D(\mathbb{C}^m;T)\times \mathrm{UnitEmb}(\mathbb{C}^m;\mathbb{C}^n)$$

$$= D(\mathbb{C}^m;T)\times U(n)/U(n-m)$$

Thus an object is a T-vector space, but a morphism is a unitary embedding

June 11, 1974

At the moment I am trying to construct a space $M(T, t_0)$ representing

$$K(Y; T, t_0) = \mathrm{Vect}(Y; T) / \mathrm{Vect}(Y; t_0)$$

where T is supposed compact connected with basepoint t_0.

I guess that $M(T, t_0)$ should be the classifying space of the following top. cat. The objects are unitary vector spaces with varying T-decomposition. The maps are unitary embeddings compatible with T-decomp. such that the complement has structure supported at the basepoint t_0.

To establish this I want to know how to think of maps of Y into the classifying space of this top. cat. I would suppose this involves stratifying Y

$$Y = \coprod_{n \geq 0} Y_n$$

and giving over Y_n a rank-n-bundle decomposed wrt $T - t_0$.

For example, consider a map $f : Y \longrightarrow SP(T) = \bigcup SP_n(T)$. Then put

$$Y_n = f^{-1}\left[SP_n(T) - SP_{n-1}(T)\right]$$

and we have a map $Y_n \longrightarrow SP_n(T - t_0)$. How well can I describe this stratification of Y?

Start with the monoid $\coprod_n SP_n(T)$, make $\coprod_n pt$ act on it. Then we get a simplicial space whose realization ought to be the desired thing over $SP(T)$, which described the stratification.

$\mathbb{N}$ acting on $\mathbb{N}$ is the ordered set $\mathbb{N}$. Its classifying space is the full simplex $\Delta(\mathbb{N})$ with vertices $0, 1, \ldots$ A map of Y into this is a partition of unity $\sum_{n \in \mathbb{N}} \rho_n = 1$ (which is finite if Y is compact), hence open sets $U_n = \rho_n^{-1}((0,1])$ in Y forming a covering.

$$\begin{tikzcd}
\coprod SP_n(T) \times \mathbb{N}^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \coprod SP_n T \times \mathbb{N} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod SP_n(T) \arrow[r] \arrow[d] & SP_\infty(T). \\
\mathbb{N} \times \mathbb{N}^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{N} \times \mathbb{N} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{N} &
\end{tikzcd}$$

Using the fact that realization commutes with products, one maybe can see that what sits over a point $\sum t_n = 1$ of $\Delta(\mathbb{N})$ is $SP_{i_0}(T)$ where i_0 is the first $n \ni t_{i_0} \neq 0$.

Let me try to understand a map $Y \longrightarrow SP_n(T)$. I will assume that I am given a neighborhood U of t_0 which is a cone

$$U = t_0 \cup_L L \times [0,1)$$

where L is the link of t_0 in T.

Let V be a vector space, and K a finite simp. cx. Then a decomposition of V with respect to K might be defined as a family F_σ of subspaces of V indexed by the simplices σ such that

(i) $$F_\sigma \cap F_\tau = \begin{cases} 0 & \text{if } \sigma \cap \tau = \phi \\ F_{\sigma\cap\tau} & \text{if } \sigma\cap\tau \neq \phi . \end{cases}$$

(ii) $$\varinjlim F_\sigma \xrightarrow{\sim} V$$

<u>Example</u> 1. Given a pt $x \in K$ put

$$F_\sigma = \begin{cases} 0 & x \notin \bar\sigma \\ V & x \in \bar\sigma \quad \text{i.e. } \mathrm{supp}(x) \subset \sigma \end{cases}$$

2. Given F_σ in V, F'_σ in V' get

$$F_\sigma \times F'_\sigma \quad \text{in} \quad V \times V'$$

3. If $V = \bigoplus_{x\in K} V_x$, put

$$F_\sigma = \bigoplus_{x \in \bar\sigma} V_x$$

Let $D(V;K)$ be the set of decompositions of V with respect to K. To make it into a poset. Idea is that if $F_\sigma(x)$ is what I defined in 1, then if x specializes to y, i.e. $\mathrm{Sup}(x) \supset \mathrm{sup}(y)$ one has

$$\mathrm{sup}(x) \subset \sigma \Longrightarrow \mathrm{sup}(y) \subset \sigma$$

or

$$F_\sigma(x) \leq F_\sigma(y)$$

Thus $D(V;K)$ becomes a poset by putting $\{F_\sigma\} \leq \{F'_\sigma\}$ if $F_\sigma \subset F'_\sigma$ for all σ. Intuitively under specialization from F to F' the number of eigenvalues in $\bar{\sigma}$ increases.

Generalization: K finite poset, have F_Z for each subcomplex Z of K such that

i) $F_\phi = 0$, $F_K = V$

ii) $Z \leq Z' \Rightarrow F_Z \leq F_{Z'}$

iii) $0 \longrightarrow F_{Z \cap Z'} \longrightarrow F_Z \oplus F_{Z'} \longrightarrow F_{Z \cup Z'} \longrightarrow 0$ exact

Example: $K =$ finite set S. Then a decomposition is just a splitting indexed by S: $V = \bigoplus_{s \in S} V_s$

Maps. Given a map $f: K \longrightarrow L$, $\sigma \mapsto f(\sigma)$, there is an induced map

$$f_* : D(V;K) \longrightarrow D(V;L)$$

defined by $$f_*(\{F_Z\})(Z') = F_{f^{-1}(Z')}.$$

Question: If f is a homotopy equiv., then is f_* also one?

Example: Given $f \leq g : K \longrightarrow L$, then for Z closed in L one has $x \in f^{-1}(Z) \Longleftrightarrow f(x) \in Z \Longleftarrow g(x) \in Z \Longleftrightarrow x \in g^{-1}(Z)$

so $\qquad f^{-1}(2) \supset g^{-1}(2)$

$$\Rightarrow \quad F_{f^{-1}(2)} \supset F_{g^{-1}(2)}.$$

Thus elements f, g in the same component of the poset $\underline{Hom}(K, L)$ induce homotopic maps. So it would seem out question reduces to

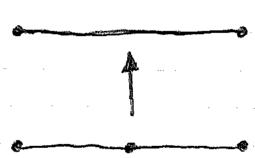
<u>Question</u>: If $f: K' \to K$ is barycentric subdivision, then is $f_*: D(V; K') \longrightarrow D(V; K)$ a homotopy equivalence?

<u>Example</u>: $K = \Delta(n)$, $\quad n = 1$ whence $\overset{0}{\bullet}\!\!\frac{\quad 01 \quad}{}\!\!\overset{1}{\bullet}$

and so an element of $D(V; \Delta(1))$ is just two subspaces V_0, V_1 of V such that $V_0 \cap V_1 = 0$.

Clearly $D(V; \Delta(n))$ has the initial object $F_Z = 0$ for all $Z < \Delta(n)$. (this argument holds more generally if K is irreducible).

Next example - subdivision.



$V_0 \subset V \supset V_1 \qquad V_0 \cap V_1 = 0$

Doesn't seem to be anyway of spreading this apart.

Example: Let V be a f.d. H.S. and let $S(V)$ be the set of its subspaces including $0, V$. Then $S(V)$ is a space and a poset, and these two structures are compatible (if $W_n \to W'$, $V_n \to V'$ and $W_n \subset V_n$, then $W' \subset V'$). ~~[illegible]~~ Make $S(V)$ into a top. cat by top. the morphisms so that $\{x \leq y\} \subset S(V) \times S(V)$ is an embedding. Claim $BS(V)$ can be identified with the space of self-adjoint operators A on V with $0 \leq A \leq 1$.

A point of $BS(V)$ is a pair consisting of a p-simplex $W_0 < \cdots < W_p$ in $S(V)$, and an interior point of $\Delta(p)$, i.e. a sequence $0 < t_1 < \cdots < t_p < 1$. To this pair associate the self-adjoint operator A having eigenvalue

$$(*) \quad \begin{array}{lll} 0 & \text{on} & W_0 \\ t_1 & \text{on} & W_1 \ominus W_0 \\ & & \\ t_p & \text{on} & W_p \ominus W_{p-1} \\ 1 & \text{on} & V \ominus W_p \end{array}$$

Conversely given A, let $t_1 < t_2 < \cdots < t_p$ be the eigenvalues of A which are not $0, 1$ and set

$$W_0 = 0 \text{ eigenspace of } A$$
$$W_i = \text{part of } V \text{ where } A \leq t_i I$$

whence we have a 1-1 correspondence between points of $BS(V)$ and operators A, $0 \leq A \leq 1$.

Define a map $BS(V) \longrightarrow \{A\}$ by sending $\{W_0 < \cdots < W_p\} \times \{0 \leq t_1 \leq \cdots \leq t_p \leq 1\}$ into the operator described by $(*)$. This is a bij. cont. map between two compact spaces, etc.

Question: Suppose V is a f.d. H.s. Then I have defined before a space $D_u(V;|K|)$ whose points are orth. decompositions (u=unitary)

$$\xi: \quad V = \bigoplus_{\lambda\in|K|} V_\lambda$$

Now I propose to map this to the simplicial gadget $D(V;K)_\lambda$ by putting

$$\xi_Z = \bigoplus_{\lambda\in Z} V_\lambda$$

Can you define a space $D_u(V;K)$, better a pospace, whose points are orthogonal decompositions

$$V = \bigoplus V_\sigma$$

such that

$$D_u(V;|K|) \longrightarrow BD_u(V;K)$$

is a heq?

<u>Idea</u>: Identify an orth decomp. $V = \oplus V_\sigma$ with a system $\xi: Z \longmapsto \xi_Z \subset V$ such that

$$\xi_{Z\cup Z'} = \xi_Z \overset{\xi_{Z\cap Z'}}{\oplus} \xi_{Z'}$$

where ξ_Z and $\xi_{Z'}$ are $\perp$ mod $\xi_{Z\cap Z'}$. Then make decompositions into a space in the obvious way: disjoint union of flag manifolds, disjoint union over functions $\sigma \mapsto n_\sigma \in \mathbb{N}$ $\ni$ $\Sigma n_\sigma = \dim V$. Finally put the ordering in by saying $\xi \leq \xi'$ if $\xi_Z \subset \xi'_Z$ for all Z.

A point of $BD_u(V;K)$ will be a pair consisting

of a p-simplex $\xi_0 < \cdots < \xi_p$ and $0 < t_1 < \cdots < t_p < 1$.
A point of $D_u(V; |K|)$ is a decomp. $V = \bigoplus_{\lambda \in |K|} V_\lambda$.

What I know already is that

$$B\,D_u(V;\ 0<1) = B(\text{ordered set of subspaces of } V)$$
$$= D_u(V;\ [0,1])$$

I also know that

$$D_u(V;\ 0 < 01 > 1) = \text{ordered set of layers in } V.$$

which has the same realization

Consider the map $D_u(V; |K|) \longrightarrow D_u(V; K)$ which sends ξ to $\xi_\sigma = \bigoplus_{\lambda \in \sigma} \xi_\lambda$. The fibre is the product over $\sigma \ni \xi_\sigma \neq 0$, of the set of possible decompositions of ξ_σ with respect to the points of σ. ?

~~Suppose that one has~~