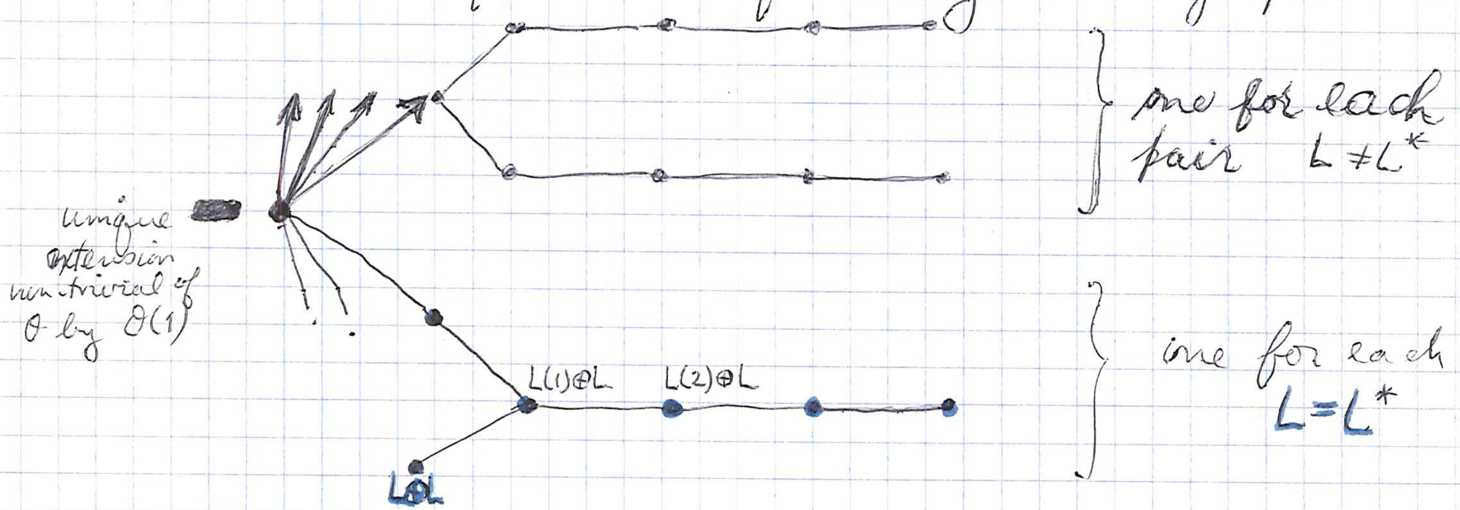
March 26, 1974 elliptic curves.

$C$ = elliptic curve over $k$ alg. closed, $\infty$ = origin
$A = \Gamma(C-\infty, \mathcal{O}_C)$.

Letting $GL_2(A)$ act on the tree $X$ of $F^2$ at $\infty$, I have found the following orbit graph:



Now I want to use this to explain the structure of $I(F^2) = I(A^2)$ as a $GL_2(A)$-module.

Recall

$$0 \longrightarrow I(F^2) \longrightarrow \bigoplus_{l \subset F^2} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

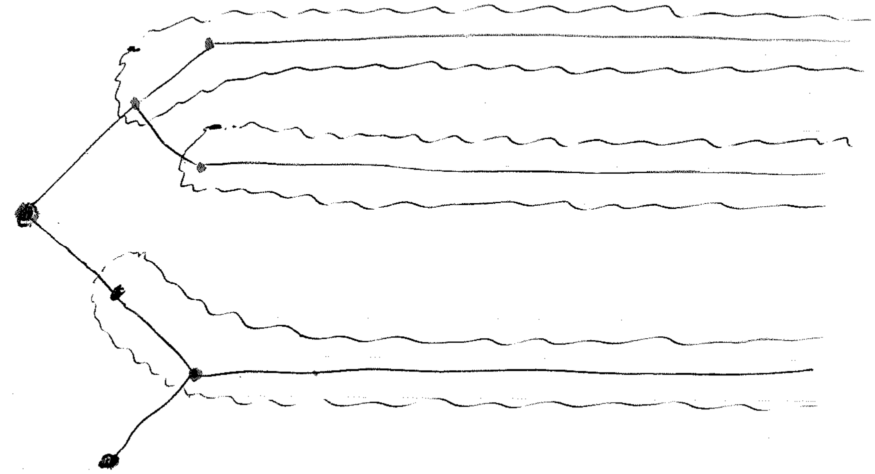
Now the idea I have is to assign to each line $l$ a tree $X_l \subset X$, such that $X_l$ collapses to $l$ and such that the different $X_l$ are disjoint. Then I will have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus_l C_1(X_l) \arrow[r] \arrow[d] & \bigoplus_l C_0(X_l) \arrow[r] \arrow[d] & \bigoplus_l \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & C_1(X) \arrow[r] \arrow[d] & C_0(X) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & C_1(X, \amalg X_l) \arrow[r] \arrow[d] & C_0(X, \amalg X_l) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

whence

$$0 \longrightarrow I(F^2) \longrightarrow C_1(X, \amalg X_\ell) \longrightarrow C_0(X, \amalg X_\ell) \longrightarrow 0$$

So for $X_\ell$ take the inverse images of the following trees



It should then be clear that $I(F^2)$ is the direct sum of three parts:

$\mathbb{Z}[GL_2 A] \otimes_{\mathbb{Z}[GL_2 k]} I_2(k)$ — one for each $L = L^*$ where $GL_2 k \hookrightarrow GL_2 A$ is obtained from an isom $L \oplus L|_{C-\infty} \simeq A \oplus A$

$\mathbb{Z}[GL_2 A] \otimes_{\mathbb{Z}[k^*]} I_2(k)$ — $k^*$ = center of $GL_2 A$ acts trivially on $I_2(k)$

$\mathbb{Z}[GL_2 A] \otimes_{\mathbb{Z}[k^* \times k^*]} \mathbb{Z}$ — one for each pair $(L, L^*)$ $L \neq L^*$, where $k^* \times k^* \hookrightarrow GL_2 A$ is obtained from an isom $L \oplus L^*|_{C-\infty} \simeq A^2$.

March 27, 1974. Curves.

$C$ curve over $k$, $E$ a vector bundle over $C$ of rank $n$. Put $\mu(E) = \deg E / \operatorname{rank}(E)$ and put

$$\mu_{\max}(E) = \sup\left\{\mu(E') \,\middle|\, \begin{array}{l} E' \text{ runs over} \\ \text{subbundles of } E \neq 0 \end{array}\right\}$$

One knows this is finite.

Recall $E$ is called <u>semi-stable</u> if $\mu(E') \leq \mu(E)$ for all subbundles $E' \neq 0$.

<u>Proposition</u>: There exists a unique filtration of $E$ by subbundles $0 < E_1 < \cdots < E_{p-1} < E_p = E$ such that $\forall i = 1, \ldots, p$ $E_i/E_{i-1}$ is semi-stable of ~~[illegible]~~ slope $\mu_i$ where

$$\mu_1 > \mu_2 > \cdots > \mu_p$$

~~[illegible] Existence [illegible] there exists [illegible]~~

<u>Lemma</u>: If $F_1, F_2 \subset E$ are subbundles $\neq 0$ of $E$ such that $\mu(F_i) = \mu_{\max}(E)$, then ~~[illegible]~~ so is $F_1 \cap F_2$ (provided this is $\neq 0$) and $F_1 + F_2$.

<u>Proof</u>: Put $\lambda = \mu_{\max}(E)$. Then we have

$$+\left(\begin{array}{l} \deg(F_1 \cap F_2) \leq \lambda \operatorname{rank}(F_1 \cap F_2) \\ = \left(+\begin{array}{l} \deg(F_1) = \lambda \operatorname{rank}(F_1) \\ \deg(F_2) = \lambda \operatorname{rank}(F_2) \end{array}\right. \\ \deg(F_1 + F_2) \leq \deg(\overline{F_1 + F_2}) \leq \lambda \operatorname{rank}(F_1 + F_2) \end{array}\right.$$

Adding as indicated we see all inequalities must be equal.

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<u>Cor.</u> $\exists$ largest subbundle $E'$ of $E$ with $\mu(E') = \mu_{\max}(E)$, and $E'$ is semi-stable.

Existence part of the proposition. Let $E_1$ be the largest subbundle of $E$ with slope $\mu_1 = \mu_{max}(E)$, $E_2/E_1$ = largest with slope $\mu_2 = \mu_{max}(E/E_1)$, etc. Have to show $\mu_1 > \mu_2$. But if one has $E_1 < F \subset E$, then

$$\deg(E_1) = \mu_1 \operatorname{rank}(E_1)$$

$$\deg(F) < \mu_1 \operatorname{rank}(F) \qquad \text{as } E_1 \text{ largest}$$

$$\Rightarrow \quad \deg(F/E_1) < \mu_1 \operatorname{rank}(F/E_1) \quad \Rightarrow \quad \mu_2 < \mu_1.$$

<u>Uniqueness</u>. Given $0 < E_1 < \cdots < E_p = E$ etc. it is enough to show that $\mu_1 = \mu_{max}(E)$ and that $E_1$ is the largest subbundle of slope $\mu_1$. Clearly $\mu_1 \leq \mu_{max}(E)$. If $F$ is a subbundle of $E$, then one has an induced filtration $0 \subset F \cap E_1 \subset \cdots\cdots \subset F \cap E_{p-1} \subset F$. and as

$$F \cap E_i / F \cap E_{i-1} \overset{\text{injects}}{\subset} E_i/E_{i-1}$$

one has $\deg(F \cap E_i / F \cap E_{i-1}) \leqslant \mu_i \operatorname{rank}(F \cap E_i / F \cap E_{i-1})$ so adding and using that $\mu_i \leqslant \mu_1$, one gets

$$\deg(F) \leq \mu_1 \operatorname{rank}(F)$$

with equality only if $F \subset E_1$. Thus $\mu_1 \geq \mu_{max}(E)$, and if $F$ has this slope then $F \subset E_1$.

Now let us fix a point $\infty \in C$ and put $C - \{\infty\} = \operatorname{Spec}(A)$, and let $M \in P(A)$ be of rank $n$, $\Gamma = \operatorname{Aut}(M)$. Let $X$ be the Tits building of $V = F \otimes_A M$ for the valuation at $\infty$. Vertices of $X$ can be identified with extensions of $M$ to a vector bundle $E$ on $C$ modulo the identifications produced by $E \longrightarrow E \otimes \mathcal{O}(1)$.

Given a proper subspace $W$ of $V$ define a full subcomplex $Y_W$ of $X$ as follows. Given a vertex of $X$ corresponding to a vector bundle $E$, this vertex will be in $Y_W$ iff $\mu_{min}(E \cap W) > \mu_{max}(E/E \cap W)$. In other words if one takes the canonical filtration of $E$:

$$0 < E_1 < \cdots < E_p = E$$

as defined above then $E \cap W = E_i$ for some $i$.

Next observe that given two proper subspaces $W_1, W_2$ one has that $Y_{W_1} \cap Y_{W_2} \neq \phi$ iff $W_1, W_2$ are related by inclusion. So therefore one finds that the nerve of the family $\{Y_W\}$ is just the Tits building of $V$.

<u>Conjecture</u>: For every $\sigma = (0 < W_1 < \cdots < W_p < V)$ in $T(V)$, $Y_\sigma = Y_{W_1} \cap \cdots \cap Y_{W_p}$ is contractible.

Consider first the case of $Y_W$. Recall that $E \in Y_W$ if $\mu_{min}(E \cap W) > \mu_{max}(E/E \cap W)$. Now I want to consider the following operation on vector bundles.

~~Given~~ [illegible] ~~has the exact sequence~~

Recall I have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow k(\infty) \longrightarrow 0$$

hence for any vector bundle $F$ an embedding [redacted] $F \hookrightarrow F(1)$. Thus given any vector bundle $E$ I can define a new one by push-out:

$$\begin{tikzcd}
0 \arrow[r] & E\cap W \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & E/E\cap W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E\cap W(1) \arrow[r] & E^* \arrow[r] & E/E\cap W \arrow[r] & 0
\end{tikzcd}$$

Since

$$\mu_{\min}([redacted]\,F(1)) = 1 + \mu_{\min}(F)$$

one sees that this operation carries $Y_W$ into itself. On the other hand it is clear that $E \subset E^*$ will give a 1-simplex in $X$.

~~I want to check now that $E \mapsto E^*$ is a simplicial mapping from~~ [redacted] ~~$Y_W$ to itself. So I suppose that I am given a simplex rep. by~~

~~$E_0 \subset \cdots \subset E_p$~~ [redacted] ~~$E_p \subset E_0(1)$.~~

~~and that each $E_i$ belongs to $Y_W$.~~ [redacted]

~~I have to show that~~

~~$\begin{matrix} E_0 & \subset \cdots \subset & E_p \\ \cap & & \cap \\ E_0^* & \subset \cdots \subset & E_p^* \end{matrix}$~~

~~is~~ [redacted] ~~a map~~ [redacted] ~~$\Delta(p) \times \Delta(1) \longrightarrow Y_W$. So you need to have $E_p^*/E_0$ killed by $\mathfrak{m}_\infty$. But~~

~~$E_p^*/E_0$~~

Better: I should remember that [redacted] for the building consisting of lattices, and not homothety classes, I was able to contract this building by choosing a sequence of lattices $\subset L_0 \subset \pi^{-1}L_0 \subset \pi^{-2}L_0 \subset \cdots$ and considering the operations

$$\varphi_n : \Lambda \longmapsto \Lambda + \pi^{-n}L_0 .$$

Then $\varphi_n$ was simplicial and given a simplex

$$\Lambda_0 \leq \cdots \leq \Lambda_p$$

one has

$$\begin{array}{ccccc} \Lambda_0 + \pi^{-n}L_0 & \leq & \cdots & \leq & \Lambda_p + \pi^{-n}L_0 \\ \wedge\!| & & & & \wedge\!| \\ \Lambda_0 + \pi^{-n-1}L_0 & \leq & \cdots & \leq & \Lambda_p + \pi^{-n-1}L_0 \end{array}$$

with

$$\Lambda_p + \pi^{-n-1}L_0 / \Lambda_0 + \pi^{-n}L_0 \twoheadleftarrow \Lambda_p/\Lambda_0 \oplus \pi^{-n-1}L_0/\pi^{-n}L_0$$

Thus one has homotopies

$$\varphi_n \Longrightarrow \varphi_{n+1} \Longrightarrow \cdots\cdots$$

and for any $\Lambda$

$$\Lambda = \varphi_n(\Lambda) \qquad n \ll 0$$

$$\varphi_n(\Lambda) = \pi^{-n}L_0 \qquad n \gg 0$$

---

Suppose now that I want to prove $Y_W$ is contractible. I consider the map which associates to $E$ the bundle $E/E\cap W$ in the building associated to the $A$-module $M/M\cap W$. This is a simplicial map from $X(M)$ to $X(M/M\cap W)$. I want to show that this induced map $Y_W \longrightarrow X(M/M\cap W)$, has contractible fibres. But this will be easy. For fix a bundle $E_0$ in $M\cap W$, and consider the operation $\varphi_n$ on $X$ which sends

$$\varphi_n(E) = E_0(n) + E$$

(Observe that on restriction to $C-\{\infty\}$, this gives $M\cap W + M = M$, hence what I am doing is to add $\pi^{-n}E_{0,\infty}$ to $E_\infty$.) Then as above, we see that there are homotopies

$$\Longrightarrow \varphi_n \Longrightarrow \varphi_{n+1} \Longrightarrow \cdots\cdots$$

and for $n$ large two bundles $E_1, E_2$ with the same image in $X(M/M\cap W)$ will have $\varphi_n(E_1) = \varphi_n(E_2)$.

---

<u>Idea</u>: Let $A$ be a d.v.r. with quotient field $F$ and let $V$ be a vector space over $F$ of dim $n$, and $\tilde{X}(V)$ the building of $A$-lattices in $V$, $X(V)$ the building of lattices modulo homothety. If $H$ is a hyperplane in $V$, one has a simplicial map $\tilde{X}(V) \longrightarrow \tilde{X}(V/H)$, compatible with action of $\pi$. Thus every simplex of $\tilde{X}(V)$, say $L_0 \subset \cdots \subset L_p$, $\pi L_p \subset L_0$ determines a simplex of $\tilde{X}(V/H)$.

<u>Assertion</u>: $X(V)$ may be identified with the subcomplex of $\tilde{X}(V)$ mapping onto a fixed lattice $\Lambda$ in $V/H$.

This is fairly clear - one replaces $L_0 \subset \cdots \subset L_p$ by $\pi L_{j+1} \subset \cdots \subset \pi L_p \subset L_0 \subset \cdots \subset L_j$ so that the image is a vertex. Get a new proof of the contractibility of $X(V)$ this way.

So it should now be clear that I can prove the conjecture on page 3. This tells me that if I now take $\bigcup Y_W$ where $W$ ranges over all proper subspaces of $V = F \otimes_A M$, then I get a complex which is of the homotopy type of $T(V)$.

Thus as a $\Gamma = \mathrm{Aut}(M)$-module

$$I(V) = \tilde{H}_{n-1}(X, \bigcup Y_W)$$

and the other homology groups are zero.

Variation for $\mathbb{Z}$: Here $M$ is a free abelian group of rank $n$ and $\tilde{X}$ is the space of pos. definite quadratic forms on $\mathbb{R} \otimes_{\mathbb{Z}} M$. $X$ is the quotient of $\tilde{X}$ by homothety and if we want we can identify $X$ with the part of $\tilde{X}$ having a fixed discriminant.

Given $q: M \longrightarrow \mathbb{R}$ I first need the analogue of the canonical filtration. The degree of the vector bundle $(M, q)$ is defined to be the volume ~~fundamental domain for M with respect to the volume determined by the metric q.~~ [illegible] of the ball $q \leq 1$ with respect to the volume on $\mathbb{R} \otimes_{\mathbb{Z}} M$ normalized so that the lattice $M$ has covolume 1. Thus

$$\{ m \mid q(m) \leq t^2 \} \sim \mathrm{vol}(q \leq 1) \cdot t^n \qquad t \nearrow \infty.$$

This compares nicely with

$$h^0(E(n)) \sim n \deg E + \mathrm{rank}(E)(1-g)$$

and $vol(E) \longleftrightarrow q^{\deg(E)}$.

Set

$$\Theta_{(M,q)} = \sum_{m\in M} e^{-\pi q(m)}$$

(the physicists way of computing the number of lattice points in the ball determined by $q$). Then we have the functional equation for $\Theta$

$$\Theta_{(M,q)} \Big/ \Theta_{(M,q)^*} = \frac{1}{disc_M(q)}$$

where

$$(M,q)^* = (M^*,q) \qquad M^* = \{v \in \mathbb{R}\otimes_{\mathbb{Z}} M \mid \underset{b(x,x)=q(x)}{b(v,m)\in\mathbb{Z}} \ \forall m\in M\}$$

and where

$$disc_M(q) = \det\{q(e_i,e_j)\} \qquad M = \mathbb{Z}e_1+\dots+\mathbb{Z}e_n$$

[redacted] Think of $\Theta_{(M,q)}$ as the analogue of $q^{h^0(E)}$, whence $\Theta_{(M,\frac{1}{t^2}q)}$ is analogous to $q^{h^0(E(n))}$ $t=q^n$.

$$\boxed{\frac{\Theta_{(M,\frac{1}{t^2}q)}}{\Theta_{(M^*,t^2q)}} = \frac{1}{disc_M(q)}\cdot t^n}$$

Think of this as [redacted] $t\nearrow\infty$ whence $\{\frac{1}{t^2}q\le 1\} = t\cdot\{q\le 1\}$. Clearly $\Theta_{(M^*,t^2q)}\longrightarrow 1$ very fast. And we get the pleasant estimate

$$\Theta_{(M,\frac{1}{t^2}q)} \sim \frac{1}{disc_M(q)}\cdot t^n$$

Enough recall.

Suppose now that I am given $M \xrightarrow{q} \mathbb{R}$, $q$ pos. definite. I want now the analogue of the canonical filtration of a vector bundle. ~~[illegible] that as [illegible] description~~ First I ought to check that $d(M,q) = \mathrm{disc}_M(q)^{-1}$ multiplies for an exact sequence. So let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of free f.t. abelian groups, and let $q$ be a pos. definite form on $M$ inducing forms $q'$ and $q''$ on $M'$ and $M''$ respectively. (Notice that given $0 \to V' \to V \to V'' \to 0$ one gets a canonical isom $\Lambda^n V \simeq \Lambda^p V' \otimes \Lambda^{n-p} V''$. Thus given volumes on $V'$, $V''$ one gets a volume on $V$ in a well-defined way. In particular the volume on $M_{\mathbb{R}}$ depends only on the volumes in $M'_{\mathbb{R}}$ and $M''_{\mathbb{R}}$, so to compute the discriminant we can split the above sequence and assume $q$ is the direct sum of $q'$ and $q''$ with respect to this splitting, in which case the formula

$$\mathrm{disc}_M(q) = \mathrm{disc}_{M'}(q')\, \mathrm{disc}_{M''}(q'')$$

becomes obvious.

Now that we have a good notion of degree we can start looking at subbundles of maximal degree. So put

$$\deg E = \log\left\{\mathrm{disc}_M(q)^{-1}\right\} \qquad \mathrm{disc}_M(q) = \mathrm{vol}_q(M_{\mathbb{R}}/M)$$

and define slope as usual. I need next to

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check that there exists a largest subbundle of the maximum slope.

First note that if $M'$ is of finite index in $M$ and $q$ is a form $\overbrace{\text{on } M_{\mathbb{R}}}^{M'_{\mathbb{R}}=}$, one has

$$\mathrm{vol}_q(M_{\mathbb{R}}/M') = [M:M']\,\mathrm{vol}_q(M_{\mathbb{R}}/M)$$

hence

$$\boxed{\deg(M,q) = \deg(M',q) + \log[M:M']}$$

---

~~Next suppose given two sub-bundles $E_1, E_2$ of $E = (M,q)$ corresponding to subspaces $W_i$ of $M_{\mathbb{Q}}$. Let $\overline{E_1+E_2}$ denote the subbundle of $E$ corresp to the subspace $W_1+W_2$. I wish to prove that~~

NO

~~$\deg(\overline{E_1+E_2}) + \deg(E_1\cap E_2) = \deg(E_1) + \deg(E_2) + [M\cap(W_1+W_2) : M\cap W_1 + M\cap W_2]$~~

~~Put $M_{12} = M\cap(W_1+W_2)$, $M_i = W_i\cap M$. One has~~

$$\begin{tikzcd} 0 \arrow[r] & M_1\cap M_2 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M_1/M_1\cap M_2 \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ 0 \arrow[r] & M_2 \arrow[r] & M_1+M_2 \arrow[r] & M_1+M_2/M_2 \arrow[r] & 0 \end{tikzcd}$$

~~and the quadratic form $q$ induces forms on $W_1/W_1\cap W_2$ and $W_1+W_2/W_2$. The only thing needing to be proved is that the two volumes correspond under the isom. $W_1/W_1\cap W_2 \simeq W_1+W_2/W_2$ for if that is true one has~~

~~$\deg(E_1\cap E_2) - \deg(E_1) = \deg(E_1/E_1\cap E_2)$~~

$\Vert$

~~$\deg(E_2) - \deg(E_1+E_2) = \deg(E_1+E_2/E_2)$~~

~~etc.~~

Suppose $E = (M, q)$ is a vector bundle, and let $E_1, E_2$ be sub-bundles corresponding to subspaces $W_i$ of $M_{\mathbb{Q}}$. Put $M_i = M \cap W_i$, $M_{12} = M \cap (W_1 + W_2)$, and denote by $\overline{E_1+E_2}$ the sub-bundle corresp. to the subspace $W_1 + W_2$.

**Lemma:** $$\deg(\overline{E_1+E_2}) + \deg(E_1 \cap E_2) \geqslant \deg(E_1) + \deg(E_2).$$

Proof: One has exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_1 \cap M_2 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M_1/M_1 \cap M_2 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & M_2 \arrow[r] & M_1 + M_2 \arrow[r] & M_1+M_2/M_2 \arrow[r] & 0 .
\end{tikzcd}$$

Thus if I equip $M_1/M_1 \cap M_2$ with the quotient form and call $E_1/E_1 \cap E_2$ the resulting bundle, I have

$$\deg(E_1) = \deg(E_1 \cap E_2) + \deg(E_1/E_1 \cap E_2).$$

Similarly if I tentatively define $E_1 + E_2$ to be $M_1 + M_2$ with the induced form, I will have

$$\deg(E_1 + E_2) = \deg(E_2) + \deg(E_1 + E_2/E_2).$$

Since $\deg(\overline{E_1+E_2}) = \deg(E_1+E_2) + \log[M_{12} : M_1 + M_2]$, the lemma will follow once I show

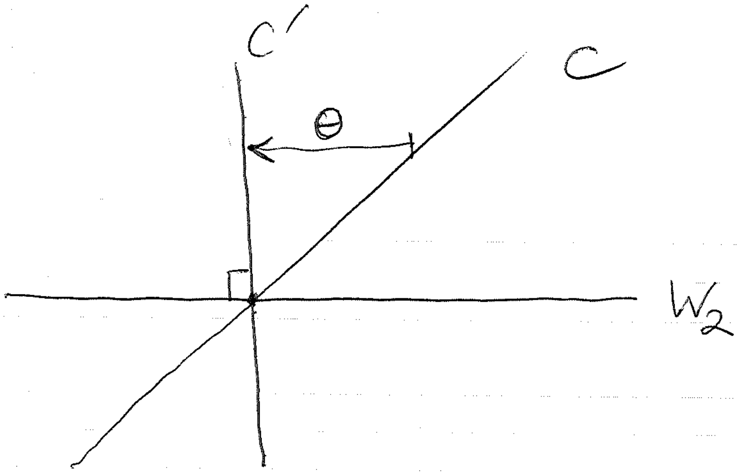
$$\deg(E_1 + E_2/E_1) \geqslant \deg(E_1/E_1 \cap E_2).$$

Now we have a canon. isom.

$$\theta : W_1/W_1 \cap W_2 \xrightarrow{\sim} W_1 + W_2/W_2$$

which preserves the lattices. What does it do to the forms? Claim $\theta$ decreases distances.

In effect, if $C$ = orthogonal complement of $W_1 \cap W_2$ in $W_1$ and $C'$ is the orthogonal complement of $W_2$ in $W_1 + W_2$, then identifying $C$ with $W_1/W_1 \cap W_2$, $C'$ with $W_1+W_2/W_2$, the forms become the restriction of $q$, and $\Theta$ becomes orth. projection onto $C'$ parallel to $W_2$.



Therefore $\Theta$ is distance decreasing. [redacted] It follows that the unit ball in $W_1+W_2/W_2$ contains more lattice points than the unit ball in $W_1/W_1 \cap W_2$, so

$$\deg(E_1+E_2/E_2) \geq \deg(E_1/E_1 \cap E_2)$$

as was to be shown.

[redacted]

<u>Refinement:</u> One has $\deg(\overline{E_1+E_2}) + \deg(E_1 \cap E_2) = \deg E_1 + \deg E_2$ if and only if [redacted] ⓘ $M_1 + M_2 = M \cap (W_1 + W_2)$ and (ii) [redacted] $W_1$ and $W_2$ intersect orthogonally.

This is clear from the preceding proof. Equality forces $M_{12} = M_1 + M_2$ and also it forces $\Theta$ to be volume-preserving. But one has only to check that a distance-decreasing volume preserving $\Theta : C \to C'$ is an isometry. To see this, one can view

$\theta$ as the identity, [redacted] and that $C$ is given two form $q, q'$ with $q \le q'$. Simultaneously diagonalization lets us write $q = \sum t_i^2$, $q' = \sum \lambda_i t_i^2$ with [redacted] $1 \le t_i$. Since volumes are equal $\prod t_i = 1$ $\Rightarrow$ all $t_i = 1$.

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<u>Consequences:</u>

Canonical filtration of a vector bundle $E = (M, q)$.

[redacted] Define

$$\mu_{max}(E) = \sup\{ \mu(W \cap E) \,/\, 0 < W \subset M_{\mathbb{Q}} \}.$$

[redacted] If now $E_1, E_2$ are sub-bundles with $\mu(E_i) = \mu_{max}(E)^{=\mu}$, then from above we get

$$\begin{array}{l} \deg(\overline{E_1 + E_2}) + \deg(E_1 \cap E_2) \le \mu[\mathrm{rank}(\overline{E_1+E_2}) + \mathrm{rank}(E_1 \cap E_2)] \\ \quad\rotatebox{90}{$\le$} \\ \deg(E_1) + \deg(E_2) = \mu[\mathrm{rank}\, E_1 + \mathrm{rank}\, E_2]. \end{array}$$

which forces [redacted] $E_1 \cap E_2$, $\overline{E_1 + E_2} = E_1 + E_2$ to have the maximal slope.

Thus one sees that one has a largest sub-bundle [redacted] of the maximum slope which is <u>semi-stable</u>.

~~That's more, we see that~~ [redacted]

To show that I have a canonical filtration of any vector bundle with the same properties I must show that if $0 \subset E_1 \subset \cdots \subset E_p = E$ with $E_i/E_{i-1}$ semi-stable of slope $\mu_i$ $\mu_1 > \cdots > \mu_p$, then $\mu_1 = \mu_{max}(E)$ and $E_1$ is the largest subbundle of slope $\mu_1$. But given $0 \ne F \subset E$ of slope $\mu_{max}(E)$, assume $j$ least $\ni F \subset E_j$. Then $E_{j-1} \subset \overline{E_{j-1} + F} \subset E_j$ so as

$E_j/E_{j-1}$ is semi-stable of slope $\mu_j$, we have

$$\begin{array}{ccc} \deg(\overline{F+E_{j-1}})-\deg(E_{j-1}) & \leqslant & \mu_j\left[rg(\overline{F+E_{j-1}})-rg(E_{j-1})\right] \\ \vee\!| & & \\ & & \mu_j\left[rg(\overset{\|}{F})-rg(F\cap E_{j-1})\right] \\ \deg(F)-\deg(F\cap E_{j-1}) & & \end{array}$$

But $\deg(F)-\deg(F\cap E_{j-1}) \geqslant \mu_{max}(E)\left[rg(F)-rg(F\cap E_{j-1})\right]$ and as $j$ is least $\Rightarrow$ $F\cap E_{j-1} < F$. Thus get $\mu_j \geq \mu_{max}(E)$, which implies $\mu_{max}(E)=\mu_1$, $j=1$. done.

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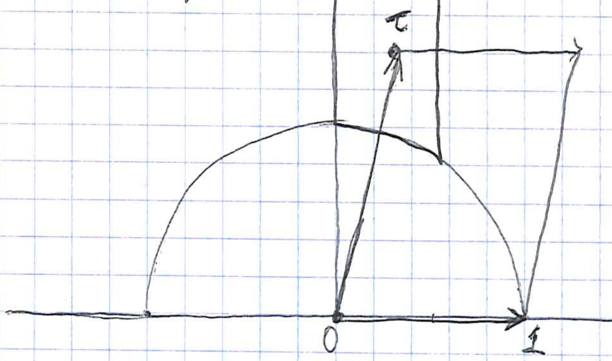
Now that I have the canonical filtration for vector bundles, I can try to describe the corners on the symmetric space.

So let $\tilde{X} = GL_n(\mathbb{R})/O_n$ be the symmetric space of positive definite forms on $(\mathbb{Z}^n)_{\mathbb{R}}$, $\Gamma = GL_n(\mathbb{Z})$. Given a proper subspace $W$ of $V=\mathbb{Q}^n$, let $\tilde{Y}_W$ be the subset of $\tilde{X}$ consisting of all $q$ such that $W$ is part of the canonical filtration of $E=(M,q)$. In other words such that the minimum slope of $E\cap W$ should be larger than the maximum slope of $E/E\cap W$.

Conjecture: $\tilde{Y}_W$ is open in $\tilde{X}$; Also $Y_W$ open in $X$.

Here $X = \tilde{X}/\mathbb{R}_+^*$.

Example: $n=2$. Here we can identify a vector bundle with a lattice $\Lambda$ in the plane $\mathbb{R}^2$. Up to a rotation and homothety we can assume $z=1$ is a minimal length vector in $\Lambda$, and there is then a unique other basis element $\tau$ in the fundamental domain $Im(\tau) > 0$, $0 \le Re(\tau) \le \frac{1}{2}$, $|\tau| \geq 1$.

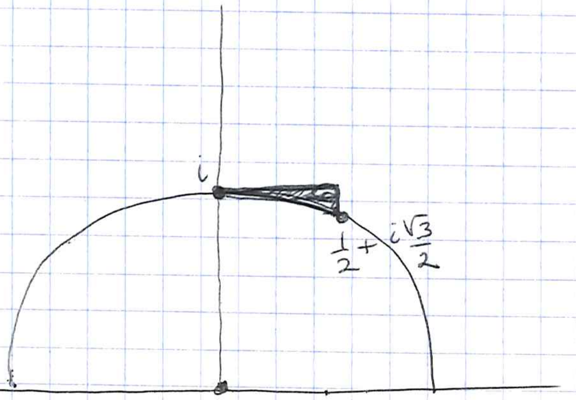


The lattice is semi-stable if on choosing a line, $L$ thru 0 passing through a lattice point, say $\Lambda \cap L = \mathbb{Z}\lambda$ then the degree of $L$ which is $\log \frac{1}{|\lambda|}$ should be $\le$ slope of $\Lambda$ which is $\frac{1}{2} \log \frac{1}{Im \tau}$. Thus one wants

$$|\lambda| \geqslant \sqrt{Im\, \tau}$$

for all $\lambda \in \Lambda$. ~~This is certainly OKAY if $Im\,\tau \le 1$,~~ [illegible]

In particular taking $\lambda = 1$, we must have $Im\,\tau \le 1$. If $Im\,\tau \le 1$, then as $|\lambda| \geq 1$ for all $\lambda$, it's OK. Thus the lattice is semi-stable iff $Im\,\tau \le 1$. So the semi-stable iso. classes look as follows:



March 30, 1974. Siegel formula

Let $C$ be a complete curve over a finite field $k$ with $q$ elements, and let $L$ be a line bundle on $C$. One considers vector bundles $E$ of rank $n$ equipped with an isomorphism $u : \Lambda^n E \simeq L$ and forms the sum (following Eisenstein)

$$\sum \frac{1}{\operatorname{aut}(E,u)}$$

where the sum is taken over the isomorphism classes, and aut denotes the order of the group of autos. The Siegel formula expresses this sum in terms of values of the $\zeta$ function of $C$.

Example: 1) Let $C = \mathbb{P}^1_k$, and take $n=2$, $L = \mathcal{O}$. The isom. classes of vector bundles [redacted] are represented by

$$\mathcal{O} \oplus \mathcal{O}, \qquad \mathcal{O}(a) \oplus \mathcal{O}(-a) \qquad a \geq 1$$

and $$\operatorname{Aut}\left[\mathcal{O}(a) \oplus \mathcal{O}(-a)\right] = \begin{pmatrix} k^{\circ} & H^0(\mathcal{O}(2a)) \\ 0 & k^{\circ} \end{pmatrix} \qquad \dim H^0(\mathcal{O}(2a)) = 2a+1.$$

The sum in question is

$$\frac{1}{|SL_2(k)|} + \sum_{a \geq 1} \frac{1}{(q-1)\, q^{2a+1}} = \frac{1}{(q^2-1)(q)} + \frac{1}{(q-1)q} \, \frac{q^{-2}}{1-q^{-2}}$$

$$= \frac{1}{(q^2-1)(q)} \left[ 1 + \frac{1}{q-1} \right]$$

$$= \frac{1}{(q-1)(q^2-1)}$$

2) [redacted] $C = \mathbb{P}^1_k$, $n=2$, $L = \mathcal{O}(1)$. [redacted]

[redacted] Then the different iso. classes are

[redacted] $$\mathcal{O}(a+1) \oplus \mathcal{O}(-a) \qquad a \geq 0$$

and the sum to evaluate is

$$\sum_{a\geq 0} \frac{1}{(q-1)\,q^{2a+2}} = \frac{1}{(q-1)q^2(1-q^{-2})} = \frac{1}{(q-1)(q^2-1)}$$

---

*Example 3.* Let $C$ be an elliptic curve with [redacted] origin $\infty$, take $L=\mathcal{O}(1)$ corresp to divisor at $\infty$, $n=2$. Then I have the following iso classes.

| | | auto gp. | contribution |
|---|---|---|---|
| (unique ext. non-trivial<br>$0\to E\to\mathcal{O}(1)$ | | $k^*$ | $1$ |
| $\forall L\in J$<br>$L^2\cong\mathcal{O}$<br>$a\geq 0$ | $L(a+1)\oplus L(-a)$ | $\begin{pmatrix} k^* & H^0(\mathcal{O}(2a+1)) \\ 0 & k^* \end{pmatrix}$ | $\frac{1}{(q-1)\,q^{2a+1}}$ |
| $\forall L\in J$<br>$L^2\not\cong\mathcal{O}$<br>$a\geq 0$ | $L(a+1)\oplus L^{*}(-a)$ | $\begin{pmatrix} k^* & H^0(L^2(2a+1)) \\ 0 & k^* \end{pmatrix}$ | $\frac{1}{(q-1)\,q^{2a+1}}$ |

So the sum is

$$1+(\operatorname{card} J)\sum_{a\geq 0}\frac{1}{(q-1)\,q^{2a+1}} = 1+(\operatorname{card} J)\frac{q}{(q-1)(q^2-1)}$$

$$= \frac{1+(-1-q+\operatorname{card} J)\,q+q^3}{(1-q)(1-q^2)}$$

But recall that $\zeta_C$ is of the form

$$\frac{1-[redacted](\alpha+\bar{\alpha})z+qz^2}{(1-qz)(1-q^2z)}$$

where $-(\alpha+\bar{\alpha}) = -1-q+\operatorname{card} J$

So this seems to check out.

Next take $L = \mathcal{O}$, and try to calculate the sum. Here the structure of the semi-stable bundles of degree zero is more complicated. We have the following classes:

auto gp.

1) $L \oplus L^*$ for each $\{L, L^*\} \in J(k)$, $L \neq L^*$. $k^* \times k^*$

2) $L \oplus L$ for each $L \subset J(k)$, $L = L^*$ $GL_2(k)$

3) $0 \to L \to E \to L \to 0$ nontrivial $k^* \times k$.

4) For each element not in the image of the Weierstrass map $x : J(k) \longrightarrow \mathbb{P}_1(k)$, i.e. a pair $\{L, L^*\}$ of conjugate bundles defined over a quadratic extension of $k$, we will get a stable bundle of degree zero, which becomes $L \oplus L^*$ over the quadratic extension. $\text{aut} = k'^*$

Count: $x : J(k) \longrightarrow \mathbb{P}_1(k)$. Let $a$ = number of $L \in J(k)$ such that $L = L^*$. Then

$$\text{card}(\text{Im}\, x) = \frac{\text{card}\, J - a}{2} + a = \frac{1}{2}\text{card}\, J + \frac{1}{2}a$$

Contributions:

1) $\dfrac{\text{card}\, J - a}{2} \, \dfrac{1}{q-1}$

2) $a \, \dfrac{1}{(q^2-1)\, q}$

3) $a \, \dfrac{1}{q}$

4) $\left(q+1 - \dfrac{\text{card}\, J + a}{2}\right) \dfrac{1}{q+1}$

Summing

$$1 + \operatorname{card} J \left[ \frac{1}{2} \cdot \frac{1}{q-1} - \frac{1}{2} \frac{1}{q+1} \right] + a \left[ -\frac{1}{2} \frac{1}{q-1} + \frac{1}{(q^2-1)q} \overset{\overset{0}{\shortparallel}}{+} \frac{1}{q} - \frac{1}{2} \frac{1}{q+1} \right]$$

$$= 1 + \frac{\operatorname{card} J}{q^2 - 1}$$

Now for the unstable bundles one has

$$\operatorname{card} J \sum_{a \geq 1} \frac{1}{(q-1)\, q^{2a}} = (\operatorname{card} J) \frac{1}{(q-1)(q^2-1)}$$

So the ~~total~~ is again

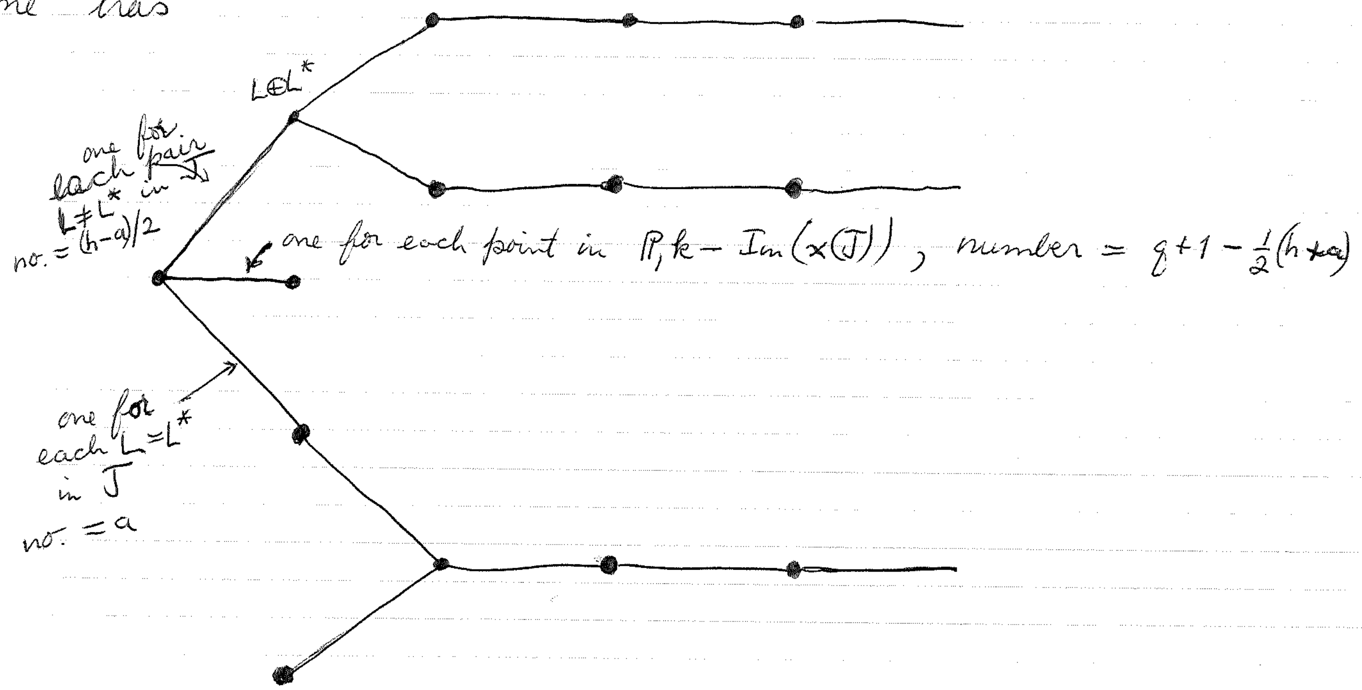
$$1 + (\operatorname{card} J) \frac{q}{(q-1)(q^2-1)}$$

---

Note: For ~~ad~~ elliptic curve over finite field $\mathbb{F}_q$, one has the following description for the quotient graph. Put

$$h = \operatorname{card} J = \operatorname{card} C(\mathbb{F}_q).$$

and let $a$ = number of points of order 2 in $C(\mathbb{F}_q)$. Then one has



Siegel formula: If $C$ is a curve over $\mathbb{F}_q$, and $\alpha \in \mathrm{Pic}(C)$ then as $E$ runs over representatives for the iso. classes of bundles of rank 2 with $c_1(E) = \alpha$, one has

$$\sum \frac{1}{\mathrm{aut}(E)} = \frac{1}{q-1} \zeta_C(q)$$

(Have checked this is the right answer for $\mathbb{P}_1$ and elliptic curves).

---

Let now $C$ be a curve over a finite field $k$ with $q$ elements, $\infty$ a rational point, $X$ the tree assoc. to a rank 2 bundle $M$ on $C - \infty$.

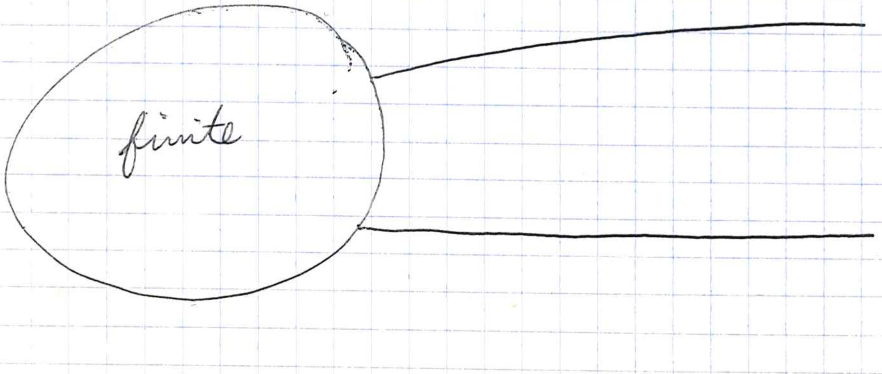
Let $\Gamma$ be a subgroup of $\mathrm{Aut}(M)$ of finite index. I claim that the sum

$$\sum_{\sigma} \frac{(-1)^{\sigma}}{\mathrm{card}(\Gamma_\sigma)}$$

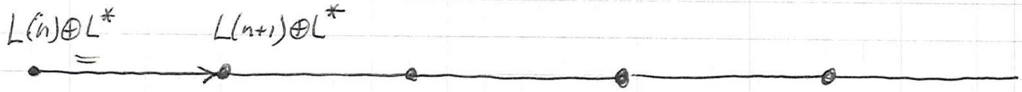
makes sense, and moreover is essentially a finite sum. Here $\sigma$ runs over the simplices of $X$ and $(-1)^{\sigma} = (-1)^{\dim \sigma}$.

~~The fact that the sum makes sense~~

Picture:



and on a cusp one has



$$\mathrm{Aut}(L(n) \oplus L^*) = \begin{pmatrix} k^* & H^0(L^2(n)) \\ & k^* \end{pmatrix}$$

$$\dim H^0(L^2(n)) = n + 1 - g \quad \text{for } n \text{ large.}$$

so by absolute convergence of

$$\sum{}' \frac{1}{q^n}$$

one gets convergence for the sum. Next thing to notice is that because on a cusp the stabilizer of an edge is the same as its smallest vertex, we have cancellation of terms in the sum. Thus the sum is really being taken over the finite core.

Alternative interpretation. To each line $\ell$ in $F^2$ we have attached the subcomplex $X_\ell$ consisting of unstable bundles whose big sub-line bundle is given by $\ell$. This gives me then an exact sequence

$$0 \longrightarrow H_1(X, \amalg X_\ell) \longrightarrow \bigoplus_\ell \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

which shows $H_1(X, \amalg X_\ell)$ is the Steinberg module $I(F^2)$.

Observe that the vertices in $X$ can be put over $n = -1, 0, 1, 2, \ldots$ as follows. A vertex of $X$ gives a vector bundle $E$ of rank 2 whose degree we can assume is either 0 or 1.

" non-semi-stable

If E is imstable, then it has a subbundle $L \ni$ the quotient E/L has smaller degree, i.e.

$$0 \to L \longrightarrow E \longrightarrow E/L \longrightarrow 0$$

$$\deg(L) > \deg(E/L).$$

It follows then that L is uniquely determined. Now the difference $\deg(L) - \deg(E/L)$ is [illegible] the thing to consider. This will be the integer I attach to E. If E is semi-stable, attach to E the integer $-1$ or $0$ depending on whether the degree of E is odd or even.

Note that what sits over $n = -1$ are the stable bundles of degree 1, and Harder tells me he can easily compute [illegible] their number from the Siegel formula.

Suppose I have now an imstable bundle

$$0 \longrightarrow L \longrightarrow E \longrightarrow E/L \longrightarrow 0$$

[illegible] with $n = \deg(L) - \deg(E/L) > 0$, and that I have a 1-simplex issuing from this vertex, i.e. a

$$\begin{tikzcd} 0 \arrow[d] \\ E' \arrow[d, phantom, "\cap"] \\ E \arrow[d] \\ k(\infty) \arrow[d] \\ 0 \end{tikzcd}$$

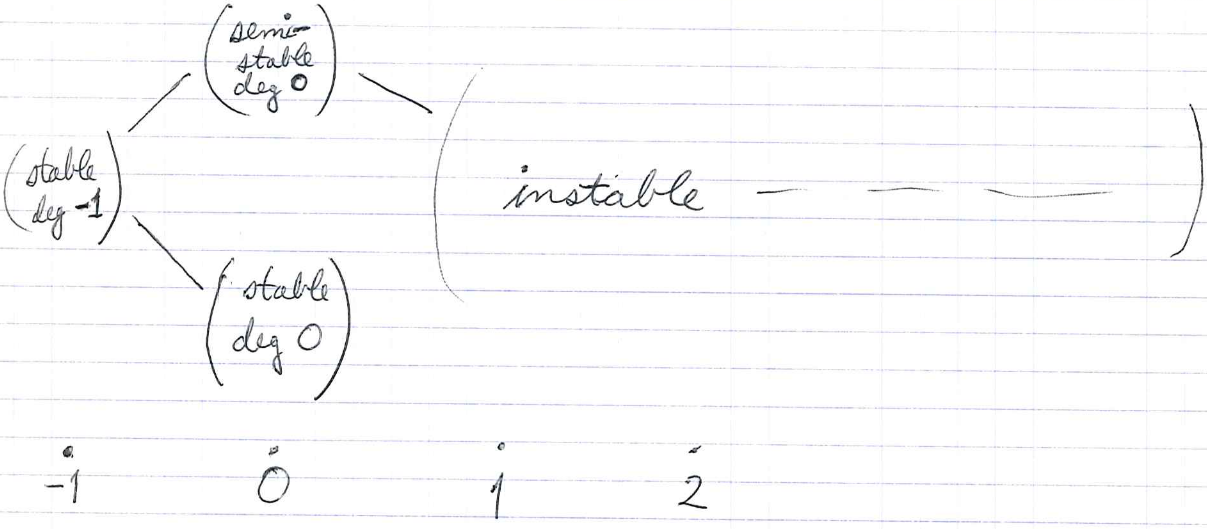
Then there are two cases: If $L \to E \to k(\infty)$ is [illegible] zero, then E' is an extension of L by E/L(-1), so it is more imstable. Otherwise E is an extension of L(-1) by E/L:

$$0 \longrightarrow L(-1) \longrightarrow E' \longrightarrow E/L \longrightarrow 0$$

and if $n \geq 2$, $E'$ is **instable**. If $n=1$, then one has $\deg L(-1) = \deg(E/L)$, so $E'$ is semi-stable.

In any case for an **instable** vertex, there is exactly one way to go to become more instable, and this is the contraction toward the cusp [redacted].

Suppose that $E$ is stable of degree $0$, and we have a vertex $E' \subset E$. Then $E$ has no sub-line-bundles of degree $0$, so neither does $E'$, and as $E'$ has deg $-1$ this means $E'$ is stable of degree $-1$. So we get the picture:



In the elliptic curve over alg cl. $k$ there were no stable bdls of degree $0$ but over a finite field there could be.

Next note that if $E$ is semi-stable of degree $0$, then either $E$ has a unique line bundle $L$ of degree $0$, or else, it is decomposable. (Recall semi-stable bundles of a given slope [redacted] form an artinean category). In the former case if $E' \subset E$ is an edge, then either

$$0 \to L(-1) \to E' \to E/L \to 0 \quad \Rightarrow \quad E' \text{ stable deg } -1.$$

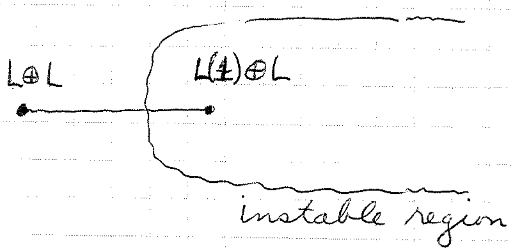
or

$$0 \to L \to E' \to E/L(-1) \to 0 \quad \Rightarrow \quad E \text{ instable}$$

So again we have a unique [illegible] instable edge starting from E.

When $E = L_1 \oplus L_2$, $L_2 \neq L_1$ there are two [illegible] edges leading to the instable region. When $L_1 = L_2$ all edges lead to the instable region.

What do these facts tell us about the Steinberg module? [illegible] Before we wrote the [illegible] instable region of X as a disjoint union of trees $X_l$ for each line $l$ in $F^2$. Now for each $l$ such that the corresponding L in J is such that $L = L^*$, we will get extremal edges [illegible]

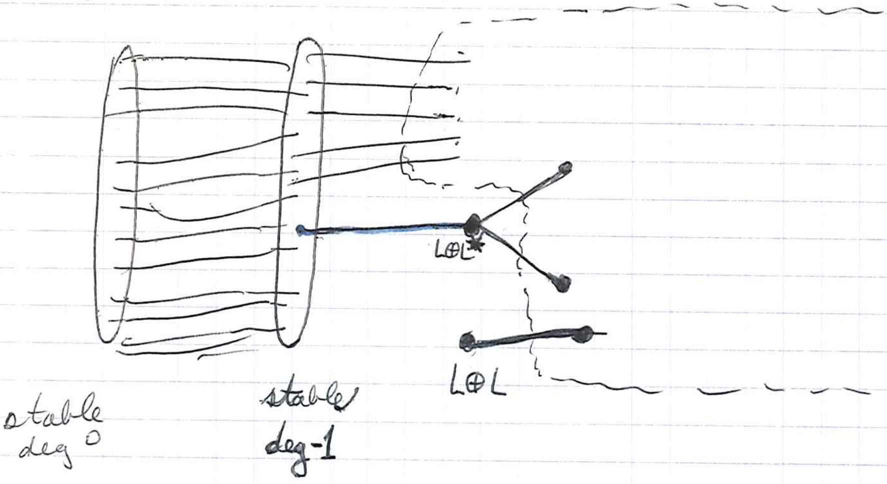


and thus for each element of order 2 in Pic(A), we get [illegible] a direct summand in $I(F^2)$ of the form $\mathbb{Z}[GL_2 A] \otimes_{\mathbb{Z}[GL_2 k]} I(k^2)$, the embedding $GL_2 k \to GL_2 A$ being obtained from an isom. $L \oplus L \simeq A^2$ over $C - \infty$.

We add to the instable region those semi-stable bundles of degree 0 having a unique subbundle of degree zero. ~~Then on the boundary of this one~~ Then in the complement of the instable region I have the following types:

$$\begin{pmatrix}\text{stable degree}\\ 0\end{pmatrix} \quad \begin{pmatrix}\text{stable}\\ \text{deg} -1\end{pmatrix} \quad \begin{cases} L \oplus L^* & L \neq L^* \text{ in } J \\ L \oplus L & L = L^* \text{ in } J. \end{cases}$$

So my picture appears



~~I am now [redacted] ready to interpret the sum~~

$$\sum \frac{(-1)^\sigma}{\mathrm{aut}(\sigma)}$$

~~as the Euler characteristic of~~ $\Gamma = GL_2(A)$ ~~acting on the Steinberg module. It is first necessary to explain what this means.~~

~~So let~~ $X_{ins}$ ~~denote the instable region of~~ $X$, ~~that is, the subcomplex consisting of vertices whose corresponding bundle are instable or [redacted] are semi-stable [redacted] and indecomposable. Then I have seen that~~ $X_{ins} = \coprod_\ell X_\ell$, ~~ℓ runs over lines in [redacted]~~ $F^2$, $X_\ell$ ~~tree [redacted] and~~

$$I(F^2) = H_1(X, X_{ins})$$

~~Denote by~~ $X_0$ ~~the subcomplex~~

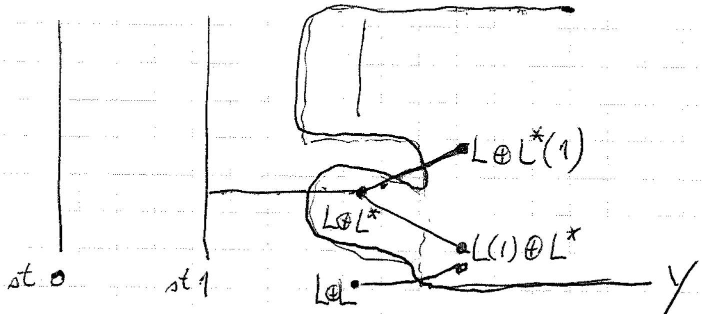
Note that what has been left after removing the part that can be pushed to $\infty$ (instable bundles and semi-stable bundles of degree 0 which are indecomposable), is the subcomplex of stable bundles and decomposable semi-stable bundles.

Denote by $X_{ins}$ the subcomplex of instable bundles, so that $X_{ins} = \coprod X_\ell$, $X_\ell$ a tree, and

$$I(F^2) = H_1(X, X_{ins})$$

Denote by $X_{sst}$ and $X_{st}$ the subcomplex consisting of the semi-stable, resp. st bundles. Finally let $Y$ denote the subcomplex consisting of $X_{ins}$, plus the indecomposable semi-stable bundles of degree zero, plus for each pair $L \neq L^*$ add in the vertices isom. to $L \oplus L^*$ and the edge $L \oplus L^* \subset L(1) \oplus L^*$.

<u>Picture</u>:



Have to choose in each $\{L, L^*\}$ a representative. The point is that for each vertex isomorphic to $L \oplus L^*$ we have two edges leading to the unstable region, and we pick one.

Then $Y$ deforms into $X_{ins}$, so

$$I(F^2) = H_1(X, Y).$$

~~Let $\sigma_1$ be a set of rep. for the pairs $L, L^* \in J \ni L \neq L^*$, and $\sigma_2$ the set of $L \in J$ such that $L = L^*$. Then as [illegible] it is clear that $X/Y$ is the wedge of $X_{sst}/X_{sst} - X_{st}$ and $S^1$ for each [illegible]~~

Notation: Let [illegible] $\sigma$ be a set of rep. for pairs $\{L, L^*\}$ in $J \ni L \neq L^*$, and for each $L \in \sigma$, let $Z_L$ denote the set of [illegible] vertices in $X$ isom to $L \oplus L^*$. Let $\tau$ be the set of $L \in J$, $L = L^*$; i.e. $\tau = {}_2J$, and

$$J = \sigma \sqcup -\sigma \sqcup {}_2J$$

$Z_L$ for $L \in \tau$ is the set of vertices isom to $L \oplus L$. Then it is clear we have

$$H_1(X, Y) = H_1(X_{sst}, X_{sst} - X_{st})$$
$$\Big(\bigoplus_{L\in\sigma} \bigoplus_{Z_L} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{L\in\tau} \bigoplus_{Z_L} I(k^2)\Big)$$

So now one is reduced to the case of the stable bundles.

---

Can one decide which stable bundles of deg $-1$ are joined to semi-stable bundles of degree $0$?

Let $E$ be stable of degree $1$ and $E' \subset E$ an edge with $E'$ semi-stable of degree $0$, i.e. $E'$ has a sub-line-bundle of degree $0$. Notice that if [illegible] $L$ is of degree zero, then $H^0(L, E)$ is at most dim. $1$. For if if one has two sections of $L^{-1} \otimes E$, they cannot be everywhere independent (for then $\mathcal{O}^2 = L^{-1}\otimes E$ – impossible as degree $L^{-1}\otimes E = \underline{1}$), thus there must be a section which is non-zero and which vanishes in some fibre, hence E would contain a subbundle of degree $>0$, which is impossible. (This works also if k not alg. closed: First if $\mathcal{O}^2 \longrightarrow E'^{\,= E\otimes L^{-1}}$ is not injective, the image would be a line subbundle of $E'$ of pos. degree. If $\mathcal{O}^2 \hookrightarrow E'$ is injective, then as degree $(E') = 1$, the cokernel would have k-dim 1, hence would be supported at a rational point P where the two sections become dependent, etc.)

Thus if E can be joined to a semi-stable bundle, E contains a line subbundle of deg 0. Conversely if E contains L of degree zero, one has

$$0 \longrightarrow L \longrightarrow E \longrightarrow L'(1) \longrightarrow 0$$

and so E contains $0 \to L \to E' \longrightarrow L' \longrightarrow 0$ which is semi-stable.

Now we have now reached the point where we see that the essential part of the Steinberg homology comes from the stable vector bundles

April 1, 1974: Siegel formula

Let $A$ be the ring of integers in a number field $F$. If $x \in A$, $x \neq 0$, one can define $Nx = \text{card}(A/xA)$; it is a theorem that this gives the norm of $x$ relative to the extension $F/\mathbb{Q}$.

Suppose we are interested in the solutions to $Nx = n$ with $x \in A$. Observe first that $A^*$ acts freely on these solutions. We would like to show that there are finitely many $A^*$-orbits. Such an orbit is the same thing as a principal ideal $\mathfrak{a}$ in $A$ such that $N\mathfrak{a} = \text{card}(A/\mathfrak{a}) = n$. So we get a bigger number by considering all ideals $\mathfrak{a}$ in $A$ with $N\mathfrak{a} = n$. Note that there are only finitely many such ideals because there are only finitely many $\mathbb{Z}$-lattices between $An$ and $A$.

Next we can introduce the generating function

$$\zeta_A(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}(\text{no. of } \mathfrak{a} \ni N\mathfrak{a} = n)$$

for the enlarged problem. (One reason for doing this is that we have an Euler formula

$$\zeta_A(s) = \prod_{\mathfrak{p}} \frac{1}{1-(N\mathfrak{p})^{-s}}$$

which relates our problem to the arithmetic in $A$). Also we have the generating function for the initial problem

$$\zeta(s,\alpha) = \sum_{\mathfrak{a} \in \alpha} \frac{1}{(N\mathfrak{a})^s} = \sum_{n=1}^{\infty}{}' \frac{1}{n^s}\left(\begin{matrix}\text{no. of } \mathfrak{a} \text{ in the class} \\ \alpha \ni N\mathfrak{a} = n\end{matrix}\right)$$

where $\alpha$ is a given ideal class.

So what I want to understand is the asymptotic behavior of $\operatorname{card}\{\mathfrak{a}\in\alpha \mid N\mathfrak{a}=n\}$ as a function of $n$. The philosophy of generating functions is that this should be reflected in the singularities of the $\zeta$-function.

Example: 1) $A=\mathbb{Z}$, here $\operatorname{card}\{\mathfrak{a} \mid N\mathfrak{a}=n\}=1$, and on the other hand $\zeta$ has a simple pole at $s=1$ with residue $1$. Reason: $\sum \frac{1}{n^s} \sim \int_1^\infty \frac{dt}{t^s} = \frac{t^{1-s}}{1-s}\Big|_1^\infty = \frac{1}{s-1}$.

2) $A=\mathbb{Z}[i]$. This is also a P.I.D. and

$$\operatorname{card}\{\mathfrak{a} \mid N\mathfrak{a}=n\} = \text{no. of integral solutions of } x^2+y^2=n \text{ divided by } 4.$$

$\zeta$ has a simple pole at $s=1$, residue $=\frac{\pi}{4}$

To determine this, let

$$\rho(r) = \operatorname{card}\{(x,y)\in\mathbb{Z}^2 \mid 0< x^2+y^2 \leq r\}$$

so that

$$\zeta(s) = \frac{1}{4}\int_{1-}^{\infty} \frac{1}{r^s}\, d\rho(r)$$

$$= \frac{1}{4}\int_1^\infty s r^{-s-1}\rho(r)\, dr \qquad \rho(r) \sim \pi r$$

$$\sim \frac{s}{4}\int_1^\infty r^{-s}\pi\, dr \sim \frac{\pi}{4}\frac{1}{s-1} \qquad \text{as } s\to 1$$

Before going on recall the formulas

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2} \qquad \theta(t) = \frac{1}{t}\,\theta\left(\frac{1}{t}\right)$$

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^{s-1}\, dt \qquad s\Gamma(s) = \Gamma(s+1)$$

$$\Gamma(1) = 1.$$

$$\pi^{-s}\Gamma(s) = 2\int_0^\infty e^{-\pi t^2}\, t^{2s}\, \frac{dt}{t} \qquad \Gamma(n) = (n-1)!$$

and the functional equation for $\zeta$:

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \sum_{n\geq 1} \cancel{\frac{1}{n^s}}\, 2\int_0^\infty e^{-\pi n^2 t^2}\, \cancel{n^s} t^s\, \frac{dt}{t}$$

$$= \int_0^\infty \left[\theta(t) - 1\right] t^s\, \frac{dt}{t}$$

$$= \int_0^1 \left[\theta(t)-1\right] t^s \frac{dt}{t} + \underbrace{\int_1^\infty \left[\theta(t)-1\right] t^s \frac{dt}{t}}_{\text{entire as } \theta(t)-1 \text{ is rapidly decreasing}}$$

$$= \int_1^\infty \Big[\underbrace{\theta\left(\tfrac{1}{t}\right)}_{\overset{\shortparallel}{t\theta(t) - t + t - 1}} - 1\Big]\, t^{-s}\, \frac{dt}{t} + \quad ''$$

$$= \underbrace{\int_1^\infty \left[\theta(t)-1\right] t^{1-s} \frac{dt}{t} + \int_1^\infty \left[\theta(t)-1\right] t^{s} \frac{dt}{t}}_{\text{entire}} + \underbrace{\int_1^\infty (t-1)\, t^{-s} \frac{dt}{t}}_{\overset{\shortparallel}{\frac{1}{s-1} - \frac{1}{s}}}$$

This is symmetric under $s \longmapsto 1-s$ so one gets the functional equation for $\zeta(s)$.

Temporary digression: [redacted] [redacted] [redacted] One can generalize the above $\zeta$ function as follows. Let $M$ be a vector bundle over $A$ and consider the sum

$$\sum_{\mathfrak{a}} \frac{1}{[M:\mathfrak{a}]^s}$$

where $\mathfrak{a}$ runs over all $A$-submodules of finite index in $M$. By Chinese Remainder theorem, this is a product of local factors

$$\sum_{\Lambda < A_{\mathfrak{q}}^n} \frac{1}{[A_{\mathfrak{q}}^n : \Lambda]} \qquad n = \operatorname{rank}(M)$$

where $\Lambda$ runs over all lattices insides of the d.v.r. $A_{\mathfrak{q}}$.

Lemma: Let $\mathcal{O}$ be a d.v.r., [redacted] $\pi$ unif., $\Lambda$ a lattice contained in $\mathcal{O}^n = \mathcal{O}e_1 + \cdots + \mathcal{O}e_n$. Then $\Lambda$ has a basis [redacted] consisting of

$$\pi^{a_1} e_1$$

$$b_{21} e_1 + \pi^{a_2} e_2$$

$$b_{n1} e_1 + b_{n2} e_2 + \cdots + \pi^{a_n} e_n$$

where the $a_1, \ldots, a_n \geq 0$ are uniquely det. integers, and where $b_{ij}$ [redacted] $\in \mathcal{O}$ is unique modulo $a_i$.

For example: $n = 2$. $\Lambda \cap \mathcal{O}e_1 = \mathcal{O}\pi^{a_1} e_1$, and projecting $\operatorname{Im}\{\Lambda \longrightarrow \mathcal{O}e_2\} = \mathcal{O}\pi^{a_2} e_2$, whence $\Lambda$ has basis $\pi^{a_1} e_1$, $b_{21} e_1 + \pi^{a_2} e_2$. Then $b_{21}$ can be modified by any element of $\mathcal{O}\pi^{a_1}$.

Thus if $O/\pi O$ has $q$ elements

$$\sum_{\Lambda\subset O^n} \frac{1}{(\operatorname{card} O^n/\Lambda)^s} = \sum_{a_1,\dots,a_n\geq 0} \frac{1}{(q^{\Sigma a_i})^s}\,(q^{a_1})^{n-1}\cdots(q^{a_{n-1}})$$

$$= \sum_{a_1,\dots,a_n\geq 0} (q^{-s+n-1})^{a_1}(q^{-s+n-2})^{a_2}\cdots(q^{-s})^{a_n}$$

$$= \frac{1}{(1-q^{-s})(1-q^{-s+1})\cdots\cdots(1-q^{-s+n-1})}$$

Thus it is the $\zeta$-function of $\mathbb{P}^{n-1}$.

$$\frac{1}{(1-z)(1-qz)\cdots(1-q^{n-1}z)}$$

Suppose $C$ is a curve over $k = \mathbb{F}_q$. Consider the series where $E_0$ is a fixed bundle of rank $n$

$$(*) \qquad \sum_{E \subset E_0}{}' \frac{1}{(\text{card } E_0/E)^s} = \sum_{E \subset E_0}{}' z^{\deg(E_0) - \deg(E)}$$

where $E$ is a sub-$\mathcal{O}_C$-module in $E_0$ such that $E_0/E$ is torsion. The above sum has an Euler product expression with local factors

$$\sum_{\Lambda \subset \mathcal{O}_P^n \otimes E_0}{}' \frac{1}{\text{card}(\mathcal{O}_P^n \otimes E_0/\Lambda)^s}$$

~~which we calculated yesterday;~~

which we found to be

$$\frac{1}{(1-(NP)^{-s}) \cdots (1-(NP)^{-s+n-1})}$$

$NP = \text{card } k(P) = q^{\deg(P)}$. Thus $(*)$ is the product

$$\zeta(s)\,\zeta(s-1) \cdots \zeta(s-n+1)$$

which shows it is independent of $E_0$.

Notice also that by duality there is a 1-1 correspondence between $E \subset E_0$ and $E_0^* \subset E^* \subset E_0^*(\eta)$.

~~Thus we also have the formula~~

Prop. Let $V$ be a vector space of dimension $n$ over $F$, and let $E_0$ be a vector bundle over $C$ with generic fibre $V$. (One might call $E_0$ a $C$-lattice in $V$). Then

$$\sum_{E_0 \subset E < V} \frac{1}{(\operatorname{card} E/E_0)^s} = \zeta_C(s)\,\zeta_C(s-1) \cdots \zeta_C(s-n+1)$$

or in terms of $z = q^{-s}$.

$$\sum_{E \ni E_0 < E \subset V} z^{\deg(E) - \deg(E_0)} = Z(z)\, Z(qz) \cdots Z(q^{n-1}z)$$

---

Now I want to break up the sum on the left over the isomorphism classes of $E$. Let $T$ be a fixed bundle of deg $n$. I want to count the number of lattices $E$ in $V$ containing $E_0$ such that $E \simeq T$. First change notation:

Let $E$ denote a fixed bundle of deg $n$ over $C$, let $\Lambda_0$ be a fixed $C$-lattice in $V$, and I want to count the number of lattices $\Lambda$ in $V$ containing $\Lambda_0$ such that $\Lambda$ is isomorphic to $E$. Given such a $\Lambda$, if I pick and isom. $\theta : \Lambda \xrightarrow{\sim} E$, I get a injection $\theta' : \Lambda_0 \hookrightarrow E$. Thus I get a map

$$\left\{ \begin{matrix} \text{set of } \Lambda \supset \Lambda_0 \\ \ni \Lambda \simeq E \end{matrix} \right\} \longrightarrow \operatorname{Inj}(\Lambda_0, E) / \operatorname{Aut}(E).$$

Note $\operatorname{Aut}(E)$ acts freely on $\operatorname{Inj}(\Lambda_0, E)$ for if $\alpha i = i$ then $\alpha$ is the identity at the generic pt; hence $\alpha =$ identity.

Conversely given $i : \Lambda_0 \hookrightarrow E$, [redacted]

$$\begin{tikzcd}
\Lambda_0 \arrow[r, hook, "i"] \arrow[d] & E \arrow[d] \\
V = \Lambda_0(\eta) \arrow[r, "\sim", "i(\eta)"'] & E(\eta)
\end{tikzcd}$$

hence there exists a unique map $E \xrightarrow{\gamma} V$ $\ni$ $\gamma i =$ inclusion of $\Lambda_0$ in $V$. Assign to $i$ the image of $\gamma$ which is a lattice containing $\Lambda_0$. This gives a map

$$\mathrm{Inj}(\Lambda_0, E)/\mathrm{Aut}(E) \longrightarrow \left\{ \begin{matrix} \text{set of } \Lambda \supset \Lambda_0 \\ \ni \Lambda \simeq E \end{matrix} \right\}$$

which is clearly inverse to the [redacted] proceding. Thus we obtain:

[redacted]

Lemma: [redacted] $\Lambda_0$ a $C$-lattice in the $F$-vectorspace $V$, $E$ a vector bundle of rank $h = \dim_F(V)$. Then

$$\left( \begin{matrix} \text{set of lattices } \Lambda \text{ in } V \\ \text{such that } \Lambda \supset \Lambda_0 \text{ and} \\ \text{such that } \Lambda \simeq E \end{matrix} \right) \simeq \mathrm{Inj}(\Lambda_0, E)/\mathrm{Aut}(E).$$

and $\mathrm{Aut}(E)$ acts freely on $\mathrm{Inj}(\Lambda_0, E)$.

So putting together the above we have

$$[redacted]\ Z_C(z) \cdot \ldots \cdot Z_C(q^{h-1} z) = \sum_E z^{\deg E - \deg(\Lambda_0)} \frac{\mathrm{card}\{\mathrm{Inj}(\Lambda_0, E)\}}{\mathrm{aut}(E)}$$

where the sum is taken over the iso. classes $E$ of bundles of rank $h$.

~~[illegible]~~ Take $\Lambda_0 = \mathcal{O}_C^r \subset F^r = V$, so that $\deg(\Lambda_0) = 0$. Break up the preceding sum according to $c_1(E)$. ~~Denote by $\Sigma^r_{(\alpha,n)}$ the set of iso classes of vector bundles of rank $r$ and determinant $\alpha$ and of degree $n$. Let $\alpha \in J = \mathrm{Pic}(C)$. Assume there exists a line bundle of deg 1 (always tr~~

If $\alpha \in J$, let ~~[illegible]~~ $\Sigma_r(\alpha, n)$ denote the set of iso classes of bundles of rank $r$ such that $\Lambda^r E \simeq \alpha(n)$. One fixes a line bundle $\mathcal{O}(1)$ of degree one. Then the preceding sum can be broken up.

$$\sum_{\alpha \in J} \sum_{n} z^n \sum_{E \in \Sigma_r(\alpha,n)} \frac{\mathrm{card}\{\mathrm{Inj}(\Lambda_0, E)\}}{\mathrm{aut}(E)}$$

---

Next we must ~~[illegible]~~ know something about $\mathrm{card}\{\mathrm{Inj}(\Lambda_0, E)\}$. ~~[illegible]~~

Now take $r = 2$, $\Lambda_0 = \mathcal{O}_C^2$.

$$\mathrm{Inj}(\mathcal{O}^2, E) \subset H^0(E)^2$$

so

$$\mathrm{card}\{\mathrm{Inj}(\mathcal{O}^2, E)\} \leq \left[q^{2(\deg E + 2(1-g))}\right]^2$$

and there is a possibility that the difference might be of smaller growth as $\deg E \to \infty$.

Example: On an elliptic curve ~~[illegible]~~ take $E = \mathcal{O}(n) \oplus \mathcal{O}(n)$. Then I wish to compute a lower bound for $\mathrm{card}\{\mathrm{Inj}(\mathcal{O}^2, E)\}$ = no of pairs $s_1, s_2 \in H^0(E)$ which are generically independent. Now $s_1$ can be chosen in $\left(\operatorname{card} H^0(E)\right) - 1 = q^{h^0(E)} - 1 = q^{2n}-1$ ways. Once $s_1$ is chosen, it determines a sub-line-bundle $\overline{\mathcal{O}s_1} \subset E$ and $s_2$ can be any section of $E$ not a section of $\overline{\mathcal{O}s_1}$. But

$$\deg\left(\overline{\mathcal{O}s_1}\right) \leq n$$

$$h^0\left(\overline{\mathcal{O}s_1}\right) \leq \deg\left(\overline{\mathcal{O}s_1}\right) + \overset{0}{\overset{\shortparallel}{1-g}} \leq n$$

so the no. of choices for $s_2$ is $q^{h^0(E)} - q^{h^0(\overline{\mathcal{O}s_1})} \geq q^{2n} - q^{n}$. Thus we get the bounds

$$\left(q^{2n}-1\right)\left(q^{2n}-q^{n}\right) \leq \operatorname{card}\left\{\operatorname{Inj}(\mathcal{O}^2, E)\right\} \leq \left(q^{h^0(E)}\right)^2 = q^{4n}$$

In particular in this case we see that as $n \to \infty$

$$\frac{\operatorname{card}\left\{\operatorname{Inj}(\mathcal{O}^2, E)\right\}}{\left(q^{h^0 E}\right)^2} = \left(1-q^{-2n}\right)\left(1-q^{-3n}\right) \longrightarrow 1.$$

Example: Take $E$ to be a rang 2 vector bundle over $C$ arbitrary, and let $\lambda$ be the maximum degree of a sub-line-bundle of $E$. Given $(s_1, s_2) \in \operatorname{Inj}\left\{\mathcal{O}^2, E(n)\right\}$, $s_1$ can be chosen in $\left(q^{h^0(E(n))} - 1\right)$-ways. Then $\deg\left(\overline{\mathcal{O}s_1}\right) \leq \lambda + n$ so

$$h^0\left(\overline{\mathcal{O}s_1}\right) \leq \lambda + n + 1 - g$$

and so once $s_1$ is chosen, $s_2$ can be chosen in

$$q^{h^0(E(n))} - q^{h^0(\overline{\mathcal{O}s_1})} \geq q^{h^0(E(n))} - q^{\lambda+n+1-g}$$

ways. Thus

$$\operatorname{card}\left\{\operatorname{Inj}\left\{\mathcal{O}^2, E(n)\right\}\right\} \geq \left[q^{h^0(E(n))} - 1\right]\left[q^{h^0(E(n))} - q^{\lambda+n+1-g}\right]$$

But for $n$ large, $h^0(E(n)) = \deg E + 2n + 2(1-g)$.

So at this point it is clear that we ought to have

<u>Proposition:</u> $E_0, E$ two vector bundles of the same rank over $C$. Then as $n \longrightarrow \infty$

$$\frac{\text{card } \mathrm{Inj}(E_0, E(n))}{\text{card } \mathrm{Hom}(E_0, E(n))} \longrightarrow 1$$

In fact this should hold for $\mathrm{rank}(E_0) \leq \mathrm{rank}(E)$.

~~Proof: Let $s = \mathrm{rank}(E_0)$, $r = \mathrm{rank}(E)$,~~ [redacted] ~~choose~~

~~$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow L \longrightarrow 0$$~~

~~with $L$ a line bundle. For $n$ large we have~~

~~$$0 \longrightarrow \mathrm{Hom}(L, E(n)) \longrightarrow \mathrm{Hom}(E_0, E(n)) \longrightarrow \mathrm{Hom}(E_1, E(n)) \longrightarrow 0$$~~

~~Given $\alpha \in \mathrm{Inj}(E_1, E(n))$, $\overline{\alpha E_1}$ is a sub-bundle of $E(n)$ of rank $s-1$, hence~~

~~$$\deg(\overline{\alpha E_1}) \leq \mu_{max}(E)(s-1)$$~~

In any case assuming [redacted] this, return to

$$Z_C(z) \cdots Z_C(q^{r-1}z) = \sum_{\alpha \in J} \sum_{n} z^n \sum_{E \in \Sigma_r(\alpha, n)}' \frac{\text{card } \mathrm{Inj}(\mathcal{O}_C^r, E)}{\mathrm{aut}(E)}$$

and use the isom. $E \longrightarrow E(1)$ between $\Sigma_r(\alpha, n)$ and $\Sigma_r(\alpha, n+r)$ to write this as

$$\sum_{\alpha} \sum_{i=0}^{r-1} \sum_{E \in \Sigma_r(\alpha, i)} \sum_{n}' z^{i+rn} \frac{\text{card } \mathrm{Inj}(\mathcal{O}_C^r, E(n))}{\mathrm{aut}(E)}$$

Lemma: Let $\frac{P(z)}{Q(z)} = \sum_{n=0}^{\infty} a_n z^n$ be a rational function of $z$ with complex coefficients such that $\lim_{\rightarrow} a_n = 0$. Then all poles of $\frac{P(z)}{Q(z)}$ are outside $|z| = 1$.

Proof: Let $\lambda$ be a pole of $P(z)/Q(z)$; whence $Q(z) = (z-\lambda)\,Q_0(z)$. Multiplying $\sum_{n=0}^{\infty} a_n z^n$ by $(z-\mu)$ replaces $a_n$ by $a_{n-1} - \mu a_n$ which still goes to zero. Thus one sees $Q_0(z)P(z)/Q(z) = P(z)/(z-\lambda)$ has the coefficients of its Taylor series going to zero. But

$$\frac{1}{z-\lambda} = -\frac{1}{\lambda\left(1-\frac{z}{\lambda}\right)} = -\frac{1}{\lambda}\sum\left(\frac{z}{\lambda}\right)^n$$

and $P(z)/(z-\lambda) = \text{poly} + P(\lambda)/z-\lambda$ has essentially the same coefficients in large degrees. Thus $|\lambda| > 1$.

Lemma: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a rational function such that $\frac{a_n}{\lambda^n} \longrightarrow \alpha$ as $n \longrightarrow \infty$. Then $f(z)$ has no poles with $|z| \leq \lambda^{-1}$ except for a simple pole at $z = \lambda^{-1}$ with residue $\alpha$.

Proof:

$$f\left(\frac{z}{\lambda}\right) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda^n} z^n$$

$$f\left(\frac{z}{\lambda}\right) - \frac{\alpha}{1-z} = \sum_{n=0}^{\infty} \left(\frac{a_n}{\lambda^n} - \alpha\right) z^n$$

has no poles in $|z| \leq 1$, hence

$$f(z) - \frac{\alpha}{1-\lambda z} \quad \text{has no poles for} \quad |z| \leq \lambda^{-1}.$$

Now let me return to the formula

$$Z_C(z)\cdots Z_C(q^{r-1}z) = \sum_{\alpha\in J}\sum_{i=0}^{r-1}\sum_{E\in\Sigma_r(\alpha,i)} z^{i+rn}\,\frac{\operatorname{card}\left(\operatorname{Inj}(\mathcal{O}_C^r, E(n))\right)}{\operatorname{aut}(E)}$$

and let me use my asymptotic formula

$$\frac{\operatorname{card}\left(\operatorname{Inj}(\mathcal{O}_C^r, E(n))\right)}{q^{h^0(E(n))r}} \longrightarrow 1 \quad \text{as } n\longrightarrow\infty.$$

$$h^0(E(n)) = \deg(E) + nr + r(1-g)$$

$$f(z) = \sum_{n=0}^{\infty}\sum_{i=0}^{r-1}(q^r z)^{i+rn}\sum_{\alpha\in J}\sum_{E\in\Sigma_r(\alpha,i)}\frac{q^{r^2(1-g)}}{\operatorname{aut}(E)}\,\frac{\operatorname{card}\{\operatorname{Inj}\,\mathcal{O}_C^r, E(n)\}}{q^{(i+rn+r(1-g))r}}$$

So it seems we get that

$$f(z) \sim \frac{1}{1-(q^r z)^r}\sum_{i=0}^{r-1} z^i \quad q^{r^2(1-g)}\sum_{\alpha,\,E\in\Sigma_r(\alpha,i)}\frac{1}{\operatorname{aut}(E)}$$

But

$$f(z) = Z_C(z)\cdots Z_C(q^{r-1}z)$$

$$= \frac{P(z)P(qz)\cdots P(q^{r-1}z)}{(1-z)(1-qz)^2\cdots(1-q^{r-1}z)^2(1-q^r z)}$$

Actually I should run the proof backward. Because $f(z)$ is rational with no poles for $|z|<q^{-r}$ except for a simple pole at $z=q^{-r}$ with residue $-\alpha$ I know that if $f(z)=\sum a_m z^m$, then

$$\frac{a_m}{q^{rm}} \longrightarrow \alpha \quad \text{as } m\longrightarrow\infty.$$

But if $m = i + rn$, then

$$a_{i+rn} = \sum_{\alpha \in J} \sum_{E \in \Sigma_r(\alpha, i)} \frac{\mathrm{card}(\mathrm{Inj}(\mathcal{O}_c^r, E(n)))}{\mathrm{aut}(E)}$$

$$\begin{array}{ccc} \dfrac{a_{i+rn}}{(q^{i+rn})^r} & = \displaystyle\sum_{\alpha \in J} \sum_{E \in \Sigma_r(\alpha, i)} & \dfrac{\mathrm{card}(\mathrm{Inj}(\mathcal{O}_c^r, E(n)))}{\mathrm{aut}(E) \quad q^{(i+rn)r}} \\ \downarrow & & \downarrow \\ \alpha & & \displaystyle\sum_{\alpha \in J} \sum_{E \in \Sigma_r(\alpha, i)} \frac{1}{\mathrm{aut}(E)} q^{r^2 (1-g)} \end{array}$$

So what is $\alpha$? Recall

$$Z(z) = \sum_L z^{\deg(L)} \frac{q^{h^0(L)} - 1}{q - 1}$$

$$= \text{poly} + \sum z^n \frac{q^{n+1-g} - 1}{q - 1} \cdot h$$

$$= \text{poly} + \frac{h q^{1-g}}{q - 1} \frac{1}{1 - qz} - \frac{h}{q-1} \frac{1}{1-z}$$

Thus it seems I get the residue

$$\boxed{Z(q^{-r}) \cdots Z(q^{-2}) \frac{h q^{1-g}}{q - 1} = \sum_{\alpha \in J} \sum_{E \in \Sigma_r(\alpha, i)} \frac{1}{\mathrm{aut}(E)} q^{r^2(1-g)}}$$

or

$$\boxed{\sum_{\alpha \in J} \sum_{E \in \Sigma_r(\alpha, i)} \frac{1}{\mathrm{aut}(E)} = Z(q^{-2}) \cdots Z(q^{-r}) \frac{h q^{(r^2-1)(g-1)}}{q - 1}}$$

With more work I might be able to get that all the sums over J are the same hence

$$\boxed{\sum_{E \in \Sigma_r(\alpha, i)} \frac{1}{\mathrm{aut}(E)} = \frac{1}{q-1} q^{(r^2-1)(g-1)} Z(q^{-2}) \cdots Z(q^{-r})}$$

~~Check this with the computations made for rank 2 bundles~~ [illegible]

Functional equation for $\zeta_2$

$$\boxed{Z_C(z) = (qz^2)^{g-1} Z_C\left(\frac{1}{qz}\right)}$$

(to remember recall
$$Z_C(z) = \frac{1 + \cdots + q^g z^{2g}}{(1-z)(1-qz)}$$
)

Plugging this in one gets the Siegel formula

$$\boxed{\sum_{E \in \Sigma_r(\alpha, i)} \frac{1}{\mathrm{aut}(E)} = \frac{1}{q-1} Z_C(q) \cdots Z_C(q^{r-1})}$$

Agrees with what I calculated for $r=2$, and curves of genus $0, 1$.