February 5, 1974. Localization

Let K = q.f. of a d.v.r. A with res. fld. k. Let V be a vector space over K of dim n. Let $X(V)$ be the building of V, that is, the simplicial complex whose simplices are chains of A-lattices $L_0 < \cdots < L_q$ such that $\pi(L_q/L_0) = 0$, π = uniformizant.

Try filtering $X(V)$ by putting $F_p X(V)$ = subcomplex consisting of $L_0 < \cdots < L_q \ni \dim(L_q/L_0) \leq p$. What is F_p/F_{p-1}, i.e. what are simplices in F_p but not in F_{p-1}? These are chains $L_0 < \cdots < L_q$ such that $\dim(L_q/L_0) = p$. ~~inclusion relations among such chains~~ One observe that each simplex $\sigma : L_0 < \cdots < L_q$ in F_p not in F_{p-1} has a least face, namely $L_0 < L_q$, with the same property. Thus I know that F_p is obtained from F_{p-1} by attaching the simplex (L, L') for each layer of length p; one thus has

$$F_p/F_{p-1} = \bigvee_{\dim(L'/L) = p} S^2 T(L'/L) \qquad S = \text{Suspension}$$

where T = Tits building.

In particular the homology of F_p/F_{p-1} is concentrated in dimension p and is a sum of the Steinberg modules of vector spaces of dim. p over the residue field k. Thus as the building is contractible I find an exact sequence

$$0 \to \bigoplus_{\dim(L'/L)=n} I(L'/L) \to \cdots \to \bigoplus_{\dim(L'/L)=p} I(L'/L) \to \cdots \to \bigoplus_{L} \mathbb{Z} \to \mathbb{Z} \to 0$$

Now use the fact that $GL_n(K)$ acts transitively in each dimension. $V = Ke_1 + \cdots + Ke_n$. Take as representative for the layers L'/L of dim p, the layer $A\pi e_1 + \cdots + A\pi e_p + Ae_{p+1} + \cdots + Ae_n \subset Ae_1 + \cdots + Ae_n$. The stabilizer is then the subgroup

$$\Gamma_{p,n} = \begin{array}{c} p \\ n-p \end{array} \left(\begin{array}{c|c} \alpha & \pi\beta \\ \hline \gamma & \delta \end{array} \right) \subset GL_n(A)$$

(Note that because A is local a matrix is invertible iff it is mod πA. Thus α, δ are invertible, so

$$\Gamma_{p,n} = \begin{bmatrix} GL_p(A) & \pi M_{p,n-p}(A) \\ M_{n-p,p}(A) & GL_{n-p}(A) \end{bmatrix} .$$

This observation should be worth something.)

So we get a spectral sequence

$$\boxed{E^1_{st} = H_t\left(\Gamma_{s,n} ; I(k^s)\right) \Longrightarrow H_*(GL_n(K)).}$$

(Actually to correspond better with what you've done for a field you might use instead the pair of lattices $Ae_1 + \cdots + Ae_n \subset A\pi^{-1}e_1 + \cdots + A\pi^{-1}e_p + Ae_{p+1} + \cdots + Ae_n$ which changes the group $\Gamma_{p,n}$ to

$$\begin{pmatrix} \alpha & \beta \\ \pi\gamma & \delta \end{pmatrix} \subset GL_n(A) .)$$

Question: Suppose I take

$$\varinjlim_{n} H_*\begin{pmatrix} 1+\pi\cdot M_{p,p}(A) & \pi\cdot * \\ * & GL_{n-p}(A)\end{pmatrix}$$

Does $GL_p(k)$ act trivially on this limit?

S

February 7, 1974: Nagao thm.

This says

$$GL_2(k[t]) = GL_2(k) *_{\begin{pmatrix} k^* & k \\ & k^* \end{pmatrix}} \begin{pmatrix} k^* & k[t] \\ & k^* \end{pmatrix}$$

Corresponding to this amalgamated product is a tree on which $GL_2(k[t])$ acts with fundamental domain a 1-simplex, whose vertices have the stabilizers $GL_2(k)$ and $\begin{pmatrix} k^* & k[t] \\ & k^* \end{pmatrix}$ resp. I want to construct that tree.

Put $\Gamma = GL_2(k[t])$; it is the group of autos. of $k[t]^2$. Γ acts transitively on $\mathbb{P}_1(k[t]) = \mathbb{P}_1(k(t))$ and $\begin{pmatrix} k^* & k[t] \\ & k^* \end{pmatrix}$ is the stabilizer of the "line" $k[t](0,1)$. no of $(1,0)$

Because $k[t]^* = k^*$ any $k[t]$-module isom. to $k[t]$ is canonical of the form $k[t] \otimes_k V$ where $\dim_k V = 1$. Thus can identify $\mathbb{P}_1(k[t])$ with 1-diml. subspaces V of $k[t]^2$ which are "unimodular" in the sense that $k[t] \otimes V \longrightarrow k[t]^2$ is injective onto a direct summand. These unimodular subspaces of dim. 1 in $k[t]$ are one type of vertices in the tree. Since Γ acts transitively on unimodular subspaces of dimension 2 and the stabilizer of $ke_1 + ke_2$ is $GL_2(k)$, the other vertices are the unimodular subspaces of dimension 2. Conclude the following must be true:

<u>Proposition</u>: <u>The ordered set of (non-zero) unimodular k-subspaces in $k[t]^2$ is contractible.</u>

I want to give a direct proof, and will

contract this to a point using a suitably distance function, that is the ~~[illegible]~~ simplicial complex will be a union of balls B_n such that B_n deforms down to B_{n-1}.

So our basepoint will be the line in $\mathbb{P}_1$ generated by $(0,1)$. Any other line will be generated by (f,g) where $f, g \in k[t]$ are relatively prime and $f \neq 0$, and (f,g) will be unique up to multiplication by k^*. We can also identify the line with the rational function g/f. Indeed once we remove a line from $\mathbb{P}_1$ the complement is $\simeq k(t)$. Now our measure of how far the line (f,g) is from $(0,1)$ will be the degree of the denominator f.

Now suppose given a line (f,g), with $\deg(f) = n$ I will prove that there is a unique pair of polynomials (f', g') such that

$$\det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - f'g = 1$$

such that $\deg(f') < \deg(f)$. ~~and that in fact $\deg(f') = n-1$.~~
The existence of some such couple is clear from the fact that f, g are relatively prime. Then by adding a multiple of (f,g) to (f', g') one can always arrange that $\deg(f') < \deg(f)$. In this case (f', g') is unique, for if $f\overline{g'} - \overline{f'}g = 0$, then f rel. prime to $g \Rightarrow f \mid \overline{f'}$ and so $\overline{f'} = 0$ if $\deg(\overline{f'}) < \deg(f)$.

So now let me define B_n to be the ordered subset of the ordered set X of non-zero unimodular subspaces consisting of those V such that for $0 \neq (f,g) \in V$, degree$(f) \leq n$. I claim then that B_{n+1} def. retracts to B_n. Suppose V is in B_{n+1} but not B_n. If $\dim(V_0) = 2$, ~~$V_0 = k(f,g)$ with deg$(f) = n+1$. According~~ then there is a unique line $L \subset V_0$ such that L is in B_n. In effect if V_0 has (f_1, g_1), (f_2, g_2) for a basis, then as $\max(\deg(f_1), \deg(f_2)) = n+1$, there is a unique-up-to-scalars non-zero linear combination of f_1, f_2 which is of degree $\leq n$. Thus we can retract V_0 into B_n by sending it to L.

If $\dim(V_0) = 1$, say $V_0 = k(f,g)$, then if V_1 is a 2-dim'l unimod. complex containing V_0, say V_1 has base (f,g) (f',g') we have $\det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} \in k^*$, and conversely. Thus the argument given above at the bottom of page 2 shows that there is a unique $V_1 > V_0$ such that V_1 is in B_{n+1}, and then we can push V_0 to V_1, which then gets pushed into the unique $L < V_1$ with $L \in B_n$.

Thus the proposition on page 1 is proved independent of Nagao's thm. In fact it is equivalent to Nagao's thm.

Conjecture: The ordered set of (non-zero) unimodular k-subspaces in $k[t]^n$ is contractible.

February 10, 1974. Nagao thm.

Let M be a free module of rank n over $k[t]$, let $X(M)$ be the poset of non-zero k-subspaces V of M which are <u>unimodular</u> in the sense that the induced map $k[t]\otimes V \to M$ is injective with image a direct summand of M.

<u>Proposition</u>: $X(M)$ is contractible ($n \geq 1$).

Proof. By induction on n. If $n=1$, then $M \simeq k[t]$ and because $k[t]^* = k^*$ the set of generators for M is a torsor under k^*. In this case $X(M)$ is reduced to a point, so it is contractible.

Suppose $n>1$ and choose an epi. $\varphi: M \twoheadrightarrow k[t]$. Put $X_s = \{V\in X(M) \mid \varphi(V) \subset k + kt + \cdots + kt^s\}$ for $s \geq -1$. Note that $X_{-1} = X(\operatorname{Ker}\varphi)$ is contractible by induction hypothesis, and that $\bigcup X_s = X(M)$. It will therefore suffice to show X_s can be deformed into X_{s-1} for each $s \geq 0$.

One way of trying to do this is to associate to $V \in X_s$ the kernel of the map

$$V \xrightarrow{\varphi} k + \cdots + kt^s \twoheadrightarrow kt^s$$

which is in X_{s-1} provided it is non-zero. This shows that if we remove from X_s those L such that $L \xrightarrow{\sim} kt^s$, then the rest deformation-retracts onto X_{s-1}. It remains therefore to show that for each such L:

$$\operatorname{Link}(L \text{ in } X_s) = \{V \in X_s \mid V > L\}$$

is contractible.

~~Fix L such that $L \xrightarrow{\sim} kt^s$, $L \in X_s$, and put $\overline{M} = M/k[t]L$. Choosing a generator for L one has $k[t] \xrightarrow{\sim} k[t]L$ and the composition $k[t]$ If we choose a generator for L, then the composition $k[t] \xrightarrow{\sim} k[t]L \subset M \xrightarrow{\varphi} k[t]$ is multiplication by a polynomial of degree s. Given $x \in \overline{M}$, one can lift it to $\tilde{x} \in M$~~

Fix $L \in X_s$ with $L \xrightarrow{\sim} kt^s$ and put $\overline{M} = M/k[t]L$. The composition $k[t]L \subset M \xrightarrow{\varphi} k[t]$ is isom. to multiplication ~~by a polynomial of degree~~ in $k[t]$ by a polynomial of degree s (namely $\varphi(l)$ where l generates L). Given $x \in \overline{M}$ it can be lifted to $\tilde{x} \in M$, and any other lifting is of the form $\tilde{x} + f(t)l$ for a unique $f(t) \in k[t]$. Applying φ, we get $\varphi(\tilde{x}) + f \cdot \varphi(l)$ so by the Euclidean algorithm in $k[t]$, there is a unique lifting $\tilde{x}$ of x such that $\varphi(\tilde{x})$ has degree $< s$. By virtue of uniqueness it follows that any k subspace W of $\overline{M}$ has a unique lifting $\tilde{W}$ to M such that $\varphi(\tilde{W}) \subset k + \dots + kt^{s-1}$. It follows that $L + \tilde{W}$ is the unique k-subspaces of M lifting W containing L such that $\varphi(L + \tilde{W}) \subset k + \dots + kt^s$.

~~Because L is unimodular in M, it~~ From

$$\begin{tikzcd}
0 \arrow[r] & k[t] \otimes L \arrow[r] \arrow[d, equal] & k[t] \otimes V \arrow[r] \arrow[d] & k[t] \otimes V/L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k[t] \otimes L \arrow[r] & M \arrow[r] & \overline{M} \arrow[r] & 0
\end{tikzcd}$$

one sees that a k-subspace V of M containing L is unimodular in M iff V/L ~~is unimodular~~ injects in

$\overline{M}$ and is a unimodular subspace of $\overline{M}$. Thus we have a map

$$\mathrm{Link}(L \text{ in } X_\delta) = \{V \in X_\delta \mid V > L\} \longrightarrow X(\overline{M})$$
$$V \longmapsto V/L = \mathrm{Im}(V \to \overline{M})$$

which is injective as there is a unique $V \subset M$ with $\varphi(V) \subset k + \cdots + kt^\delta$ and a given image in $\overline{M}$. On the other hand the map is onto, for if $W \in X(\overline{M})$, then $L + \tilde{W} \in X_\delta$, $L + \tilde{W} > L$, and $L + \tilde{W}/L = \mathrm{Im}(\tilde{W} \to \overline{M}) = W$. By induction $X(\overline{M})$ is contractible, hence the proof is complete.

<u>Lemma</u>: Given $L \in X_\delta$ such that $L \xrightarrow{\sim} kt^\delta$, then one has an isom.

$$\mathrm{Link}(L \text{ in } X_\delta) = \{V \in X_\delta \mid V > L\} \xrightarrow{\ \sim\ } X(\overline{M})$$
$$V \longmapsto \mathrm{Im}\{V \to \overline{M}\}$$

where $\overline{M} = M/k[t]L$.

February 11, 1974. Nagao thm. (applications).

Recall that if M is a free $k[t]$-module of rank $n \geq 1$, and if $X(M)$ is the poset of non-zero unimodular subspaces of M, then we have proved $X(M)$ is contractible. On the other hand, Since given $V \in X(M)$, $\{W \in X(M) \mid W < V\} = T(V)$ is a bouquet of $\dim(V)-2$ spheres, we get à la Lusztig an exact sequence:

$$\cdots\cdots \longrightarrow \bigoplus_{\substack{W \in X(M) \\ \dim(W)=2}} I(W) \longrightarrow \bigoplus_{\substack{W \in X(M) \\ \dim(W)=1}} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Since $GL_n(k[t])$ acts transitively the $V \in X(M)$ of dimension s, and the stabilizer of $ke_1 + \cdots + ke_s \subset k[t]^n$ is

$$\begin{pmatrix} GL_s(k) & * \\ 0 & GL_{n-s}(k[t]) \end{pmatrix} \qquad * = k[t]\text{-matrices}$$

one has then for any $GL_n(k[t])$ module A

$$H_*\Big(GL_n(k[t]), \bigoplus_{\substack{W \in X(k[t]^n) \\ \dim(W)=s}} I(W) \otimes A\Big) = H_*\left(\begin{pmatrix} GL_s k & * \\ & GL_{n-s}(k[t]) \end{pmatrix}, I_s \otimes A\right)$$

and so we have a spectral sequence

$$\boxed{E^1_{st} = H_t\left(\begin{pmatrix} GL_s(k) & * \\ & GL_{n-s}(k[t]) \end{pmatrix}, I_s \otimes A\right) \Longrightarrow 0}$$

Now recall that because $k[t]$ is a Dedekind domain the ordered set of non-zero direct summands of $k[t]^n$ gives us an exact sequence

$$\cdots \longrightarrow \bigoplus_{E \in G_2(M)} I(E) \longrightarrow \bigoplus_{E \in G_1(M)} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $I(E) =$ Steinberg module of $k(t) \otimes_{k[t]} E$. We have a functor from $X(M)$ to this ordered set sending V to $k[t] \otimes V$. This induces a map from the exact sequence on page 1 to this one, hence a map of spectral sequences

$$H_t\left(\begin{pmatrix} GL_s(k[t]) & * \\ 0 & GL_{n-s}(k[t]) \end{pmatrix}, I_s(k[t]) \otimes A\right) \Longrightarrow 0$$

$$\uparrow$$

$$H_t\left(\begin{pmatrix} GL_s(k) & * \\ 0 & GL_{n-s}(k[t]) \end{pmatrix}, I_s(k) \otimes A\right) \Longrightarrow 0$$

which will enable us to prove:

<u>Proposition</u>: For any $GL_n(k[t])$-module A we have an isomorphism

$$H_*(GL_n k, I_n(k) \otimes A) \xrightarrow{\sim} H_*(GL_n(k[t]), I_n(k[t]) \otimes A).$$

But it seems that the good result is even stranger:

Theorem:
$$\mathbb{Z}(GL_n(k[t])) \otimes_{\mathbb{Z}[GL_n k]} I_n(k) \xrightarrow{\sim} I_n(k[t]).$$

Proof. By induction on n. What this says simply is that $I(M)$ is the direct sum of $I(V)$ where V runs over the maximal unimodular subspaces of M. The proof is by induction on n using the map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{V \in X_n(M)} I(V) \arrow[r] \arrow[d] & \bigoplus_{V \in X_{n-1}(M)} I(V) \arrow[r] \arrow[d] & \cdots\cdots \\ 0 \arrow[r] & I(M) \arrow[r] & \bigoplus_{E \in G_{n-1}(M)} I(E) \arrow[r] & \cdots\cdots \end{tikzcd}$$

The induction hypothesis implies that we have isos. in dimensions $< n$. In effect to get V in $X_p(M)$ is the same as given $E = k[t] \otimes V \subset G_p(M)$ & a maximal unimodular subspace of E.

Thm: $H_*(GL_n(k)) \xrightarrow{\sim} H_*(GL_n(k[t]))$ mod p torsion p = char. exp. of k. This is an isom. if char$(k) > 0$ & k contains inf. many roots of unity, or if char$(k) = 0$.

Proof: One proceeds by induction on n. There is a map of the Lusztig sequence for k^n to the one for $k[t]^n$ which induces a map of spectral sequences

$$H_t\left(\begin{pmatrix} GL_s k & \cdot \\ & GL_{n-s} k \end{pmatrix}, I_s(k)\right) \longrightarrow H_t\left(\begin{pmatrix} GL_s k & * \\ & GL_{n-s}(k[t]) \end{pmatrix}, I_s(k[t])\right)$$

Now the unipotent parts can be cancelled if either we work mod p torsion, or if char$(k) > 0$ & k contains ∞ roots of 1. By induction $GL_{n-s}(k) \longrightarrow GL_{n-s}(k[t])$ induces h-isos. for $s > 0$. Since $H_*(GL_s k, I_s(k)) \xrightarrow{\sim} H_*(GL_s k[t], I_s k[t])$ it follows $E^1_{s*} \xrightarrow{\sim} {}'E^1_{s*}$ for $s > 0 \Rightarrow$ also for $s = 0$, QED

February 11, 1974. $GL_2(\mathbb{Z})$.

Think of $\mathbb{Z}$ as being analogous to $k[t]$ and try to generalize the above ideas to $GL_2(\mathbb{Z})$.

Let L be a line in $\mathbb{Z}^2$, say L is generated by (x,y). If L' is a line complementary to L, then L' has a unique generator (x',y') such that

$$xy' - x'y = 1$$

We weight the different lines in $\mathbb{P}_1\mathbb{Q}$ according to $|x|$. Then by the Euclidean algorithm, given (x,y) with $x>0$, there are two solutions to the above equation such that $|x'|<|x|$, provided $|x|>1$. In effect, there is the solution with $0<x'<x$ (note $x'=0 \Rightarrow xy'=1 \Rightarrow x=1$), and the solution $x'-x$ which is $\ni$ $|x'-x| = x-x' < x$.

~~So now let X be the poset consisting of lines in and~~

Observe that if L, L' are complementary lines in $\mathbb{Z}^2$ then there are exactly 2 lines which are complementary to both L and L'. In effect changing basis so that $L = \mathbb{Z}(1,0)$, $L' = \mathbb{Z}(1,0)$ for $L'' = \mathbb{Z}(x,y)$ to both L and L' means $x = \pm 1$, $y = \pm 1$, so there are the two possibilities $\mathbb{Z}(1,1)$, $\mathbb{Z}(1,-1)$.

~~Let X be the poset consisting of subsets σ of $\mathbb{P}_1\mathbb{Z}$ of the following type. Either σ consists of a single line or it consists of three lines each of which is complementary to the other. I want to show X is contractible, so I filter X by putting in X_0 all σ whose~~

~~lines L are of the form $\mathbb{Z}(x,y)$ with $|x| \le s$.~~

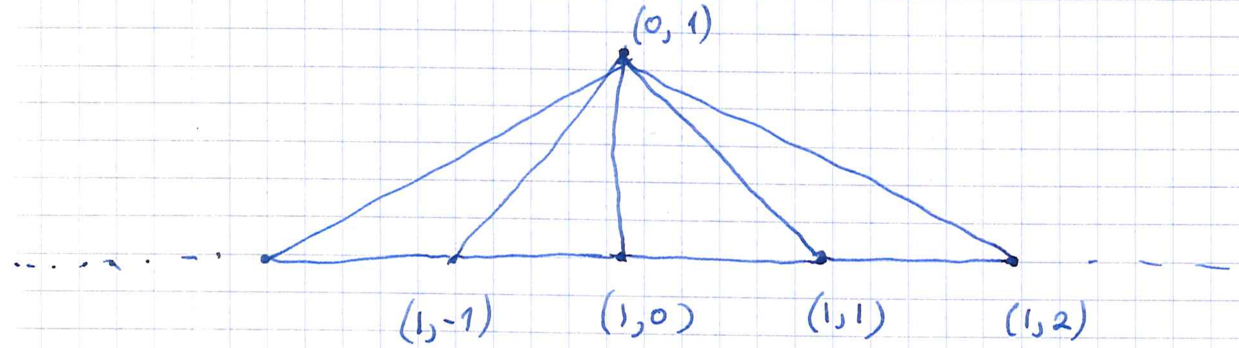
~~X_0 consists of the single line $\mathbb{Z}(0,1)$.~~

~~X_1 consists of the lines $\mathbb{Z}(0,1)$, $\mathbb{Z}(1,y)$ $y \in \mathbb{Z}$ and of the line triples $\{(0,1),(1,y),(1,y+1)\}$ for each $y \in \mathbb{Z}$.~~

Let X be the simplicial complex consisting of nonempty subsets σ of $\mathbb{P}_1\mathbb{Z}$ such that any two lines in σ are complementary. According to the preceding σ is of dimensin 2. I want to show X is contractible, so I filter X by letting X_s be the subcomplex consisting of lines $\mathbb{Z}(x,y)$ with $|x| \le s$.

X_0 consists of the single vertex $\mathbb{Z}(0,1)$.

X_1 consists of the lines $\mathbb{Z}(0,1)$, $\mathbb{Z}(1,y)$ $y \in \mathbb{Z}$ and is the 2-complex:



which is clearly contractible.

Finally if $s \ge 2$ we try to contract X_s to X_{s-1} ~~as follows. Given σ and if $\mathbb{Z}(x,y), \mathbb{Z}(x',y') \in \sigma$ with $|xy'-x'y| = 1$, then $|x|, |x'| = s$ is impossible; hence~~ by removing those vertices $L = \mathbb{Z}(x,y)$ with $|x| = s$. It is necessary to show the link of L in X_s is contained in X_{s-1} and that it is contractible.

First note that if $L = \mathbb{Z}(x,y)$ $L' = \mathbb{Z}(x',y')$ $xy' - x'y = 1$ and $|x| = |x'| = a$, then $a = 1$; hence if $|x| = s > 1$, the link of L in X_s is contained in X_{s-1}. But we have shown above that there are two solutions to $xy' - x'y = 1$ with $|x'| < |x|$ and that these two solutions are of the form (x', y'), $(x'-x, y'-y)$ hence determine complementary lines. Thus we see that the link of L is a 1-simplex and the proof is complete.

Implications: One gets an exact sequence

$$0 \longrightarrow \bigoplus_{d(\sigma)=2} \mathbb{Z}(\sigma) \longrightarrow \bigoplus_{d(\sigma)=1} \mathbb{Z}(\sigma) \longrightarrow \bigoplus_{L} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

from the complex of chains on X. This gives use a short exact sequence

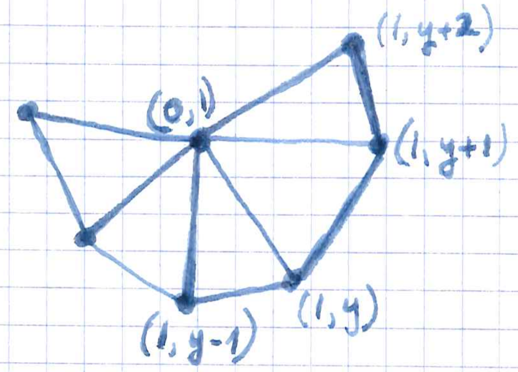
$$0 \longrightarrow \bigoplus_{d(\sigma)=2} \mathbb{Z}(\sigma) \longrightarrow \bigoplus_{d(\sigma)=1} \mathbb{Z}(\sigma) \longrightarrow I(\mathbb{Z}^2) \longrightarrow 0$$

~~Now the stabilizer of a pair of lines~~ Now $Gl_2(\mathbb{Z})$ acts transitively on the simplices of each dimension. The stabilizer of the pair $\mathbb{Z}(1,0), \mathbb{Z}(0,1)$ is the group $\Sigma_2 \tilde{\times} (\mathbb{Z}^*)^2$ isomorphic to the dihedral group of order 8. The stabilizer of a triple of lines is a group of order 12 which is the trivial central extension of Σ_3 by $\mathbb{Z}/2$. (To see this I consider the triple of lines $(\mathbb{Z}(1,0), \mathbb{Z}(0,1), \mathbb{Z}(1,1))$. The matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ produces a cyclic permutation of these lines

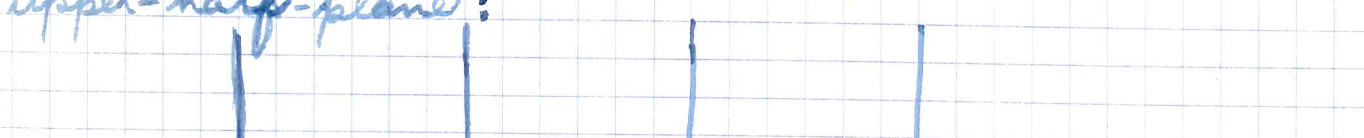
and ~~moreover this matrix has order 6. Etc.)~~
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ transposes the lines $(1,0)$, $(0,1)$ fixing the third. The Sylow 2 subgroup is gen. by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\begin{pmatrix}-1&\\&-1\end{pmatrix}$ + is elementary abelian.) So one gets an exact sequence

$$\longrightarrow H_q\left(\Sigma_3\times\{\pm 1\}, \mathbb{Z}^{sgn}\right) \longrightarrow H_q\left(\Sigma_2 \tilde{\times} \{\pm 1\}^2, \mathbb{Z}^{sgn}\right) \longrightarrow H_q\left(GL_2(\mathbb{Z}), I_2(\mathbb{Z})\right) \longrightarrow \cdots$$

Curious thing about X is that every edge belongs to exactly 2 2-simplices, hence X is a surface of some sort. The link of a vertex is a staircase:



There is probably some relation between this X and the upper-half-plane:



Another version of X: Because the link of any vertex is contractible (see page 2 for the Link of $\mathbb{Z}(0,1)$), we can remove these vertices from the barycentric subdivision of X. Then we get the ordered set consisting of the 1 and 2 simplices in X, which is a tree on which $GL_2(\mathbb{Z})$ acts with fundamental domain a simplex. This leads to an amalgamated product:

$$GL_2(\mathbb{Z}) = \left(\Sigma_2 \tilde{\times} (\mathbb{Z}^*)^2\right) *_{\text{[redacted]}\,(\Sigma_2 \times \text{cent}\,\mathbb{Z}^*)} \left(\Sigma_3 \times \text{cent}\,\mathbb{Z}^*\right)$$

which must be the one consistent with the standard one:

$$SL_2(\mathbb{Z}) = (\mathbb{Z}/4) *_{(\mathbb{Z}/2)} (\mathbb{Z}/6)$$

In fact you can now give a proof of this.

Next I want to see to what extent I can handle $GL_3(\mathbb{Z})$ by a similar method. So set $M = \mathbb{Z}^3$ and let $\varphi : M \twoheadrightarrow \mathbb{Z}$. Let $L \in \mathbb{P}_1 M$ and suppose $\varphi(L) = s\mathbb{Z}$ $s \geq 0$. What I want to understand first of all is the relation between lines L' independent of L such that $|\varphi(L')| \leq s$ and lines in M/L. Here I put $|\varphi(L')| = t$ if $\varphi(L) = t\mathbb{Z}$, $t \geq 0$.

Now as before one has

$$\begin{tikzcd} L = \mathbb{Z} \arrow[d] \arrow[dr, "s"] & \\ M \arrow[r, two heads, "\varphi"'] \arrow[d] & \mathbb{Z} \\ M/L & \end{tikzcd}$$

so one sees that given $x \in M/L$ we can lift it to $\tilde{x} \in M$, whence all other liftings are of the form $\tilde{x} + nl$ (l base for $\mathbb{Z} \ni \varphi(l) = s$). There is a unique value of n such that $0 \leq \varphi(\tilde{x}) + ns < s$. Thus one sees that any element x of M/L lifts uniquely to $\tilde{x} \in M$ such that $0 \leq \varphi(\tilde{x}) < s$. Now given a line $\bar{L} \in M/L$ it has generators $\pm x$ and so one has the two liftings $\tilde{x}$, $l - \tilde{x}$ with sizes $\varphi(\tilde{x})$, $s - \varphi(\tilde{x})$ if $0 < \varphi(\tilde{x}) < s$ and otherwise one lifting $\pm\tilde{x}$ if $\varphi(\tilde{x}) = 0$.

<u>Lemma</u>: Let $X(\mathbb{Z}^n)$ be the simplicial complex whose simplices are non-empty subsets of $\mathbb{P}_1(\mathbb{Z}^n)$ which are "in general position" in the sense that every subset of card $\leq n$ is independent. Then $X(\mathbb{Z}^n)$ is of dimension n and $GL_n(\mathbb{Z})$ acts transitively on the simplices of a given dimension.

<u>Proof</u>: Let $\sigma = (L_1, \ldots, L_q)$ be a maximal simplex. If $q < n$, then because $\mathbb{Z}^n / L_1 \oplus \cdots \oplus L_q \simeq \mathbb{Z}^{n-q}$, we can enlarge σ. Thus $q \geq n$ and I can find an auto. of $\mathbb{Z}^n$ which transforms L_i into $\mathbb{Z}e_i$ for $i = 1, \ldots, n$. Now let $L = \mathbb{Z}(\sum x_i e_i)$. In order that $\sum x_i e_i$ be independent of $\mathbb{Z}e_1, \ldots \widehat{\mathbb{Z}e_i} \ldots \mathbb{Z}e_n$, it is necessary and sufficient that $|x_i| = \pm 1$. Thus if σ is maximal by changing e_i to $-e_i$ if necessary we can assume that $L_1, \ldots, L_n, L_{n+1}$ have bases $e_1, \ldots, e_n, \sum e_i$ respectively. It is clear $GL_n(\mathbb{Z})$ acts transitively on these guys, so really the only thing left to show is that we can't add another line to the n-simplex

$$\sigma = (\mathbb{Z}e_1, \ldots, \mathbb{Z}e_n, \mathbb{Z}(\sum e_i))$$

But if $\mathbb{Z}(\sum x_i e_i)$ is a line independent of each face of σ, then we have $x_i = \pm 1$ as before. Not all the x_i can be of the same sign, so we can suppose by perm. coordinates that $x_1 = +1$, $x_2 = -1$. Then working mod. $e_3, \ldots, e_n$ we have that $(1, -1)$ has to be indep. of $(1, 1)$ which is a contradiction.

Question: Is $X(\mathbb{Z}^n)$ contractible for $n \geq 2$?

I have proved this for $n=2$, and am now in the process of going to $n=3$. In this case $X(\mathbb{Z}^3)$ is a 3-complex. Again I put $M = \mathbb{Z}^3$ and consider an epim. $\varphi : M \twoheadrightarrow \mathbb{Z}$ and let X_s be the subcomplex of X with the vertices $L \ni |\varphi(L)| \leq s$. X_0 is the subcomplex cons. of L such that $\varphi(L) = 0$; it is the complex $X(\mathrm{Ker}\,\varphi)$ which is contractible by induction hypothesis.

So it is necessary to examine X_1 which is the subcomplex consisting of line $L \ni \varphi(L) = \mathbb{Z}$ or 0. In this case we can have 4 lines in general position mapping onto $\mathbb{Z}$ via φ. e.g. if $\varphi(x,y,z) = x - y + z$ and we have the lines with generators $e_1, e_2, e_3, e_1+e_2+e_3$. ~~that gives a σ consisting of 4 lines all mapped onto $\mathbb{Z}$ by φ, these~~

Assume for the moment that X_1 is contractible and let's try to go on.

February 13, 1974 Sylvester thm.

Recall that if $\mathcal{H}$ is a family of subgroups of G ordered by inclusion, then the fibre of the map

$$\underset{\overrightarrow{H\in\mathcal{H}}}{\mathrm{holim}}\ BH \longrightarrow BG$$

can be identified with the poset of left cosets of members of $\mathcal{H}$. And more generally if $i \mapsto H_i$ is a functor from a cat I to subgroups of G, then the fibre of $\underrightarrow{\mathrm{holim}}\ BH_i \longrightarrow BG$ may be identified with the cofibred category over I associated to the functor $i \mapsto G/H_i$.

<u>Example:</u> If H_j $j\in J$ are subgroups of G put I = non-empty subsets α of J and $H_\alpha = \bigcap_{j\in\alpha} H_j$. Then the fibre of

$$\underrightarrow{\mathrm{holim}}\ BH_\alpha = \bigcup_{j\in J} BH_j \subset BG$$

is the following simplicial complex. A vertex is a coset gH_j and a simplex is a collection of such cosets with non-empty intersection. Thus the simplicial is the nerve of the covering of G given by the different left cosets of the subgroups $\{H_j, j\in J\}$.

Conversely suppose G acts on a contractible simplicial complex X having as fundamental domain a simplex σ_0, where I mean fundamental domain in the strict sense: every σ in X is conjugate to a unique face of σ_0. Then if H_j are the stabilizers of the different vertices j of σ_0, we know that

$$\underset{\alpha}{\underrightarrow{\mathrm{holim}}}\, H_\alpha = \bigcup BH_\alpha \longrightarrow BG$$

is a homotopy equivalence.

We now take $G = GL_n(k[t])$, and let X be the simplicial complex associated to the unimodular subspaces of $k[t]^n$ which are non-zero. Let σ_0 be the simplex

$$ke_1 < ke_1 + ke_2 < \cdots\cdots < ke_1 + \cdots + ke_n$$

so that every simplex of X is conjugate to exactly one face of σ_0. From the contractibility of X I conclude that

$$\underrightarrow{\mathrm{holim}}\, BH_\alpha = BGL_n(k[t])$$

where

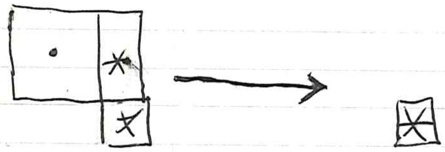
$$H_j = \text{stabilizer of } ke_1 + \cdots + ke_j \qquad 1 \leq j \leq n$$

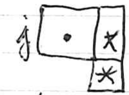
$$= \begin{pmatrix} GL_j k & * \\ 0 & GL_{n-j}(k[t]) \end{pmatrix}$$

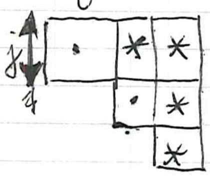
Put another way $GL_n(k[t])$ is the homotopy amalgamation of its subgroups

where $\cdot$ denotes entries in k and $*$ entries in $k[t]$.

But by applying this to the group

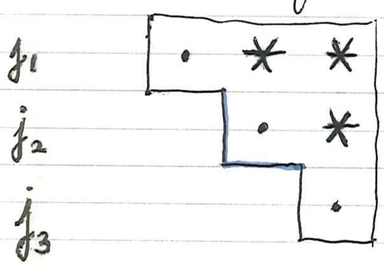


one sees that the group  is the homotopy amalgamation of the subgroups



and so continuing we get the

<u>Sylvester theorem:</u> $GL_n(k[t])$ is the homotopy inductive limit of its subgroups H [redacted] π



where π = a decomposition $n = j_1 + j_2 + \cdots\cdots$ with $j_1, j_2, \ldots > 0$.

February 17, 1974. Waldhausen

Example of his theory: $R = \mathbb{C}[t]$.

Let $\mathcal{A}$ be the category of diagrams in $\mathcal{P}(\mathbb{C})$ of the form

$$\left[M_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M_0 \right]$$

with the obvious notion of exact sequence. Using the exact filtration

$$0 \longrightarrow [0 \rightrightarrows M_0] \longrightarrow \left[M_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M_0 \right] \longrightarrow [M_1 \rightrightarrows 0] \longrightarrow 0$$

the "exact seq. thm" implies one has

$$K_*(\mathcal{A}) \xrightarrow{\sim} K_*(\mathbb{C}) \oplus K_*(\mathbb{C})$$

induced by the two forgetful functors

$$[M_1 \rightrightarrows M_0] \longmapsto M_0 \quad , \quad M_1$$

Now we will identify an object $[M_1 \rightrightarrows M_0]$ of $\mathcal{A}$ with the $\mathbb{C}[t]$-module map

$$\alpha + \beta t \;:\; M_1[t] \longrightarrow M_0[t] \qquad M[t] = M \otimes_{\mathbb{C}} \mathbb{C}[t]$$

We are now going to apply the resolution thm. to replace $\mathcal{A}$ by various subcategories with the same K-theory. For memory suppose we have an exact sequence

(*) $\quad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$

in $\mathcal{A}$ whence a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M_1'[t] & \longrightarrow & M_1[t] & \longrightarrow & M_1''[t] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'+\beta't} & & \downarrow{\scriptstyle \alpha+\beta t} & & \downarrow{\scriptstyle \alpha''+\beta''t} & & \\ 0 & \to & M_0'[t] & \longrightarrow & M_0[t] & \longrightarrow & M_0''[t] & \longrightarrow & 0 \end{array}$$

Consider the following conditions on an object $M = [M_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M_0]$ of $\mathcal{A}$:

(1) $\alpha+\beta t : M_1[t] \longrightarrow M_0[t]$ is injective

(2) Coker $(\alpha+\beta t)$ is in $\mathcal{P}(C[t])$

(3) The composite map $M_0 \hookrightarrow M_0[t] \twoheadrightarrow \mathrm{Coker}(\alpha+\beta t)$ is injective.

Denote by $\mathcal{A}_i$ the full subcategory of $\mathcal{A}$ consisting of those M satisfying conditions (1) thru (i), so that

$$\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A}_0 = \mathcal{A}$$

One sees by diagram chasing (serpent lemma) that each category $\mathcal{A}_i$ is closed under extensions in $\mathcal{A}$ and that given an exact sequence (*) in $\mathcal{A}$ one has

$$M \in \mathcal{A}_i \, , \, M'' \in \mathcal{A}_{i-1} \implies M' \in \mathcal{A}_i$$

(For example if $M'' \in \mathcal{A}_1$ then $\alpha''+\beta''t$ is injective, so cokernel $\alpha''+\beta''t \in \mathcal{P}_1(C[t])$. If $M \in \mathcal{A}_2$, then $\mathrm{Coker}(\alpha+\beta t) \in \mathcal{P}(C[t])$ maps onto $\mathrm{Cok}(\alpha''+\beta''t)$ with $\mathrm{Ker} = \mathrm{Cok}(\alpha'+\beta't)$ so the latter is in $\mathcal{P}(C[t])$ by Shanuel's lemma.)

Next we ~~show~~ notice that any $M_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M_0^*$ is a quotient of

$$[M_1 \underset{in_2}{\overset{in_1}{\rightrightarrows}} M_1 \oplus M_1] \oplus [0 \rightrightarrows M_0^*]$$

Indeed:

$$\begin{tikzcd} M_1 \arrow[r, shift left, "in_1"] \arrow[r, shift right, "in_2"'] \arrow[d, equal] & M_1 \oplus M_1 \oplus M_0^{\blacksquare} \arrow[d, "\alpha+\beta / id_{M_0^{\blacksquare}}"] \\ M_1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & M_0^{\blacksquare} \end{tikzcd}$$

Moreover clearly $[0 \rightrightarrows M_0^{\blacksquare}] \in \mathcal{a}_3$ and one has for $[M_1 \overset{in_1}{\underset{in_2}{\rightrightarrows}} M_1 \oplus M_1]$

$$\begin{array}{ccccccc} 0 \to M_1[t] & \longrightarrow & M_1[t] \oplus M_1[t] & \xrightarrow{\theta} & M_1[t] \to 0 \\ f & \longmapsto & (f, tf) & & \\ & & (f, g) & \longmapsto & tf - g \end{array}$$

Clearly θ is injective on $M_1 \oplus M_1$, so this gadget belongs to $\mathcal{a}_3$. Thus I know by the resolution thm. that I have

$$K_*(\mathcal{a}_3) \xrightarrow{\sim} K_*(\mathcal{a}_2) \xrightarrow{\sim} K_*(\mathcal{a}_1) \xrightarrow{\sim} K_*(\mathcal{a})$$

<u>Variation</u>: suppose we have a free product situation: $R = A *_C B$ e.g. $\mathbb{Z}[G] = \mathbb{Z}[G_1 *_H G_2] = \mathbb{Z}[G_1] *_{\mathbb{Z}[H]} \mathbb{Z}[G_2]$

Here we take $\mathcal{a}$ to be the category of diagrams

$$\begin{tikzcd} & M_A \\ M_C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & M_B \end{tikzcd}$$

where $M_C \in \mathcal{P}(C)$, $M_B \in \mathcal{P}(B)$, $M_A \in \mathcal{P}(A)$ and α, β are C-module homomorphisms. ~~Applied to~~ By the filtration

$$0 \to \left[0 \begin{smallmatrix} \nearrow M_A \\ \searrow M_B \end{smallmatrix}\right] \longrightarrow \left[M_C \begin{smallmatrix} \xrightarrow{\alpha} M_A \\ \xrightarrow[\beta]{} M_B \end{smallmatrix}\right] \longrightarrow \left[M_C \begin{smallmatrix} \nearrow 0 \\ \searrow 0 \end{smallmatrix}\right] \to 0$$

one deduces

$$K_*(\mathcal{a}) = K_*(A) \oplus K_*(B) \oplus K_*(C)$$

To the diagram $\begin{tikzcd}[row sep=tiny] & M_A \\ M_C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & M_B \end{tikzcd}$ one associates the map of R-modules

$$R\otimes_C M_C \xrightarrow{(\bar\alpha,\bar\beta)} (R\otimes_A M_A)\oplus(R\otimes_B M_B)$$

and one considers the following conditions:

(1) $(\bar\alpha,\bar\beta)$ is injective $\left(\Rightarrow f(\alpha,\beta) = \mathrm{Cok}(\bar\alpha,\bar\beta) \in \mathcal{P}_1(R)\right)$

(2) $f(\alpha,\beta) \in \mathcal{P}(R)$.

(3) $0 \to M_C \xrightarrow{(\alpha,\beta)} M_A \oplus M_B \xrightarrow{\text{can.}} f(\alpha,\beta)$ is exact

One defines again the full subcategories $\mathcal{A}_i$ $i = 0,1,2,3$.
Check last condition:

$$\begin{tikzcd}
0 \arrow[r] & M_C' \arrow[r] \arrow[d] & M_C \arrow[r] \arrow[d] & M_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_A'\oplus M_B' \arrow[r] \arrow[d] & M_A\oplus M_B \arrow[r] \arrow[d] & M_A''\oplus M_B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f(\alpha',\beta') \arrow[r] & f(\alpha,\beta) \arrow[r] & f(\alpha'',\beta'') \arrow[r] & 0
\end{tikzcd}$$

← arrows inject as $M_A \hookrightarrow R\otimes_A M_A$ etc.

rows exact

So by diagram-chasing one sees that

vert. ex. at $M_A'\oplus M_B'$ & at $M_A''\oplus M_B''$ $\Longrightarrow$ same at $M_A\oplus M_B$

vert. ex. at $M_A'\oplus M_B'$ $\Longleftarrow$ exact at $M_A\oplus M_B$

Thus again each $\mathcal{A}_i$ is closed under extensions and given $0\to M'\to M\to M''\to 0$ exact in $\mathcal{A}$ with $M\in\mathcal{A}_i$, $M''\in\mathcal{A}_{i-1}$ $\Longrightarrow$ $M'\in\mathcal{A}_i$. Check also that any object $M = \{M_A, M_B, M_C, \alpha, \beta\}$ is a quotient of an object in $\mathcal{A}_3$: Clearly it is a quotient of $\left[M_C \begin{smallmatrix} \xrightarrow{1\otimes} A\otimes_C M_C \\ \xrightarrow{1\otimes} B\otimes_C M_C \end{smallmatrix} \right] \oplus \left[0 \begin{smallmatrix} \to M_A \\ \to M_B \end{smallmatrix} \right]$ and the latter is obviously in $\mathcal{a}_3$. As for the former

$$\begin{array}{ccccccc} & & R\otimes_C M & \longrightarrow & R\otimes_A A\otimes_C M_C \oplus R\otimes_B B\otimes_C M_C & \longrightarrow & f(1\otimes, 1\otimes) \\ & & \| & & \| & & \\ 0 & \longrightarrow & R\otimes_C M & \xrightarrow{\Delta} & R\otimes_C M_C \oplus R\otimes_C M_C & \xrightarrow{\text{diff.}} & R\otimes_C M_C \to 0 \end{array}$$

so it is clear that one has conditions (i) and (ii). As for (iii) we must check that

$$0 \longrightarrow M_C \longrightarrow A\otimes_C M_C \oplus B\otimes_C M_C \xrightarrow{-} R\otimes_C M_C$$

is exact, i.e. that $C\otimes_C M_C = (A\otimes_C M_C)\cap(B\otimes_C M_C)$ in $R\otimes_C M_C$. This will be true if R, A, B are flat over C and $A\cap B = C$ in R.

So therefore in this case it follows by resolution that

$$K_*(\mathcal{a}_3) = K_*(\mathcal{a}_2) = K_*(\mathcal{a}_1) = K_*(\mathcal{a})$$

Let $k \to A$ be a ring homom. which is injective and such that $\bar{A} = A/k$ is right-flat over k. Suppose $k \to B$ the same, and let $R = A *_k B$ be the free product of these extensions.

I will be interested in diagrams

$$X = [L_1 \xleftarrow{\alpha} L_0 \xrightarrow{\beta} L_2]$$

where L_0, L_1, L_2 are resp. k, A, B modules, and where α, β are k-modules homoms. To such a diagram I associate the induced R-module map

$$R \otimes_k L_0 \longrightarrow R \otimes_A L_1 \oplus R \otimes_B L_2 .$$

I call X a <u>presentation</u> of the R-module M if this map is injective with cokernel M.

Given the diagram X as above, define L_1^* by the cocartesian square

$$(*) \qquad \begin{array}{ccc} A \otimes_k L_0 & \longrightarrow & A \otimes_k L_2 \\ \downarrow & & \downarrow u \\ L_1 & \longrightarrow & L_1^* \end{array}$$

and let X^* be the diagram $[L_1^* \xleftarrow{\alpha^*} L_2 \xrightarrow{id} L_2]$ where α^* comes from the map u.

<u>Lemma</u>: <u>If</u> X <u>is a presentation of</u> M <u>with injective arrows</u> α, β, <u>then so is</u> X^*.

<u>Proof</u>. Since A is right flat over k and L_0 injects into L_2, L_1 injects into L_1^* and $A \otimes_k (L_2/L_0) \xrightarrow{\sim} L_1^*/L_1$. Thus extending

(*) to a map of short exact sequences running horizontally ~~[crossed out]~~ and tensoring with R over A, we get

$$
\begin{tikzcd}
 & 0 \arrow[d, dotted] & & \\
\mathrm{Tor}_1^A(R, A\otimes_k(L_2/L_0)) \arrow[r] \arrow[d, "\wr"] & R\otimes_k L_0 \arrow[r] \arrow[d] & R\otimes_k L_2 \arrow[r] \arrow[d] & R\otimes_k(L_2/L_0) \arrow[r] \arrow[d, "\wr"] & 0 \\
\mathrm{Tor}_1^A(R, L_1^*/L_1) \arrow[r] & R\otimes_A L_1 \arrow[r] & R\otimes_A L_1^* \arrow[r] & R\otimes_A(L_1^*/L_1) \arrow[r] & 0
\end{tikzcd}
$$

Now add in $R\otimes_B L_2$:

$$
\begin{tikzcd}
R\otimes_A L_1 \oplus R\otimes_B L_2 \arrow[r] \arrow[d, dotted] & R\otimes_A L_1^* \oplus R\otimes_B L_2 \\
M \arrow[d, dotted] & \\
0 &
\end{tikzcd}
$$

and diagram chase, and one sees that X^* is a presentation of M. To finish we must show that L_2 injects into L_1^*. But this is clear from

$$
\begin{tikzcd}
0 \arrow[r] & L_0 \arrow[r] \arrow[d, hook] & L_2 \arrow[r] \arrow[d, hook] & L_0/L_2 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & L_1^* \arrow[r] & A\otimes_k L_0/L_2 \arrow[r] & 0
\end{tikzcd}
$$

~~[crossed out]~~ where the injection on the left comes from the fact A/k is right k-flat.

~~Lemma 2: Assume In the situation of the preceding lemma, assume that $L_1/L_0, L_2/L_0$ are k-projective and that $\bar{A} = A/k$ is k-projective. Then $L_1^*/L_0^* = L_1^*/L_2$ and L_2^*/L_0^* are k-projective, and $L_1^*/L_1, L_2^*/L_2, L_0^*/L_0$ are resp. A, B, and k-projective.~~

~~Proof. One has $0 \to L_1/L_0 \to L_1^*/L_2 \to \bar{A}\otimes_k L_0/L_2 \to 0$, and $\bar{A}\otimes_k L_0/L_2$ is a direct summand of $\bar{A}\otimes_k k^{(I)} = \bar{A}^{(I)}$, hence is~~

~~k-proj. As $L_0^* = L_2^* = L_2$ one has $L_2^*/L_0^* = L_2^*/L_2 = 0$. $L_0^*/L_0 = L_2/L_0$ is k-proj. And $L_1^*/L_1 \cong A \otimes_k (L_2/L_0)$ is A-projective.~~

We shall call X^* the prolongation of X in the A-direction. One defines analogously the prolongation in the B-direction. By prolonging alternately one gets a sequence of presentations of M

$$
\begin{tikzcd}
X: & L_1 \arrow[d, hook] & L_0 \arrow[l] \arrow[r] \arrow[d, hook] & L_2 \arrow[d, "\wr"] \\
X^{(1)}: & L_1^{(1)} \arrow[d, "\wr"] & L_0^{(1)} \arrow[l] \arrow[r, "\sim"] \arrow[d, hook] & L_2^{(1)} \arrow[d, hook] \\
X^{(2)}: & L_1^{(2)} & L_0^{(2)} \arrow[l, "\sim"'] \arrow[r] & L_{(2)}^{(2)}
\end{tikzcd}
$$

$\cdots$

Taking the inductive limit we get a presentation $X^{(\infty)}$: $L_1^{(\infty)} \leftarrow L_0^{(\infty)} \rightarrow L_2^{(\infty)}$ of M in which the arrows are isomorphisms. In other words we have a presentation of the form $[L \xleftarrow{id} L \xrightarrow{id} L]$ where L has A and B module structures coinciding over k. As $R = A *_k B$, L has a unique R-module structure.

But one has always an exact sequence

$$R \otimes_k L \longrightarrow R \otimes_A L \oplus R \otimes_B L \longrightarrow L \longrightarrow 0$$

(apply the functor $\operatorname{Hom}_R(?, M)$.)

Thus one sees $L = M$. So we get

<u>Prop</u>: i) If $X = [L_1 \leftarrow L_0 \rightarrow L_2]$ is <u>presentation of M with injective arrows</u>, then the limit of the sequence

of prolongations of X is the presentation $[M \xleftarrow{id} M \xrightarrow{id} M]$

ii) The maps $L_1 \to M$, $L_2 \to M$ are injective and the square

$$\begin{array}{ccc} L_0 & \longrightarrow & L_2 \\ \downarrow & & \downarrow \\ L_1 & \longrightarrow & M \end{array}$$

is cartesian

Proof. The only point remaining to be proved is the last assertion. But we've seen above that

$$\begin{array}{ccc} L_0 & \longrightarrow & L_2 \\ \downarrow & & \downarrow \\ L_1 & \longrightarrow & L_1^* \end{array}$$

is cartesian, and that $L_1^* \hookrightarrow M$.

If we identify L_i with its image in M, then we have

$$L_1^* = AL_2 + L_1 \qquad\qquad L_0 = L_1 \cap L_2$$

$$L_1^*/L_1 + L_2 \cong \bar{A} \otimes_k (L_2/L_0) \qquad\qquad L_1 + L_2/L_1 = L_2/L_0$$

Thus our sequence of prolongations appears:

$$L_1 \longleftarrow L_0 \longrightarrow L_2$$

$$AL_2 + L_1 \longleftarrow L_2 \Longrightarrow L_2$$

$$AL_2 + L_1 \Longleftarrow AL_2 + L_1 \longrightarrow BL_1 + BAL_2$$

and we get a filtration of M:

$$L_0 \quad L_1 \subset AL_2 + L_1 \subset BL_1 + BAL_2 \subset ABL_1 + ABAL_2$$

$$A \otimes L_2/L_0 \qquad\qquad B \otimes (AL_2 + L_1/L_2)$$

Proposition: Let $[L_1 \hookleftarrow L_0 \hookrightarrow L_2]$ be a presentation of M with injective arrows, and identify L_i with its image in M. Then the sequence of A-modules

$$L_1 \subset L_1 + AL_2 \subset ABL_1 + AL_2 \subset \cdots$$

exhausts M and has the sequence of quotients

$$A \otimes_k (L_2/L_0), \quad A \otimes_k \bar{B} \otimes_k (L_1/L_0), \quad A \otimes_k \bar{B} \otimes_k \bar{A} \otimes_k (L_2/L_0), \ldots$$

Similarly the sequence of B-submodules

$$L_2 \subset BL_1 + L_2 \subset BL_1 + BAL_2 \subset \cdots$$

exhausts M and has quotients

$$B \otimes (L_1/L_0), \quad B \otimes \bar{A} \otimes (L_2/L_1), \ldots$$

Also one has an exhaustive filtration by k-modules.

$$L_0 \subset L_1 + L_2 \subset BL_1 + AL_2 \subset ABL_1 + BAL_2 \subset \cdots$$

with quotients

$$\begin{array}{ccc} L_1/L_0 & \bar{B} \otimes (L_1/L_0) & \bar{A} \otimes \bar{B} \otimes (L_1/L_0) \\ \oplus & \oplus & \oplus \\ L_2/L_0 & \bar{A} \otimes (L_2/L_0) & \bar{B} \otimes \bar{A} \otimes (L_2/L_0) \end{array}$$

This is more or less clear.

Corollary: For the filtration of R:

$$k < A + B \subset AB + BA \subset ABA + BAB \subset \cdots$$

one has the quotients $\bar{A} \oplus \bar{B}$, $\bar{A} \otimes \bar{B} \oplus \bar{B} \otimes \bar{A}$, ...

In effect apply preceding to the presentation $[A \leftarrow k \rightarrow B]$ of R.

<u>Corollary</u>: *Assume* L_1/L_0, L_2/L_0, $\bar{A}$, $\bar{B}$ *are projective over* k; *then* M/L_1, M/L_2, M/L_0 *are respectively projective* A, B, k*-modules.*

<u>Proof</u>: To show M/L_1 is projective it suffices in virtue of the preceding filtration to show that the quotients $A\otimes(L_2/L_0)$, $A\otimes\bar{B}\otimes(L_1/L_0)$, etc are A-proj., i.e. that L_2/L_0, $\bar{B}\otimes(L_1/L_0)$, $\bar{B}\otimes\bar{A}\otimes(L_2/L_0)$ are k-proj. Thus enough to show that V k-proj. $\Rightarrow$ $\bar{A}\otimes_k V$ and $\bar{B}\otimes_k V$ are k-projective, which is clear as $\bar{A}$ and $\bar{B}$ are and V is a direct summand of $k^{(I)}$.

Let us now consider presentations $X = [L_1 \xleftarrow{\alpha} L_0 \xrightarrow{\beta} L_2]$ in which α, β are not both injective. Assuming $\mathrm{Ker}(\alpha) \neq 0$, let us define K, $\bar{L}_2$ by

$$0 \longrightarrow K \longrightarrow B\otimes_k \mathrm{Ker}(\alpha) \xrightarrow{\cdot\beta} L_2 \longrightarrow \bar{L}_2 \longrightarrow 0$$

whence we get

$$\begin{tikzcd}
 & & & 0 & & \\
 & & & M \arrow[u] & & \\
0 \arrow[r] & R\otimes_B K \arrow[r] & R\otimes_k \mathrm{Ker}(\alpha) \arrow[r] & \begin{matrix} R\otimes_B L_2 \\ \oplus \\ R\otimes_A L_1 \end{matrix} \arrow[r] \arrow[u] & \begin{matrix} R\otimes_B \bar{L}_2 \\ \oplus \\ R\otimes_A L_1 \end{matrix} \arrow[r] & 0 \\
 & 0 \arrow[r] & R\otimes_k \mathrm{Ker}(\alpha) \arrow[r] \arrow[u, "\wr"] & R\otimes_A L_0 \arrow[r] \arrow[u] & R\otimes_k L_0/\mathrm{Ker}(\alpha) \arrow[r] \arrow[u] & 0 \\
 & & & 0 \arrow[u] & &
\end{tikzcd}$$

Diagram-chasing shows $R\otimes_B K = 0 \Longrightarrow K = 0$, and that $\bar{X} = [L_1 \longleftarrow \bar{L}_0 \longrightarrow \bar{L}_2]$ is also a presentation of M, where $\bar{L}_0 = L_0/\mathrm{Ker}(\alpha)$.

Lemma: If $X = [L_1 \xleftarrow{\alpha} L_0 \longrightarrow L_2]$ is a presentation of M, then one has $B \otimes_k \mathrm{Ker}(\alpha) \hookrightarrow L_2$, and $\bar{X} = [L_1 \leftarrow \bar{L}_0 \to \bar{L}_2]$ is also a presentation, where $\bar{L}_0 = L_0 / \mathrm{Ker}(\alpha)$, $\bar{L}_2 = L_2 / B \cdot \mathrm{Ker}(\alpha)$.

Now given X a pres. of M one can alternately construct a sequence of presentations of M by alternately killing $\mathrm{Ker}(\alpha)$, $\mathrm{Ker}(\beta)$:

$$\begin{tikzcd}
L_1 \arrow[d, equal] & L_0 \arrow[l] \arrow[r] \arrow[d, two heads] & L_2 \arrow[d, two heads] \\
L_1^{(1)} \arrow[d, two heads] & L_0^{(1)} \arrow[l, hook'] \arrow[r] \arrow[d, two heads] & L_2^{(1)} \arrow[d, equal] \\
L_1^{(2)} & L_0^{(2)} \arrow[l] \arrow[r, hook] & L_2^{(2)}
\end{tikzcd}$$

and in the limit one obtains a presentation with injective arrows. Thus we have

Prop: Given a presentation $X = [L_1 \longleftarrow L_0 \longrightarrow L_2]$ of M, then putting $X^+ = [\mathrm{Im}(L_1 \to M) \longleftarrow \mathrm{Im}(L_0 \to M) \longrightarrow \mathrm{Im}(L_2 \to M)]$ we obtain a presentation of M with injective arrows. The kernel of $X \to X^+$ is a presentation of zero, and any presentation of zero has an exhaustive filtration with quotients of the form

$$[A \otimes K \longleftarrow K \longrightarrow 0] \quad \text{or} \quad [0 \longleftarrow K \longrightarrow B \otimes K].$$

Remark: Presentations of zero form an abelian category. In effect if one has a map $X \longrightarrow Y$ of presentations of zero it is clear that the image computed component-wise is necessarily a presentation of zero:

$$\begin{tikzcd} \bullet \arrow[r, "\sim"] \arrow[d, tail] & \bullet \arrow[d, tail] \\ \bullet \arrow[r] \arrow[d, two heads] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r, "\sim"] & \bullet \end{tikzcd}$$

forces the middle ~~to~~ be an isomorphism.

Now given an R-module M, by an <u>admissible presentation</u> of M I will mean a presentation $X = [L_1 \leftarrow L_0 \rightarrow L_2]$ with injective arrows such that L_0, L_1, L_2 are in $P(k)$, $P(A)$, $P(B)$ respectively, and such that L_1/L_0, L_2/L_0 are k-projective. I let $T(M)$ be the ordered set of admissible presentations of M, where $X \leq X'$ if $L_i \subset L_i'$.

Assume that $\bar{A}, \bar{B}$ are projective over k. It follows then from the corollary on page 6 that M/L_1 and M/L_1' are A-projective, so from

$$0 \rightarrow L_1'/L_1 \longrightarrow M/L_1 \longrightarrow M/L' \rightarrow 0$$

one sees that L_1'/L is projective hence in $P(A)$. But it is clear that if we only assume $\bar{A}, \bar{B}$ k-flat, then we could still conclude M/L_1 is A-flat, and so L_1'/L_1 would be flat of finite presentation, hence projective. So we have

<u>Lemma</u>: <u>If $X \leq X'$ in $T(M)$, then the inclusions $L_1 \subset L_1'$, $L_0 \subset L_0'$, $L_2 \subset L_2'$ are admissible monos. in $P(A)$, $P(k)$, and $P(B)$ respectively.</u>

March 6, 1974. Localization.

Walldhausen has nice ways of thinking about localization. Suppose $\mathcal{B}$ is a Serre subcat. of the abelian category $\mathcal{A}$. Then for each $p \geq 0$ I consider the exact category of diagrams in $\mathcal{A}$

$$A_0 \longrightarrow \cdots \longrightarrow A_p$$

such that the arrows are mod $\mathcal{B}$ isomorphisms. Call this exact category $\mathcal{C}_p$.

Consider the case $p=1$. Let $\mathcal{C}_p'$ be the subcategory consisting of those $[A_0 \to A_1]$ such that $A_0 \to A_1$ is injective. $\mathcal{C}_1'$ closed under extensions and

$$\begin{array}{ccccccccc} 0 \to & X' & \to & X & \to & X'' & \to 0 & \quad \Rightarrow & X' \in \mathcal{C}_1' \\ & & & \cap & & \cap & & & \\ & & & \mathcal{C}_1' & & \mathcal{C}_1 & & & \end{array}$$

~~Further any $[A_0 \xrightarrow{u} A_1]$ is a quotient of $[A_0 \xrightarrow{\binom{1}{u}} A_0 \oplus A_1]$. Thus resolution says that $Q(\mathcal{C}_p') \sim Q(\mathcal{C}_p)$. Now on $\mathcal{C}_p'$ we have exact functor~~

$$\cancel{\mathcal{C}_p' \longrightarrow \mathcal{A} \times \mathcal{B}}$$

Given $[A_0 \xrightarrow{u} A_1]$ in $\mathcal{C}_1$, can one write it as a quotient of something in $\mathcal{C}_1'$?

March 6, 1974 Localization

Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$. One knows then that for any $V \in \mathcal{A}/\mathcal{B}$ the category consisting of $(A, \lambda A \simeq V)$ ($\lambda : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ canon. functor) is contractible. Thus the category consisting of $Ob(\mathcal{A})$ and mod $\mathcal{B}$ isos. is homotopy equivalent to $\mathrm{Is}(\mathcal{A}/\mathcal{B})$. Similarly, the category consisting of exact sequences in $\mathcal{A}$ and maps between exact sequences which are mod $\mathcal{B}$-isos. is equivalent to $\mathrm{Is}((\mathcal{A}/\mathcal{B})^{(2)})$.

Assertion: ~~___~~ Let C_q be the category consisting of q-filtered objects $0 \subset M_1 \subset \cdots \subset M_q$ of $\mathcal{A}$ and maps between them which are mod $\mathcal{B}$ isos. (precisely one has $M \xrightarrow{\varphi} M'$ $\ni$ $\varphi(M_i) \subset M_i'$ and such that $M_i \xrightarrow{\varphi} M_i'$ is a mod $\mathcal{B}$ isom. for each i). Then C_q is homotopy equivalent to the groupoid of q-filtered objects of $\mathcal{A}/\mathcal{B}$.

Proof: Will show that ~~___~~ $\forall$ $0 \subset V_1 \subset \cdots \subset V_q$ in $(\mathcal{A}/\mathcal{B})^{(q)}$, the category of $(M, \lambda M \simeq V)$ is filtering. Given $\lambda M \simeq \lambda M'$ this isomorphism is realized by two maps $\alpha : M \to N$, $\beta : M' \to N$ ~~___~~ both arrows being mod $\mathcal{B}$ isos. Can ~~consist~~ consider $\alpha(M_i) + \beta(M_i') = N_i$. Then N becomes filtered and α, β become maps in C_q.

Next given $M \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M'$ ~~___~~ which becomes equal in $\mathcal{A}/\mathcal{B}$ we let $M'' = M'/\mathrm{Im}(\alpha - \beta)$ and equip it with image filtration. Clearly equalizes.

Now to prove the localization one lets $N_p(C_q)$ be the groupoid consisting of the p-simplices in C_q and there isos. i.e.

$$\begin{tikzcd}
(M)_0 \arrow[r, phantom, "\subset \cdots \subset"] & (M)_p \\
(M_q)_0 \arrow[u, phantom, "\wr"] \arrow[r, dotted] & (M_q)_p \\
(M_1)_0 \arrow[u, dotted, phantom, "\cup \vdots \cup"] \arrow[r, dotted] & (M_1)_p \arrow[u, phantom, "\cup \vdots \cup"]
\end{tikzcd}$$

where the horizontal arrows are mod $\mathcal{B}$-isomorphisms. If I fix q what I have is the simplicial groupoid of simplices in C_q, which I know to be homotopy equivalent to C_q by

<u>Lemma</u>: Let C be any category and let $\mathcal{G}_p$ be the groupoid of p-simplices in C. Then the total category $\mathcal{G}$ is homotopy equivalent to C.

<u>Proof</u>: One has a functor $f: \mathcal{G} \to C$ sending $X_0 \to \cdots \to X_p$ to its last face X_p. Given $X \in C$, f/C ~~is the groupoid consisting~~ consisting of $Y_0 \to \cdots \to Y_q$ simplices $X_0 \to \cdots \to X_p$ + a map $X_p \to X$, when the maps come from those in $\mathcal{G}$. But

$$\begin{array}{c}
(X_0 \to \cdots \to X_p \,,\; + \;\; X_p \to X) \\
\downarrow \\
(X_0 \to \cdots \to X_p \to X \,,\;\; X \xrightarrow{id} X) \\
\uparrow \\
(X, \; X \xrightarrow{id} X)
\end{array}$$

give a contraction to a point.

What this proves is that the bisimplicial groupoid I have defined contracts horizontally to $q \mapsto C_q$ which I know is heq to $q \mapsto Is\,(\mathcal{A}/\mathcal{B})_{(q)}$, which gives the K-theory of $\mathcal{A}/\mathcal{B}$.

On the other hand, if I fix p I have the simplicial groupoid of filtered objects in the exact category consisting of chains

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_p$$

of mod $\mathcal{B}$ isos. in $\mathcal{A}$. Call this exact category $\mathcal{X}_p$.

<u>Conjecture:</u> $\mathcal{X}_p$ has the same K-theory as the subcategory $\mathcal{X}'_p$ consisting of chains, where the arrows are monos. in $\mathcal{A}$. The exact functor

$$(A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_p) \mapsto (A_0, A_1/A_0, \ldots, A_p/A_{p-1})$$

$$\mathcal{X}'_p \longrightarrow \mathcal{A} \times \underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{p \text{ times}}$$

induces isos. on K-groups.

Last assertion follows from exactness thm.

Look at $\mathcal{X}_1$: it consists of arrows $A_0 \longrightarrow A_1$ in $\mathcal{A}$ which are mod $(\mathcal{B})$ isomorphisms. Now one would like to apply the resolution theorem to $\mathcal{X}'_1 \longrightarrow \mathcal{X}_1$. The only thing giving trouble is to show that any $X = [A_0 \hookrightarrow A_1]$ is a quotient of something in $\mathcal{X}'$. Observe that if we have

$$\begin{tikzcd}
 & & \arrow[d] & \arrow[d] & & \\
 & 0 \arrow[r] & L_0' \arrow[r, dashed] \arrow[d] & L_1' \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & 0 \arrow[r] & L_0 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & D \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] & A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

$F, D, C, K \in \mathcal{B}$

then we get

$$0 \to K \to F \to D \to C \to 0$$

and so the element of $\operatorname{Ext}^2_{\mathcal{A}}(C, K)$ represented by $0 \to K \to A_0 \to A_1 \to C \to 0$ comes (probably) from an element of $\operatorname{Ext}^2_{\mathcal{B}}(C, K)$. Thus if

$$\operatorname{Ext}^2_{\mathcal{B}}(C, K) \longrightarrow \operatorname{Ext}^2_{\mathcal{A}}(C, K)$$

isn't onto what we want to do is impossible.

On the other hand the conjecture seems to be true for $\mathcal{X}_1$ is an abelian category with Serre subcategory arrows in $\mathcal{B}$ and quotient $\mathcal{A}/\mathcal{B}$, so one should have a fibration

$$Q(\mathcal{B}) \times Q(\mathcal{B}) \longrightarrow Q(\mathcal{X}_1) \longrightarrow Q(\mathcal{A}/\mathcal{B})$$

which is consistent with

$$\begin{array}{c} Q(\mathcal{X}_1') \\ \parallel \\ Q(\mathcal{A}) \times Q(\mathcal{B}) \end{array}$$

and the conjecture.

Example: Let $\mathcal{A}$ be abelian, $\mathcal{B}$ a Serre subcategory, and let $\mathcal{C}$ be the abelian category of arrows $[A_1 \to A_0]$ in $\mathcal{A}$ which are isos. modulo $\mathcal{B}$. Let us identify the subcategory of isos with $\mathcal{A}$, whence we have $\mathcal{A} \hookrightarrow \mathcal{C}$, $A \mapsto [A \xrightarrow{id} A]$. Show then that the quotient K-theory of that of $\mathcal{C}$ by that of $\mathcal{A}$ is equivalent to the K-theory of $\mathcal{B}$.

This should be true because I am fairly certain that the K-theory of $\mathcal{C}$ is the same as that of the full sub $\mathcal{C}^{inj}$ consisting of inj. arrows, and by the exactness thm one has a fibration $Q(\mathcal{A}) \longrightarrow Q(\mathcal{C}^{inj}) \longrightarrow Q(\mathcal{B})$.

March 7, 1974: vector bundles over an elliptic curve C

Recall in general for a vector bundle E over a curve one can consider

$$\mu_{max}(E) = \sup\{\mu(E') \mid 0 < E' \leq E\}$$

$$\mu(E') = \frac{\deg(E')}{rg(E')}$$

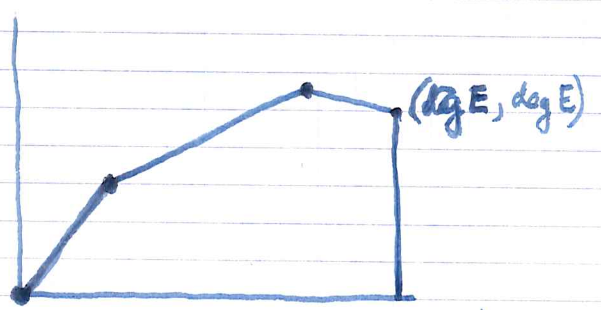
and there exists of unique $\neq 0$ maximum subbundle F_1 of E with $\mu(F) = \mu_{max}(E)$. In fact we get a canonical filtration

$$0 \subset F_1 \subset F_2 \subset \cdots\cdots \subset F_k = E$$

with F_i/F_{i-1} semi-stable of rank μ_i

$$\mu_{max}(E) = \mu_1 > \cdots\cdots > \mu_k = \mu_{min}(E).$$

Picture:



It seems that $(rg(E'), \deg(E'))$ is below this polygon for every E' which injects into E.

Lemma: Given $0 \to E' \to E \to E'' \to 0$ if

$$\mu_{min}(E') > \mu_{max}(E'') + 2g-2$$

then the sequence splits.

Proof: $H^1(\underline{Hom}(E'', E'))$ is dual to $H^0(\underline{Hom}(E', E''\otimes\Omega))$. If this is non-zero we get $E' \twoheadrightarrow Q \hookrightarrow E''\otimes\Omega$ $\quad Q \neq 0$. Then

$$\mu_{min}(E') \leq \mu(Q) \leq \mu(E''\otimes\Omega) = \mu(E'') + (2g-2).$$

QED.

Cor. Over an elliptic curve, every indecomposable vector bundle is semi-stable.

To classify indecomp. v.b. of deg d, rank r; call the set of these iso classes $\mathcal{E}(d,r)$. Tensoring with a fixed line bundle of degree 1 gives a bijection

(1) $$\mathcal{E}(d,r) \xrightarrow{\sim} \mathcal{E}(d+r,r).$$
$$E \longmapsto E\otimes L \qquad \deg L = 1.$$

so one can assume $0<d\leq r$. If $d>0$, then as E is semi-stable of slope $\mu = \frac{d}{r} > 0$, $\mathrm{Hom}(E,\mathcal{O}) = 0 \Longrightarrow H^1(E)$ (dual to $H^0(E^*)$) $= 0$. Thus $h^0(E) = d$ by R-R, and we get $\mathcal{O}^d \to E$ inducing iso on H^0. If this fails to be a subbundle injection, then there is a non-zero section of E which vanishes in some fibre, hence E has a sub-line-bundle L of deg >0. As E is semi-stable this happens only when $d=r$.

If $0<d<r$, then $\mathcal{O}^d$ is a subbundle of E. Let E' = quotient. Then

$$H^1(E') = 0.$$
$$H^0(E') \xrightarrow{\sim} H^1(\mathcal{O}^d) \simeq H^1(\mathcal{O})\otimes H^0(E).$$

Now for any E' with $H^1(E')=0$ one has univ. extension of E' by $\mathcal{O}\otimes V$:

$$0 \longrightarrow H^0(E')\otimes\mathcal{O} \longrightarrow (E')^+ \longrightarrow E' \longrightarrow 0$$

chararacterized by the fact that $H^1(E'^+) = 0$ and $H^0(E') \simeq H^0(E'^+)$.

It is clear then that in the case $0<d<r$, $E=(E')^+$ and that E' is also indecomposable by functorality. Thus for $0<d<r$ one has

$$(2) \qquad \mathcal{E}(d,r) \simeq \mathcal{E}(d,r-d)$$

Finally note that if one has

$$0 \to E' \to E \to E'' \to 0$$

with E', E, E'' semi-stable of slope μ, E' stable, then if E is indecomposable $\mathrm{Hom}(E',E'') \neq 0 \Rightarrow E' \hookrightarrow E''$ as a subbundle. Continuing one sees that E is the tensor product of E' and an indecomposable extension of $\mathcal{O}$'s. There is only one of these J_r of each rank, so one gets a bijection

$$(3) \qquad \begin{aligned} C &\xrightarrow{\sim} \mathcal{E}(d,d) \\ P &\longmapsto \mathcal{L}(P) \otimes J_d . \end{aligned}$$

From (1), (2), (3) one concludes a bijection

$$C \simeq \mathcal{E}(d,r)$$

for any r, d.

Now since one has a map

$$\begin{aligned} \mathcal{E}(d,r) &\longrightarrow \mathcal{E}(kd,kr) \\ E &\longmapsto E \otimes J_d \end{aligned} \qquad (d,r)=1$$

which is non-trivial (take determinants), it seems more or less clear that the stable bundles on C are those indecomposable bundles with (d,r) rel. prime.

March 8, 1974. Sylvester's theorem

k field, $X(k[t]^n)$ = ordered set of non-zero unimodular subspaces of $k[t]^n$. Recall that we have proved this is contractible.

Let $M = k[t]^n$ and let $Y(M)$ consist of the following kinds of data: a flag $0 < M_1 < \cdots < M_p = M$ in M (each M_i is a direct summand of M) together with a maximal unimodular subspace V_i of M_i/M_{i-1} for each $i = 1, \ldots, p$. Given another such thing $\{M_j', V_j'\}$ say that $\{M_j', V_j'\} \leq \{M_i, V_i\}$ if the flag $\{M_j'\}$ refines $\{M_i\}$, that is

$$M_i = M'_{\tau(i)} \qquad \{j\} \xleftarrow{\tau} \{i\}$$

and if the filtration

$$M_{i-1} = M'_{\tau(i-1)} < \cdots < M'_{\tau(i)} = M_i$$

induces on V_i a filtration consistent with the V_j' for $\tau(i-1) < j \leq \tau(i)$.

Thus for each i, I give myself a flag in V_i which induces one in M_i. Putting these together I get the flag M_j'. So one has that

$$\{\{M_*', V_*'\} \leq \{M_*, V_*\}\} = \prod_{i=1}^{p} Flag(V_i)$$

Assertion: $Y(M)$ is contractible.

Proof: Consider the map $Y(M) \longrightarrow X(M)$ which sends $\{M_*, V_*\}$ into V_1. This is a functor. The fibre over a $V \in X(M)$ is clearly $Y(M/k[t]V)$ which will be contractible by induction. (If V is maximal the fibre is a point.) So we only have to show this functor is fibred.

But if I give $V' \subset V$ and $\{M_*, V_*\}$ with $V_1 = V$, then those $\{M'_*, V'_*\} \leq \{M_*, V_*\}$ with $V'_1 = V_1$ are the same as flags in V_* starting with V_1. Clearly there is a minimal such refinement namely the flag

$$V' \subset V_1, V_2, \ldots, V_p.$$

So the fact the functor is fibred is now clear.

<u>Cor.</u> $GL_n(k[t])$ is the homotopy-amalgamation of the subgroups

$$\begin{pmatrix} GL_{a_1}k & & * \\ & GL_{a_2}k & \\ 0 & & \ddots \\ & & & GL_{a_p}k \end{pmatrix} \qquad \begin{array}{l} a_1 + \cdots + a_p = n \\ a_i > 0 \end{array}$$

$*$ denotes entries in $k[t]$.

~~Assume I know that for the mod p homology that I_p is a projective fin. type $\Gamma_{p,\alpha}$-module, whence I know that~~

Assume: exactness of complex

1)
$$s \longmapsto \bigoplus_\alpha H_0(\Gamma_{s,\alpha}, I_s)$$

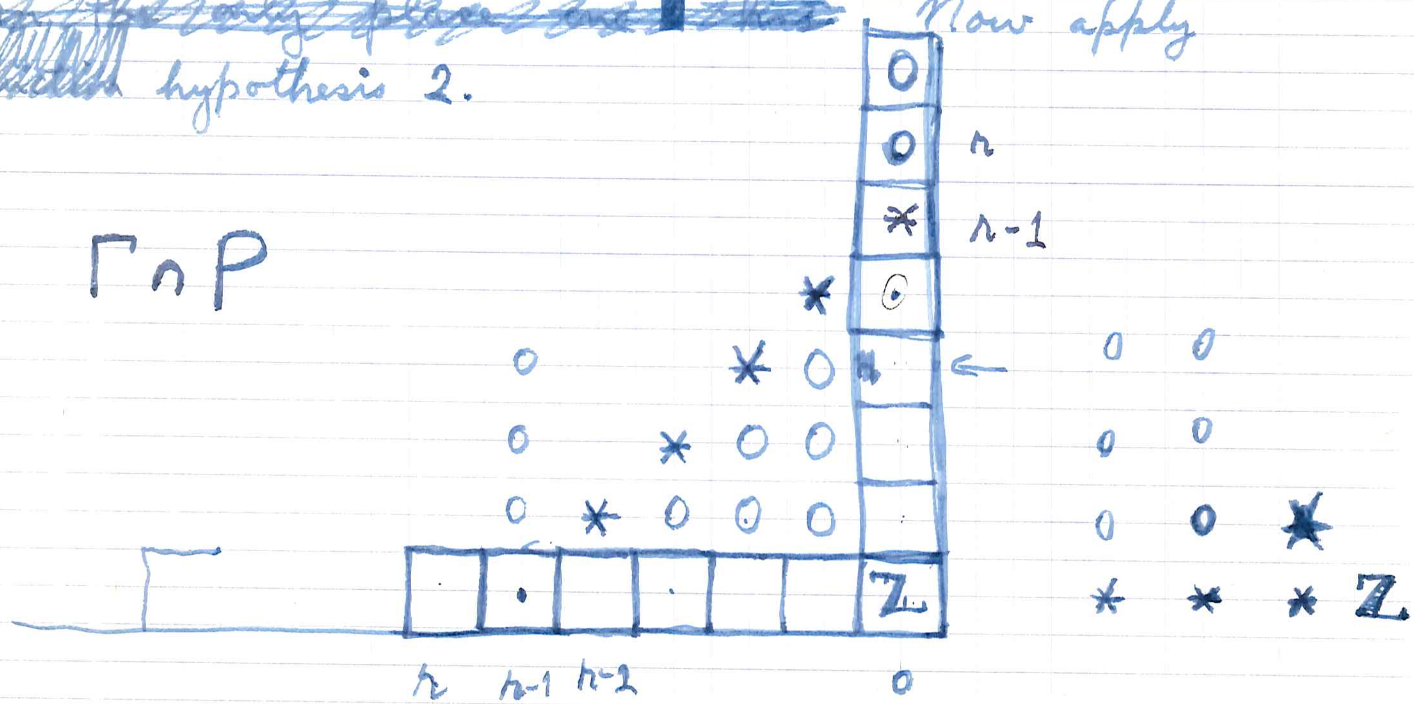
2)
$$\tilde{H}_i(\Gamma_{s,\alpha}) = 0 \qquad i \neq s-1 \qquad \text{for } s < r.$$

Now consider the spectral sequence

$$E^1_{st} = \begin{cases} \bigoplus_\alpha H_0(\Gamma_{s,\alpha}; I_s) \otimes H_t(\Gamma_{r-s,\beta\alpha}) & 0 < s < r \\ H_t(\Gamma_{r,\beta}) & s = 0 \\ H_0(\Gamma_{r,\beta}, I_r) & s = r,\ t = 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now apply hypothesis 2.

$\Gamma \cap P$



$r = 6$

C = complete smooth curve over a field $k = H^0(C, \mathcal{O}_C)$, F = the function field $k(C)$, ∞ = a (closed) point of C which is assumed to be rational for simplicity, $\mathcal{O}(1)$ = line bundle associated to the divisor ∞, $A = \Gamma(C - \infty, \mathcal{O}_C)$.

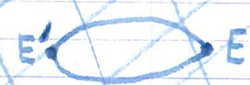
M = rank 2 projective A-module, $\Gamma = GL_2 A$. To simplify, suppose A free whence $\Gamma = GL_2 A$.

X = Tits building associated to the F-vector space $V = F \otimes_A M$ and the valuation v_∞ on F. X is the simplicial complex defined as follows. A vertex is an equivalence classes of $\mathcal{O}_\infty$-lattices in V, where Λ and Λ' are equivalent if $f\Lambda = \Lambda'$ for some $f \in F^*$. Such a lattice Λ may be interpreted as an extension of M, considered as a vector bundle over $C - \infty$, to a vector bundle E on C, and two extensions E, E' represent the same vertex of X iff $E = E'(n)$, where $E'(n) = E \otimes \mathcal{O}(1)^{\otimes n}$, for some n. Since $\deg E(n) = \deg E + 2n$, every vertex corresponds to an extension E of M such that $\deg E = 0$ or 1. Two vertices form a 1-simplex if ....

~~The group Γ acts on X. A Γ orbit of vertices can be identified with an isom. class of extensions of M to a vector bundle E on C such that $\deg E = 0$ or 1.~~

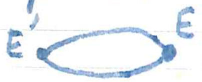
The group Γ acts on X, hence one has a quotient graph X/Γ. A vertex of X/Γ may be identified with an isom. class of vector bundles E on C such that E restricted to $C-\infty$ becomes isom. to M, and such that $\deg E = 0$ or 1. An edge ~~in an~~ ~~[illegible] of inclusions [illegible]~~ joining $cl(E')$ to $cl(E)$, ~~[illegible] may be identified with a class of~~ where $\deg(E') = 0$, $\deg(E) = 1$, may be identified with an equivalence class of injections $\alpha : E' \longrightarrow E$ whose cokernel $\simeq k(\infty)$, two such injections being equivalent if they are conjugate by the actions of ~~$\mathrm{aut}(E')$ and $\mathrm{aut}(E)$~~ $\mathrm{Aut}(E)$ and $\mathrm{Aut}(E')$.

~~Question: Is X/Γ a simplicial complex? The problem is that we might have in X/Γ the subgraph:~~



~~that is, one might have two subsheaves $E' \subset E$ $E'' \subset E$ with E/E', $E/E'' \simeq k(\infty)$ such that $E' \simeq E''$ &c.~~

<u>Question</u>: Is X/Γ a simplicial complex? The problem is that we might have in X/Γ a subgraph:



that is, one might have two subsheaves $E' \subset E$ $E'' \subset E$ with E/E', $E/E'' \simeq k(\infty)$ and such that $E' \simeq E''$ but such that there is no auto. of E carrying E' into E''.

Cusps of X/Γ:

Given a vector bundle E over C one defines its <u>slope</u> to be

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E}.$$

and one says that E is <u>semi-stable</u> (resp. <u>stable</u>) if for any proper sub-bundle E' of E one has $\mu(E') \leq \mu(E)$ (resp. $\mu(E) < \mu(E)$). I will call E <u>instable</u> if it is not semi-stable. If $\operatorname{rank} E = 2$, this means there exists a sub-line-bundle L of E such that $\deg(L) > \deg(E/L)$. ~~It is clear that L is unique. Supposing L chosen to be of maximum degree, let L′ be another sub-line-bundle of the same degree. Then L′ ⊂ E → E/L must be zero, hence L′ ⊂ L, so L′ = L as they have the same degree.~~ Let L' be another sub-line-bundle of E such that $\deg(L') > \deg(E/L)$. Then $L' \subset E \to E/L$ must be zero, so $L' \subset L$, hence $L' = L$ if $\deg(L') \geq \deg(L)$. This proves:

<u>Prop</u>: If E is instable, it has a unique line sub-bundle L such that $\deg(L) > \deg(E/L)$.

More generally, I guess I want to consider E such that if $d_1 = \max\{\deg(L) \mid L \text{ subbundle } E\}$ then there exists a unique L with $\deg(L) = d_1$. The instable bundles belong to this class. But also let E be semi-stable but not stable and indecomposable. ~~Then it has an L ∋ μ(L) = μ(E) and if it had another we would have a map L ⊕ L′ ≅ E since it~~

the semi-stable bundles of a given slope μ form an abelian category. Thus E has a line subbundle of the same slope, and it can't have more than one without being decomposable.

So now let E be a rank 2 bundle having a unique line subbundle L of the maximum degree. Put $n = \deg(L) - \deg(E/L)$. Let $E' \subset E$ be such that $E/E' \simeq k(\infty)$. Then there are two cases:

<u>Case 1</u>: $L \subset E'$. Clearly L is the unique line subbundle of E' of maximum degree. Also $\deg(L) - \deg(E'/L) = n+1$.

<u>Case 2</u>: $L \not\subset E'$, so that E' is an extension of $L(-1)$ by E/L. Then E' has no line subbundles of degree $= \deg(L)$. In effect if $L_1 \subset E'$, then $L_1 \subset E$ and the line subbundle of E generated by L_1 has degree $\geq \deg(L) \Rightarrow L_1 = L$ which is impossible. Thus the integer n for E', $n' = \deg L(-1) - \deg(E/L) = n-1$.

So we have proved.

<u>Definition</u>: $n(E) = \deg(L) - \deg(E/L)$ where L is a line subbundle of E of maximum degree.

<u>Prop</u>. If E is rank 2 bundle having a unique line subbundle of maximum degree, then among the $E' \subset E$ with $E/E' \simeq k(\infty)$, there is exactly one with $n(E') = n(E)+1$, and all the others have

$n(E') = n(E) - 1.$

Calculation of the minimum value for $n(E)$.

If $\deg E = 2g-1$, then R.R. $h^0(E) \geqslant \deg E + 2(1-g)$ $\Rightarrow h^0(E) > 0 \Rightarrow \deg(L) \geqslant 0$ for a maximal L, hence

$\deg E$ odd $\left[\; n(E) = 2\deg L - \deg E \geqslant -(2g-1) \right.$

This bound is achieved for $g=1$.

If $\deg E = 2g$, then here we get $h^0(E) \geqslant 2$, so we get 2 sections which must vanish somewhere (assuming $g \geqslant 1$, k alg. closed), so get $\deg L \geqslant 1$ and

$\deg E$ even
$$\left[\begin{array}{ll} n(E) \geqslant 2-2g & (k \text{ alg. closed}, g \geqslant 1) \\ n(E) \geqslant -2g & (\text{in general}) \end{array} \right.$$

These bounds are realized for elliptic curves.

Also I should check that $n(E)$ can change at most one. But given L maximal, $E' \subset E \to k(\infty)$, if one has $0 \to L \to E' \to E/L(-1) \to 0$, then $n(E') = n(E)+1$. And if $0 \to L(-1) \to E' \to E/L \to 0$ and if $L' \subset E'$ is maximal, one has $\deg L' = \deg L$ or $\deg L - 1$ whence

$$n(E') = 2\deg L' - \deg E' = \left| \begin{array}{l} 2\deg L \\ 2\deg L - 2 \end{array} \right. - \deg E + 1 = \left| \begin{array}{l} n(E)+1 \\ n(E)-1 \end{array} \right.$$

Thus this is clear.

The cusps:

~~Let X' be the full subcomplex of X consisting of vertices whose associated vector bundles have a unique line bundle of the maximum degree. We have seen that starting from each E in X' there is a unique edge $E'E$ such that $n(E') = n(E)+1$, and moreover that if L is the unique maximum line subbundle of E, then L is also the max. line subbundle of E'. (Provided we regard E' as contained in E with $E/E' \cong k(\infty)$).~~

Let $\ell \in P_1(V)$, $(V = F \otimes_A M)$, and let X_ℓ^* be the full subcomplex of X consisting of those E such that the line subbundle $E \cap \ell$ of E is the unique line subbundle of maximum degree.

<u>Claim:</u> X_ℓ^* is a tree.

Proof. Given E in X_ℓ we have seen that there is a unique edge in X starting from E whose endpoint E' is such that $n(E') = n(E)+1$. In fact E' is given by pull-back:

$$\begin{tikzcd}
0 \arrow[r] & E\cap\ell \arrow[r] & E \arrow[r] & E/E\cap\ell \arrow[r] & 0 \\
0 \arrow[r] & E\cap\ell \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u, phantom, "\cup" description] & E/E\cap\ell(-1) \arrow[r] \arrow[u, phantom, "\cup" description] & 0
\end{tikzcd}$$

I know that $E' \cap \ell = E \cap \ell$ is the unique maximal subbundle of E', hence $E' \in X_\ell$.

The process I have given associates to each E in X_ℓ a unique path in X_ℓ

$$E \cdots E_1 \cdots E_2 \cdots\cdots$$

such that $n(E_i) = n(E) + i$. To finish the proof all I have to show is that given two vertices E, E' in X_ℓ with $n(E) = n(E')$, then $E_n = E'_n$ for n large.

I can normalize things so that $\deg(E\cap\ell) = \deg(E'\cap\ell) = 0$, whence $E\cap\ell = E'\cap\ell$ ($E\cap\ell$ is simply an extension of $M\cap\ell$ to a line bundle on C, hence it is determined by its degree). One has

$$E(-n) \subset E'$$

for n large, hence $E_n = E\cap\ell + E(-n) \subset E'$. But

$$\begin{tikzcd}
0 \arrow[r] & E\cap\ell \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d, phantom, "\cap"] & E/E\cap\ell(-n) \arrow[r] \arrow[d, leftrightarrow, "\theta"] & 0 \\
0 \arrow[r] & E'\cap\ell \arrow[r] & E' \arrow[r] & E'/E'\cap\ell \arrow[r] & 0
\end{tikzcd}$$

By assumption $n(E') = \deg(E'\cap\ell) - \deg(E'/E'\cap\ell) = n(E)$, whence $\deg(E'/E'\cap\ell) = \deg(E/E\cap\ell)$. Thus because $E/E\cap\ell(-n)$ and $E'/E'\cap\ell$ are both extensions of $M/M\cap\ell$, one has $E/E\cap\ell(-n) \xrightarrow{\sim} E'/E'\cap\ell(-n)$, hence $E_n = E'_n$.

Set $X' = \coprod X_\ell \subset X$. The complementary subcomplex consists of all bundles having more than one line bundle of maximal degree. It consists of stable bundle and bundles $L \oplus L^*$ where $\deg L = \deg L^* = 0$

Note X' contains all E with $n(E) > 0$, i.e. all unstable bundles.

Def: X_ℓ = cusp assoc. to $\ell \in P_1(V)$.

March 9, 1974. elliptic curves.

C elliptic curve over k alg. closed. Let ∞ be the basepoint and A the coordinate ring of $C - \{\infty\}$. To understand $Gl_2(A)$, Serre makes it act on the building at ∞. (Since this is a tree this gives a gen. free product structure on $Gl_2(A)$ in the sense of Waldhausen.)

I want now to find the simplest possible model for $Gl_2(A)$, as a category of groups which are algebraic over k, which one can obtain from $\Gamma = Gl_2(A)$ acting on the building X.

Recall that the vertices of X are extensions of A^2 to a vector bundle on C modulo equivalence of E with $E(n) = E \otimes \mathcal{O}(n)$, $\mathcal{O}(1) = \mathcal{L}(\infty)$. The Γ iso. classes of vertices are thus iso classes of E extending the trivial 2-plane bundle over A. Since $\mathrm{Pic}(A) = C$, i.e. in going from $K_0(C)$ to $K_0(A)$ we only kill the degree, the condition that $E|C-\{\infty\} \simeq A^2$ is equivalent to $\Lambda^2 E \simeq \mathcal{O}(d)$, $d = \deg E$. Up to equiv. $\therefore$ $\Lambda^2 E = \mathcal{O}$ or $\mathcal{O}(1)$.

Can classify Γ-orbits on vertices as follows:
If $\Lambda^2 E = \mathcal{O}(1)$, then two cases:

E indecomposable $\Rightarrow$ E is the unique non-trivial ext.
$$0 \to \mathcal{O} \to E \to \mathcal{O}(1) \to 0$$
and $\mathrm{Aut}(E) = k^*$.

E decomp. $\Rightarrow$ $E \simeq L(n+1) \oplus L^*(-n)$ where $L \in \mathrm{Jac}$ and $n \geq 0$ are uniquely determined.
Here $\mathrm{Aut}(E) = \begin{pmatrix} k^* & H^0(L^{\otimes 2}(2n+1)) \\ 0 & k^* \end{pmatrix}$

If $\Lambda^2 E = 0$, then

E indecomp $\Longrightarrow$ $E \simeq L \otimes U_1$, where $L \in \mathrm{Jac}$, $L = L^*$ and U_1 is the unique non trivial extension of $\mathcal{O}$ by $\mathcal{O}$. Here

$$\mathrm{Aut}(E) = k^* \times \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

E decomp. with max. subbundle L of degree 0 $\Longrightarrow$ $E \simeq L \oplus L^*$.

$$\mathrm{Aut}(E) = k^* \times k^* \quad \text{if} \quad L \neq L^*$$
$$= GL_2(k) \quad \text{if} \quad L = L^*.$$

E decomp. with max. subbundle L of degree $n > 0$ $\Longrightarrow$ $E = L(n) \oplus L^*(-n)$

$$\mathrm{Aut}(E) = \begin{pmatrix} k^* & H^0(L^{\otimes 2}(2n)) \\ 0 & k^* \end{pmatrix}$$

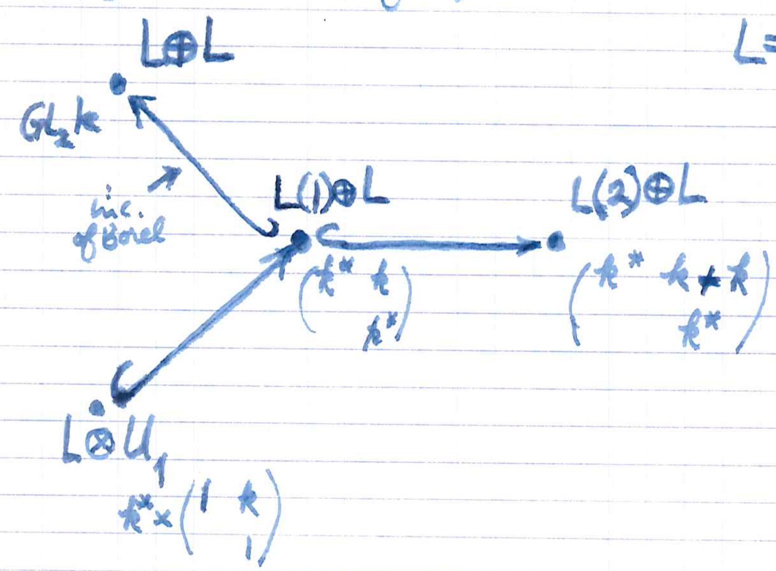
Now given an E representing a vertex of X/Γ, an edge issuing from E is the same as an orbit of $\mathrm{Aut}(E)$ on the lines in $E \otimes k(\infty)$.

Start with typical decomp. E, say $L(n) \oplus L^*$, $n > 0$ whose auto group is $\begin{pmatrix} k^* & H^0(L^{\otimes 2}(n)) \\ 0 & k^* \end{pmatrix}$. Now for $n \geq 2$, $L^{\otimes 2}(n)$ is spanned by its sections, hence we can find a section $s \in H^0(L^{\otimes 2}(n))$ not vanishing at ∞. This implies the auto group maps onto the Borel subgroup of $GL(E(\infty))$, hence there are two edge in the quotient graph issuing from E. One obvious goes to $L(n+1) \oplus L^*$ and the other to $L(n-1) \oplus L^*$. One has the picture

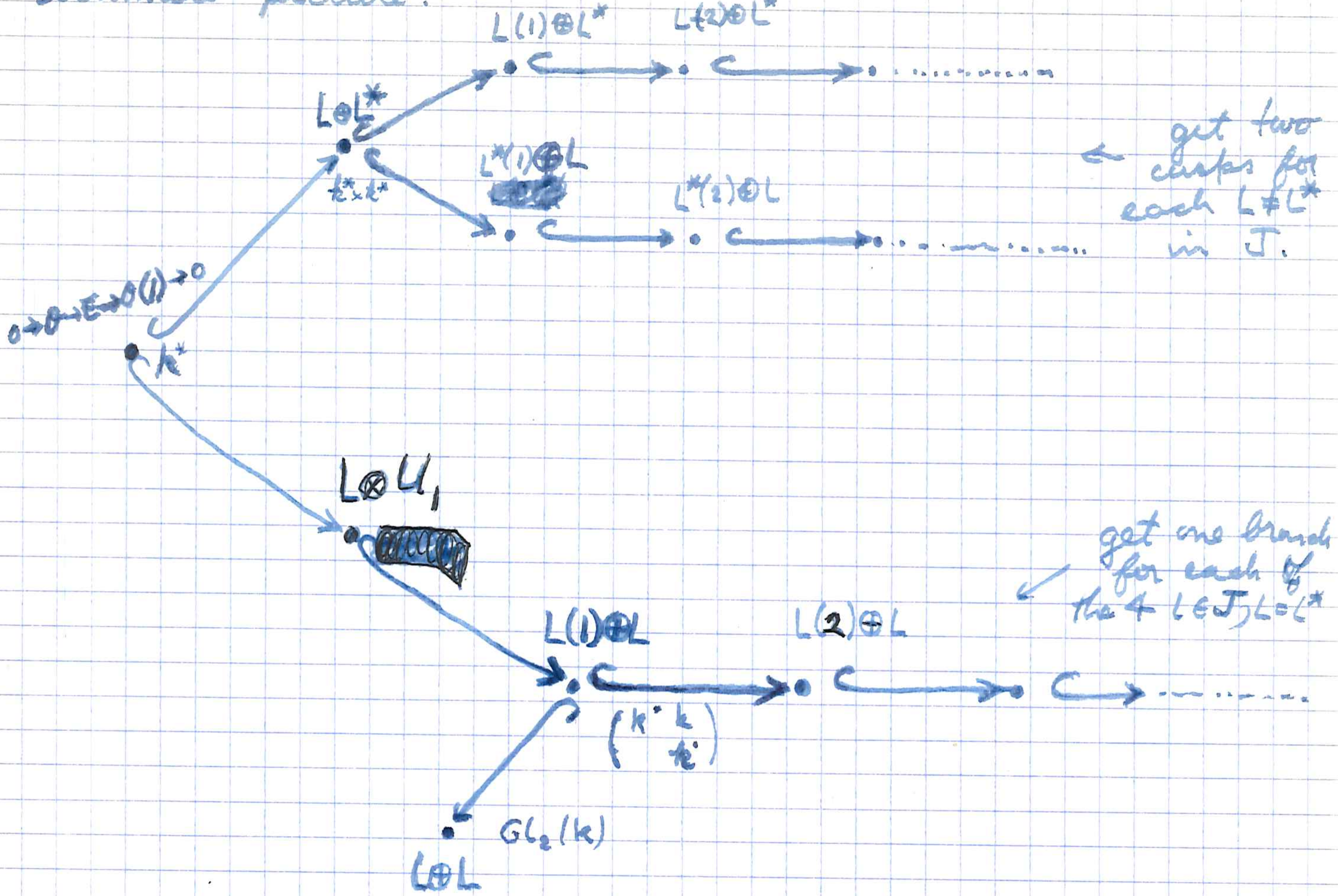
$$\begin{array}{ccccc} L(n-1)\oplus L^* & \hookrightarrow & L(n)\oplus L^* & \hookrightarrow & L(n+1)\oplus L^* \\ \begin{pmatrix} k^* & H^0(L^2(n-1)) \\ 0 & k^* \end{pmatrix} & & \begin{pmatrix} k^* & H^0(L^2(n)) \\ 0 & k^* \end{pmatrix} & & \begin{pmatrix} k^* & H^0(L^2(n+1)) \\ & k^* \end{pmatrix} \end{array}$$

If $n=1$, then $\exists!$ section $\mathcal{O} \xrightarrow{s} L^2(1)$ and so $L^2(1) = \mathcal{L}(P)$ for some! P on C. If $P \neq \infty$ s doesn't vanish at 0 and we have the same situation as before. If on the other hand $L^2 = \mathcal{O}$, then $\begin{pmatrix} k^* & H^0(\mathcal{O}(1)) \\ 0 & k^* \end{pmatrix}$ maps onto the torus and there are three edges issuing from our vertex.

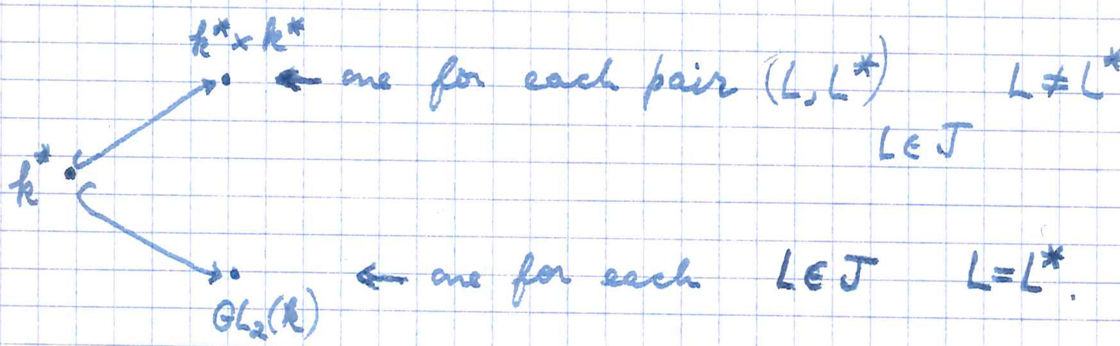
$L = L^*$



Ultimate picture:



From the point of view of mod l homology, this simplifies to



Here J = Jacobian = line bundles of degree 0.

March 10, 1974.   d.v.r.

A d.v.r., quotient field F, res. field k.

Question: Is there a nice relation between $H_*(GL_2 F, I(F^2))$ and the corresponding things for A, k?

Recall the model I have for $GL_2(F)$. I let $GL_2(F)$ act on the simplicial complex of lattices in F^2, where the simplices are chains $L_0 < L_1 < \dots L_q$ such that L_q/L_0 is a k-module. The resulting cofibred category over $GL_2(F)$ has four iso classes represented by

vertices:	L		$\sim GL_2(A)$
1-simp	$L_0 < L_1$	$\ni \dim(L_1/L_0) = 1$	$\sim \begin{pmatrix} * & * \\ \equiv 0 & * \end{pmatrix}$
1-simp	$L_0 < L_1$	$\ni \dim(L_1/L_0) = 2$	$\sim GL_2(A)$
2-simp	$L_0 < L_1 < L_2$	$\ni \dim(L_2/L_0) = 2$	$\sim \begin{pmatrix} * & * \\ \equiv 0 & * \end{pmatrix}$

So we get a diagram of groups, where $C = \begin{pmatrix} * & * \\ \equiv 0 & \end{pmatrix}$

$$\begin{tikzcd} & GL_2 A \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] & GL_2 A \\ C \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & C \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

And also I have an exact sequence

$$0 \longrightarrow \bigoplus_{\substack{L_0 < L_1 \\ \text{cod } 2}} I(L_1/L_0) \longrightarrow \bigoplus_{\substack{L_0 < L_1 \\ \text{cod } 1}} \mathbb{Z} \longrightarrow \bigoplus_{L} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

which comes from filtering the ordered set of layers (L_0, L_1) killed by π according to $\dim(L_1/L_0)$.

This leads to a spectral sequence

$$E^1_{st} = H_t\left(\begin{pmatrix} GL_s(A) & \\ \equiv 0 & GL_{n-s}A \end{pmatrix}, I_s(k) \right) \Longrightarrow H_*(GL_n F).$$

Now one should ask what happens when one brings in the Borel subgroup: $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $* \in F$.

So therefore one has to compute B orbits on lattices.

Claim B acts transitively on lattices. In effect B is the stabilizer of the line V' ($V' = Fe_1$) in V. If L is a lattice then $L \cap V'$ is a direct summand of L, so one basis elt of L is $\pi^n e_1$. Thus by moving $\pi^n e_1$ to e_1 and the other basis element to e_2, we move L to $Ae_1 + Ae_2$ by a transformation in B.

Similarly there are two B orbits on (L_0, L_1) such that L_1/L_0 has dim. 1.

~~Instead of working with the simplicial complex of lattices, use layers. Then our category has three objects~~

~~$\dim(L_1/L_0)=0$~~	~~$GL_2 A$~~
~~1~~	~~C~~
~~2~~	~~$GL_2 A$~~

~~with arrows:~~

~~?~~

<u>March 17, 1974.</u> Waldhausen (Continued)

$R = A *_k B$ where A, B are "pure" extensions of k, i.e. $A = k \oplus A'$ as k-bimodules.

I consider diagrams

$$L: \quad (L_A \xleftarrow{\alpha} L_k \xrightarrow{\beta} L_B)$$

where L_A is an A-module, etc, $\alpha\ \beta$ are k homos. If M is an R-module, then by a <u>presentation</u> of M I will mean a diagram L as above + maps $L_A \to M$ $/A$ and $L_B \to M$ $/B$ compatible with α, β $\ni$

$$0 \to R \otimes_k L_k \to R \otimes_A L_A \oplus R \otimes_B L_B \to M \to 0$$

is exact.

<u>Proposition</u>: Let L be a presentation of M such that α, β are direct injections over k, and assume that A, B are pure extensions of k. Then $L_A \to M$, $L_B \to M$, $L_k \to M$ are direct injections over A, B, k resp. and $L_k \xrightarrow{\sim} L_A \times_M L_B$.

Proof. Prolongation process going from $L = (L_A \leftarrow L_k \to L_B)$ to $L^* = (L_A^* \longleftarrow L_B \longrightarrow L_B)$ defined by

$$\begin{array}{ccc} A \otimes_k L_k & \longrightarrow & A \otimes_k L_B \\ \downarrow & & \downarrow \\ L_A & \longrightarrow & L_A^* \end{array} \qquad \begin{array}{c} \text{direct inj. over } A \\ \Downarrow \\ \text{direct inj. over } A. \end{array}$$

From

$$\begin{array}{ccccccccc} 0 & \to & L_k & \longrightarrow & L_B & \longrightarrow & L_B/k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_A & \longrightarrow & L_A^* & \longrightarrow & A \otimes_k (L_B/k) & \longrightarrow & 0 \end{array}$$

and fact that $k \to A$ has retraction as a k bimodule, one sees $L_B \to L_A^*$ is a direct injection $/k$.

Iteration of this process leads to M (see March 3)

$\mathcal{D}$ = category of diagrams L where L_A, L_B, L_k are free fin. type.

$\mathcal{D}^a$ = exact subcategory such that $L_k \to L_A$, $L_k \to L_B$ are direct injections, and such that L is a pres. of a free R-mod.

One has exact filtration of objects of $\mathcal{D}$

$$0 \to (L_A \leftarrow 0 \to L_B) \to (L_A \leftarrow L_k \to L_B) \longrightarrow (0 \leftarrow L_k \to 0)$$

hence $\quad K_*\mathcal{D} = K_*\mathcal{F}_A \times K_*\mathcal{F}_B \times K_*\mathcal{F}_k$. (exactness thm)

One has exact resolutions of objects of $\mathcal{D}$ by objects in $\mathcal{D}^a$:

$$0 \to (A\otimes_k L_k \leftarrow 0 \to B\otimes_k L_k) \longrightarrow \begin{matrix} (A\otimes_k L_k \leftarrow L_k \to B\otimes_k L_k) \\ \oplus \\ (L_A \leftarrow 0 \to L_B) \end{matrix} \longrightarrow (L_A \leftarrow L_k \to L_B) \to 0$$

This ~~[illegible]~~ leads to

$$K_*\mathcal{D}^a = K_*\mathcal{D}$$

by exactness thm.

$\mathcal{D}^b$ = exact subcat. of $\mathcal{D}$ ~~[illegible]~~ consisting of L which are presentations of 0.

<u>GOAL</u>. ~~The following~~ One has a fibration

$$\begin{array}{ccc} Q(\mathcal{D}^b) \longrightarrow & Q(\mathcal{D}) & \\ & \wr\!\!| & \\ & Q(\mathcal{D}^a) & \longrightarrow Q(\mathcal{F}_R) \end{array}$$

<u>Test point:</u> Let $\mathcal{C}$ be the exact category of exact sequences in $\mathcal{D}$

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

such that $L'' \in \mathcal{D}^b$. From the exactness thm. get

$$K_*(\mathcal{C}) \xrightarrow{\sim} K_*(\mathcal{D}) \oplus K_*(\mathcal{D}^b)$$
$$(0 \to L' \to L \to L'' \to 0) \longmapsto (L', L'')$$

Let $\mathcal{C}^a$ be the sub-exact-cat of $\mathcal{C}$ consisting of $L' \rightarrowtail L \twoheadrightarrow L''$ such that $L', L \in \mathcal{D}^a$. Then if we denote by $0 \to L_1 \to L_0 \to L \to 0$ the canonical resolution of $L \in \mathcal{D}$ by L_1, L_0 in $\mathcal{D}^a$, one has

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L_1 \arrow[r, equal] \arrow[d] & L_1 \arrow[d] & & \\
0 \arrow[r] & \tilde{L}' \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If I can show $\tilde{L}' \in \mathcal{D}^a$, then I get an exact functorial resolution of an object of $\mathcal{C}$ by objects $\mathcal{C}^a$. So I need.

<u>Lemma:</u> $0 \to L' \to L \to L'' \to 0$ exact in $\mathcal{D}$, $L'' \in \mathcal{D}^b$, $L \in \mathcal{D}^a$ $\Longrightarrow L' \in \mathcal{D}^a$.

<u>Proof:</u>

$$\begin{tikzcd}
0 \arrow[r] & L'_k \arrow[r, "\alpha"] \arrow[d, "\gamma"] & L_k \arrow[r] \arrow[d, "\beta"] & L''_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'_A \arrow[r] & L_A \arrow[r] & L''_A \arrow[r] & 0
\end{tikzcd}$$

α, β direct injections $\Longrightarrow \gamma$ is.

Clearly L' presents the same module as L.

<u>Next point</u>: Let $\mathcal{C}_n$ be the exact category of n-chains of admissible monos

$$L_0 \rightarrowtail L_1 \rightarrowtail \cdots \rightarrowtail L_n$$

in $\mathcal{D}$ such that the quotients L_i / L_{i-1} are in $\mathcal{D}^b$ for $i = 1, \dots, n$. Let $\mathcal{C}_n^a$ be the subcat of ones with $L_i \in \mathcal{D}^a$ for all $i = 0, \dots, n$. I want to check that $\mathcal{C}_n^a \to \mathcal{C}_n$ is a K-equivalence. So if the exact resolution functor of objects of $\mathcal{C}_n$ by $\mathcal{C}^a$ is

$$0 \longrightarrow \varphi_1(L) \longrightarrow \varphi_0(L) \longrightarrow L \longrightarrow 0$$

then I have an exact sequence

$$\begin{array}{ccccc}
0 & & & & 0 \\
\downarrow & & & & \downarrow \\
\varphi_1(L_n) & = \quad = & \varphi_1(L_n) & = & \varphi_1(L_n) \\
\downarrow & & \downarrow & & \downarrow \\
L_0 \times_{L_n} \varphi_0(L_n) & \rightarrowtail \cdots \rightarrowtail & L_{n-1} \times_{L_n} \varphi_0(L_n) & \rightarrowtail & \varphi_0(L_n) \\
\downarrow & & \downarrow & & \downarrow \\
L_0 & \rightarrowtail \cdots \rightarrowtail & L_{n-1} & \rightarrowtail & L_n \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

and by the lemma since

$$0 \longrightarrow L_j \times_{L_n} \varphi_0(L_n) \longrightarrow \underset{\mathcal{D}^a}{\underset{\cap}{\varphi_0(L_n)}} \longrightarrow \underset{\mathcal{D}^b}{\underset{\cap}{L_n / L_j}} \longrightarrow 0$$

one has $L_j \times_{L_n} \varphi_0(L_n) \in \mathcal{D}^a$.

So now we can take up the proof of the GOAL on page 2. Following Waldhausen's techniques consider the bicategory consisting of objects of $\mathcal{D}^a$ with Q-morphisms vertically and $\mathcal{D}$-admissible monos. with quotients in $\mathcal{D}^b$ horizontally.

Thus a bimorphism is a Q-morphism is the exact category $\mathcal{C}_1$ is a diagram in $\mathcal{D}$ with objects in $\mathcal{D}^a$:

$$\begin{tikzcd} \cdot \arrow[r, hook] & \cdot \\ \cdot \arrow[r, hook] \arrow[u, tail] \arrow[d] & \cdot \arrow[u, tail] \arrow[d] \\ \cdot \arrow[r, hook] & \cdot \end{tikzcd}$$

and horizontally the quotients are in $\mathcal{D}^b$, etc.

Maybe it would be better to form a bisimplicial groupoid which in degree p,q would be a $(p+1,q)$ bifiltered object

$$\begin{tikzcd} \cdot \arrow[r, hook] & \cdot \arrow[r, hook] & \cdot & q \\ \cdot \arrow[r, hook] \arrow[u, tail] & \cdot \arrow[r, hook] \arrow[u, tail] & \cdot \arrow[u, tail] & 1 \\ 0 & 1 & p & \end{tikzcd}$$

with objects in $\mathcal{D}^a$, where the horizontal arrows are $\mathcal{D}$-admissible monos with quotients in $\mathcal{D}^b$, where the vertical arrows are $\mathcal{D}^a$-admissible monos; ~~with quotients~~ finally the squares have to be good so that on passing the vertical quotients, one gets a $\mathcal{D}$-admissible mono with horizontal $\mathcal{D}^b$ quotients.

Denote this $\mathcal{F}_q(\mathcal{C}_p)$. For fixed p, it is the simp. groupoid of admissibly filtered objects in $\mathcal{C}_p$.

Using the heq $\quad \Delta(q \mapsto \mathcal{F}_q(\mathcal{C}_p)) \longrightarrow Q(\mathcal{C}_p)$ and the heq $\quad Q(\mathcal{C}_p) \longrightarrow Q(\mathcal{D}) \times Q(\mathcal{D}^b)^p$

$$L_0 \hookrightarrow \cdots \hookrightarrow L_p \longmapsto (L_0, L_1/L_0, \ldots, L_p/L_{p-1})$$

and Segal-Waldhausen theory one sees that this bisimplicial groupoid gives the relative K-theory of $\mathcal{D}^{\cdot}$ modulo $\mathcal{D}^b$.

The remaining point is to show now that the evident horizontal augmentation $\mathcal{C}_p \longrightarrow \mathcal{F}_R$ leads to a heq

$$\Delta\big(p \mapsto \mathcal{F}_q(\mathcal{C}_p)\big) \longrightarrow \mathcal{F}_q(\mathcal{F}_R)$$

For $q = 1$, this means I want to show that if I make $\mathcal{D}^a$ into a category using the arrows $\hookrightarrow$ ($\mathcal{D}$-admiss. monos. with quotients in $\mathcal{D}^b$), then I get $\mathcal{I}so(\mathcal{F}_R)$.