May 1, 1974. Lang problem

I seem to have arrived at the following situation. I recall that I can identify the category $P(k[F])$ with the cat. whose objects are k-vector spaces V and whose morphisms are algebraic k_0-linear maps. Also $P(k[F,F^{-1}])$ then becomes the category where I invert the maps $F: V \to \Phi V$.

I already understood that if I have an alg. k_0-linear map $\theta: V \to W$ between spaces of the same dimension which is onto (i.e. $S(V^*) \leftarrow S(W^*)$ injective finite), then $\mathrm{Ker}\,\theta \in P(k_0)$. In this way I got the residue map for the skew-field D of fractions of $k[F,F^{-1}]$ relative to $k[F,F^{-1}]$. What I want now is to do the same for $k[F,F^{-1}]$ relative to $k[F]$, but to have the residue in $P(k_0)$, not just $P(k)$.

The idea will be to use transversality. Given $\theta: V \to W$ algebraic k_0-linear bijective, I want to perturb it slightly into $\theta - \varepsilon$ which will be separable. Then $\mathrm{Ker}(\theta-\varepsilon)$ will be in $P(k_0)$.

An essential feature of the perturbation ε will be that it is trivial near infinity, hence ~~also will be properly homotopic to zero~~ $\theta, \theta-\varepsilon$ will be properly homotopic in some sense.

__Examples:__ Suppose I start with θ a Frobenius map $\theta: V \to W$, $\theta(\lambda v) = \lambda^q \theta(v)$, θ bijective.

Then I can take ε to be an isomorphism $\varepsilon : V \xrightarrow{\sim} W$. So in this case $\operatorname{Ker}(\theta - \varepsilon)$ is a k_0-subspace of V freely generating V.

Next suppose that I take ε to be ~~an isom.~~ of the form $V \underset{\sim}{\xrightarrow{\sigma}} \Phi W \underset{bij}{\xleftarrow{F}} W$

$$\varepsilon = F^{-1}\sigma$$

$$\theta - \varepsilon = \theta - F^{-1}\sigma : V \longrightarrow W$$

$$F(\theta - \varepsilon) = F\theta - \sigma : V \longrightarrow FW$$

$$\operatorname{Ker}(\theta - \varepsilon) = \operatorname{Ker} F(\theta - \varepsilon) = \operatorname{Ker}(F\theta - \sigma)$$

is a vector space over $\mathbb{F}_{q^2}$:

$$(\theta - \varepsilon)(\lambda v) = \lambda^{q}\theta(v) - \lambda^{-q}\varepsilon(v)$$
$$= \lambda^{-q}\left(\lambda^{2q}\theta(v) - \varepsilon(v)\right).$$

So obviously this kind of perturbation is not allowed. Screwy situation. (The problem is that F^{-1} is not smooth, i.e. it doesn't have a derivative, hence this perturbation will not be etale.).

~~The problem then will be to understand finitely~~

~~So next suppose~~

Thus I can conclude that if θ is homogeneous of degree q, then the possible perturbations ε are the different isomorphisms of V with W. This seems to generalize: If one writes $\theta = \sum a_n F^n$ $\quad a_n : \Phi^n V \to W$ and if $a_n = 0$ $\quad n \leq 0$, then $\varepsilon = \varepsilon_0 + \varepsilon_1 F + \cdots$

can be arbitrary $\ni$ i) ε_0 is an isom. ($\Leftrightarrow$ Θ-ε stable) ii) ε dominated by Θ so that $\frac{\varepsilon}{\Theta}$ negligible at ∞. Condition ii) is linear, so that ε should be retractible to ε_0 directly. Thus it would seem again that the "space" of perturbations allowed is the different isos. of V and W.

(Digression. You seem to be assuming that one can identify the solutions of

$$x^{q^2} + t x^q - \varepsilon x = 0 \qquad \varepsilon \text{ fixed } \neq 0$$

for different values of t.

Take $\varepsilon = 1$. Want $y = f(x)$, so that

$$y^{q^2} + t y^q - y = f(x)^{q^2} + t f(x)^q - f(x) \quad (\equiv 0 \bmod x^{q^2} - x)$$

$$(f(x) = \alpha x + \beta x^q) \quad = \alpha^{q^2} x^{q^2} + \beta^{q^2} x^{q^3} + t\alpha^q x^q + t\beta^q x^{q^2} - (\alpha x + \beta x^q)$$

$$\equiv x(\alpha^{q^2} + t\beta^q - \alpha) + x^q(\beta^{q^2} + t\alpha^q - \beta)$$

$\therefore$ want

$$\alpha^{q^2} - \alpha + t\beta^q = 0$$
$$\beta^{q^2} - \beta + t\alpha^q = 0$$

These solutions form an $\mathbb{F}_{q^2}$ vector space of dim. 1. Thus to get from $t=0$ to $t=t_0$, one would need to choose a solution of $x^{q^2} + t_0 x^q - x$, algebraically in terms of t_0. IMPOSSIBLE

~~Consider now the critical case, i.e. whe~~

Psychology: I can replace $GL_n(k[F,F^{-1}])$ by the category of k-vector spaces of dim. n and algebraic k_0-linear bijections $\theta: V \to W$. From this perturbation process I want to associate to each such arrow a k_0-vector space depending additively as θ composes.

Thus suppose I have

$$U \xrightarrow{\varphi} V \xrightarrow{\theta} W$$

Now if φ' is a perturbation of φ, and θ' is a perturb. of θ, then one has an exact sequence

$$0 \to \mathrm{Ker}(\varphi') \longrightarrow \mathrm{Ker}(\theta'\varphi') \longrightarrow \mathrm{Ker}\,\theta' \longrightarrow 0.$$

So I need a way of identifying $\theta'\varphi'$ with $(\theta\varphi)'$. e.g. if $U=V=W$ and φ, θ have no constant term, I then have to relate

$$(\theta-1)(\varphi-1) \quad \text{with} \quad \theta\varphi - 1$$

Formula:

$$\begin{aligned}(\theta-1)(\varphi-1) &= \theta\varphi - \theta - \varphi + 1 \\ &= (\theta\varphi-1) - (\theta-1) - (\varphi-1).\end{aligned}$$

Since we don't yet know how to remove lower terms from a perturbation, this will not go anywhere.

Consider next the critical case of interest in which $\theta: V \longrightarrow W$ is linear, i.e. of the form $\theta = a_0 + a_1 F$. I know then that there is a canonical splitting of θ as a direct sum of $\theta': V' \longrightarrow W'$, $\theta'': V'' \longrightarrow W''$ such that a_0' is a unit, a_1'' is a unit, ~~I want to classify the possible perturbations of ε: V~~ and $(a_0')^{-1} a_1'$ is nilpotent, $(a_1'')^{-1} a_0''$ is nilp. (not exactly: $1 - aF$ has an inverse $1 + aF + a^{1+\sigma} F^2 + \ldots$ which is a poly in F when $a a^{\sigma} a^{\sigma^2} \ldots = 0$. ~~diagonally~~ Triangularizing a, this implies the diagonal entries are 0, hence a is nilpotent. The converse is ~~also probably~~ true, for if a is nilp, its kernel?).

I want to classify the possible perturbations $^{\varepsilon}$ of θ. Now as ε will be dominated by θ, this means first of all that on V' where a_0 is a unit.

It will first be necessary to get the basic splitting result in shape.

$$(a_0 - a_1 F)\left(\sum b_n F^n\right) = 1$$

$$a_0 b_n - a_1 b_{n-1}^{\sigma} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\left(\sum b_n F^n\right)(a_0 - a_1 F) = 1$$

$$b_n a_0^{\sigma^n} - b_{n-1} a_1^{\sigma^{n-1}} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

So one gets

$$
\begin{aligned}
b_0 a_0 b_n &= b_0 a_1 b^{\sigma}_{n-1} \\
&= b_1 a_0^{\sigma} b^{\sigma}_{n-1} = b_1 a_1^{\sigma} b^{\sigma^2}_{n-2} = \cdots = 0 \qquad n<0 \\
&= b_n a_0^{\sigma^n} b_0^{\sigma^n} \qquad n \geq 0 \\
&= b_n a_1^{\sigma^n} b_{-1}^{\sigma^{n+1}} + b_n \\
&= b_{n+1} a_0^{\sigma^{n+1}} b_{-1}^{\sigma^{n+1}} + b_n = \cdots = b_n \qquad n \geq 0.
\end{aligned}
$$

$$
\therefore \quad \boxed{b_0 a_0 b_n = \begin{cases} b_n & n \geq 0 \\ 0 & n<0 \end{cases}}
$$

similarly

$$
\boxed{b_n (a_0 b_0)^{\sigma^n} = \begin{cases} b_n & n \geq 0 \\ 0 & n<0 \end{cases}}
$$

Thus $b_0 a_0$ is a projector on V
$a_0 b_0$ ——————— W

and as $a_1 b_0^{\sigma} a_0^{\sigma} = a_0 b_1 a_0^{\sigma} = a_0 b_0 a_1$, one has

$$
\boxed{(a_0 - a_1 F)\, b_0 a_0 = a_0 b_0 (a_0 - a_1 F)}
$$

Thus we get a splitting $V = V' \oplus V''$
$W = W' \oplus W''$

stable under $\Theta = a_0 - a_1 F$. Observe that

$$
V' = \mathrm{Im}(b_0) \qquad W'' = \mathrm{Ker}(b_0)
$$

~~What happens: this is that A~~ Now after we get this splitting, we can look separately at the two pieces.

~~On the~~ $V = V'$; here $b_0 a_0 = id_V$, so a_0 is an isomorphism with inverse b_0 identifying V' and W' via this iso. amounts to replacing $a_0 - a_1 F$ by $1 - a_0^{-1} a_1 F$. For $1 - \alpha F$ to have an inverse of the form $\sum'_{n \geq 0} b_n F^n$ means that

$$(\alpha F)^n = \alpha \alpha^\sigma \alpha^{\sigma^2} \cdots \alpha^{\sigma^{n-1}} F^n$$

is zero for n large.

$V = V''$; here $b_n = 0$ $n \leq 0$ so $a_0 - a_1 F$ has an inverse $\sum b_n F^n$ $n < 0$. Thus a_1 is an isomorphism with inverse $-b_{-1}^{\sigma}$.

Question: What was Nil in Waldhausen's description? A diagram $V_{-1} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V_0$ such that $\alpha - F^{-1}\beta$ is an isomorphism.

Thus one sees that a Nil object is the same thing as a degree ≤ 1 bijection $\theta = a_0 - a_1 F : V \to W$, and that the K-theory of such things is two copies of that of k corresponding to V', V''.

To trivialize this object of Nil, I have to write it as a quotient of something where a_0, a_1 are injective.

Summary:

Consider maps $\theta : V \to W$ where θ is k_0-linear and bijective. I think of θ as a fibre heq between the sphere bundles of V and W. Such a map θ can be identified with an isom. of $k[F,F^{-1}]$-mods

$$\theta^t : k[F,F^{-1}]\otimes_k V^* \xleftarrow{\sim} k[F,F^{-1}]\otimes_k W^*.$$

~~[illegible]~~

~~Consider instead maps $\theta : V \to W$ which are alg. k_0-linear and bijective. These correspond to maps~~

~~$$k[F]\otimes_k V^* \longleftarrow k[F]\otimes_k W^*$$~~

~~which are injective and whose cokernel is F-nilpotent~~

If I try to lift θ^t to be an isom. of $V + W$, I meet two obstructions - one to extending θ^t to a map $k[F]\otimes V^* \longleftarrow k[F]\otimes W^*$, and the other to extending it to a map $k[F^{-1}]\otimes V^* \longleftarrow k[F^{-1}]\otimes W^*$.

To simplify I consider θ of the form $a_0 - a_1 F : V \to W$. Then I get exactly the Nil category of Waldhausen's theory. Top. to consider bundles V, W and a fheq $\theta : V \to W$ between them amounts to looking at the fibre of $BU \times BU \to BG$ which is $BU \times G/U$. In this algebraic game $G/U \sim BU$ so that the map to $G/U \to BU$ is $\Psi^q - 1$. Thus I get ~~two~~ copies of the K-theory of k which checks.

Now if I have a nil object $a_0 - a_1 F : V \to W$, I have seen that it splits into a prime part where a_0 is an isomorphism and $(a_0^{-1}a_1)^{1+\sigma+\dots+\sigma^n} = 0$ n large, and into a double prime part where $a_1 : \Phi V \to W$ is an isomorphism, ..etc. Thus analysis of nil gives me that any $\theta : V \to W$ determines $\begin{cases} V = V' \oplus V'' \\ W = V' \oplus \Phi V''. \end{cases}$

Now to get down to the finite field k_0, I have in addition to gives an isomorphism between V and W

<u>Question</u>: Consider the category of k-vector spaces V equipped with a self-bijection of the form $\theta = a_0 - a_1 F : V \to V$. This is an Artinean abelian category. What are the simple objects?

This looks a bit too rigid, like asking for the structure of vector spaces with two non-commuting idempotents.

Now suppose we allow stabilization, what I mean, is that I will try to find exact sequences

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "id"] & Z \arrow[r] \arrow[d, "\tilde{\theta}"] & V \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Z \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

?

What the answer should involve is the following examples:

α) $\theta =$ linear isom. : $V \to V$ i.e. $a_1 = 0$

β) $\theta = -a_1 F$: $V \to V$ i.e. $a_0 = 0$.

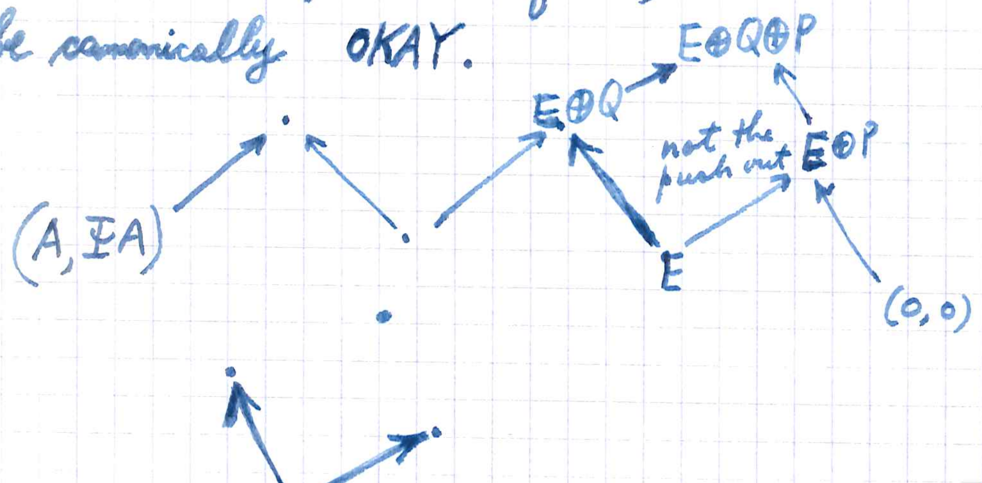
γ) hyperbolic case:

$$V = V' \oplus V''$$
$$= W' \oplus W''$$

I want $V' = W''$, $W' = V''$.

So here's where I am. I start with a $\theta : V \to W$ in Nil, and then I want to find exactly what I need in the way of an isomorphism of V with W so that θ takes a standard form and I can get a k_0-vector space of finite dimension over k_0.

Suppose $\theta : V \to W$. What would I mean by a stable isomorphism of V, W? Want elem. autos. to be canonically OKAY.



May 7, 1974:

Review the structure of the Grassmannian of p planes in $(p+q)$-space. If W is a fixed q plane I get strata

$$Z_h = \{A^p \mid \dim(A \cap W) \geq h\}$$

$$Z_0 \supset Z_1 \supset Z_2 \supset \cdots\cdots$$

and the normal space to Z_h at A is $\mathrm{Hom}(A\cap W, V/A+W)$.

One has a map

$$\begin{array}{ccc} Z_h - Z_{h+1} & \longrightarrow & G_h(W) \times G_h(V/W) \\ A & \longmapsto & (A\cap W,\ V/A+W) \end{array}$$

and the fibre thru A is the affine space of splittings of

$$0 \longrightarrow W/A\cap W \longrightarrow A+W/A\cap W \longrightarrow A+W/W \longrightarrow 0$$

If I choose a splitting $V = W \oplus (V/W)$, then A can be viewed as a correspondence from V/W to W, hence as a map from $A+W/W$ to $W/A\cap W$.

May 9, 1974

Given two vector bundles E, F over a smooth manifold X, say rank $E =$ rank F, I can choose a generic map $f: E \to F$. (Generic means that if one stratifies $\mathrm{Hom}(E,F)$ according to the action of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ on the fibres, then f as a section of $\mathrm{Hom}(E,F)$ is transversal to the strata.) This means that at each point x of X the derivative map

$$T(x) \longrightarrow \mathrm{Hom}(\mathrm{Ker}\, f(x), \mathrm{Cok}\, f(x))$$

is onto.

From f we get a stratification $X = X_0 \supset X_1 \supset \cdots\cdots$ with $X_p = \{x \mid \dim \mathrm{Ker}\, f(x) \geq p\}$. Over $X_p - X_{p+1}$ we have that f has constant rank p, hence classified by a map into $BU_p \times BU_{r-p} \times BU_p$; these pieces corresp. to Ker, Im, Cok of f.

Idea over $BU_r \times BU_r$ I form the fibre space with fibre $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r)$, and I stratify this fibre space according to the rank stratification of $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r)$. I want to recover the stratified space as some sort of topological category which might make sense for discrete rings.

Question: Suppose one has $Y \overset{i}{\subset} X \overset{j}{\supset} U$ as usual. Then is there some analogue of Artin's theorem but for homotopy types? What I want is something which recovers X up to homotopy from Y, U and something additional.

The extra thing needed is something like a boundary for $\mathcal{U}$. Thus if Y is a submanifold of X, then a tubular neighborhood N of Y is X plays the role of a $\partial\mathcal{U}$, and X is up to homotopy the pushout

$$\begin{tikzcd} \partial\mathcal{U} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

Maybe I should think of $\partial\mathcal{U}$ as a correspondence between Y and $\mathcal{U}$.

Now can such a pushout square be constructed equivariantly?

Let E, F now denote two complex vector spaces of dimension n, and let $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ act on $X = \mathrm{Hom}(E,F)$ in the usual way. Then we stratify X according to dim Ker.

$$X = X_0 \supset X_1 \supset \cdots\cdots \supset X_n = 0,$$

hence I get a stratification of the classifying topos of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ acting on X.

<u>Question</u>: What is the homotopy significance of a stratification?

In the case of $Y \subset X$, we get the following exact sequences in coh.

$$\longrightarrow H^*(X,\mathcal{U}) \longrightarrow H^*(X) \longrightarrow H^*(\mathcal{U}) \longrightarrow$$
$$\longrightarrow H^*(X,Y) \longrightarrow H^*(X) \longrightarrow H^*(Y) \longrightarrow$$

<u>Example</u> $n=1$. Here E, F are lines $\mathrm{Aut}(E) = \mathrm{Aut}(F) = \mathbb{C}^*$.
$X = \mathrm{Hom}(E,F) \simeq \mathbb{C}$ with $\mathbb{C}^* \times \mathbb{C}^*$ acting $(\lambda,\mu)u = \mu u \lambda^{-1}$.
$Y = 0$, $U = \mathrm{Iso}(E,F)$. So the possibilities are

$$X_G \sim BU_1 \times BU_1$$

$$U_G \sim BU_1 \qquad Y_G \sim BU_1 \times BU_1$$

The inclusions
$U_G \subset X_G$ is $\sim \Delta$, and of $Y_G \hookrightarrow X_G$ is identity.

$$\begin{tikzcd} & U_G \arrow[d, phantom, "\cap"] \\ Y_G \arrow[r, hook] & X_G \end{tikzcd} \qquad \begin{tikzcd} & BU_1 \arrow[d] \\ BU_1 \times BU_1 \arrow[r] & BU_1 \times BU_1 \end{tikzcd}$$

So it is clear that $(\partial U)_G$ has to be $\sim BU_1$.

<u>Next</u> $n=2$. $X = X_0 \supset X_1 \supset X_2 = \{0\}$ Here $X - X_2$ is 4-space $\mathrm{Hom}(E,F)$ minus the origin, hence $\sim S^7$. $X_1 - X_2$ is the set of maps of rank 1 $\sim P_1E \times S^1 \times P_1F$.

$$\begin{tikzcd} & (X - X_1)_G \arrow[d] \\ (X_1 - X_2)_G \arrow[r] & (X - X_2)_G \end{tikzcd}$$

[redacted]

$X_1 - X_2$ = maps $u : E \to F$ of rank 1 is a submanifold of codimension 1 in $X - X_2$, [redacted] ~~and the normal bundle~~ [redacted] whose normal space at u is $\mathrm{Hom}(\mathrm{Ker}(u), \mathrm{Cok}(u))$. Note that the stabilizer of u in $G = \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ acts transitively on the non-zero elements of $\mathrm{Hom}(\mathrm{Ker}(u), \mathrm{Cok}(u))$. The stabilizer of (u, v), where $v : \mathrm{Ker}(u) \xrightarrow{\sim} \mathrm{Cok}(u)$, is the subgroup of autos. of

$$0 \to \mathrm{Ker}\, u \to E \to \mathrm{Im}\, u \to 0$$

$$0 \to \mathrm{Im}\, u \to F \to \mathrm{Cok}\, u \to 0$$

inducing the identity on $\mathrm{Ker}(u)$, $\mathrm{Im}(u)$, $\mathrm{Cok}(u)$. which is a unipotent group.

Now $(X_1 - X_2)_G$ is a submanifold of $(X - X_2)_G$, hence there has to be a map up to homotopy from the normal [redacted] bundle minus zero section to $(X - X_1)_G$. Now

$$(X - X_1)_G \sim BU_2$$

and [redacted] $\nu - 0_{\text{section}}$ = pairs (u, v) where $u \in X_1 - X_2$ and v is an isomorphism of $\mathrm{Ker}(u)$ with cokernel u.

$\cong$ G/stabilizer of [redacted] a fixed pair (u_0, v_0).

~~So in this example it appears that the normal~~ [redacted] ~~bundle - zero section~~ [redacted] ~~is homotopy equivalent to $X_0 - X_1$~~

Check this carefully.

Let $X = \mathrm{Hom}(E,F)$ stratified by

$$X_p = \{u \mid \dim \mathrm{Ker}(u) \geq p\}.$$

Now $X_1 - X_2$ is a codimension 1 submanifold of $X - X_2$. Let ν be the normal bundle, and ν' the normal bundle minus zero section. Up to homotopy one has a map

$$\nu' \longrightarrow (X - X_1)$$

Description of the map. An element of ν' is a pair (u,v) where $u: E \to F$ has $\dim(\mathrm{Ker}\, u) = 1$ and where $v: \mathrm{Ker}(u) \to \mathrm{Cok}(u)$ is an isomorphism. Now choosing hermitian positive definite forms on E, F we can select continuously in u an orthogonal complement to $\mathrm{Ker}(u)$ in E and $\mathrm{Im}(u)$ in F, and hence we can extend v to a map $\tilde{v}: E \to F$, and we can then form $u + v$ which is an isomorphism of E with F.

The map in the other direction: Given metrics on E and F, and an isom. $\theta: E \to F$, one can speak of the eigenvalues of the form $e \mapsto \|\theta e\|^2$, and restrict to the open subset of isos. where there is a single minimum eigenvalue. This gives us a splitting $E = L \oplus L^\perp$, from which we can get a pair (u,v).

[redacted]

Summary: Here $G = \mathrm{Aut}\,E \times \mathrm{Aut}\,F$ acts on $X = \mathrm{Hom}(E,F)$ and I want to understand the homotopy types of [redacted] $X - X_2$. Now $X_1 - X_2$ is a submanifold of codim 1; it consists of maps $u : E \to F$ [redacted] with $\dim \mathrm{Ker}(u) = 1$. ν = normal bundle of $X_1 - X_2$ in $X - X_2$; the normal space at $u \in X_1 - X_2$ is $\mathrm{Hom}(\mathrm{Ker}(u), \mathrm{Cok}(u))$. [redacted] The non-zero part of ν consists of pairs (u, v) with $u : E \to F$ in $X_1 - X_2$ and $v \in \mathrm{Isom}(\mathrm{Ker}\,u, \mathrm{Cok}\,u)$. Over this, denote it ν', we can consider the bundle $\tilde{\nu}'$ consisting of u, v, and complements to $\mathrm{Ker}\,u$, $\mathrm{Im}\,u$. Then $\tilde{\nu}'$ is an affine bundle over ν' and we have a map

$$\begin{aligned} \tilde{\nu}' &\longrightarrow X - X_1 \\ u, v &\longmapsto u + v \end{aligned}$$

So what we seem to get for the homotopy type is the following (after forming fibre over BG):

$$\nu'_G \sim \tilde{\nu}'_G \sim \left(P_1 E \times \mathrm{Iso}(E,F)\right)_G = \left(\frac{U(n)}{U(1) \times U(n-1)} \times U(n)\right)_{U(n) \times U(n)}$$

$$\sim \underset{\mathrm{Ker}\,u \sim \mathrm{Cok}\,u}{BU_1} \times \underset{\mathrm{Im}\,u}{BU_{n-1}}$$

$$(X_1 - X_2)_G \sim \left([redacted] \frac{U(n)}{U(1)} \overset{U(n-1)}{\times} U(n)/U(1) \right)_{U(n) \times U(n)}$$

$$\sim \underset{\mathrm{Ker}\,u}{BU_1} \times \underset{\mathrm{Im}\,u}{BU_{n-1}} \times \underset{\mathrm{Cok}\,u}{BU_1}$$

Therefore what I seem to get is the picture

$$\begin{tikzcd} BU_1 \times BU_{n-1} \arrow[r, "\oplus"] \arrow[d, "\Delta \times id"] & BU_n \arrow[d] \\ BU_1 \times BU_1 \times BU_{n-1} \arrow[r] & (X - X_2)_G \end{tikzcd}$$

push-out

Suppose now I take a point u in $X_1 - X_2$, and a θ in $X - X_1$. Is there a good notion of specialization map from θ to u?

One possibility is to split E into $L \oplus L'$ with L of dimension 1, such that $\operatorname{Ker} u = L$ and $u =$ the composite $E \xrightarrow{pr} L' \xrightarrow{\theta} F$. Note that u itself specifies $\operatorname{Ker}(u)$ and $\operatorname{Im}(u)$. Hence

$$L' = \theta^{-1} \operatorname{Im}(u)$$

So start again. Suppose given $\theta : E \xrightarrow{\sim} F$ and $u : E \longrightarrow F$ with $\dim \operatorname{Ker}(u) = 1$. Suppose that the composite

$$\operatorname{Ker}(u) \longrightarrow E \xrightarrow{\theta} F \longrightarrow \operatorname{Cok}(u)$$

is an isomorphism. Then $\theta^{-1}\{\operatorname{Im}(u)\}$ is a complement for $\operatorname{Ker}(u)$, for if $\alpha \in \operatorname{Ker}(u) \cap \theta^{-1}\{\operatorname{Im}(u)\}$, then $\theta(\alpha) \in \operatorname{Im}(u)$ so its proj. in $\operatorname{Cok}(u)$ is $0 \Rightarrow \alpha = 0$. Thus we get splittings

$$E = \operatorname{Ker}(u) \oplus \theta^{-1}\{\operatorname{Im}(u)\}$$

$$F = \theta \operatorname{Ker}(u) \oplus \operatorname{Im}(u)$$

stable under θ, and one can ask now that

$u = \theta$ on $\theta^{-1}\{\mathrm{Im}(u)\}$. It follows that $\theta^{-1}u$ is a projection operator, in fact that

$$\boxed{u\theta^{-1}u = u}$$

which forces ~~both~~ $\theta^{-1}u$ and $u\theta^{-1}$ to be projectors. Conversely assume that $u\theta^{-1}u = u$, whence we get

$$\mathrm{Ker}(u) \subseteq \mathrm{Ker}(\theta^{-1}u) \subseteq \mathrm{Ker}(u)$$

$$\mathrm{Im}(u) \subseteq \mathrm{Im}(u\theta^{-1}) \subseteq \mathrm{Im}(u)$$

hence direct sum decompositions

$$E = \mathrm{Ker}(u) \oplus \overset{\mathrm{Im}(\theta^{-1}u)}{\overset{\parallel}{\theta^{-1}\mathrm{Im}\,u}}$$

$$F = \underset{\theta\,\mathrm{Ker}(u)}{\underset{\parallel}{\mathrm{Ker}(u\theta^{-1})}} \oplus \mathrm{Im}\,u$$

etc.

It appears therefore that the set of specializations of an isomorphism θ, are the same as the set of projectors in E.

Summary: Let E, F be two bundles over X of the same rank. I can then form over X the bundle with fibre $\mathrm{Hom}(E,F)$, and I can stratify this according to dim Ker. In this way I get over X various spaces and pairs of spaces whose homotopy type it would be nice to know. [redacted]

~~If $X = BU_n \times BU_n$ (and E, F) are the canonical bundles, then~~

So call Z the total space and let

$$Z = Z_0 \supset Z_1 \supset Z_2 \supset \cdots\cdots$$

be the stratification. Then I will consider the problem of constructing inductively $Z - Z_p$. The first point is that $Z_p - Z_{p+1}$ is a submanifold of $Z - Z_{p+1}$ and hence from the homotopy viewpoint, one has a pushout square

$$\begin{array}{ccc} \nu' & \xrightarrow{\alpha} & Z - Z_p \\ \downarrow & & \downarrow \\ Z_p - Z_{p+1} & \longrightarrow & Z - Z_{p+1} \end{array}$$

where ν' = normal bundle minus zero section.

Suppose now that I try to understand α which is defined only up to homotopy in some senses. [redacted]
Recall $Z_p - Z_{p+1} = \{u : E \to F \mid \dim(\mathrm{Ker}\, u) = p\}$, and that a normal vector to [redacted] $Z_p - Z_{p+1}$ at u is a homo. [redacted] $v : \mathrm{Ker}(u) \to \mathrm{Cok}(u)$. Now [redacted] to the pair u, v I want to associate $\alpha(u,v) \in Z - Z_{p+1}$.

~~The method will be to~~ [redacted] ~~choose complements for Ker(u) and~~ [redacted] Im ~~(u) and, then to lift v to a map~~

~~First let $W = \mathrm{Ker}(v$ on $\mathrm{Ker}(u))$, $W' = \mathrm{Im}(v)/\mathrm{Im}(u)$ in $\mathrm{Cok}(u)$. Then replacing E by E/W, F by W' we will arrange that $\alpha(u,v)$ is an isom.~~ [redacted]

Choose complements $E = C \oplus \mathrm{Ker}(u)$, $F = C' \oplus \mathrm{Im}(u)$ and [redacted] let $\tilde{v}$ be the lifting of v such that $\tilde{v}(C) = 0$ and $\mathrm{Im}(\tilde{v}) \subset C'$. Put $\alpha(u,v) = u + \tilde{v}$.
Note that $\mathrm{Ker}\,\alpha(u,v) = \mathrm{Ker}(v$ on $\mathrm{Ker}(u))$, $\mathrm{Im}\,\alpha(u,v) =$ inverse image of $\mathrm{Im}(v)$ in $\mathrm{Cok}(u)$. Thus [redacted] no matter how we choose C, C', $\alpha(u,v)$ [redacted] will be an isomorphism of $E/\mathrm{Ker}(v$ on $\mathrm{Ker}(u))$ with the inverse image of $\mathrm{Im}(v)$. Put $\mathrm{Ker}(u,v)$ and $\mathrm{Im}(u,v)$ for these.

Observe that since the possible choices for C, C' are affine spaces, the map α is unique up to homotopy. [redacted]

Next step is to get the inductive construction under control.

This requires me maybe to make some choices, e.g. normal tubes.

Be careful: Given (u,v) where $v : \mathrm{Ker}\,u \xrightarrow{\sim} \mathrm{Im}\,u$. I want to understand all $\theta : E \xrightarrow{\sim} F$ which reduce to (u,v), i.e. [redacted] $\theta = u \oplus \tilde{v}$, where $\tilde{v}$ is the extension of v which results by choosing complements for $\mathrm{Ker}(u)$ + $\mathrm{Im}(u)$: [redacted]
I have seen [redacted] that θ determines these complements, for [redacted] they are $\mathrm{Ker}(\theta - u)$, $\mathrm{Im}(\theta - u)$ resp.

Thus it should be clear that the set of possible θ's is an [redacted] orbit in $\mathrm{Isom}(E,F)$ for the unipotent subgroup of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ fixing $\mathrm{Ker}(u)$, [redacted] $\mathrm{Coim}(u)$, $\mathrm{Im}(u)$, $\mathrm{Cok}(u)$ and inducing the identity on these spaces.

So now I must check transitivity. Assume that I have three maps [redacted] $\theta, \theta', \theta''$ from E to F. Assume

$$\mathrm{Ker}\,\theta \supset \mathrm{Ker}\,\theta' \supset \mathrm{Ker}\,\theta''$$

$$\mathrm{Im}\,\theta \subset \mathrm{Im}\,\theta' \subset \mathrm{Im}\,\theta''$$

that

$$\left\{\begin{array}{l} \mathrm{Ker}\,\theta' \oplus \mathrm{Ker}(\theta'-\theta) = \mathrm{Ker}\,\theta \\ \mathrm{Im}\,\theta \oplus \mathrm{Im}(\theta'-\theta) = \mathrm{Im}\,\theta' \end{array}\right.$$

$$\left\{\begin{array}{l} \mathrm{Ker}\,\theta'' \oplus \mathrm{Ker}(\theta''-\theta) = \mathrm{Ker}\,\theta' \\ \mathrm{Im}\,\theta' \oplus \mathrm{Im}(\theta''-\theta') = \mathrm{Im}\,\theta'' \end{array}\right.$$

Thus I seem to get this. Given $u : E \to F$ and a normal vector v: [redacted] $\mathrm{Ker}(u) \longrightarrow \mathrm{Cok}(u)$ to the stratum containing u, then I have a contractible set [redacted] $\alpha(u,v)$ of maps from E to F which I might call the image of the normal vector v at u. I have seen that any θ in $\alpha(u,v)$ determines

?

May 11, 1974

Let E, F be two vector spaces over $\mathbb{C}$ of dim n equipped with inner products, and let $X = \mathrm{Hom}(E,F)$ be stratified as usual. I want to describe this stratification in detail. First some remarks.

Let $u: E \longrightarrow F$ be an isomorphism. Then $(ux, ux) = (u^*ux, x)$ is a pos. def. herm. form on E, hence it is diagonalizable: $u^*u = \alpha^2$ where $\alpha > 0$. (means α pos. def. self adjoint op. on E). Hence $u\alpha^{-1}: E \to F$ is unitary, so $u = (u\alpha^{-1})\alpha$ is the polar decomp. of u. Let

$$E = \bigoplus_\lambda E_\lambda$$

be the eigenspace decomposition of E for α, and let $F = \bigoplus_\lambda F_\lambda$ be the corresponding decomposition with respect to the unitary operator $u\alpha^{-1}$. Thus $F_\lambda = u\alpha^{-1}E_\lambda = u(E_\lambda)$. The map u splits as an orthogonal direct sum of maps

$$u_\lambda : E_\lambda \longrightarrow F_\lambda$$

such that $u_\lambda = \lambda$ times a unitary isom. of E_λ with F_λ. So

<u>Lemma</u>: Any isomorphism $u: E \longrightarrow F$ is uniquely decomposable as an orthogonal direct sum $u = \oplus \lambda \theta_\lambda$ where the λ are pos. real numbers and θ_λ is a unitary isomorphism.

If $u: E \to F$ is arb. we can write u as an orth. direct sum of a zero map and an isomorphism.

<u>Remark</u>: Given a complex of vector spaces with inner products, one can decompose orthogonally the complex according to the eigenvalues of the Laplacean. This writes the complex as a direct sum of a complex with zero differentials, and of elementary complexes of the form $E \xrightarrow{\lambda\theta} F$, where $\lambda > 0$ and θ is unitary.

$\mathrm{Hom}(E,F)$ has the inner product $(a,b) = \mathrm{tr}(ab^*)$. If $u \in \mathrm{Hom}(E,F)$, I can identify the normal space to the stratum through u with $\mathrm{Hom}(\mathrm{Ker}\, u, \mathrm{Cok}\, u)$. The tangent space to the stratum is

$$\mathrm{Hom}(E, \mathrm{Im}(u)) + \mathrm{Hom}(E/\mathrm{Ker}\, u, F) \subset \mathrm{Hom}(E,F).$$

Thanks to the inner product & the linear structure, I can lift $v \in \mathrm{Hom}(\mathrm{Ker}\, u, \mathrm{Cok}\, u)$ to a vector $\tilde{v}$ orthogonal to the stratum, i.e. such that

$$\mathrm{tr}(\tilde{v} b^*) = 0 \qquad \forall \quad b : E/\mathrm{Ker}\, u \longrightarrow F$$

$$\Longrightarrow \quad \tilde{v}\, (\mathrm{Ker}\, u)^{\perp} = 0$$

$$\mathrm{tr}(a^* \tilde{v}) = 0 \qquad \forall \quad a : E \longrightarrow \mathrm{Im}(u)$$

$$\Longrightarrow \quad \mathrm{Im}\, \tilde{v} \subset \mathrm{Im}(u)^{\perp}.$$

Conclude:

If I use the linear structure & inner product on Hom(E, F) to identify the normal space to the stratum through u with a subspace of Hom(E, F), then this identification is given by the map

$$\mathrm{Hom}(\mathrm{Ker}\, u, \mathrm{Cok}\, u) \longrightarrow \mathrm{Hom}(E, F)$$

$$v \longmapsto \tilde{v}$$

where $\tilde{v}$ is the composite

$$E \xrightarrow[\text{proj}]{\text{orth}} \mathrm{Ker}\, u \xrightarrow{\ v\ } \mathrm{Cok}(u) \xrightarrow[\substack{\text{via} \\ (\mathrm{Im}\, u)^{\perp}}]{\text{lifting}} F \quad .$$

I should now be ready to take apart the space Hom(E, F). For each integer p I want U_p to be a normal tube around the stratum with dim Ker $= p$. Let

$$U_p = \left\{ u : E \to F \;\middle|\; \text{there exists an } \lambda_0 > 0 \text{ such that the eigenspaces of } u \text{ with eigenvalues} < \lambda_0 \text{ have total dimension } p \right\}$$

Let Y_p be the strata of $u \ni$ dim Ker $= p$. Then we have a map

$$U_p \longrightarrow Y_p$$

which sends u to its direct summand with eigenvalues $\geq \lambda_0$ as above. I also have that U_p is an open disk bundle in the normal bundle to Y_p in Hom(E, F). Namely, we have all pairs (u, v)

where the maximum eigenvalue of v is less than the minimum eigenvalue of u.

Check: $\mathcal{U}_0 = \mathrm{Iso}(E,F)$, $\mathcal{U}_n = \mathrm{Hom}(E,F)$; for u to belong to $\mathcal{U}_p$ means that if $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of u, then $\lambda_p < \lambda_{p+1}$ (λ_0 is defined to be zero, $\lambda_{n+1} = \infty$).

Suppose now we compute the intersections:

$$\mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_a} = \text{those } u \ni$$
$$\lambda_{i_0} < \lambda_{i_0+1}, \ldots, \lambda_{i_a} < \lambda_{i_a+1}$$

So now maybe I can describe how to inductively put the space $\mathrm{Hom}(E,F)$ together. I will construct $V_p = \{u \mid \dim \mathrm{Ker}(u) \leq p\}$ inductively. Recall Y_p is the stratum of $u \ni \dim \mathrm{Ker}(u) = p$, and that $\mathcal{U}_p$ is the open tube around Y_p consisting of pairs (u,v), where $v : \mathrm{Ker}(u) \longrightarrow \mathrm{Cok}(u)$ is such that the max eigenvalue of v is less that the minimum eigenvalue of u. Now then I have

$$V_p = V_{p+1} \underset{(\mathcal{U}_p - Y_p)}{\cup} \mathcal{U}_p$$

Thus

$$V_p = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_p$$

i.e. if u has $\leq p$ eigenvalues zero then for some $0 \leq i \leq p$ one has $\lambda_i < \lambda_{i+1}$.

Now perhaps I am ready to describe the topological category I am looking for.

So I am now going to form over $BU_n \times BU_n$ the fibre bundle with fibre $\mathrm{Hom}(E,F)$.

$\| $

$B(\mathrm{Aut}\,E) \times B(\mathrm{Aut}\,F)$. Put $G = \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ where here Aut = unitary maps, and denote by subscript the assoc. fibre bundle. Then I have the natural stratification

$$(Z_n)_G \subset \cdots\cdots \subset (Z_0)_G$$

$$Y_p = Z_p - Z_{p+1}$$

$$V_p = Z_0 - Z_{p+1}$$

and I want to describe the resulting homotopy types.

Guess that the thing to look at is the nerve of the covering

$$(V_p)_G = (U_0)_G \cup \cdots \cup (U_p)_G .$$

So fix $0 \leq i_0 < \cdots < i_a \leq p$ and try to determine the homotopy type of

$$(U_{i_0} \cap \cdots \cap U_{i_p})_G$$

Now $$U_{i_0} \cap \cdots \cap U_{i_a} = \left\{ u : E \to F \;\middle|\; \begin{matrix} \lambda_{i_0} < \lambda_{i_0+1} \\ \cdots \\ \lambda_{i_a} < \lambda_{i_a+1} \end{matrix} \right\}$$

and this may be described as the following set:

An object is a family consisting of a

$$w_a : E \longrightarrow F \qquad \dim \operatorname{Ker} w_a = i_a$$

$$w_{a-1} : \operatorname{Ker} w_a \longrightarrow \operatorname{Cok} w_a \qquad \dim \operatorname{Ker} w_{a-1} = i_{a-1}$$

$$w_{a-2} : \operatorname{Ker} w_{a-1} \longrightarrow \operatorname{Cok} w_{a-1} \qquad \cdots$$

$$w_{0} : \operatorname{Ker} w_{1} \longrightarrow \operatorname{Cok} w_{1} \qquad \dim (\operatorname{Ker} w_0) = i_0$$

$\dim i_1$

I am probably going to want to think of this when [redacted] $i_0 = 0$, as some sort of 1-parameter family

$$\sum_{i=0}^{a} w_i t^{a-i} = w_a + t w_{a-1} + \cdots + t^a w_0$$

of isomorphisms from E to F with the indicated asymptotic expansion as $t \longrightarrow 0$.

Anyway what is the homotopy type of $(\mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_a})_G$.
It is

$$(B\mathcal{U}_{i_0})^2 \times B\mathcal{U}_{i_1 - i_0} \times \cdots \times B\mathcal{U}_{i_a - i_{a-1}}$$

In effect given u with $\lambda_{i_0} < \lambda_{i_0+1}, \ldots, \lambda_{i_a} < \lambda_{i_a+1}$ then up to homotopy we can make :

$$\lambda_1 = \cdots = \lambda_{i_0} = 0$$

$$\lambda_{i_0+1} = \cdots = \lambda_{i_1} = 1$$

$$\lambda_{i_{a-1}+1} = \cdots = \lambda_{i_a} = a$$

Want the homotopy type of $(U_{i_0}\cap\cdots\cap U_{i_a})_G$. Now given u with eigenvalue jumps $\lambda_{i_0}<\lambda_{i_0+1}, \dots, \lambda_{i_a}<\lambda_{i_a+1}$ one can up to an equivariant homotopy wrt G replace $\lambda_1\dots\lambda_{i_0}$ by zero, $\lambda_{i_0+1},\dots,\lambda_{i_1}$ by 1, $\dots$; $\lambda_{i_{a-1}+1}\dots\lambda_{i_a}$ by a, and $\lambda_{i_a+1},\dots,\lambda_n$ by $a+1$. Then the u satisfying these conditions are all conjugate under G and the stabilizer is

$$U_{i_0}^2 \oplus \Delta U_{i_1-i_0} \oplus \cdots \oplus \Delta U_{n-i_a} \subset U_n^2$$

don't confuse U's These are unitary

Thus we get

<u>Lemma:</u> $(U_{i_0}\cap\cdots\cap U_{i_a})_G \sim BU_{i_0}^2 \oplus \Delta BU_{i_1-i_0} \oplus\cdots\oplus \Delta BU_{n-i_a}$

Now one finds the face maps of the simplicial space $a \longmapsto \coprod_{0\le i_0<\cdots<i_a\le n} (U_{i_0}\cap\cdots\cap U_{i_a})_G$ to correspond to the obvious diagonal action of $\coprod_n BU_n$ on $\coprod BU_n^2$.

Thus the nerve of the covering $(U_i)_G$ of Z_G is homotopy equivalent to the simplicial space

$$\begin{array}{c}\rightarrow\\\rightarrow\\\rightarrow\end{array} \coprod_{0\le i_0\le i_1\le n} BU_{i_0}^2 \times BU_{i_1-i_0} \times BU_{n-i_1} \rightrightarrows \coprod_{0\le i_0\le n} BU_{i_0}^2 \times BU_{n-i_0}$$

Make $\mathrm{Hom}(E,F)$ into a poset as follows. Call $u \leq v$ if $\mathrm{Ker}(u) \supset \mathrm{Ker}(v)$, $\mathrm{Im}(u) \subset \mathrm{Im}(v)$ and if in the diagram

$$\begin{tikzcd}
E/\mathrm{Ker}(v) \arrow[r, "\bar{v}", "\sim"'] \arrow[d, "p"'] & \mathrm{Im}(v) \\
E/\mathrm{Ker}(u) \arrow[r, "\bar{u}", "\sim"'] & \mathrm{Im}(u) \arrow[u, hook, "i"']
\end{tikzcd}$$

we have

$$p\,\bar{v}^{-1} i\,\bar{u}(\alpha) = \alpha. \qquad \forall \alpha \in E/\mathrm{Ker}(u).$$

I want to be sure that this notion is transitive

$$\begin{tikzcd}
E/\mathrm{Ker}(w) \arrow[r, "\bar{w}"] \arrow[d, "p'"'] & \mathrm{Im}(w) \\
E/\mathrm{Ker}(v) \arrow[r, "\bar{v}"] \arrow[d, "p"'] & \mathrm{Im}(v) \arrow[u, "i'"'] \\
E/\mathrm{Ker}(u) \arrow[r, "\bar{u}"] & \mathrm{Im}(u). \arrow[u, "i"']
\end{tikzcd}
\qquad p'\,\bar{w}^{-1} i'\,\bar{v} = id$$

Then

$$p p'\,\bar{w}^{-1} i' i\,\bar{u}(\alpha) = p\left[p'\,\bar{w}^{-1} i'\,\bar{v}\right]\bar{v}^{-1} i\,\bar{u}(\alpha)$$
$$= p\bar{v}^{-1} i\,\bar{u}(\alpha) = \alpha.$$

Thus the notion is transitive and it seems that one gets a good notion of specialization

Another way of expressing $u \leq v$ is to say $uv^{-1}u = u$ as correspondences. For the fact that $uv^{-1}u$ is everywhere defined implies $\mathrm{Im}\, u \subset \mathrm{Im}\, v$, and the fact that it is single-valued implies $\mathrm{Ker}\, v \subset \mathrm{Ker}\, u$.

If $uv^{-1}u = u$, $vw^{-1}v = v$, then as $uv^{-1}v = u$ and $vv^{-1}u = u$, one has $uw^{-1}u = (uv^{-1}v)w^{-1}(vv^{-1}u)$
$= uv^{-1}vv^{-1}u = uv^{-1}u = u.$

So now we can let $\operatorname{Aut} E \times \operatorname{Aut} F$ act on the poset $\operatorname{Hom}(E,F)$ and take the associated cofibred category.

Up to equivalence I get the category whose objects are maps $u : E \to F$ between vector spaces of dim. n in which a morphism from $u' : E' \to F'$ is a pair of isos.

$$
\begin{tikzcd}
E \arrow[r, "u"] & F \\
E' \arrow[u, "\alpha"', "\wr"] & F' \arrow[u, "\wr"', "\beta"]
\end{tikzcd}
$$

such that $(\beta u' \alpha^{-1}) = (\beta u' \alpha^{-1}) u^{-1} (\beta u' \alpha^{-1})$ or more simply

$$\boxed{u' = u'(\alpha^{-1} u^{-1} \beta) u'}$$

as correspondences.

Go back - given $u \leq v$ in $\operatorname{Hom}(E,F)$, i.e. $u v^{-1} u = u$, then in what sense might u be a direct summand of v? If v is an isomorphism this is indeed the case - for

~~$v^{-1}u$ and uv^{-1} are projectors and~~

$v^{-1}u$ is a projector in E, $v(v^{-1}u)v^{-1} = uv^{-1}$ is the image of this projector under the iso. v, and $u = v(v^{-1}u)$ is the corresponding direct summand of v.

(Another version of $u \leq v$ when v is an isom. is that $u = ve$ where e is a projector.)

If $uv^{-1}u = u$, where v is not nec. an isomorphism, then we have $(v^{-1}u)(v^{-1}u) = v^{-1}u$ as correspondences; and also $(uv^{-1})(uv^{-1}) = uv^{-1}$. Furthermore $v^{-1}u$ is defined everywhere, while uv^{-1} has trivial indeterminacy.

What is an idempotent correspondence α on a vector space E? Let $D =$ domain(α), $N =$ indeterminacy(α), so that α is really a map

$$E \supset D \xrightarrow{\bar\alpha} E/N \twoheadleftarrow E$$

~~Suppose first that α is a map~~ Now I am told that $\alpha^2 = \alpha$. To say α^2 has same domain + indet. means one has dotted arrows

$$\begin{tikzcd}
 & & & & D \arrow[r, phantom, "\subset"] \arrow[dl, dashed] \arrow[d, "\bar\alpha"] & E \\
 & D \arrow[r, two heads] \arrow[d, "\bar\alpha"'] & \frac{D}{D\cap N} \arrow[r, "\theta", "\sim"'] \arrow[dl, dashed] & \frac{D+N}{N} \arrow[r, phantom, "\subset"] & E/N & \\
E \arrow[r, two heads] & E/N & & & &
\end{tikzcd}$$

And that $\alpha^2 = \alpha$ means the comp. of the dotted arrows is $\bar\alpha$. We see that $\bar\alpha$ factors

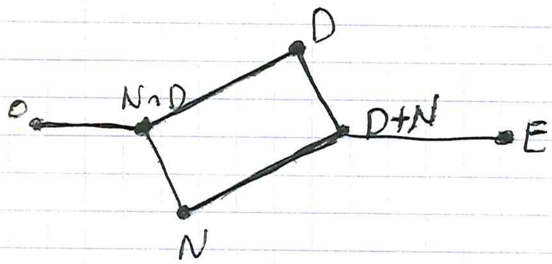
$$D \twoheadrightarrow D/D\cap N \xrightarrow{\tilde\alpha} \frac{D+N}{N} \hookrightarrow E/N$$

and if we identify $D/D\cap N$ with $\frac{D+N}{N}$ via θ, then $\tilde\alpha$ is probably indempotent. This is clear, for this amounts to $\tilde\alpha\,\theta^{-1}\tilde\alpha = \tilde\alpha$, which is immediate from the above diagram.

Thus we obtain:

__Lemma:__ Let α be an idempotent correspondence on E given by $E \supset D \xrightarrow{\tilde{\alpha}} E/N \twoheadleftarrow E$. Then $\tilde{\alpha}$ factors $D \twoheadrightarrow D/D\cap N \xrightarrow{\tilde{\alpha}} D+N/N \hookrightarrow E/N$ where if θ denotes the canon. isom $D/D\cap N \xrightarrow{\sim} D+N/N$, then $\tilde{\alpha}\,\theta^{-1}\tilde{\alpha} = \tilde{\alpha}$.

Thus one gets a picture:



and the layer $D+N/N \simeq D/D\cap N$ has been split in two.

__Special cases:__

1) $N\cap D = 0$, $D+N = E$; here $E = D \oplus N$ and D has been split according to the idempotent $\theta^{-1}\tilde{\alpha}$ on D.

2) $D=E$; here one is just giving a splitting of E/N.

3) $N=0$; here one splits the submodule D

So when $u \leq v$ in $\mathrm{Hom}(E,F)$, one knows that $\mathrm{Ker}\, u \supset \mathrm{Ker}\, v$, $\mathrm{Im}\, u \subset \mathrm{Im}\, v$, and that $v = ue$ where e is an idempotent correspondence with domain E, and indeterminacy contained in $\mathrm{Ker}(u)$.

So again we can let $G = \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ act on the poset $\mathrm{Hom}(E,F)$ and obtain a category C. The objects are maps $u: E \longrightarrow F$. A morphism from u to v is an element g of G such that $g(u) \leq v$.

Given u, v then

$$\mathrm{Hom}(u,v) = \{g \mid g(u) \leq v\}$$

$\mathrm{Aut}(v) = \{g \mid gv = v\}$ acts to the left, and $\mathrm{Aut}(u)$ to the right on $\mathrm{Hom}(u,v)$. The actions are without fixpts.

$$\mathrm{Hom}(u,v)/\mathrm{Aut}(u) \xrightarrow{\sim} \{v' \mid v' \leq v \text{ and } v' \text{ isom to } u\}.$$

$$\mathrm{Aut}(v)\backslash \mathrm{Hom}(u,v) \xrightarrow{\sim} \{u' \mid u' \geq u \text{ and } u' \text{ isom to } v\}.$$

Question: Does $\mathrm{Aut}(u) \times \mathrm{Aut}(v)$ act transitively on $\mathrm{Hom}(u,v)$?

This seems to be equivalent to whether $\mathrm{Aut}(v)$ acts transitively on $\{v' \mid v' \leq v \text{ and } v' \simeq u \text{ i.e. have } \dim \mathrm{Ker}(v') = \dim \mathrm{Ker}(u)\}$. which is something I think is true.

Problem: Given u show that the ordered set $\{v \mid v > u\}$ gets higher and higher connected.

Note that if C denotes the category in question, then $u \backslash C$ is the ordered set of $v \geq u$, up to equivalence.

What is the effect of replacing the ordered set $\mathrm{Hom}(E,F)$ by its nerve considered as a simplicial complex? Suppose

$$u_0 < u_1 < \cdots < u_p$$

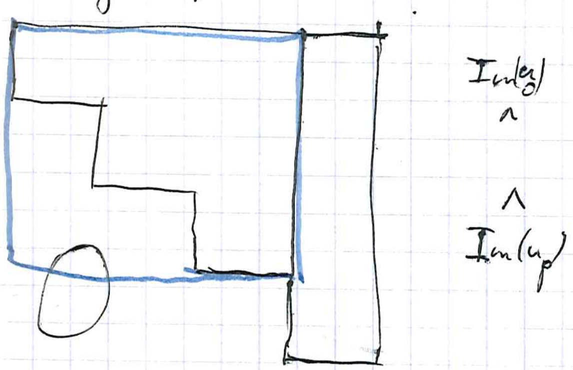
is a p-simplex. This is the same as giving a direct sum decomposition of $E/\mathrm{Ker}(u_p) \simeq \mathrm{Im}(u_p)$.

$$\begin{tikzcd}
\mathrm{Im}(u_0) \arrow[r, phantom, "<"] & \mathrm{Im}(u_1) \arrow[r, phantom, "<\cdots<"] & \mathrm{Im}(u_p) \\
\mathrm{Coim}(u_0) \arrow[u, "\wr"] & \mathrm{Coim}(u_1) \arrow[u, "\wr"] \arrow[l, two heads] & \cdots \arrow[l, two heads] & \mathrm{Coim}(u_p) \arrow[ull, "\wr"] \arrow[l, two heads]
\end{tikzcd}$$

Inside $\mathrm{Aut}(F)$ the stabilizer of

$$\mathrm{Im}(u_0) < \cdots < \mathrm{Im}(u_p)$$

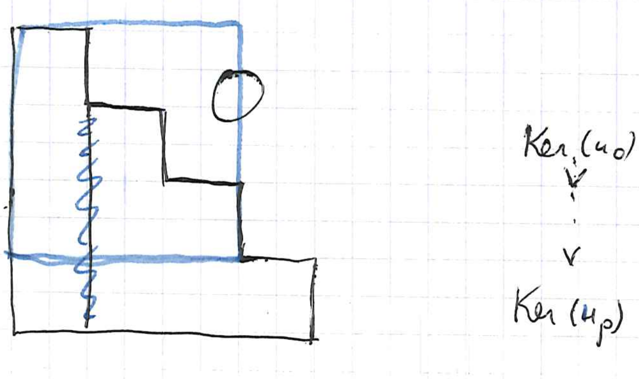
is the subgroup



Inside of $\mathrm{Aut}(E)$, the stabilizer of

$$\mathrm{Ker}(u_0) > \mathrm{Ker}(u_1) > \cdots > \mathrm{Ker}(u_p)$$

is the subgroup



So one gets for the stabilizer the subgroup of

$$\left[\mathrm{Aut}(\mathrm{Cok}\,u_p) \ltimes \mathrm{Hom}(\mathrm{Cok}\,u_p, \mathrm{Im}\,u_p)\right] \rtimes \mathrm{Aut}'(\mathrm{Im}\,u_p) \ltimes \left[\mathrm{Hom}(\mathrm{Im}\,u_p, \mathrm{Ker}\,u_p) \rtimes \mathrm{Aut}(\mathrm{Ker}\,u_p)\right]$$

where Aut′ denotes the subgroup preserving the splitting.

May 13, 1974: The Grassmannian

Let V be a vector space of dim $p+q$, W a subspace of dim q, and X the Grassmannian of all p planes in V. Put

$$X_k = \{A \mid \dim A\cap W \geq k\}$$

so that one has

$$X = X_0 \supset X_1 \supset \cdots\cdots$$

and put

$$V_k = X - X_{k+1} = \{A \mid \dim(A\cap W) \leq k\}$$

$$Y_k = X_k - X_{k+1} = k\text{-th stratum}$$

The normal bundle to Y_k at A is $\mathrm{Hom}(A\cap W, V/A+W)$; denote it ν_k, so that one has up to homotopy a pushout

$$\begin{tikzcd} \nu_k' \arrow[r, "\alpha"] \arrow[d] & V_{k-1} \arrow[d] \\ Y_k \arrow[r] & V_k \end{tikzcd}$$

I have now to describe the map α which will be defined using a metric on V.

Fix $A \in Y_k$, and let $v: A\cap W \longrightarrow V/A+W$ be a normal vector. Then $\alpha(A,v)$ should be in the neighborhood of A which is isomorphic to $\mathrm{Hom}(A, A^\perp)$ by the graph map

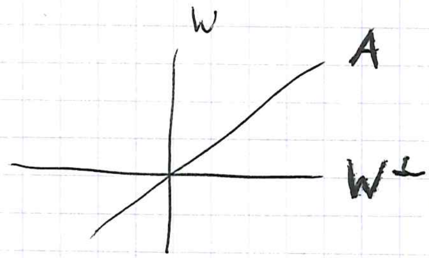
$$\mathrm{Hom}(A, A^\perp) \subset X.$$

$$\theta \longmapsto \{(a+\theta a) \mid a \in A\}.$$

Now by virtue of the metric, as $Hom(A \cap W, V/A+W)$ which is a quotient of $Hom(A, V/A)$, one can lift v to an element of $Hom(A, A^\perp)$.

Question: Can you see a tubular nbd. of Y_k this way?

Guess I should think of A as being a correspondence from [redacted] $W^\perp = V/W$. to W. Thus $X - X_1 = V_0 = \{A \mid A \cap W = 0\}$ can be identified with $Hom(W^\perp, W)$



So now change notation maybe: [redacted] $W^\perp = E$, [redacted] $W = F$. Then using the metrics on E and F, I can [redacted] gives the eigenvalue decomposition of the correspondence A, namely, it will be an orthogonal direct sum of pieces with eigenvalues $0 \le \lambda \le \infty$, and where for these two extremes one has no unitary isomorphism [redacted].

Now [redacted] for the correspondence α to belong to Y_k means exactly k of the eigenvalues are infinity. ~~Suppose now [illegible] the other [illegible]~~ So it is clear what the normal [redacted] disk around α has to be [redacted].

~~But to get simpler formulas, I will~~ [redacted] ~~want to think of A as a correspondence~~ [redacted]

[redacted]

$A \subset W^{\perp} \oplus W \qquad \dim A = \dim W^{\perp}$.

can be interpreted as a correspondence from $W^{\perp}$ to W. Precisely put $D = \mathrm{Im}\,\{A \xrightarrow{pr_2} W^{\perp}\}$ and $N = A \cap W$, so that

$$D = \mathrm{Im}\,\{A \xrightarrow{pr_2} W^{\perp}\} = \mathrm{Im}\,\{A \to V/W\} = \frac{A+W}{W}$$

$$\begin{tikzcd} & A/A\cap W \arrow[d, "\wr"] \arrow[drr, "\text{proj on } W"] & & \\ W^{\perp} \supset & D & & W/N & W \arrow[l, two heads] \end{tikzcd}$$

we get a corresp. with domain D and indeterminacy N. [redacted] Y_k is the set of these correspondences with $\dim\,(W^{\perp}/D) = \dim\,(N) = k$.

Suppose now I try to find the normal tube U_k around Y_k. Given $\alpha \in Y_k$ given by

$$W^{\perp} \supset D_{\alpha} \xrightarrow{\bar{\alpha}} W/N_{\alpha} \twoheadleftarrow W$$

a normal vector to Y_k at α is an element of

$$\mathrm{Hom}\,(A \cap W,\ V/A+W) = \mathrm{Hom}\,(N_{\alpha}, D_{\alpha}^{\perp})$$

where $D_{\alpha}^{\perp}$ is the orthogonal complement of D_{α} in $W^{\perp} = E$. Thus somehow a normal vector v at α is a way of [redacted] extending the domain of α and its indeterminacy?

Try the following - Given a pair E, F of vector spaces, note that a correspondence between E and F is the same thing as an isomorphism of a subquotient of E with a subquotient of F, hence it is the same as a diagram in the Q-category of the form

$$\begin{tikzcd} E & & F \\ & T \arrow[ul] \arrow[ur] & \end{tikzcd}$$

mod isos of T. One should see if this is compatible with composition of correspondences in the transversal case. Thus suppose we have two transversally composable correspondences

$$\begin{tikzcd} T'' \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & T' \arrow[r, "q'"] \arrow[d, "p'"] \arrow[dr, phantom, "\alpha'"] & G \\ T \arrow[r, "q"'] \arrow[d, "p"'] \arrow[dr, phantom, "\alpha"] & F & \\ E & & \end{tikzcd}$$

where the square is transversal cartesian. The subquotient iso. defined by p, q is

$$S_\alpha = \frac{\mathrm{Im}(p)}{p(\mathrm{Ker}\, q)} \simeq \frac{\mathrm{Im}(q)}{q(\mathrm{Ker}\, p)} .$$

What I want to show is that one has a comm. diag.

$$\begin{tikzcd} E & & F & & G \\ & S_\alpha \arrow[ul] \arrow[ur] & & S_{\alpha'} \arrow[ul] \arrow[ur] & \\ & & S_{\alpha''} \arrow[ul] \arrow[ur] & & \end{tikzcd}$$

in the Q-category.

FALSE

$$\begin{tikzcd} & F & \\ \dfrac{q(T)}{q(\mathrm{Ker}\, p)} \arrow[ur] & & \dfrac{p'(T)}{p'(\mathrm{Ker}\, q')} \arrow[ul] \end{tikzcd}$$

$$\frac{q\, pr_1(T'')}{q\, pr_1(\mathrm{Ker}\, q' pr_2)}$$

So this doesn't work (probably for the usual reason) e.g. suppose p and q' are zero (whence p', q are injective) and suppose that $T''=0$ so that

$$F = q(T) \oplus p'(T).$$

Return ~~to~~ earlier problem. I have the Grassmannian X of planes in V of the same dimension as $W^{\perp} = E$ stratified according to $\dim(A \cap W) = \operatorname{cod}(\operatorname{Im} A \to W^{\perp})$. Y_k = p-planes A $\ni$ $\dim(A \cap W) = k$. Thinking of such a plane α as a map

$$E \supset D_\alpha \xrightarrow{\bar{\alpha}} F/N_\alpha \twoheadleftarrow F$$

up to homotopy $\bar{\alpha}$ can be homotoped to zero, hence up to homotopy

$$Y_k \sim G'_k(E) \times G_k(F)$$

where the prime denotes quotient spaces of dim k. Now I want to describe a normal tube U_k around Y_k.

Now for fun [redacted] let α, β be two correspondences between E and F of index 0. [redacted] I want to define when α is a "specialization" of β. Clearly I want

$$D_\alpha \leqslant D_\beta \qquad\qquad N_\alpha \geqslant N_\beta$$

and hence I can form

$$\begin{tikzcd}
E \arrow[d, phantom, "\parallel"] & \supset & D_\alpha \arrow[r, "\bar{\alpha}"] \arrow[d, phantom, "\cap"] & F/N_\alpha & F \arrow[l] \arrow[d, phantom, "\parallel"] \\
E & \supset & D_\beta \arrow[r, "\bar{\beta}"] & F/N_\beta \arrow[u, two heads] & F \arrow[l]
\end{tikzcd}$$

and require that $\bar{\beta}$ induce $\bar{\alpha}$ in the obvious sense.

Now how does this correspond to what happens normally at α.
$\alpha \Longleftrightarrow A \subset E \times F$

$$D_\alpha = \mathrm{Im}\{A \longrightarrow E\} \qquad\qquad \frac{A+W}{W} \subset V/W$$

$$N_\alpha = A \cap F. \qquad\qquad A \cap W.$$

The normal space [redacted] to Y_k at α is $\mathrm{Hom}($[redacted] $N_\alpha, E/D_\alpha)$. If v is a normal vector, [redacted] and β is the image of α, then D_β/D_α ought to be [redacted] $\mathrm{Im}(v)$, and [redacted] N_β ought to be $\mathrm{Ker}(v)$, and v will induce an isomorphism

$$D_\beta/D_\alpha \simeq N_\alpha/N_\beta.$$

Thus we ought to have

$$\begin{tikzcd}
D_\alpha \arrow[r, "\bar{\alpha}"] \arrow[d, phantom, "\cap"] & F/N_\alpha \\
D_\beta \arrow[r, "\bar{\beta}"] \arrow[d] & F/N_\beta \arrow[u, two heads] \\
D_\beta/D_\alpha & N_\alpha/N_\beta \arrow[l, "\simeq"', "v"] \arrow[u, hook]
\end{tikzcd}$$

and β should induce ν in the following sense: $\mathrm{Ker}(\bar\alpha) = \mathrm{Ker}(\bar\beta)$, $\mathrm{Cok}(\bar\alpha) = \mathrm{Cok}(\bar\beta)$, and

$$\begin{tikzcd} D_\beta/\mathrm{Ker}\,\bar\beta \arrow[r, "\sim", "\bar{\bar\beta}"'] \arrow[d, two heads] & \mathrm{Im}\,\bar\beta \\ D_\beta/D_\alpha & N_\alpha/N_\beta \arrow[l, "\nu"] \arrow[u, hook] \end{tikzcd}$$

should commute. [redacted]

Thus [redacted] given two correspondences α, β from E to F say that $\alpha \leq \beta$ if

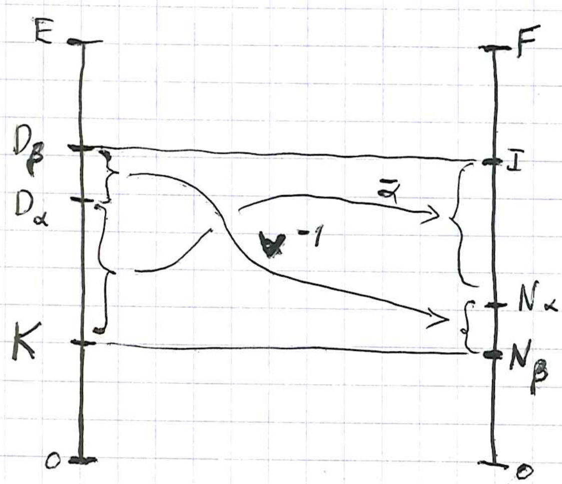
$$D_\alpha \subset D_\beta \qquad \mathrm{Ker}(\bar\alpha) = \mathrm{Ker}(\bar\beta)$$
$$N_\alpha \supset N_\beta \qquad \mathrm{Cok}(\bar\alpha) = \mathrm{Cok}(\bar\beta)$$

and if

$$\begin{tikzcd} D_\alpha \arrow[r, "\bar\alpha"] \arrow[d, phantom, "\cap"] & F/N_\alpha \\ D_\beta \arrow[r, "\bar\beta"] & F/N_\beta \arrow[u, two heads] \end{tikzcd}$$

commutes. One gets a partially ordered set in this way.

Picture: Put $K = \mathrm{Ker}(\bar\alpha) = \mathrm{Ker}(\bar\beta) \subset E$, $C = \mathrm{Coker}(\bar\alpha) = \mathrm{Cok}(\bar\beta)$ as a quotient of F, $C = F/I$:



Stratification of the Grassmannian. I consider the Grass. of planes $A \subset E \times F$ with $\dim(A) = \dim(E)$. Such an A I can think of as the graph of a correspondence α from E to F with domain $D_\alpha = \mathrm{Im}\{A \xrightarrow{p_1} E\}$ indeterminacy $N_\alpha = A \cap F$, and map $\bar{\alpha}$

$$E \supset D_\alpha = A+F/F \xrightarrow{\sim} A/A\cap F \xrightarrow{p_2} F/\underset{\underset{N}{\|}}{A\cap F} \twoheadleftarrow F.$$

induced by $p_2 : A \longrightarrow F$. ($p_i$ = rest. of pr_i to A)

(Note before going on that if $V = E \times F$, then D_α, N_α depend only on $F \subset V$, and not $pr_2 : V \rightarrow F$, whereas $\bar{\alpha}$ does depend on the choice of complement for F. Thus this interpretation in terms of correspondences has defects.)

Suppose A is on the stratum Y_k, i.e. $k = \dim N_\alpha = \operatorname{codim} D_\alpha$. The tangent space to X at A is canonically $\mathrm{Hom}(A, V/A)$; the normal space to Y_k at A is $\mathrm{Hom}(N_\alpha, E/D_\alpha)$

Y_k

One has exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_\alpha \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & D_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F/N_\alpha \arrow[r] \arrow[d] & V/A \arrow[r] \arrow[d] & E/D_\alpha \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Given a tangent vector $v \in \mathrm{Hom}(A, V/A)$ we can take its image in the normal space, which is the induced map $N_\alpha \longrightarrow E/D_\alpha$. If this is zero, one gets maps $N_\alpha \longrightarrow F/N_\alpha$, $D_\alpha \longrightarrow E/D_\alpha$ which represent the moving of the planes N_α in F, D_α in E. If this motion is zero, one gets a map $D_\alpha \longrightarrow F/N_\alpha$ which is motion of the map $\bar{\alpha}$. By virtue of metrics, one thus decomposes the tangent space $\mathrm{Hom}(A, V/A)$ into 4 pieces

$\mathrm{Hom}(N_\alpha, E/D_\alpha)$	normal motion to Y/k
$\mathrm{Hom}(D_\alpha, E/D_\alpha)$	motion of $D_\alpha \subset E$
$\mathrm{Hom}(N_\alpha, F/N_\alpha)$	$N_\alpha \subset F$
$\mathrm{Hom}(D_\alpha, E/D_\alpha)$	motion of $\bar{\alpha}$

Now given $v \in \mathrm{Hom}(N_\alpha, E/D_\alpha)$ one uses the maps

$$\mathrm{Hom}(N_\alpha, E/D_\alpha) \xrightarrow[\text{orthogonal section}]{} \mathrm{Hom}(A, V/A) \xhookrightarrow[\substack{\text{graph of} \\ \text{map from } A \text{ to } A^\perp}]{} X$$

Here's how this works. Given $v : N_\alpha \longrightarrow E/D_\alpha \simeq D_\alpha^\perp$ in E. One extends v to $\tilde{v} : A \longrightarrow A^\perp$ as follows.

$$A \xrightarrow[\text{proj}]{\text{orth}} N_\alpha \xrightarrow{\ \ v\ \ } E/D_\alpha \xrightarrow[\text{section}]{\text{orth}} V/A \simeq A^\perp$$

Now α' is the correspondence with graph

$$A' = \{a + \tilde{v}(a) \mid a \in A\}.$$

What is D'_α? Project onto E. Let C be the orth. complement of N_α in A, so that $C \subset A \twoheadrightarrow D_\alpha$ is an isom. Then $\tilde{v} = 0$ on C, hence $D'_\alpha \supset D_\alpha$. $A = C \overset{\perp}{\oplus} N_\alpha$, so $A' = C + \tilde{v} N_\alpha$; projecting into E gives $D_\alpha + \mathrm{Im}\, v$. Thus $D_{\alpha'} = D_\alpha \oplus \mathrm{Im}(v)$ $\quad \mathrm{Im}(v) \subset D_\alpha^{\perp}$ in E.

What is $N_{\alpha'}$? When does $a + \tilde{v}(a) \in F$, i.e. when does it get killed by projection into E; necessary that $a \in N_\alpha$ and that $v(a) = 0$. Thus one sees that $D_\alpha \subset D_{\alpha'}$ with $D_{\alpha'}/D_\alpha = \mathrm{Im}(v)$, $N_{\alpha'} \subset N_\alpha$ with $N_\alpha / N_{\alpha'} = \mathrm{Coim}(v)$. And it is fairly certain that the maps $\bar{\alpha}'$ is the direct sum

$$\begin{array}{ccc} D_\alpha & \xrightarrow{\bar{\alpha}} & F/N_\alpha \\ \oplus & & \oplus \\ \mathrm{Im}\, v & \xrightarrow[\sim]{v^{-1}} & \mathrm{Coim}(v) \end{array}$$

Granted that this is true, when v is small, one sees that v^{-1} has large eigenvalues. Thus I can define a tubular nbd U_k of Y_k by saying $\alpha \in U_k$ if the following hold:

$p = \dim N_\alpha \leq k$

next let $0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-p}$

$n = \dim(E)$ be the eigenvalue sequence of

$$\bar{\alpha} : D_\alpha \longrightarrow F/N_\alpha$$

Then I want to know that $\lambda_{n-k} < \lambda_{n-k+1}$ for α to be in U_k.

Thus given a correspondence α from E to F whose graph has dim $n = \dim(E)$, I get a sequence of eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \infty$, and the number of infinite eigenvalues is the dimension of N_α. Thus $\alpha \in Y_k \Leftrightarrow \lambda_{n-k} < \lambda_{n-k+1} = \infty$.
I will define U_k to be the set of α such that there is a jump at the point $n-k$ in the eigenvalue sequence, i.e. $\lambda_{n-k} < \lambda_{n-k+1}$. The map of U_k to the normal bundle around Y_k is clear.

So we have a covering U_k of X $k = 0, 1, 2, \ldots, n$ and I want to know the homotopy type of the intersections

$$U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_\ell} \qquad 0 \leq i_0 < \cdots < i_\ell \leq n.$$

An α in this intersection has jumps in its eigenvalue sequence

$$0 \leq \cdots \leq \lambda_{i_0} < \lambda_{i_0+1} \leq \cdots \leq \lambda_{i_1} < \lambda_{i_1+1} \cdots \leq \lambda_{i_\ell} < \lambda_{i_\ell+1} \leq \cdots \infty$$

These eigenvalues can be homotoped to

0 —— 0		(i_0)-times
1 —— 1		$(i_1 - i_0)$-times
ℓ - - - ℓ		$(i_\ell - i_{\ell-1})$-times
∞ ∞		$(n - i_\ell)$-times

in which case α has been reduced to some sort of flag.

e.g. U_1

Homotopy type of U_k: since $\lambda_{n-k} < \lambda_{n-k+1}$ we can push all λ_i to zero for $i \le n-k$ and to infinity for $i \ge n-k+1$. Then α is

$$E \supset D_\alpha \xrightarrow{\ 0\ } F/N_\alpha \twoheadleftarrow F$$

and so

$$U_k \sim G_k'(E) \times G_k(F). \qquad (\sim BU_k \times BU_k \text{ as } n \to \infty)$$

If $0 \le j < k \le n$, then we have $\lambda_{n-k} < \lambda_{n-k+1} \le \lambda_{n-j} < \lambda_{n-j+1}$ so we can push λ_i to

$$\begin{array}{ll} 0 & i \le n-k \\ 1 & n-k < i \le n-j \\ \infty & n-j < i \end{array}$$

Then one finds $\bar{\alpha}$ split in two. [redacted]

On $\underbrace{E \supset D_\alpha}_{[redacted]\ j} \supset \underbrace{\mathrm{Ker}(\bar{\alpha})}_{k-j} \supset \underbrace{0}_{n-k}$.

$$F \underset{j}{\twoheadrightarrow} F/N_\alpha \underset{k-j}{\longrightarrow} F/\mathrm{Cok}\,\bar{\alpha}$$

$$0 \underset{j}{\subset} N_\alpha \underset{k-j}{\subset} \mathrm{Im}\,\bar{\alpha} \subset F$$

$$U_j \cap U_k \sim \text{[redacted]}$$

$$\big(E \to E/K \to E/D,\ N \subset I \subset F;\ \underbrace{D/K \simeq I/N}_{k-j}\big)$$

[redacted]

$$\sim \underset{\substack{\uparrow \\ E/D}}{BU_j} \times \underset{D/K \simeq I/N}{BU_{k-j}} \times \underset{(N \subset F)_j}{BU_j}$$

[redacted] The two maps

$$\begin{array}{ccc} U_j \cap U_k & \longrightarrow & U_k \\ \downarrow & & \\ U_j & & \end{array}$$

are respectively diagonal addition $(BU_j)^2$ [redacted] $\times \Delta BU_{k-j} \to (BU_k)^2$ and projection.

Fredholm operator. Suppose E, F are sep. Hilbert spaces and X is the space of Fredholm operators from E to F of index zero stratified according to the dim of Ker; thus $Y_k = \{\alpha \in X \mid \dim \mathrm{Ker}(\alpha) = k\}$. One has a map

$$Y_k \longrightarrow G_k(E) \times G_k(F)$$

defined by sending α to $\mathrm{Ker}(\alpha)$, $\mathrm{Ker}(\alpha^*) =$ orth. complement of $\mathrm{Im}(\alpha)$. The fibre is the space of isom. of $E/\mathrm{Ker}(\alpha)$ with $\mathrm{Im}(\alpha)$. According to Kuiper's theorem it is contractible.* Since E, F are Hilbert spaces $G_k(E) \sim BU_k$, $G_k(F) \sim BU_k$.

* strictly speaking, one first has to note that any isomorphism $u: E' \longrightarrow F'$ factors $u = \sigma\rho$, where ρ is positive bdd away from 0 and ∞ ($\rho = (u^*u)^{1/2}$). Then ρ can be pushed to 1, and this constitutes a def. of u to σ which is unitary. Kuiper $\Rightarrow$ Unitary group is contractible.

Now one can define U_k to consist of all $\alpha : E \rightarrow F$ such that if we arrange the eigenvalues of $\alpha^*\alpha$ as

$$\lambda_1 \leq \lambda_2 \leq \cdots$$

then $\lambda_k < \lambda_{k+1}$. (This implies that $\dim \mathrm{Ker}\, \alpha \leq k$). What does this mean iff $\alpha^*\alpha$ has continuous spectrum? Thus we must be precise and say that for $\alpha \in U_k$ means that the first k

eigenvalues of $\alpha^*\alpha$ are discrete and that the spectrum of the rest of $\alpha^*\alpha$ is $> \lambda_k$. Better there exist a $\delta > 0$ not in the spectrum of $\alpha^*\alpha$ such that the multiplicity of eigenvalues $< \delta$ is k. Then it is clear that U_k is a normal tube around Y_k.

<u>Conjecture:</u> Make the pair category of unitary vector spaces over $\mathbb{C}$ into a topological category. Then its classifying space is the space of Fredholm operators via Kuiper's theorem.

So again I consider the Grassmannian X of p planes in $p+q$, but this time I take W to be of dimension $d \leq q$, and stratify according to $\dim(A \cap W)$. Here

$$X_0 - X_1 = Y_0 = \{A \mid A \cap W = 0\}$$

$$\sim G_p(V^{p+q}/W^d) = G_{q-d}(V^{p+q}/W^d)$$

$$\sim BU_{q-d} \qquad \text{as} \quad p \to \infty.$$

and
$$Y_k = \{A \mid \dim(A \cap W) = k\}.$$

Again put $E = W^\perp$, $F = W$, so that A can be viewed as a correspondence α from E to F with $D_\alpha = \operatorname{Im}\{A \to E\}$, $N_\alpha = A \cap F$. Here $k = \dim N_\alpha$ so from

$$0 \longrightarrow N_\alpha \longrightarrow A^p \longrightarrow D_\alpha \longrightarrow 0$$

one gets

$$\dim(D_\alpha) = p - k \qquad \begin{aligned} \dim E &= p+q-d \\ \dim F &= d \end{aligned}$$

Thus using the homotopy of $\bar{z}$ to 0 one has

$$\begin{aligned} Y_k &\sim G'_{(p+q-d)-(p-k)}(E) \times G_k(F^d) \\ \alpha &\longmapsto (E/D_\alpha \quad , \quad N_\alpha) \\ &\sim BU_{q-d+k} \times BU_k \qquad \text{as } p,d \to \infty \end{aligned}$$

So this ought to be equivalent to the component of degree $q-d$ of the <u>pair category</u>.

Resemblance between correspondences $E \overset{\alpha}{\dashrightarrow} F$ of index t (means $\dim(E/D_\alpha) - \dim(N_\alpha) = t$) and stable bundles of degree t.

May 15, 1974

Two models for BU: Grassmannians and Fredholm operators of degree zero. Notice the similarities:

In the case of a Fredholm operator $u: F \to E$ in the stratum Y_k one has the pair $(\mathrm{Ker}(u), \mathrm{Cok}(u)) \in G_k(F) \times G'_k(E)$, and the iso $F/\mathrm{Ker}(u) \to \mathrm{Im}\, u$ which is negligible by Kuiper's theorem. The normal space to the stratum at u is $\mathrm{Hom}(\mathrm{Ker}(u), \mathrm{Cok}(u))$.

In the case of a correspondence α with graph $A \subset E \times F$ of the same dimension as E with $\dim (N = A \cap F) = k$

$$E \supset D_\alpha \xrightarrow{\bar{\alpha}} F/N_\alpha \twoheadleftarrow F$$

one has the pair $(N_\alpha, E/D_\alpha) \in G_k(F) \times G'_k(E)$ and the homomorphism $\bar{\alpha}$, which is negligible up to homotopy. The normal space to Y_k at α is $\mathrm{Hom}(N_\alpha, E/D_\alpha)$.

Thus we maybe ought to think of a correspondence $\alpha: E \to F$ with $\dim(N_\alpha) - \dim(E/D_\alpha) = t$ as the analogue of a Fredholm operator $f: F \to E$ with $\mathrm{Ker}(f) = N_\alpha$, $\mathrm{Im}(f) = D_\alpha$.

Suppose now I try to make a category out of pairs of vector spaces (L, M) of the same dimension. To go from (L, M) to (L', M') I will want to give an map $L \xrightarrow{u} M$ plus an isom of L' with the kernel and L' with the cokernel, i.e. an exact sequence

$$0 \to L' \to L \xrightarrow{u} M \to M' \to 0$$

?

Idea: To a correspondence from E to F I should be able to associate a path joining E to F in the Q-category. Maybe I can make correspondences from E to F into a space, and this space would map to the space of paths from E to F. The problem: Make correspondences between E and F into a space, which maps to paths in Q between E and F.

Recall that I already have a model for the space of paths between 0 and E. Namely I consider epis $E \overset{p}{\twoheadleftarrow} T$, so I consider paths from E to zero of the form

$$E \overset{surj}{\dashrightarrow} T \overset{inj}{\dashleftarrow} 0$$

Then I localize so as to make the operation $\oplus Z$ on $\operatorname{Ker} p$ invertible.

Denote by $\Omega(E,0)$ the space of paths in Q from E to 0. For me to interpret a correspondence from E to F as a path from E to F, would mean that I have a functor $\Omega(F,0) \longrightarrow \Omega(E,0)$, presumably compatible with composition. In particular for a map $f: E \longrightarrow F$ we would get two functors

$f^*: \Omega(F,0) \longrightarrow \Omega(E,0)$ here think of f as $E = E \xrightarrow{f} F$

$f_*: \Omega(E,0) \longrightarrow \Omega(F,0)$ ——————— $F \leftarrow E \Longrightarrow E$

satisfying the habitual relations. Now I already have in an obvious way i^* for i injective and

p_* for p surjective.

As with the model $f^{-1}\tilde{Q} \to Q$ I will trivialize things with respect to i^*. Thus I will think of $\Omega(0,0)$ as the space of stable vector bundles and will identify $\Omega(E,0)$ with $\Omega(0,0)$ via i_E^*, where $i_E : 0 \hookrightarrow E$. Note this forces $f^* = \mathrm{id}$ as we have

$$\begin{tikzcd} & E \arrow[dd, "f"] \\ 0 \arrow[ur, hook, "i_E"] \arrow[dr, hook, "i_F"'] & \\ & F \end{tikzcd} \qquad f i_E = i_F$$

And at the same time f_* becomes the operation of addition by the stable bundle $\mathrm{Ind}(f) = \mathrm{Ker}(f) - \mathrm{Cok}(f)$ on $\Omega(0,0)$.

~~Now for a correspondence α from E to F~~

$$\begin{tikzcd} & A \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ F & & E \end{tikzcd}$$

~~I will want to understand what should be meant by $\alpha_* : \Omega(E,0) \longrightarrow \Omega(F,0)$~~ $\alpha_* = p_{2*}p_1^*$.

Now suppose we have a correspondence between E and F

$$\begin{tikzcd} A \arrow[r, "p_2"] \arrow[d, two heads] \arrow[ddl, "p_1"'] & F \arrow[d, two heads] \\ A/N \arrow[r] \arrow[d, equal] & F/N \\ E \supset D & \end{tikzcd} \qquad A \subset E \times F$$

and we want to associate to it the map $p_{1*}p_2^*$. Then this clearly amounts to multiplying by

$$\mathrm{Ind}(p_1) = [N] - [E/D] \quad \text{on } \Omega(0,0).$$

(Questions – to remember.

1) Bicategory of f_*, f^* – show this has homotopy type of Q. In the realization of this bicategory one has obvious paths associated to correspondences.

2) Take an infinite dimensional vector space E and consider the monoid of correspondences from E to itself

$$\begin{array}{c} E_0 \subset E \\ p \downarrow \quad \\ E \quad\;\; \end{array}$$

such that p has finite dimensional kernel. Does this monoid have the homotopy type of $\Omega Q(k)$? Obviously not, but maybe this monoid has the homotopy type of $Q(k)$. Call this monoid M, and introduce the fibred category C over M whose fibre over E is the groupoid of epis $E \overset{f}{\twoheadrightarrow} E$ with finite dimensional kernel. Then a map in C looks like

$$\begin{array}{ccc} E'' & \hookrightarrow & E' \\ \downarrow & \text{cart.} & \downarrow \\ E_0 & \hookrightarrow & E \\ \downarrow & & \\ E & & \end{array}$$

~~whence it should be clear that C is equivalent to pairs~~

whence C is equivalent to pairs $(K \subset E)$ where K is finite dimensional and the maps $(K' \subset E') \longrightarrow (K, E)$ are injections $E' \hookrightarrow E$, $K' \supset K$. Now C is contractible for one can deform down to

$$(K \subset E) \longmapsto (0 \subset E)$$

to the category of injections which is contractible. Wrong direction.

Instead introduce the subdivision of M whose objects are such arrows $E \xleftarrow{p} E_0 \subset E$. Then one has a functor $Sub(M) \longrightarrow Q$ defined by sending the object to $Ker(p)$.

Better I will let C be the following cat. Its objects are epis $\begin{array}{c} X \\ \downarrow\!\!\downarrow \\ E \end{array}$ with finite kernel; a map will be a diagram

$$\begin{tikzcd} X' \arrow[d, two heads] & \\ X_0 \arrow[r, hook] \arrow[d, two heads] & X \arrow[d, two heads] \\ E_0 \arrow[r, hook] \arrow[d, two heads] & E \\ E' & \end{tikzcd}$$

where the square is cartesian. In other words, I consider for each E the epis $X \twoheadrightarrow E$ with ~~finite~~ kernel with maps

$$\begin{tikzcd} X' \arrow[rd, two heads] \arrow[dd, two heads] & \\ & X \arrow[ld] \\ E & \end{tikzcd}$$

and let this be the fibre C_E over E; note C_E has final object $E \twoheadrightarrow E$. Thus C is homotopy equivalent to M. One has an evident functor $C \to Q^\circ$ such that the essential fibre over V is the categ of $V \hookrightarrow E$ in which the maps are

functor is cofibred

$$\begin{tikzcd} & V \arrow[ld] \arrow[rd, hook] & \\ E \arrow[rr, hook] & & E' \end{tikzcd}$$

This is evidently contractible by the one construction.

Enough digression. But this example raises the following problems.

Example: Suppose I have a space X. Then I can consider the singular complex of X as a simp. set or as a simplicial space. This doesn't affect homotopy type.

Example: Let R be a simplicial ring which is connected. Then $GL(R)$ already has the right homotopy type for K-theory.

Example: $F \longrightarrow E \longrightarrow B$ maps of simplicial spaces such that the composite is the basepoint map, and such that $F_q = \text{fibre } E_q \to B_q$, $\forall q$. If B_q is connected $\forall q$, then $|F| \to |E| \to |B|$ is a fibration.

Corollary: M connected topological monoid, then $M \to pt \to BM$ is a fibration.

Example: Anderson's simple way of getting a cohomology theory out of a permutative category.

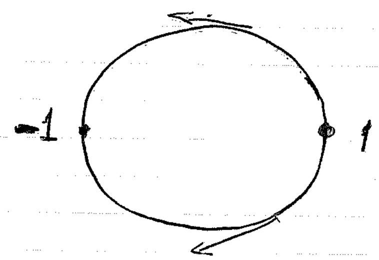
March 17, 1974

Interpretation of Q category for topological K-theory:

Stratify the unitary group U_n by putting

$$Y_k = \{\, \theta \in U_n \mid \dim \operatorname{Ker}(\theta - 1) = k \,\}$$

Now Y_0 is the open subset of θ not having the eigenvalue 1. Given a θ in Y_0 we can deform it to $-\mathrm{id}$ by keeping the same eigenspaces but pushing all of the eigenvalues to -1.



Thus Y_0 is contractible.

As for Y_k, one can map it $\theta \mapsto \operatorname{Ker}(\theta-1) \in G_k(\mathbb{C}^n)$ and the fibre is the Y_0 for the orthogonal complement, hence one has

$$Y_k \sim G_k(\mathbb{C}^n)$$

Note

$$\dim(Y_k) = 2k(n-k) + (n-k)^2 = n^2 - k^2$$

so Y_k is of codimension k^2 in U_n.

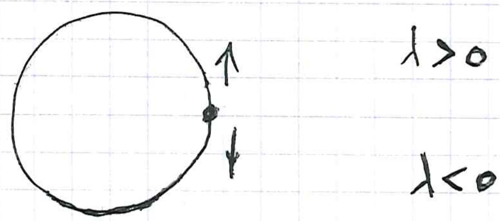
Given $\theta \in Y_k$ and a tangent vector v to U_n at θ, one can first see what v does ~~to~~ $\operatorname{Ker}(\theta-1) \in G_k(\mathbb{C}^n)$, i.e. the induced map

$$\operatorname{Ker}(\theta-1) \longrightarrow \operatorname{Cok}(\theta-1).$$

If θ is normal to Y_k, then this map is zero, i.e. v preserves $\operatorname{Ker}(\theta-1)$. Also it induces 0 map on $\operatorname{Im}(\theta-1) = [\operatorname{Ker}(\theta-1)]^{\perp}$. Thus the normal space to Y_k at θ

can be identified with the skew-hermitian matrices $v: \mathrm{Ker}(\theta-1) \circlearrowleft$.

Such a v is a way of pushing the 1 eigenvalues of θ off 1. Now v has eigenvalues of the form $i\lambda$, λ real. Hence the non-trivial eigenspace of v splits into two parts representing the two motions



We have a map $Y_k \longrightarrow G_k$, a heq in fact. And we have a stratification of the normal bundle to Y_k which meshes with the exponential map and the stratification of $V_k = \bigcup_{i \le k} Y_i$. Thus fixing θ the normal stratified space may be identified with the ordered space of subquotients of $\mathrm{Ker}(\theta-1)$.

Stratified space $X = \bigcup_S Y_\alpha$. First of all this should sit over the ordered set of strata $\alpha \le \beta \Leftrightarrow Y_\alpha \subset \overline{Y}_\beta$. Thus we have a map $X \longrightarrow S$ with fibre Y_α over α. Next given $\alpha < \beta$ we must give the normal sphere bundle of Y_α in Y_β, call this $\nu'_{\alpha<\beta}$

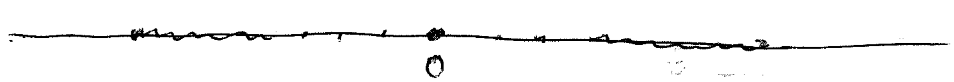
March 19, 1974: stratifications

I have already noted that if U_n is stratified with $Y_k = \{\theta \mid \theta$ has the eigenvalue 1 k-times$\}$, then $Y_k \sim G_k(\mathbb{C}^n)$, ~~in fact we can identify $G_k(\mathbb{C}^n)$ with the~~ and we obtain this homotopy equivalence by pushing all eigenvalues $\neq 1$ into -1. The normal space to Y_k at θ is the space of skew-adjoint maps v from $\mathrm{Ker}(\theta-1)$ to itself, the exponential of v being $e^{iv}\theta$. Thus going from $\theta \in Y_k$ to $e^{iv}\theta \in Y_\ell$ one passes from $\mathrm{Ker}(\theta-1)$ to the kernel of v. Breaking v up into eigenspaces

$$\mathrm{Ker}(\theta-1) = \underbrace{\qquad}_{\substack{i\lambda \\ \lambda<0}}\ \underbrace{\qquad}_{0}\ \underbrace{\qquad}_{\substack{i\lambda \\ \lambda>0}}$$

shows that the passage from θ to $e^{iv}\theta$ corresponds to a Q-map to $\mathrm{Ker}(\theta-1) \in G_k(\mathbb{C}^n)$ from $\mathrm{Ker}(e^{iv}\theta-1)$ in $G_\ell(\mathbb{C}^n)$.

Another model: Consider the space of self-adjoint Fredholm operators. The spectrum of such an A looks like



and we can deform A so that its eigenvalues are $-1, 0, 1$. This space has two components which are contractible – namely where there are finitely many neg. or pos. eigenvalues. We forget these and call the other component R. Stratify R according to the dimension of the kernel. Then R_k maps to $G_k(H)$ by taking the kernel. The fibre over K deforms to the space of orthogonal decompositions $H/K = V_1 \oplus V_2$ where V_1, V_2 are $\simeq H$. This space is contractible by Kuiper's theorem. Thus $R_k \sim G_k(H) \simeq BU_k$.

Again the normal bundle to A in R_k is the self-adjoint ops $\vee$ on $\mathrm{Ker}(A)$, and given $\vee$ it splits $\mathrm{Ker}(A)$ into three pieces $-, 0, +$, so one gets the Q-category.

Equivalence of R with U. Replace R by the homotopy equivalent space R' of self adjoint operators whose spectrum lies in $-1 \leq \lambda \leq 1$ and whose continuous spectrum is $\{-1, 1\}$. In other words, the spectrum piles up on $-1, 1$. and R' is the closure of the operators having a finite no. of eigenvalues in $[-1, 1]$ with $-1, 1$ having infinite multiplicity.

Now one has the exponential $2\pi i$ map from R' to unitary operators with -1 as continuous spectrum, i.e. unitary ops. of form $-1 +$ compact. Thus we get the space U. The exponential map is a homotopy equivalence, since it is so stratum by stratum – again using Kuiper's thm.

May 20, 1974. K-theory of quadratic modules.

Similarity: ① Let V be a non-deg. quadratic space over a field. To any subspace W of V we get a "splitting" of V into quadratic spaces

$$V \sim \underbrace{W \cap W^\perp \oplus V/W+W^\perp}_{\text{hyperbolic}} \oplus \left(W+W^\perp / W\cap W^\perp\right)$$

$$\left(W+W^\perp / W\cap W^\perp\right) = W/W\cap W^\perp \oplus W^\perp/W\cap W^\perp$$

Basically two types of elementary "splitting"

a) orthogonal direct sum

b) $V \sim H(L) \oplus L/L^\perp$ if $L \subset L^\perp$, i.e. L subLagrangian

② On the other hand consider the relations between subquotients $A \subset B$ of a vector space V. One has two basic relations

a) boxing

b) $(A, C+A) \sim (C \cap A, C)$ congruence

③ Consider quadratic space over $A \times A^0$; it is of the form $M \oplus M^*$ and it has two basic kinds of subobjects

a) non-degenerate: Corresponds to giving $L \underset{i}{\overset{p}{\rightleftarrows}} M$ a direct injection.

b) sub-Lagrangian: Corresponds to giving a subquotient of M.

<u>Goal</u>: Construct a K-theory of Quadratic modules. It should be represented by an H-space generated by Quadratic modules with relations coming from a) and b).

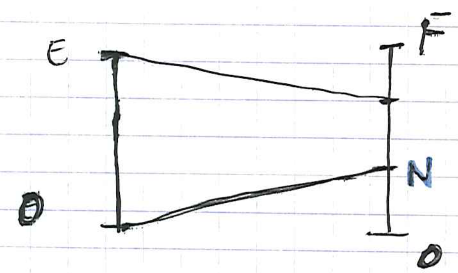
~~Geometric situation in K-theory — One has for~~

Go back to the problem of making a good space out of the correspondences between E and F. Certain correspondence namely

$$E = D = A/N \xrightarrow{\sim} I/N \subset F/N \twoheadleftarrow F$$

can be interpreted as morphisms from E to F in the Q-category. The picture:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & F \\ E & \end{tikzcd}$$



I recall that the stable object attached to the correspondence should be $[N] - [E/D] \in K_0$, which we can think of as $\mathrm{Ind}(\alpha)$, α running from F to E

Now somehow a correspondence in general should give me some sort of path from F to E with invariant $[N] - [E/D]$. Furthermore it should be compatible with composition, when composition is defined.

<u>Problem</u>: Suppose for every pair E, F we consider the set of correspondences $A \subset E \times F$, and for every triple E_0, E_1, E_2 we consider the composable correspondences:

$$\begin{tikzcd}
A \times_{E_1} B \arrow[r] \arrow[d] & B \arrow[r, dash] \arrow[d] & E_2 \\
A \arrow[r] \arrow[d] & E_1 & \\
E_0 & &
\end{tikzcd}$$

etc.

In this way we get a partial category. Is this partial category equivalent to Q? If we consider the partial monoid of correspondences of E with itself, where E is large, do we get a good approximation to the Q-category?