April 3, 1974

Recall that if X is the tree of extensions of a rank 2 bundle M over $C-\infty$ to C, and $\Gamma = \mathrm{Aut}(M)$, then we have seen that

$$I(M) = H_1(X, X_{inst})$$

where X_{inst} is the subcomplex consisting of vertices of X which are instable bundles. When the ground field is finite, this implies that one can define the Euler characteristic $\chi(\Gamma, I)$ as follows.

Since the isom. classes of stable and semi-stable bundles is finite, one can find a normal subgroup Γ' of finite index in Γ which acts freely on $X - X_{inst}$. Thus in

$$0 \longrightarrow I(M) \longrightarrow C_1(X, X_{inst}) \longrightarrow C_0(X, X_{inst}) \longrightarrow 0$$

the $C_i(X, X_{inst})$ are free $\mathbb{Z}[\Gamma']$-modules of finite type, hence $I(M)$ is $\mathbb{Z}[\Gamma']$-projective of finite type. Also

$$H_0(\Gamma', I(M)) = H_1(X/\Gamma', X_{inst}/\Gamma').$$

So it is clear how to define χ of Γ' on $I(M)$:

$$\chi(\Gamma', I(M)) = -\chi(X/\Gamma', X_{inst}/\Gamma')$$

and then one can define

$$\chi(\Gamma, I(M)) = \frac{-1}{[\Gamma : \Gamma']}\, \chi(X/\Gamma', X_{inst}/\Gamma').$$

Alternative version:

$$\chi(\Gamma, I(m)) = \sum_{\sigma \in X/\Gamma} \frac{(-1)^{\sigma}}{\mathrm{aut}(\sigma)}$$

This infinite sum converges. In effect, if I take the sum over the vertices of even degree (resp. odd degree), this series converges by the Siegel formula. As for the one-simplices, way out in the tree the auto. group is the same as for one of its vertices.

It is clear that there is a lot of cancellation in the above sum, because once in the unstable region, the term for a vertex is cancelled by the unique simplex in the cusp which follows this vertex.

I can compute this Euler characteristic using the Siegel formula. In effect, having chosen Γ' acting freely on the region X_n defined by $\mu_{max} \leq n$ $(\deg = 0, -1)$; i.e. X_n is bounded by vertices $\simeq L(n) \oplus L^*(-n)$, let $Y_n = X_n/\Gamma$. Note that everywhere except at ∂Y_n, there are $(q+1)$ edges coming into each vertex. At each point of ∂Y_n, there are q edges coming in. Thus one finds that

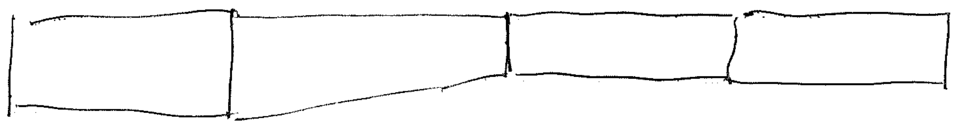
$$2\begin{pmatrix}\text{number of edges} \\ \text{in } Y_n - \partial Y_n\end{pmatrix} = (q+1)\begin{pmatrix}\text{no. of vertices} \\ \text{in } Y_n - \partial Y_n\end{pmatrix}$$

(count pairs $\sigma \subset \sigma_1$ in two ways). Not quite for each edge going to ∂Y_n belongs to only one vertex. Thus

$$(q+1)\begin{pmatrix}\text{no. of vertices}\\ \text{in } Y_n - \partial Y_n\end{pmatrix} = 2\begin{pmatrix}\text{no. of edges}\\ \text{in } Y_n - \partial Y_n\end{pmatrix} - \operatorname{card}(\partial Y_n)$$

Too complicated. Return to your picture.

Recall one has the following picture of X/Γ:



stables bundles deg 0	stable bundles deg -1	semi-stable bundles deg 0 not stable	instable $\mu_{max}-\mu_{min}=1$	instable $\mu_{max}-\mu_{min}=2$

Call these sets

B_{-2} $\quad$ B_{-1} $\quad$ B_0 $\quad$ B_1 $\quad$ B_2

Now from each element of B_n $\;n \geq 1$ there are q edges going to B_{n-1} and 1 edge going to B_{n+1} of the sort that

$$\chi(B_n,\Gamma) = q\,\chi(B_{n+1},\Gamma) \qquad n \geq 1.$$

The Euler char I want is

$$\chi(B_{-2},\Gamma) - (q+1)\chi(B_{-2},\Gamma)$$
$$+\chi(B_{-1},\Gamma)$$
$$\chi(B_0,\Gamma) - (q+1)\chi(B_0,\Gamma) \qquad = -q\,\chi(B_{-2},\Gamma) + \chi(B_1,\Gamma) - q\,\chi(B_0,\Gamma)$$
$$0\left\{\begin{array}{l}\chi(B_1,\Gamma)\\ \chi(B_2,\Gamma) - (q+1)\chi(B_2,\Gamma)\end{array}\right.$$

$$0\ \Big\{$$

The Siegel formula gives

$$\chi(B_{-2},\Gamma) + \chi(B_0,\Gamma) + \chi(B_2,\Gamma) + \cdots = \frac{1}{q-1} Z_C(q).$$

$$\chi(B_{-1},\Gamma) + \chi(B_1,\Gamma) + \chi(B_3,\Gamma) + \cdots = \frac{1}{q-1} Z_C(q)$$

Thus one gets

Thm. $$\boxed{\chi(\Gamma, I(M)) = + Z_C(q)}$$

Notice also that we can easily compute the number of stable bundles of degree -1 up to isom. Since B_{2n+1} for n large consists of iso classes $L(n) \oplus L^*(-n-1)$ and

$$\mathrm{Aut}\left(L(n) \oplus L^*(-n-1)\right) = \begin{pmatrix} k^* & H^0(L^2(2n+1)) \\ & k^* \end{pmatrix}$$

$$h^0\left(L^2(2n+1)\right) = 2n+1+1-g.$$

$$\mathrm{aut}\left(L^2(2n+1)\right) = (q-1)^2 \, q^{2n+2-g}$$

$\therefore$

$$\chi(B_{2n+1},\Gamma) = \frac{h}{(q-1)^2 \, q^{2n+2-g}} \qquad n \geq 0$$

$\therefore$

$$\chi(B_1,\Gamma) + \chi(B_3,\Gamma) + \cdots = \frac{h}{(q-1)^2} \frac{1}{q^{2-g}} \sum_{n \geq 0} \frac{1}{q^{2n}}$$

$$= \frac{h}{(q-1)^2} \frac{1}{q^{2-g}} \frac{1}{1-\frac{1}{q^2}}$$

$$= \frac{h q^g}{(q-1)^2} \frac{1}{(q^2-1)}$$

Therefore

$$\chi(B_{-1}, \Gamma) = \frac{1}{q-1} Z_C(q) - \frac{hq^g}{(q-1)^2} \frac{1}{q^2-1}$$

or

$$\boxed{\begin{pmatrix}\text{no. of stable rg 2} \\ \text{bundles of deg } -1 \\ \text{with given det.}\end{pmatrix} = Z_C(q) - \frac{hq^g}{(q-1)(q^2-1)}}$$

<u>Check</u>: $g=0$. get 0

$g=1$ $\qquad \dfrac{1+(-1-q+\cancel{h})q+q^3 \cancel{-hq}}{(q-1)(q^2-1)} = 1$

Next try to compute the number of stable bundles of degree 0. It is necessary to divide the set B_0 of semi-stable non-stable vertices into the decomposable and non-decomposable groups:

$$B_0 = B_0^{dec} \amalg B_0^{ind}.$$

and to divide J into $J_2 \amalg J_+ \amalg J_-$ where $J_2 = \{L \mid L = L^*\}$, and $J_+^* = J_-$; corresponding to this we have $h = h_0 + 2h_+$. We know that through each element of B_0^{ind} there are q vertices going toward B_{-1}, and 1 toward B_1. From something of the form

$L \in J_2$	$L \oplus L$	have	$q+1$ edges to B_{+1}	, none to B_{-1}
$L \in J_+$	$L \oplus L^*$	—	2 ——— B_1	, $q-1$ to B_{-1}

So now count the number of edges from B_0 to B_1. On one hand there are

$$q\,\chi(B_1,\Gamma) = \frac{h}{(q-1)^2\,q^{1-g}}$$

On the other we have

$$= \frac{\mathrm{card}(J_2)}{(q^2-1)(q^2-q)}(q+1) + \frac{\mathrm{card}(J_+)}{(q-1)^2}\,2 + \chi(B_0^{ind},\Gamma)$$

But

$$0 = \chi(B_0^{dec},\Gamma) - \frac{\mathrm{card}(J_2)}{(q^2-1)(q^2-q)} - \frac{\mathrm{card}(J_+)}{(q-1)^2}$$

~~Thus~~ Now

$$\chi(B_{-2},\Gamma) + \chi(B_0,\Gamma) + \underbrace{\frac{h}{(q-1)^2}\sum_{n=1}^{\infty}\frac{1}{q^{2n+1-g}}}_{=\ \frac{h\,q^{g-1}}{(q-1)^2(q^2-1)}} = \frac{1}{q-1}Z_c(q)$$

But

$$\frac{h}{(q-1)^2 q^{1-g}} = \frac{\mathrm{card}(J_2)}{(q^2-1)(q-1)} + \frac{\mathrm{card}(J_+)}{(q-1)^2} + \chi(B_0,\Gamma)$$

So

$$\chi(B_{-2},\Gamma) + \frac{h\,q^{1+g}}{(q-1)^2(q^2-1)} = \frac{\mathrm{card}(J_2)}{(q^2-1)(q-1)} + \frac{\mathrm{card}(J_+)}{(q-1)^2} + \frac{1}{q-1}Z_c(q)$$

Check: take $g=0$

$$\chi(B_{-2},\Gamma) + \frac{q}{(q-1)^2(q^2-1)} = \frac{1}{(q^2-1)(q-1)} + 0 + \frac{1}{(q-1)^2(q^2-1)} \Rightarrow \chi(B_{-2},\Gamma) = 0 \ ✓$$

Now what I really want to get at is the Euler char of the stable part which is

$$\chi(B_{-2},\Gamma) - q\,\chi(B_{-1},\Gamma)$$

$$= \cancel{-\frac{hq^{1+g}}{(q-1)^2(q^2-1)}} + \frac{\mathrm{card}(J_2)}{(q^2-1)(q-1)} + \frac{\mathrm{card}(J_+)}{(q-1)^2} + \frac{1}{(q-1)}Z_C(q)$$

$$\cancel{+\frac{hq^{1+g}}{(q-1)^2(q^2-1)}} - \frac{q}{q-1}Z_c(q)$$

$$= -Z_c(q) + \frac{\mathrm{card}(J_2)}{(q^2-1)(q-1)} + \frac{\mathrm{card}(J_+)}{(q-1)^2}$$

I want to find a formula for $\chi(X/\Gamma)$. Clearly

$$\chi(X_{inst}/\Gamma) = \mathrm{card}\, J$$

because it is clear that the deformation to the cusps proceeds Γ-equivariantly. On the other hand

$$\chi(X_{inst};\Gamma) = 0$$

as we have seen.

$$\frac{1}{q-1}\chi(X/\Gamma, X_{inst}/\Gamma) = \chi(X, X_{inst};\Gamma) + \sum_{\sigma\in(X-X_{inst})/\Gamma}\left(\frac{1}{q-1} - \frac{1}{\mathrm{aut}(\sigma)}\right)(-1)^{\sigma}$$

What are the simplices whose auto groups exceed k^*?

a) semi-stable stable ind. bundles of degree 0 + edge leading to B_1. | terms cancel

b) $L\oplus L$, $L\in J_2$; $\quad \mathrm{aut} = (q^2-1)(q^2-q)$

edge leading from $L\oplus L$ to B_1, aut $(q-1)^2 q$

contribution

$$\left[1-\frac{(q-1)}{(q^2-1)(q^2-q)}\right]-\left[1-\frac{(q-1)}{(q-1)^2 q}\right]$$

$$=\frac{1}{(q-1)\,q}-\frac{1}{(q^2-1)(q)}=\frac{1}{q(q-1)}\left[1-\frac{1}{q+1}\right]=\frac{1}{q^2-1}$$

Thus get

$$\mathrm{card}(J_2)\ \frac{1}{q^2-1}\left(\cancel{\frac{1}{q}-\frac{1}{q^2-1}}\right).$$

c) $L\oplus L^*$, $L\in J_+$. aut $=(q-1)^2$

2 edges leading from $L\oplus L^*$ to B_1 with aut$=(q-1)^2$

contribution

$$\cancel{\frac{2}{q-1}-\frac{1}{q-1}=\frac{1}{q-1}}\qquad\left(1-\frac{1}{q-1}\right)-2\left(1-\frac{1}{q-1}\right)$$

Thus get

$$\mathrm{card}(J_+)\cdot\frac{-(q-2)}{q-1}.$$

d) indecomposable bundles of degree 0 which decompose into $L\oplus L^*$ over $\mathbb{F}_{q^2}$. Here aut$=q^2-1$. Let α be the number of these. Then we have the contribution

$$\alpha\left(1-\frac{1}{q+1}\right)=\frac{\alpha q}{q+1}$$

Therefore it seems that

$$\chi(X/\Gamma, X_{inst}/\Gamma)=-(q-1)Z_C(q)+\alpha\frac{q}{q+1}$$
$$+\ \mathrm{card}(J_2)\frac{1}{q^2-1}\left(\cancel{\frac{1}{q}-\frac{1}{q^2+1}}\right)+\mathrm{card}(J_+)\frac{-(q-2)}{q-1}$$

To check for an elliptic curve, put $a = \mathrm{card}(J_2)$,
$h = \mathrm{card}\, J$, $\mathrm{card}(J_+) = \frac{1}{2}(h-a)$, $\mathrm{card}(\mathrm{Im}\, x) = \frac{1}{2}(h+a)$

$$\alpha = q+1 - \mathrm{Im}\{x : J \to \mathbb{P}^1\}$$
$$= q+1-\frac{1}{2}(h+a).$$

Then

$$\left[q+1-\frac{1}{2}(h+a)\right]\frac{q}{q+1} + a\,\frac{1}{q^2-1} + \frac{1}{2}(h-a)\,\frac{(q-2)}{q-1}$$

$$= q + h\left(-\frac{1}{2}\frac{q}{q+1} - \frac{1}{2}\frac{q-2}{q-1}\right) + a\left(-\frac{1}{2}\frac{q}{q+1} + \frac{1}{q^2-1} + \frac{1}{2}\frac{q-2}{q-1}\right)$$

$$-\frac{h}{2}\left(\frac{q^2-q + \overbrace{(q+1)(q-2)}^{q^2-q-2}}{q^2-1}\right) \qquad \cancel{-\frac{q(q-1)+2+(q-2)(q+1)}{q^2-1}} = 0$$

$$= q - h\,\frac{q^2-q-1}{q^2-1} = q - h + \frac{hq}{q^2-1}$$

add this to

$$-(q-1)Z_C(q) = -\frac{1+(-1-q+h)q+q^3}{q^2-1} = -(q-1) - \frac{hq}{q^2-1}$$

and one gets

$$\chi(X/\Gamma, X_{mixt}/\Gamma) = 1-h$$

showing that X/Γ is a tree.

Thus we do have the formula on the bottom of page 8.

~~Prop: $h = \mathrm{card}(J)$, $a = \mathrm{card}(J_0) = \mathrm{card}\{L \in J \mid L = L^*\}$~~
~~number of $L \in J(\overline{\mathbb{F}}_q)$ such that $L^\sigma = L^*$ minus~~

To compute α now.

α = number of $L \in J(\overline{\mathbb{F}}_q)$ such that $L^\sigma = L^*$ minus a divided by 2.

Put
$$b = \#\ \mathrm{Ker}\{1+\sigma \text{ on } J(\overline{\mathbb{F}}_q)\} \qquad \alpha = \frac{b-a}{2}$$
$$h = \#\ \mathrm{Ker}\{1-\sigma \ \text{———}\ \} \qquad (\mathrm{card}\, J_+) = \frac{h-a}{2}$$

Then

$$h + \alpha \frac{q}{q+1} + a\frac{1}{q^2-1} + \mathrm{card}(J_+)\left(\frac{2-q}{q-1}\right)$$

$$= h + \left(\frac{b-a}{2}\right)\left(1-\frac{1}{q+1}\right) + a\left(\frac{1}{q^2-1}\right) + \left(\frac{h-a}{2}\right)\left(\frac{2-q}{q-1}\right)$$

$$= h\left[1+\frac{1}{2}\,\frac{2-q}{q-1}\right] + a\left(-\frac{1}{2}+\frac{1}{2}\,\frac{1}{q+1}+\frac{1}{q^2-1}-\frac{1}{2}\,\frac{2-q}{q-1}\right)$$

$$+\, b\left(\frac{1}{2}-\frac{1}{2(q+1)}\right)$$

$$= \frac{h}{2}\,\frac{q}{q-1} + \frac{b}{2}\,\frac{q}{q+1}$$

So we have

$$\chi(X/\Gamma) = -(q-1)Z_c(q) + \frac{h}{2}\,\frac{q}{q-1} + \frac{b}{2}\,\frac{q}{q+1}$$

But if the numerator of $Z_c(z)$ is $1+c_1z+\dots+c_{2g}z^{2g}$ then $h = 1+c_1+\dots+c_{2g}$ $\quad b = 1-c_1+c_2\dots+c_{2g}$

so

$$\frac{h}{2}\,\frac{q}{q-1} + \frac{b}{2}\,\frac{q}{q+1} = \frac{q}{2(q^2-1)}\left[(q+1)(1+c_1+\dots+c_{2g}) + (q-1)(1-c_1+\dots)\right]$$

$$= \frac{q}{(q^2-1)}\; q(1+c_2+\dots+c_{2g}) + (c_1+\dots+c_{2g-1})$$

so

$$\chi(X/\Gamma) = \frac{-(1+c_1q+\dots+c_{2g}q^{2g}) + q^2(1+c_2+c_4+\dots)+q(c_1+\dots)}{q^2-1}$$

$$= 1 - qc_3 + \frac{q^2-q^4}{q^2-1}c_4 + \frac{q-q^5}{q^2-1}c_5 + \dots$$

$$\boxed{\chi(X/\Gamma) = 1 - qc_3 - q^2c_4 - q(q^2+1)c_5 - \dots}$$

April 6, 1974. Euler characteristic formula

In the case of symmetric spaces the Euler measure is a multiple of the volume. Want the p-adic analogue.

So let X be Tits building belonging to a F-vector space of dimension r, F field with a discrete valuation and residue field $\mathbb{F}_q$. Let Γ be a group acting freely on X and such that X/Γ is compact. I want a formula for $\chi(X/\Gamma)$ in terms of $\mathrm{vol}(X/\Gamma)$ = number of vertices in X/Γ of a given type. Put $Y = X/\Gamma$.

Example: $r = 2$. Here Y is of dim. 1

$$\#\{\text{pairs } (\sigma_0,\sigma_1),\ \sigma_0 < \sigma_1\} = 2\,\#\{\sigma_1\}$$

$$\shortparallel$$

$$(q+1)\,\#\{\sigma_0\}$$

Hence

$$\chi(Y) = \#\{\sigma_0\} - \#\{\sigma_1\} = \left(1 - \frac{q+1}{2}\right)\#\{\sigma_0\}$$

$$= (1-q)\,\frac{\#\{\sigma_0\}}{2}$$

Example:

$$\#\{\sigma_1 \mid \sigma_0 < \sigma_1\} = \#\,G_1(\bar V) + \cdots + \#\,G_{r-1}(\bar V)$$

$$\#\{\sigma_p \mid \sigma_0 < \sigma_p\} = \sum_{\substack{i_1+\cdots+i_p<r \\ i_j>0}}{}' \#\,G_{i_1,\ldots,i_p}(\bar V)$$

$$\#\{\sigma_p\} = \frac{1}{p+1}\sum_{i_1+\cdots+i_p<r} \#\,G_{i_1\cdots i_p}(\bar V)\ \cdot\ \#\{\sigma_0\}$$

$$\sum(-1)^p\,\#\{\sigma_p\} = \left[\sum_{p=0}^{r-1}\frac{(-1)^p}{p+1}\sum_{i_1+\cdots+i_p<r}\#\,G_{i_1\cdots i_p}(\bar V)\right]\#\{\sigma_0\}$$

For $r = 3$ one gets

$$\left[1 - \tfrac{1}{2}(1+q+q^2 + 1+q+q^2) + \tfrac{1}{3}(1+q)(1+q^2)\right] \#\{\sigma_0\}$$

$$= (1-q)(1-q^2)\,\frac{\#\{\sigma_0\}}{3}$$

<u>Conjecture</u>: For general r one gets

$$\sum (-1)^p \# \{\sigma_p\} = (1-q)\cdots(1-q^{r-1})\,\frac{\#\{\sigma_0\}}{r+1}.$$

To prove this it will be necessary to do the combinatorics <u>intelligently</u>.

So now return to curves, where X represents extensions of a given vector bundle [redacted] M on $C-\infty$ and $\Gamma = \mathrm{Aut}(M)$. Here I want to compute the sum

$$\sum_{\sigma \in X/\Gamma} \frac{(-1)^\sigma}{\mathrm{aut}(\sigma)}$$

which I claim converges absolutely. To see this it suffices to note that the sum over the vertices converges by the Siegel formula, [redacted] hence because one has maps

[redacted]

$$\begin{array}{ccc} \{(\sigma_0,\sigma_i) \mid \sigma_0 < \sigma_i\} & \longrightarrow & \{\sigma_0\} \\ \big\downarrow {\scriptstyle \text{each fibre has } i+1 \text{ elts.}} & & {\scriptstyle \text{each fibre has } m \text{ elements}} \\ \{\sigma_i\} & & \end{array}$$

[redacted] one finds

$$(i+1)\sum_{\sigma_i \in X/\Gamma} \frac{1}{\mathrm{aut}(\sigma_i)} = m \sum_{\sigma_0 \in X/\Gamma} \frac{1}{\mathrm{aut}(\sigma_0)}$$

So it is now clear to me that I will get in this way the formula

$$\sum_{\sigma \in X/\Gamma} \frac{(-1)^\sigma}{\mathrm{aut}(\sigma)} \overset{?}{=} (1-q)\cdots(1-q^{r-1}) \sum_{\Lambda^r E \in \alpha} \frac{1}{\mathrm{aut}(E)}$$

This part is only combinatorial.

But the next thing to will be to identify

$$\sum_{\sigma \in X/\Gamma} \frac{(-1)^\sigma}{\mathrm{aut}(\sigma)} \overset{?}{=} \chi(\Gamma; X, X_{inst})$$

which means one has lots of cancellation in the infinite sum.

<u>April 6, 1974.</u> Remarks on $\chi(\Gamma)$.

For an honest arithmetic group Serre has defined $\chi(\Gamma)$ as $\frac{1}{[\Gamma:\Gamma']}\chi(\Gamma')$ where Γ' is a torsion-free subgroup of finite index and $\chi(\Gamma') = \chi(B\Gamma')$ is defined because $B\Gamma'$ has the homotopy type of a finite complex.

The question arises of how to make sense of this when Γ is arithmetic in the function field sense. To fix the ideas let $\Gamma = GL_2(A)$, $A = \Gamma(C, \mathcal{O}_C - \infty)$, and let X be the associated tree.

The good number to call $\chi(\Gamma)$ seems to be

$$\chi(\Gamma;\, X \bmod X_{inst})$$

for the following reasons.

1) $$\chi(\Gamma;\, X \bmod X_{inst}) = \sum_{\sigma\in X/\Gamma} (-1)^{\sigma} \frac{1}{\#\Gamma_\sigma}$$

2) $$\chi(\Gamma',\, X \bmod X_{inst}) = [\Gamma':\Gamma]\,\chi(\Gamma,\, X \bmod X_{inst})$$

3) $\chi(\Gamma;\, X \bmod X_{inst})$ is given by a ζ function value

4) If Γ' acts freely on $X - X_{inst}$, then

$$\chi(\Gamma',\, X \bmod X_{inst}) = \chi(X/\Gamma',\, X_{inst}/\Gamma')$$
$$= \chi_c(X/\Gamma')$$

(This last formula shows that we should think of $\chi(\Gamma)$ as depending on (X,Γ) with compact support.)

The other thing one has to examine is $\chi(X/\Gamma')$. Clearly

$$\chi(X/\Gamma') = \chi_c(X/\Gamma') + h'$$

where $h' = P_1(F)/\Gamma'$ is the number of Γ'-cusps. But h' does not depend multiplicatively in Γ', e.g. if

$$P_1 F = \coprod_{i=1}^{h} \Gamma/\Gamma_i$$

and if Γ' is normal of finite index in Γ, then

$$P_1 F/\Gamma' = \coprod_i \Gamma/\Gamma'\Gamma_i$$

$$1 \longrightarrow \underset{\substack{\mathrel{\wr\!\shortparallel} \\ \Gamma_i/\Gamma_i \cap \Gamma'}}{\Gamma'\Gamma_i/\Gamma'} \longrightarrow \Gamma/\Gamma' \longrightarrow \Gamma/\Gamma'\Gamma_i \longrightarrow 0$$

so

$$\frac{h'}{[\Gamma:\Gamma']} = \sum_{i=1}^{h} \frac{1}{[\Gamma_i : \Gamma_i \cap \Gamma']}$$

This changes as Γ' does. Notice however that as Γ' gets smaller and smaller, then the denominators on the right get larger. Thus

$$\varinjlim_{\Gamma'} \frac{\chi(X/\Gamma')}{[\Gamma:\Gamma']} = \chi(\Gamma, X \bmod X_{inst})$$

Compare with the arithmetic case $\Gamma = SL_2(\mathbb{Z})$. Here it should be true that

$$\chi(X/\Gamma') = \chi_c(X/\Gamma') + h'\,\overset{0}{\overset{\shortparallel}{\chi(S^1)}}$$

because the cusps are circles. Also by the Borel-Serre duality thm. one has

$$H_i(\Gamma'; I) = H^{d-i}(\Gamma'; \mathbb{Z})$$

so the Steinberg Euler characteristic should be the same as the Euler char.

April 6, 1974. Upper half plane and quadratic forms.

I want the formulas giving the isomorphism of H with pos. def. quadratic binary forms.

$$SL_2(\mathbb{R})/SO_2(\mathbb{R}) \xrightarrow{\sim} H$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} SO_2(\mathbb{R}) \longmapsto \frac{ai+b}{ci+d}$$

On the other hand

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (ax+by)^2 + (cx+dy)^2$$
$$= (a^2+c^2)x^2 + 2(ab+cd)xy + (b^2+d^2)y^2$$

gives an isomorphism

$$SL_2(\mathbb{R})/SO_2(\mathbb{R}) \xrightarrow{\sim} \text{pos. def. forms on } \mathbb{R}^2 \text{ of discriminant } 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right|^2 = (x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

So now if $z = \alpha + i\beta \in H$, $\beta > 0$, then

$$\begin{pmatrix} \beta^{1/2} & \alpha\beta^{-1/2} \\ 0 & \beta^{-\frac{1}{2}} \end{pmatrix}(i) = \frac{\alpha\beta^{-\frac{1}{2}}i + \beta^{\frac{1}{2}}}{\beta^{-\frac{1}{2}}} = \alpha i + \beta$$

and the corresponding quadratic form is

$$\left(\beta^{\frac{1}{2}}x + \alpha\beta^{-\frac{1}{2}}y\right)^2 + \left(\beta^{-\frac{1}{2}}y\right)^2 = \beta x^2 + 2\alpha xy + \frac{\alpha^2+1}{\beta}y^2$$

Rule

$$\boxed{\alpha + i\beta \longleftrightarrow \begin{pmatrix} \beta & \alpha \\ \alpha & \frac{\alpha^2+1}{\beta} \end{pmatrix}}$$

<u>April 7, 1974.</u>

$k = \mathbb{F}_q$ $\quad q$ large.

C curve over k, ∞ a rational point $A = \Gamma(C-\infty, \mathcal{O}_C)$.

X the building associated to a proj. module M of rank r over A. $\quad \Gamma = \mathrm{Aut}(M)$.

If X_{inst} is the full subcomplex of X containing those vertices whose corresponding bundles are instable I believe I can show that

$$H_i(X, X_{inst}) = \begin{cases} 0 & i \neq r-1 \\ I & i = r-1 \end{cases}$$

where I = Steinberg module associated to $F \otimes_A M$.

But if now Γ' is a subgroup of finite index of Γ such that Γ' acts freely on the semi-stable region, then each of the complexes $C_q(X, X_{inst})$ is a free Γ'-module of finite type. It follows from the exact sequence

$$0 \longrightarrow I \longrightarrow C_{r-1}(X, X_{inst}) \longrightarrow \cdots\cdots \longrightarrow C_0(X, X_{inst}) \to 0$$

that I is a projective $\mathbb{Z}[\Gamma']$-module of finite type, which is even stably-free.

Hence

$$0 \to I_{\Gamma'} \longrightarrow C_{r-1}(X, X_{inst})_{\Gamma'} \to \cdots \to C_0(X, X_{inst})_{\Gamma'} \to 0$$

is exact.

$$\begin{array}{ccc} \| & & \| \\ C_{r-1}(X/\Gamma', X_{inst}/\Gamma') \to \cdots \to & & C_0(X/\Gamma', X_{inst}/\Gamma') \to 0 \end{array}$$

And so we see that we have

$$H_i(X/\Gamma', X_{inst}/\Gamma') = \begin{cases} 0 & i \neq r-1 \\ I_{\Gamma'} & i = r-1. \end{cases}$$

We even know the rank of $I_{\Gamma'}$ over $\mathbb{Z}$ since we have the Harder formula for $\chi(\Gamma; X \bmod X_{inst})$. For example, in the rank 2 case I have

$$\chi(\Gamma; X \bmod X_{inst}) = -Z_C(q).$$

April 8, 1974

It will be necessary to find proofs of various things I believe. It is probably enough to understand completely bundles of rank 3.

Usual notations, $C, \infty, A, M, \Gamma, X, V$.

Maybe I should think of a point of X as an equivalence class of $\mathcal{O}_C$-lattices Λ in M, where Λ and Λ' are equivalent iff $\Lambda = \Lambda'(n)$ for some n.

X_{ins} = full subcomplex whose vertices are the instable Λ. From the theory of canonical filtrations of vector bundles I know already that X_{ins} has a covering whose nerve is the Tits complex $T(V)$ as follows. For each W proper subspace of V, I let X_W be the full subcomplex of X whose vertices are those Λ such that $\mu_{min}(\Lambda \cap W) > \mu_{max}(\Lambda/\Lambda \cap W)$, or equivalently such that $\Lambda \cap W$ is part of the canonical filtration of Λ. Then

$$X_{ins} = \bigcup_W X_W$$

and

$$X_{W_0} \cap \cdots \cap X_{W_p} \neq \phi \iff \{W_0, \ldots, W_p\} \text{ is a simplex of } T(V).$$

Put

$$X_\tau = \bigcap X_{W_i} \qquad \text{if } \tau = \{W_0, \ldots, W_p\}$$

The first thing I want to understand well is why X_τ is contractible. Also I want to show that the weighted Euler characteristic

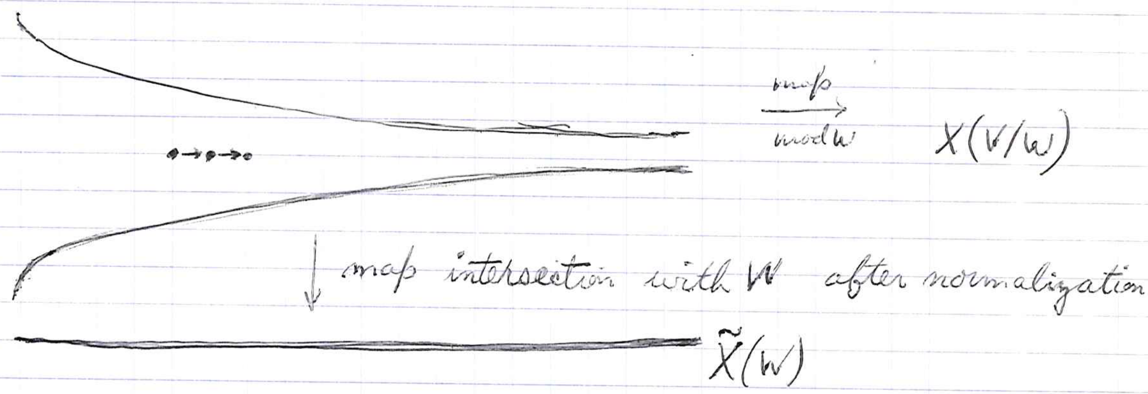
$$\sum_{\sigma \subset X_\tau} (-1)^\sigma \frac{1}{\mathrm{aut}(\sigma)}$$

is zero. Also it would be nice to be able to enlarge X_{ind} so that it includes semi-stable indecomposable bundles.

First recall what happens in rank 2 case. Here W is a line in V and I can normalize my lattice Λ such that $\Lambda + W/W$ is a fixed lattice in V/W. Then I have as basic operation which increases the size of $\Lambda \cap W$ by 1.

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \cap W \arrow[r] \arrow[d, hook] & \Lambda \arrow[r] \arrow[d, hook] & \Lambda/\Lambda \cap W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda \cap W(1) \arrow[r] & \Lambda^* \arrow[r] & \Lambda/\Lambda \cap W \arrow[r] & 0.
\end{tikzcd}$$

This is really the only possibility and gives the picture:



~~What point is~~ ~~Why does this prove $X(V)$ is contractible? Because if I take a finite subcomplex K of $X(V)$, then I can contract~~

Let X be the Tits building of lattices in an F-vector space V, lattices for a d.v.r. $\mathcal{O}$. Let rank $V = r$, whence $\dim X = r-1$. I recall that the link of any point of X is a bouquet of spheres of dimension $r-2$. In more detail, let σ be a p-simplex of X. Realize σ by a chain of lattices $L_0 < L_1 < \cdots < L_p$; where $L_p < \pi^{-1}L_0$. Any vertex in the link of L_0 may represented by a unique lattice L such that $L_0 < L < \pi^{-1}L_0$, and the vertex is in the link of σ if L refines the chain $L_0 < \cdots < L_p < \pi^{-1}L_0$; and is not in this chain. One concludes that

$$\operatorname{Link}\sigma = \left|\{L \mid L_0 < L < L_1\}\right| * \cdots * \left|\{L \mid L_p < L < \pi^{-1}L_0\}\right|$$

where $|S|$ denotes the simplicial complex belonging to the poset S. Hence if $r_0 = \dim(L_1/L_0), \ldots, r_p = \dim(\pi^{-1}L_0/L_p)$, $\sum_{i=0}^{p} r_i = r$, and

$$\begin{aligned}
\operatorname{Link}\sigma &= T(L_1/L_0) * \cdots * T(L_p/\pi^{-1}L_0) \\
&= \left(\bigvee S^{r_0-2}\right) * \cdots * \left(\bigvee S^{r_p-2}\right) \\
&= \bigvee S^{r-2p-2+p} = \bigvee S^{r-p-2}
\end{aligned}$$

Hence

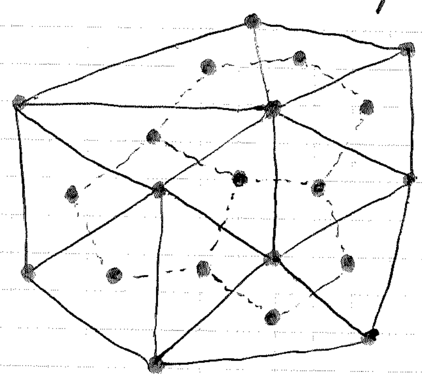
$$\begin{aligned}
\operatorname{Link}(b_\sigma) &= \partial\sigma * \operatorname{Link}\sigma \\
&= S^{p-1} * \bigvee S^{r-p-2} \\
&= \bigvee S^{r-2}
\end{aligned}$$

$p-1+r-p-2+1$
$= r-2$

as claimed. Note also that the number of these spheres is $\sum_{i=0}^{p} \frac{1}{2} r_i(r_i-1)$

q

Dual complex. Recall that there is something called the dual cell complex to a PL manifold X:



To obtain this, consider the ordered set of simplices S of X, but with the reverse ordering, so that now one has the the dimension function

$$c(\sigma) = n - \dim(\sigma).$$

Then one [redacted] attaches to σ [redacted] the ordered set

$$S_{<\sigma} = \{\tau \mid \tau > \sigma\}$$

which is the link of σ, and, which by the assumption [redacted] that X is a PL-manifold, [redacted] is homotopy equivalent to a sphere of dimension $c(\sigma) - 1$. Thus by filtering S by

$$S_{\leq p} = \{\sigma \in S \mid c(\sigma) \leq p\}$$

one gets [redacted] the skeletal filtration of a CW complex. In effect [redacted] in passing from $S_{\leq p-1}$ to $S_{\leq p}$, I add those σ with $c(\sigma) = p$, and σ is attached by putting a cone on $\{\tau \mid \tau > \sigma\}$ which is a [redacted] $(p-1)$-sphere.

Now consider the case where X is a simplicial complex such that the link at each point is (homologically) a bouquet of $(n-1)$-spheres. Then again I can let S be the set of simplices with the ordering reverse to inclusion, and $c(\sigma) = n - \dim(\sigma)$.

$$S_{\leq p} = \{\sigma \in S \mid c(\sigma) \geq p\}.$$

Again I get a spectral sequence

$$E^1_{pq} = H_{p+q}(S_{\leq p}, S_{<p}) = \bigoplus_{c(\sigma)=p} H_{p+q}(\{\tau \geq \sigma\}, \underbrace{\{\tau > \sigma\}}_{\substack{= \mathrm{Link}(\sigma) \\ \simeq \\ \bigvee S^{\cancel{n-\dim\sigma}-1}_{\;p}}})$$

$$\Longrightarrow H_*(X).$$

$$H_{p+q}(\{\tau \geq \sigma\}, \{\tau > \sigma\}) = \begin{cases} 0 & q \neq 0 \\ I(\sigma) & q = 0. \end{cases}$$

So we get the complex

$$\cdots\cdots \longrightarrow \bigoplus_{d(\sigma)=n-1} I(\sigma) \longrightarrow \bigoplus_{d(\sigma)=n} I(\sigma)$$

whose homology is that of X.

---

no ~~Now I wanted to apply this to show the Steinberg module is finite type projective over Γ' when Γ' is net, (net means that every element of finite order has order a power of p). The idea is to use the fact that for each σ, Γ'_σ is a p-group, hence $I(\sigma)$ is a $\mathbb{Z}[\Gamma'_\sigma]$- free module of finite type. There are~~

some problems at the boundary.

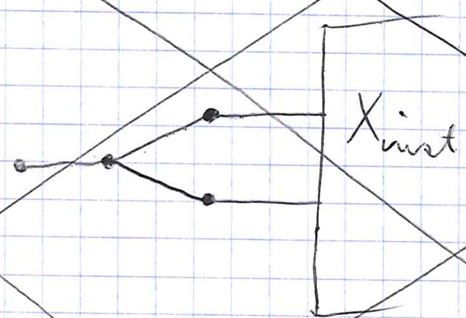
Example: $r = 2$. Here

$$I = H_1(X, X_{inst})$$

One has

$$0 \to I \longrightarrow \bigoplus_{\substack{\sigma \in X - X_{inst} \\ d(\sigma)=0}} I(\sigma) \longrightarrow \bigoplus_{\substack{\sigma \in X - X_{inst} \\ d(\sigma)=1}} \begin{cases} \mathbb{Z} & \text{if } \partial\sigma \in X - X_{inst} \\ 0 & \text{if not.} \end{cases} \to 0$$

Picture:



So what we want works here.

Conjecture: If Γ' is net (every element of finite order has order a power of p), then the Steinberg module is projective of finite type over $\mathbb{Z}[\Gamma']$.

I know the conclusion is true when Γ' acts freely on $X - X_{inst}$, because then $C_*(X, X_{inst})$ is free fin. type over $\mathbb{Z}[\Gamma']$.

In the case $r=2$, I can enlarge X_{inst}, and split away the decomposable semi-stable bundles of degree 0, in which case Γ' will act freely. $\therefore$ The conjecture is true for $r=2$.

<u>April 10, 1974</u>

F = function field of a curve C, V vector space rank r over F, we consider vector bundles E over C with generic fibre V; we can think of E as a lattice inside of V.

I have to recall first the canonical filtration of E which is the unique subbundle filtration $0 < E_1 < \cdots < E_p = E$ such that the quotients are semi-stable with strictly decreasing slopes. The uniqueness comes as follows. Let $F \subset E$, and consider the induced filtration $F_i = F \cap E_i$. Since $F \cap E_i / F \cap E_{i-1}$ embeds in E_i / E_{i-1} its slope (when defined, ie $F \cap E_{i-1} \neq F \cap E_i$) is $\leq \mu_i = \text{slope}(E_i/E_{i-1})$.

$$\deg(F) = \sum \deg(F \cap E_i / F \cap E_{i-1})$$
$$\leq \sum \mu_i \operatorname{rank}(F \cap E_i / F \cap E_{i-1})$$

$$\operatorname{rank}(F) = \sum_i \operatorname{rank}(F \cap E_i / F \cap E_{i-1})$$

Thus if $\mu(F) = \mu_{max}(E) \implies F \subset E_1$.

Assume F is a subbundle of E such that $\mu_{min}(F) > \mu_{max}(E/F)$. Let

$$0 < F_1 < \cdots < F_{p-1} < F_p = F$$
$$\mu_1 > \cdots\cdots > \mu_p$$

and

$$F < F_{p+1} < \cdots < F_{m-1} \leq E$$
$$\mu_{p+1} > \cdots\cdots > \mu_m$$

be the canonical filtrations. Then $\mu_{p+1} = \mu_{max}(E/F)$ and $\mu_p = \mu_{min}(F)$. Thus $\mu_p > \mu_{p+1}$ and it follows that F is part of the canonical filtration of E.

Next assume only that $\mu_{min}(F) = \mu_{max}(E/F)$ i.e. $\mu_p = \mu_{p+1}$. Then the canonical filtration of E is

$$0 < F_1 < \cdots < F_{p-1} < F_{p+1} < \cdots < F_m = E$$

with the sequence of slopes $\mu_1 > \cdots > \mu_{p-1} = \mu_{p+1} > \cdots > \mu_m$. And one sees that F corresponds to a division of the semi-stable chunk F_{p+1}/F_{p-1} of E into two semi-stable bundles of the same slopes.

Given E then, I want to consider those proper subspaces W of V such that

$$\mu_{min}(E \cap W) \geq \mu_{max}(E/E \cap W).$$

(These will be like faces of a convex body?). These subspaces form a simplicial complex.

<u>Philosophy</u>: For each simplex σ in $T(V)$ I will associate a contractible subcomplex X_σ of X, such that $\sigma \subset \tau \Longrightarrow X_\tau \subset X_\sigma$. Then one will have a category Y over X homotopy equivalent to $T(V)$. Over most of the points of X, the fibre will be contractible.

Let W be a proper subspace of V. Define X_W to be the full subcomplex of X whose vertices E are <u>nice</u> with respect to W in the sense that

$$\mu_{min}(E\cap W) \geq \mu_{max}(E/E\cap W).$$

Given a simplex σ in X, I wish to consider the full subcomplex of $T(V)$ consisting of those W such that $\sigma \in X_W$. Call this complex S_σ. Clearly

$$S_\sigma = \bigcap_{E\in\sigma} S_E$$

where $S_E = \{W \mid W \text{ nice w.r.t. } E\}$.

~~Assume that σ contains an instable vertex E~~
~~[illegible]~~

Consider the case where σ is a vertex E, in which case we want to know about the full subcomplex S_E of T_E whose vertices are those W nice with respect to E. If E is instable, let its canonical filtration be $0 < E\cap Z_1 < \cdots < E\cap Z_p < E$. Then we have seen that any W in S_E is compatible with $Z_1, \ldots, Z_p$, hence S_E is contractible.

If E is stable, then there are no W nice with respect to E, hence S_E is empty.

April 11, 1974:

Lemma: Let $E' \subset E$ be a 1-simplex, and W a proper subspace of V such that

$$\mu_{min}(E \cap W) \geqslant \mu_{max}(E/E \cap W) + d_\infty$$

Then
$$\mu_{min}(E' \cap W) \geqslant \mu_{max}(E'/E' \cap W).$$

Proof: Choose $W_2 < W_1$ and $W_3 > W_1$ such that

$$\mu(E' \cap W_1 / E' \cap W_2) = \mu_{min}(E' \cap W_1)$$

(0)

$$\mu(E' \cap W_3 / E' \cap W_1) = \mu_{max}(E'/E' \cap W_1)$$

The because $E' \cap W_1 / E' \cap W_2$ embeds in $E \cap W_1 / E \cap W_2$ and the cokernel is killed by m_∞, I know that

$$\mu(E' \cap W_1 / E' \cap W_2) \geqslant \mu(E \cap W_1 / E \cap W_2) - d_\infty$$

(1)

$$\geqslant \mu_{min}(E \cap W_1) - d_\infty$$

And because $E' \cap W_3 / E' \cap W_1$ embeds in $E \cap W_3 / E \cap W_1$, I know that

(2) $$\mu(E' \cap W_3 / E' \cap W_1) \leqslant \mu_{max}(E/E \cap W_1).$$

Then subtracting (2) from (1) and using (0), I get

$$\boxed{\begin{aligned} &\mu_{min}(E' \cap W_1) - \mu_{max}(E'/E' \cap W_1) \\ &\qquad \geqslant \mu_{min}(E \cap W_1) - \mu_{max}(E/E \cap W_1) - d_\infty \end{aligned}}$$

To each W proper, let X_W° be the region of X consisting of those open simplexes σ such that W is nice with respect to each vertex of σ, and such that at least one vertex E of σ has W as part of its canonical filtration. Thus X_W° is the open part of X_W obtained by removing those E such that $\mu_{min}(E \cap W) = \mu_{max}(E/E\cap W)$; equiv. such that W is not part of the canonical filtration of E.

Given a simplex σ of X, let $\sigma \in X_W^\circ$, $\sigma \in X_{W'}^\circ$ where $\dim(W) \leq \dim(W')$. By assumption σ contains a vertex E such that W is part of the canonical filtration of E. But ~~then~~ because W' is nice with respect to each vertex of σ, in part. σ, I know that $W \subset W'$. Hence the set of W such that $\sigma \in X_W^\circ$ is either empty or contractible.

Assuming I can prove for each simplex τ of $T(V)$ that $\bigcap_{W\in\tau} X_W^\circ$ is contractible, this implies that $\bigcup_{W \text{ proper}} X_W^\circ$ has the homotopy type of $T(V)$.

Now the question arises as to what simplices are contained in $\bigcup_W X_W^\circ$. In particular I want to show that there are only finitely many Γ-classes of simplices not in this union. However if σ is a simplex containing a vertex E such that $\exists\, W$ with $\mu_{min}(E\cap W) - \mu_{max}(E/E\cap W) \geq d_\infty$, then the preceding lemma shows that W is nice with respect to all vertices of E, hence $\sigma \in X_W^\circ$.

Case of $\mathbb{P}^1$, $d_\infty = 1$: Here I know, because there are no stable bundles besides line bundles, the slope of any semi-stable bundle is integral. Hence there are no 1-simplices between two semi-stable bundles. Thus in this case $\bigcup_W X_W^\circ$ contains all simplices except for vertices isomorphic to $\mathcal{O}^r$. Thus one finds my old formula

$$I(F^r) = H_{r-1}\Big(X, \bigcup_W X_W^\circ\Big)$$
$$= \bigoplus_N I(N)$$

where N runs over all unimodular subspaces of dim r in V.

April 12, 1974. Counting stable bundles (after Harder and Nara–...).

One starts with the Siegel formula

$$\sum_{\Lambda^2 E \cong L_0} \frac{1}{\mathrm{aut}(E)} = \gamma_r$$

where L_0 is a given element of $\mathrm{Pic}(C)$, and the sum is taken over iso. classes of E of rank r such that $\Lambda^r E \cong L_0$. Here γ_r is a constant $\left(= \frac{1}{q-1} Z(q) \cdots Z(q^{r-1})\right.$ I think).

$r =$ rank $= 2$. Remove the instable bundles from the above sum. These are exactly

$$\sum_{\substack{E \supset E_1 \\ \Lambda^2 E \cong L_0 \\ \mu(E_1) > \mu(E)}} \frac{1}{\mathrm{aut}(E \supset E_1)}$$

In effect, E instable $\Longrightarrow$ E contains a unique sub-line bundle E_1 with $\mu(E_1) > \mu(E)$, and $\mathrm{aut}(E) = \mathrm{aut}(E \supset E_1)$. Better: The map $(E \supset E_1) \longmapsto E$ from the groupoid of $(E \supset E_1)$ $\Lambda^2 E \cong L_0$, $\mu(E_1) > \mu(E)$ to the groupoid of E $\ni$ $\Lambda^2 E \cong L_0$ is fully faithful with image the unstable E.

Thus

$$\sum_{\substack{\Lambda^2 E \cong L_0 \\ E \text{ semi-stable}}} \frac{1}{\mathrm{aut}(E)} = \sum_{\Lambda^2 E \cong L_0} \frac{1}{\mathrm{aut}\, E} - \sum_{\substack{E \supset E_1 \\ \mu(E_1) > \mu(E) \\ \Lambda^2 E \cong L_0}} \frac{1}{\mathrm{aut}(E \supset E_1)}$$

To evaluate this last sum we use the functor

$$(E \supset E_1) \longmapsto (E_1,\ E/E_1)$$

which is cofibred in groupoids. Thus if I fix A, B, then

$$\sum_{\substack{E_1 \simeq A \\ E/E_1 \simeq B}} \frac{1}{\operatorname{aut}(E \supset E_1)} = \frac{1}{\operatorname{aut}(A)\operatorname{aut}(B)} \sum_{E \in \operatorname{Ext}^1(B,A)} \frac{1}{\operatorname{aut}(E)}$$

Now

$$\operatorname{Ext}^1(B,A) = H^1(\operatorname{Hom}(B,A))$$

And if E is any extension of B by A

$$\operatorname{Aut}(E) = H^0(\operatorname{Hom}(B,A)).$$

Thus

$$\sum_{E \in \operatorname{Ext}^1(B,A)} \frac{1}{\operatorname{aut}(E)} = \frac{\# H^1(\operatorname{Hom}(B,A))}{\# H^0(\operatorname{Hom}(B,A))}$$

$$= q^{-(\deg \operatorname{Hom}(B,A) + \operatorname{rank} \operatorname{Hom}(B,A)(1-g))}$$

In the case at hand, B, A are line bundles. So

$$\sum_{\substack{E \supset E_1 \\ \mu(E_1) > \mu(E) \\ \Lambda^2 E \simeq L_0}} \frac{1}{\operatorname{aut}(E \supset E_1)} = \frac{1}{(q-1)^2} \sum_{\substack{L \in \operatorname{Pic}(C) \\ \deg(L) > \mu(E)}} q^{-(2\deg(L) - \deg E)}$$

$$\deg(\operatorname{Hom}(E/L, L)) = 2\deg L - \deg E$$

$$= \frac{h}{(q-1)^2} \sum_{n \geq 1} q^{-2n+1} = \frac{h\,q}{(q-1)^2(q^2-1)} \qquad \mu(E) = \frac{1}{2}$$

$$= \frac{h}{(q-1)^2} \sum_{n \geq 1} q^{-2n} = \frac{h}{(q-1)^2(q^2-1)} \qquad \mu(E) = 0$$

$\underline{n=3.}$ Here we count the instable bundles according to canonical filtration. Category:

$$(E \supset E_1) \quad \left\{ \begin{array}{l} E_1 \text{ line bundle} \\ E/E_1 \text{ semi-stable} \\ \mu(E_1) > \mu(E/E_1) \end{array} \right.$$

Then the functor

$$(E \supset E_1) \longmapsto E$$

is fully faithful with image those instable bundles whose canonical filtration reduces to a line. So if I denote by

$$f_2(\mu) = \sum_{\substack{E \text{ semi-stable} \\ \text{rank } 2 \\ \text{slope } \mu \\ \Lambda^2 E \simeq L_0(2\mu)}} \frac{1}{\operatorname{aut}(E)}$$

the answer obtained in the rank 2 case, then

$$\sum_{\substack{E \supset E_1 \\ E_1 \text{ line bundle} \\ E/E_1 \text{ semi-stable} \\ \deg(E_1) > \mu(E),\ \Lambda^3 E \simeq L_0(3\mu)}} \frac{1}{\operatorname{aut}(E \supset E_1)} = \sum_{\deg(E_1) > \mu(E)} \frac{1}{q-1} \cdot \sum_{\substack{E/E_1 \text{ semi-stable} \\ \text{of } \deg = \\ \deg E - \deg E_1}} \frac{1}{\operatorname{aut}(E/E_1)} \qquad ?$$

$\boxed{\text{net conjecture:}}$ Let Γ' be a net subgroup of Γ [redacted] of finite index in Γ, [redacted] i.e. every torsion element of Γ' is a p-torsion element. Then I is a projective $\mathbb{Z}[\Gamma']$-module.

Let $Y = \{\sigma \in X \mid \Gamma'_\sigma \neq 1\}$. Since $\sigma < \tau \Rightarrow \Gamma'_\sigma \supset \Gamma'_\tau$, Y is a subcomplex of X. Since Γ' is of finite index in Γ, and since there are only finitely many σ in X/Γ with $\operatorname{ant}(\sigma) \leq N$, it follows that Y/Γ' contains all but finitely many simplices of X/Γ'.

Conjecture 1: Y has the homotopy type of $T(V)$.

Observe that if this is true then $I = H_{r-1}(X, Y)$ and $H_j(X, Y) = 0 \quad j < r-1$, hence we will have a resolution

$$0 \to I \to C_{r-1}(X, Y) \to \cdots \to C_0(X, Y) \to 0.$$

But because every σ in $X - Y$ has $\Gamma'_\sigma = 1$, $C_j(X, Y)$ is a free $\mathbb{Z}[\Gamma']$-module of finite type, hence $I \in \mathcal{P}(\mathbb{Z}[\Gamma'])$.

Proof of conjecture 1 for $r = 2$: If $\sigma \in Y$, then we consider the action of Γ'_σ on V. As Γ'_σ is a p-group, [redacted] there is a line L in V invariant under Γ'_σ, in fact a unique line as $\Gamma'_\sigma \neq 1$. Thus one has

$$Y = \coprod Y_L \qquad L \in \mathbb{P}_1(V)$$

where $Y_L = \{\sigma \in Y \mid \Gamma'_\sigma \text{ leaves } L \text{ invariant}\}$. But now if Γ'_σ leaves L fixed, then the shift in the L-direction

will carry σ to σ^* such that $\Gamma'_\sigma \subset \Gamma'_{\sigma^*}$. *
Thus Y_L is stable under the L-shift. As Y_L is non-empty for each L it is contractible.

---

* incomplete. Not clear that [redacted] $\sigma^* \in Y_L$

In the general case, one might try to cover Y by

$$Y_W = \{\sigma \mid \Gamma'_\sigma \text{ leaves } W \text{ invariant}\}$$

or more generally for each $\tau \in T(V)$, we [redacted] can put

$$Y_\tau = \{\sigma \in Y \mid \Gamma'_\sigma \text{ leaves } \tau \text{ invariant}\}$$

whence Y_τ is open in [redacted] Y. This leads to the following question:

[redacted] <u>Question</u>. [redacted] Let a p-group G act non-trivially on V. Is the subcomplex of $T(V)$ consisting of the invariant subspaces contractible?

<u>Yes</u>. [redacted] Since the action is non-trivial V^G is a proper subspace. Now given $0 < W < V$ stable under G, $W^G = W \cap V^G$ is non-zero. Thus we get the [redacted] deformation [redacted]

$$W \geq W^G \leq V^G$$

of $T(V)^G$ to a point.

So at this point [redacted] it is clear that I ought to be able to prove the net conjecture.

Study of the net conjecture.

Γ' net subgroup of [redacted] Γ. To simplify suppose $[\Gamma:\Gamma'] < \infty$ and we are over a finite field. Put $Y = \{\sigma \mid \Gamma'_\sigma \neq 1\}$. Then Y is a [redacted] subcomplex of X.

? <u>Proposition:</u> Y has the homotopy type of $T(V)$.

<u>Proof:</u> [redacted] Let Z be the set of pairs (σ,τ) where $\sigma \in Y$ and $\tau \in T(V)$ and where Γ'_σ leaves τ invariant. We will consider the two projections

$$\begin{array}{ccccc} Y & \xleftarrow{\pi_1} & Z & \xrightarrow{\pi_2} & T(V) \\ \sigma & \longleftarrow & (\sigma,\tau) & \longmapsto & \tau . \end{array}$$

[redacted] Equip Z with the ordering $(\sigma',\tau') \leq (\sigma,\tau)$ if $\sigma' \leq \sigma$ and $\tau' \leq \tau$.

Note that if $\sigma' \leq \sigma$, then $\Gamma'_{\sigma'} \supset \Gamma'_\sigma$, hence $(\sigma',\tau') \in Z \Rightarrow (\sigma,\tau') \in Z$. Thus [redacted]

$$\{(\sigma,\tau) \in \pi_1^{-1}(\sigma) \mid (\sigma',\tau') \leq (\sigma,\tau)\}$$

has a least element, namely, (σ,τ'). Thus π_1 is cofibred with cobase-change $(\sigma',\tau') \longmapsto (\sigma,\tau')$.

If $\tau' \leq \tau$, then $(\sigma,\tau) \in Z \Rightarrow (\sigma,\tau') \in Z$. Thus

$$\{(\sigma',\tau') \in \pi_2^{-1}(\tau') \mid (\sigma,\tau') \leq (\sigma,\tau)\}$$

has a largest element, namely (σ,τ'). Hence π_2 is fibred with base-change $(\sigma,\tau) \longmapsto (\sigma,\tau')$.

To proof the proposition, it suffices therefore to show that the fibres of π_1, π_2 are contractible.

$$\pi_1^{-1}\{\sigma\} = \{\tau \in T(V) \mid \Gamma'_\sigma \text{ fixes } \tau\}$$

and we have seen that because Γ'_σ is a p-group acting non-trivially on V this is contractible. Now

$$\pi_2^{-1}\{\tau\} = \{\sigma \in X \mid \Gamma'_\sigma \neq 1,\ \Gamma'_\sigma \text{ fixes } \tau\}.$$

ordered by inclusion. (This is open in Y but not in X.) Let $\tau = 0 < W_1 < \cdots < W_p < V$. To simplify notation put

$$Y_\tau = \{\sigma \in Y \mid \Gamma'_\sigma \text{ fixes } \tau\} = \pi_2^{-1}\{\tau\}.$$

I want to show that translation with respect to W_i maps Y_τ into itself, and that this translation is homotopic to the identity of Y_τ. Let $\sigma = (E_0 \subset \cdots \subset E_q) \in Y_\tau$. Then

$$T_W(\sigma) = E_0 + E_0 \cap W(1) \subset E_1 + E_1 \cap W(1) \subset \cdots \subset E_q + E_q \cap W(1).$$

If Γ'_σ fixes W, then it is clear that Γ'_σ fixes $E_j + E_j \cap W(1)$ hence

$$\Gamma'_\sigma \subset \Gamma'_{T_W(\sigma)}$$

and therefore $T_W(\sigma) \in Y$. But it is not clear that any auto. of $T_W(\sigma)$ leaves W fixed.

?

It seems to be important to know that W is part of the canonical filtration of $T_W(\sigma)$.

In view of the above problems it is essential that I understand the case of rank 2 and $d_\infty > 1$, and that I check the net conjecture carefully in this case.

For example, take $C = \mathbb{P}^1_k$, and let ∞ be a point of degree d_∞, $A = \Gamma(C-\infty, \mathcal{O}_C)$. Then one has

$$K_0\, k(\infty) \longrightarrow \tilde{K}_0 C \longrightarrow \tilde{K}_0 A \longrightarrow 0$$

hence $\mathrm{Pic}\, A \cong \mathbb{Z}/d_\infty \mathbb{Z}$, $h = d_\infty$. Consider rank 2 bundles reducing to A^2 over $C-\infty$. The isomorphism classes are

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \qquad a \geq b \, , \quad a+b \equiv 0 \ (d_\infty)$$

as usual. Here $\mathcal{O}(\infty) = \mathcal{O}(d_\infty)$, so modulo homothety the iso. classes are

$$\mathcal{O}(a) \oplus \mathcal{O}(-a) \qquad a \geq 0$$

$$\mathcal{O}(d_\infty + a) \oplus \mathcal{O}(-a) \qquad a \geq -\frac{d_\infty}{2}$$

Now there are d_∞-cusps represented by the chains (take $d_\infty = 3$)

$$\mathcal{O}(1)\oplus\mathcal{O}(-1) \subset \mathcal{O}(4)\oplus\mathcal{O}(-1) \subset \mathcal{O}(4)\oplus\mathcal{O}(-4) \subset \cdots\cdots$$

$$\mathcal{O}(2)\oplus\mathcal{O}(1) \subset \mathcal{O}(2)\oplus\mathcal{O}(-2) \subset \mathcal{O}(5)\oplus\mathcal{O}(-2) \subset \mathcal{O}(5)\oplus\mathcal{O}(-5) \subset \cdots\cdots$$

$$\mathcal{O}(3)\oplus\mathcal{O}(-3) \subset \mathcal{O}(6)\oplus\mathcal{O}(-3) \subset \mathcal{O}(6)\oplus\mathcal{O}(-6) \subset \cdots\cdots$$

$$\mathrm{Aut}\,(\mathcal{O}(1)\oplus\mathcal{O}(-1)) = \begin{pmatrix} k^* & H^0(\mathcal{O}(2)) \\ 0 & k^* \end{pmatrix}$$

$$\mathrm{Aut}\,(\mathcal{O}\oplus\mathcal{O}) = GL_2(k)$$

Question: how does $GL_2(k)$ act on $\mathbb{P}_1(k(\infty))$? No longer transitively, so $\mathcal{O}\oplus\mathcal{O}$ is no longer extremal in the quotient graph.

April 13, 1974 net conjecture (cont.)

New idea: Define a map from the set of σ in X such that $\Gamma'_\sigma \neq 1$ to the set of W, $0 < W < V$ by

$$f(\sigma) = H^0(\Gamma'_\sigma, V).$$

Since Γ'_σ is a p-group acting non-trivially on V, this is indeed a proper subspace of V. Also if $\sigma' \subset \sigma$, then $\Gamma'_{\sigma'} \supset \Gamma'_\sigma$ hence

$$H^0(\Gamma'_{\sigma'}, V) \subset H^0(\Gamma'_\sigma, V)$$

and so f is a map of posets.

Conjecture: f is a homotopy equivalence (assuming Γ' is of finite index in Γ).

This is true for $r=2$. In effect we showed that $f^{-1}(L) = Y_L$ is stable under the deformation T_L. The point was that if Γ'_σ fixes L, then L is unique. Thus since $\Gamma'_\sigma \subset \Gamma'_{T_L(\sigma)}$ the latter must also fix L.

The analogue of this argument for $r=3$, would be to consider for any line L the fibre $f^{-1}(L)$ which is a subcomplex; that is, if L is the invariant subspace of Γ'_σ it must also be for any $\sigma' \subset \sigma$, as well as for $\Gamma'_{T_L(\sigma)} \supset \Gamma'_\sigma$. Thus $f^{-1}(L)$ is stable under T_L.

What can I say about the quotient complex, when C is an elliptic curve, $r=3$, $d_\infty = 1$. I can try to classify vertices according to slopes. Thus we get the following scheme:

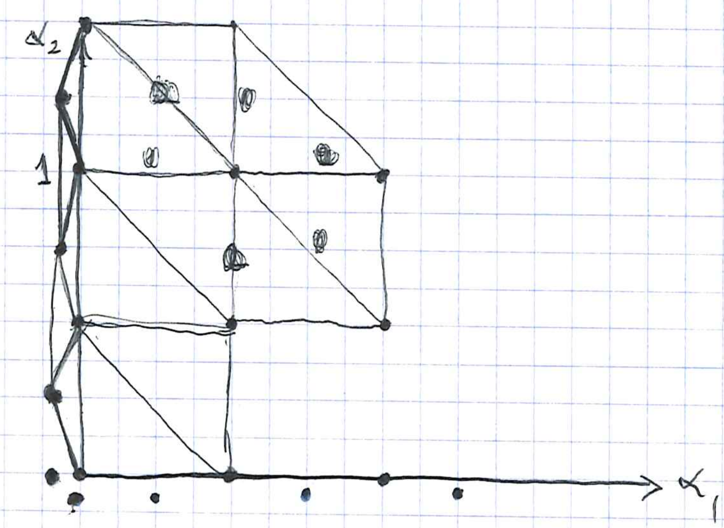
Canonical filtration is a flag: here α_1, α_2 are pos. integers

" " is a line: here $\alpha_1 > 0, \alpha_2 = 0$ and α_1 is half-integral when the quotient is stable.

" " " " plane: $\alpha_1 = 0, \alpha_2 > 0$ and α_2 is half-integral when the [redacted] plane bundle is stable

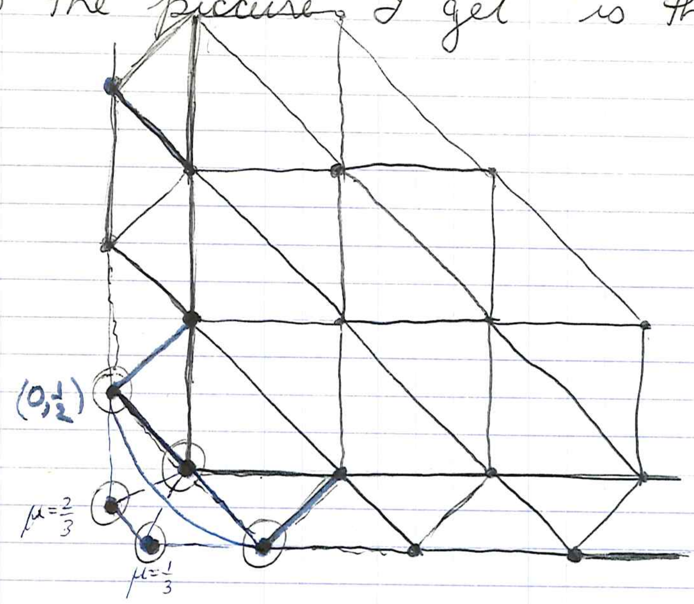
no canonical filtration: $\alpha_1 = 0 = \alpha_2$. Here the bundle is semi-stable. There are two stable bundles of [redacted] slopes $\frac{1}{3}, \frac{2}{3}$.

Picture:



I have drawn the dots corresponding to stable parts [redacted] as being slightly negative.

So the picture I get is this.



~~I have drawn the stuff that is definitively attached to some part of the building at ∞.~~

The problem begins with the point $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$ which looks like . It's neighbors:

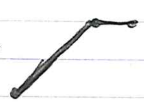
$$0 \to F \to E \to L \to 0$$

stable deg 1.

<u>down</u>

 $\alpha_1 = 0,\ \alpha_2 = \frac{3}{2}$

 $\alpha_1 = \alpha_2 = 0.$

<u>up</u>:  $\alpha_1 = 0,\ \alpha_2 = 1$

now if $E \subset E'$ has cokernel killed by $k(\infty)$, and $E \twoheadrightarrow L$ induced $E' \twoheadrightarrow L(1)$, then one has

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, hook] & L \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & F \arrow[r] & E' \arrow[r] & L(1) \arrow[r] & 0 \end{tikzcd}$$

where $\deg(E') = 2$, $\mu(E') = \frac{2}{3}$. If E' has a $\neq 0$ subbundle of slope 1, it must map $\xrightarrow{\sim} L(1)$, so $E' = F \oplus L(1)$, and [redacted] $E = F \oplus L$. [redacted] sequence splits for an elliptic curve. Thus we get for E' the diagram

i.e. $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 0$.

This gives me an example of a 1-simplex made out of instable bundles.

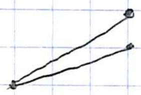
Another possibility is that E' doesn't contain a subbundle of slope 1, hence it is stable of slope $\frac{2}{3}$ with diagram. E.g.

$$\mathrm{Ext}^1(L(1), F^*) = H^1(\mathrm{Hom}(L, F^{*-1}(-1))) \cong k$$

by R-R since $\mathrm{Hom}(L(1), F^*) = 0$.

<u>Next</u>. Take a stable E of slope 2. The possible E' of ~~[illegible]~~ colength 1 in E are

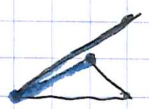
$\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$



$\mu = \frac{1}{3}$ stable.

And the possible E'' of colength 2 in E are

$\alpha_1 = 0$, $\alpha_2 = 0$

Observe that  is impossible because the quotient line bundle jumps in degree by 2.

So if I am not mistaken there is a map of the quotient complex to the simplicial complex I have drawn on page 3, at least for elliptic curves.

For a general curve C [redacted] but still with $d_\infty = 1$, we perhaps get the same classification. Again we can **classify** the vertices according to slope

semi-stable : slopes $0, \frac{1}{3}, \frac{2}{3}$. $\qquad \alpha_1 = \alpha_2 = 0$

canonical line : [redacted] $\qquad \alpha_1 \in \mathbb{Z}\frac{1}{2} > 0, \; \alpha_2 = 0$

canon. 2 plane : $\qquad \alpha_1 = 0 \;, \; \alpha_2 \in \mathbb{Z}\frac{1}{2} > 0$

canonical flag : $\qquad \alpha_1, \alpha_2 \in \mathbb{Z} \; > 0$.

Suppose we are in an <u>interior point</u> E, with $\alpha_1, \alpha_2 > 0$. and let the canonical filtration be $0 < L < W < V$. [redacted] Let $E' \subset E$ be of colength one. Then we have exactly one spot in the canonical filtration where E' first differs from E. Better to denote by $0 < E_1 < E_2 < E$ the canonical filtration. Then if $E' \subset E$ is of colength one, there is exactly one i for which $E' \cap E_{i-1} = E_{i-1}$, $\quad E' \cap E_i < E_i$. Picture

$i = 1 \qquad\qquad i = 2 \qquad\qquad i = 3$

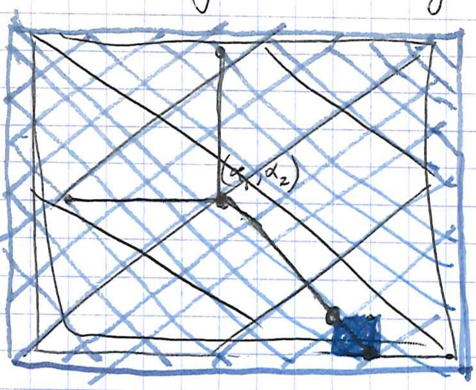
$(\alpha_1 - 1, \alpha_2) \qquad (\alpha_1 + 1, \alpha_2 - 1) \qquad (\alpha_1, \alpha_2 + 1)$

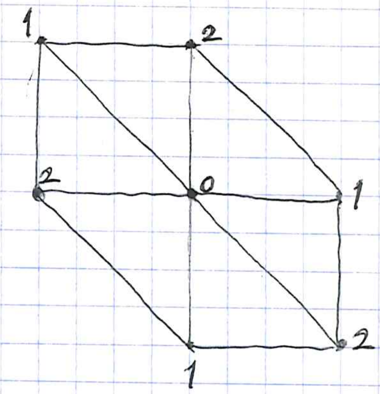
If $E'' \subset E$ is of colength 2, then there are two places where E/E'' gets distributed.

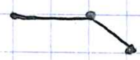
$(\alpha_1, \alpha_2 - 1) \qquad (\alpha_1 - 1, \alpha_2 + 1) \qquad (\alpha_1 + 1, \alpha_2)$

$(1,2) \qquad\qquad (1,3) \qquad\qquad (2,3)$

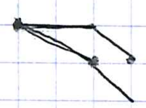
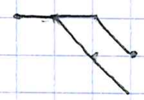
and so we get the typical interior hexagon



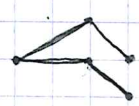


Next consider the situation where the canonical filtration is a 2-plane $0 < E_2 < E$, i.e. $\alpha_1 = 0$, $\alpha_2 \in \mathbb{Z}\frac{1}{2} > 0$. Suppose also $\alpha_2 \geq 1$.  or 

Given E' of colength one, there are two possibilities according to whether E' contains E_2 or not.

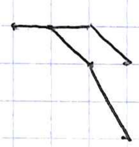
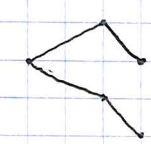
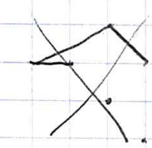
  or 

$(0, \alpha_2 + 1)$ $\quad$ $(0, \alpha_2 - \frac{1}{2})$ $\quad$ $(1, \alpha_2 - 1)$ when α_2 is integral

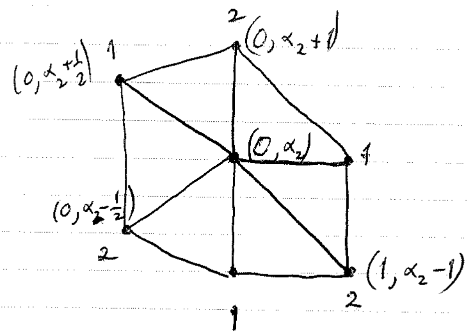
$(0, \alpha_2 + 1)$ $\quad$ $(0, \alpha_2 - \frac{1}{2})$

And if E'' is colength two, there are two possibilities

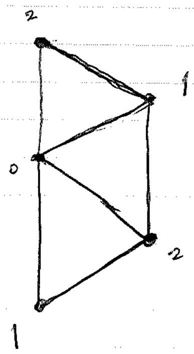
     

$(0, \alpha_2 - 1)$ $\quad$ $(0, \alpha_2 + \frac{1}{2})$ $\quad$ $(1, \alpha_2)$ $\quad$ $(0, \alpha_2 - 1)$ $\quad$ $(0, \alpha_2 + \frac{1}{2})$

So for $(0, \alpha_2)$, α_2 integral > 0 I get the hexagon



And for $(0, \alpha_2)$ α_2 halfintegral $\geq \frac{3}{2}$ I get



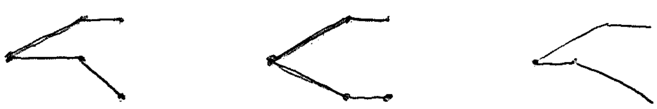
Now for the interesting vertices:

$(0, \frac{1}{2})$ 

For E', we have the possibilities:

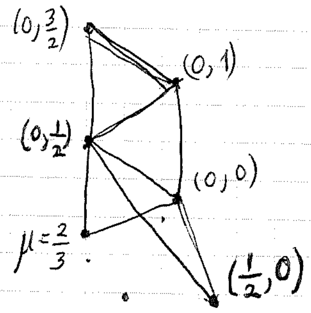
$(0, \frac{3}{2})$ $\quad$ $(0,0)$

For E'', we have the possibilities



$(0, 1)$ $\quad$ [redacted]stable $\mu = \frac{2}{3}$ $\quad$ $(\frac{1}{2}, 0)$

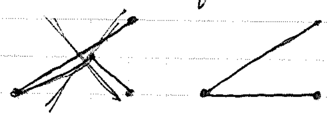
Thus we get the picture:



stable $\mu=\frac{2}{3}$: possibilities for E'

$\mu=\frac{1}{3}$ or $(0,\frac{1}{2})$

possibilities for E'':



$(0,0)$

Conclusion: The complex on page 3 should be the same for any curve.

What is the region I know has the homotopy type of the building at ∞? I am happy about any simplex containing a vertex with either α_1 or $\alpha_2 \geq 1$.

<u>Example:</u> Take the simplex

[redacted]

$$\mathcal{O}(1) \oplus \mathcal{O}(-1) \subset \mathcal{O}(1) \oplus \mathcal{O}(2)$$

This joins two different [redacted] cusps. This is an example of what one might call a stable simplex, because there is no line nice with respect to the vertices.

Now [redacted] I should check the net conjecture in the case [redacted] that $d_\infty > 1$. Γ' net subgroup of Γ of finite index, $Y = \{\sigma \mid \Gamma'_\sigma \neq 1\}$. I saw that Γ'_σ fixes a unique line L in V, hence

$$Y = \coprod Y_L \qquad Y_L = \{\sigma \in Y \mid \Gamma'_\sigma \text{ fixes } L\}$$

Now I have to show that translation T_L with respect to L carries Y_L into itself. [redacted] It is clear that if $\sigma \in Y_L$, then $T_L(\sigma) \in Y$ and that $\Gamma'_\sigma \subset \Gamma'_{T_L(\sigma)}$. Moreover the unique line fixed by $\Gamma'_{T_L(\sigma)}$ must be L clearly.

April 15, 1974.

The net conjecture for $r=3$, $d_\infty = 1$.

Denote by Z the subcomplex of X obtained by removing the open stars of the following simplices.

i) stable vertices

ii) direct sum of stable bundles of degree zero.

iii) edges of the form $F(-1)\oplus L \subset F\oplus L$ where L is of degree 0 and F is stable of rank 2 and deg 1.

I want to show that Z has the homotopy type of $T(V)$. So for each proper subspace W of V put

$$Z_W = \{\sigma \in Z \mid W \text{ nice wrt } \sigma\}$$

where nice means nice with respect to each vertex E of σ, i.e. $\mu_{\min}(E\cap W) \geq \mu_{\max}(E/E\cap W)$. Observe that Z_W is closed in Z.

Check

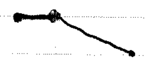
$$(*) \qquad \bigcup_W Z_W = Z$$

Let σ belong to Z. Then σ has no stable vertices. If some vertex E of σ, has a root ≥ 1, [redacted] and if W is the corresponding subspace, I know that [redacted] W is nice with respect to each of the other vertices, so σ is in Z_W. So I can suppose no root is ≥ 1.

If σ consists of a vertex E, then E is not stable, so $\exists\, W \ni W$ nice wrt E, so $\sigma \in Z_W$. So I can suppose $\operatorname{card}(\sigma) \geq 2$.

Then at least one vertex [redacted] has deg $\equiv 1$ or $2 \pmod 3$, so as this vertex is not stable, it must be non-semi-stable.

Call this vertex E, so that either $\alpha_1(E)$ or $\alpha_2(E) > 0$. Take the case $\alpha_1(E) > 0$, whence we have a line subbundle E_1 of E as part of the canonical filtration of E; suppose $E_1 = E \cap L$, L line in V, and $\deg E_1 = 0$. Since $\alpha_1(E) < 1$, geometry shows E/E_1 is stable of degree -1.



If L is not nice for σ, there exists another vertex E' such that L is not nice with respect to σ. Assuming $E' \subset E$ with cokernel killed by $m(\infty)$, then it must be that $E' \cap L = E_1(-1)$. In effect $\mu(E' \cap L) \le \mu_{max}(E') \le \mu_{max}(E)$, so if $E' \cap L$ had degree 0, it would have to be part of the canonical filtration of E'.

Since L is not nice with respect to E' and E' is not stable, the only possibility is $\mu_{max}(E') = -\frac{1}{2}$. (i.e. $\mu_{max}(E') > -1$, so the possibilities are $-\frac{1}{2}, -\frac{2}{3}$). Let W be $\ni$ $\mu(E' \cap W) = -\frac{1}{2}$. Then $E' \cap W = E \cap W$. So now consider the map

$$E \cap W \hookrightarrow E \longrightarrow E/E \cap L$$

of stable bundles of slope $\frac{1}{2}$. Has to be an isom, hence $E = E\cap L \oplus E \cap W$. Hence $E' = (E \cap L)(-1) \oplus E \cap W$ and we have conveniently reached that $(E' \subset E) \subset \sigma$ is of the type iii) of things we have removed.

The case $\alpha_2(E) > 0$ is similar.

Thus $(*)$ has been proved.

Next one must show that $\forall \sigma \in Z$.

$$S_\sigma = \{ W \mid W \text{ nice wrt } \sigma \}$$

is contractible.

As before there is no problem if some root is ≥ 1, because then I get a subspace W nice with respect to σ such that any other W' nice with respect to σ either includes or is included in W. In fact, once I know that $\exists$ W nice wrt σ such that W belongs to the canonical filtration of some vertex, the contractibility of S_σ is clear.

First worry about S_σ being empty. This can only happen if σ contains a stable vertex, or if σ contains the example $F(-1)\oplus L \subset F \oplus L$.
Thus the operating lemma appears to be:

<u>Lemma</u>: For $r=3$, $d_\infty = 1$, the only σ for which $S_\sigma = \{ W \mid W \text{ nice wrt } \sigma \}$ is empty, are the σ's containing a stable vertex, or an edge $F(-1) \oplus L \subset F \oplus L$ where F is stable of $\mu = \frac{1}{2}$, L of deg 0.

Review the situation. I want to prove that if Γ' is a subgroup of finite index which is net, then I is a projective finite type $\mathbb{Z}[\Gamma']$-module. To do this I want to enclose the set of σ such that $\Gamma'_\sigma \neq 1$ into a complex Y, having the homotopy type of $T(V)$. It will then follow that $I = H_{r-1}(X, Y)$, and $H_i(X,Y) = 0$ $i < r-1$, and on the other hand the groups $C_\bullet(X,Y)$ will be $\mathbb{Z}[\Gamma']$-free.

Now I can start by letting Y be the region consisting of all simplices σ such that some vertex has a root ≥ 1. Better, I call a proper subspace W of V <u>nice</u> wrt E if $\mu_{min}(W \cap E) \geq \mu_{max}(E/E \cap W)$. Then I let Y consist of all σ such that there exists at least one space W with W nice wrt all E in σ and W part of the canonical filtration of some vertex of σ. Such σ I can call <u>instable</u>, and it determines a simplex in $T(V)$, namely its canonical filtration $=$ those W nice with respect to all E in σ, very nice with respect to one.

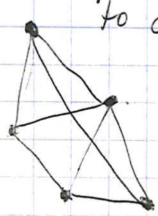
In the case $r=3$, $d_\infty = 1$, I have classified all simplices which are not instable.

i) semi-stable vertices

ii) simplices containing a stable vertex

iii) simplices containing the edge $F \oplus L \supset F(-1) \oplus L$

2 simplices to come out



1 simp

0 simp

Thus one adds to those simplices where some root is always ≥ 1 the region

<u>Digression</u>: In the general situation where d_∞ is not necessarily one, the natural thing to consider is $U = \{\sigma \mid \sigma$ contains a vertex with a root $\geq d_\infty\}$. More precisely, such that the canonical filtration has a slope change of at least $\geq d_\infty$. Check this: Let σ be $E_0 < \cdots < E_p = E$, and suppose that

$$\mu_{min}(E \cap W) \geq \mu_{max}(E/E\cap W) + d_\infty$$

Then one has for $E' = E_i$ that

$$\mu_{min}(E' \cap W) \geq \mu_{min}(E\cap W) \dot{-} d_\infty$$

$$\mu_{max}(E'/E'\cap W) \leq \mu_{max}(E/E\cap W).$$

$$\therefore \quad \mu_{min}(E'\cap W) \geq \mu_{max}(E'/E'\cap W)$$

and so W is nice w.r.t. E'.

<u>Good object</u>: U = open region containing those vertices having a slope change $\geq d_\infty$.