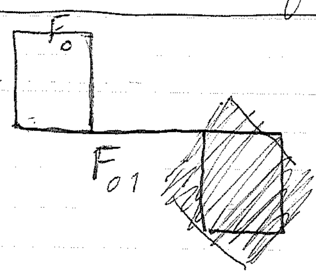
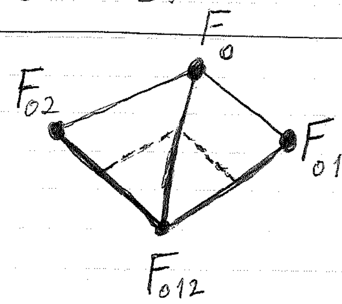
So the basic question is why, if you replace F_σ by $F'_\sigma = \underset{\sigma\le\tau}{\underrightarrow{\text{holim}}}\, F_\tau$ then form the twisting of F'_σ with the covering U_σ, why is this good from the homotopy viewpoint? So because things are local I can assume I have $U_{\sigma_0} = X$, in which case ~~one has maps~~ the space constructed is the same as that constructed from from F'_σ for $\sigma \geq \sigma_0$, hence one has a map to $X \times F'_{\sigma_0}$ which I want to show is an equivalence.

Example: 



So I form

$$\bigcup_{\sigma_0\le\sigma} X_\sigma \times \bigcup_{\sigma\le\tau_0<\cdots<\tau_k} \Delta(k)^\circ \times F_{\tau_k}$$

Still I didn't get an example

$\tilde{Q}$
$\downarrow$
Q

K
$\downarrow$
E
$\downarrow$
P

make M act on the fibre + divide.

Thus I want a map over P to be

$$\begin{array}{ccc} E & \simeq & E' \oplus \mathbb{Z} \\ \downarrow & \swarrow & \\ P & & \end{array}$$

So the category is going to be extensions of E with direct action of P. To prove contractible.

Take then finite subcat

$\mathcal{U}$ $\quad \mathfrak{U}_{dir}$

$\uparrow$

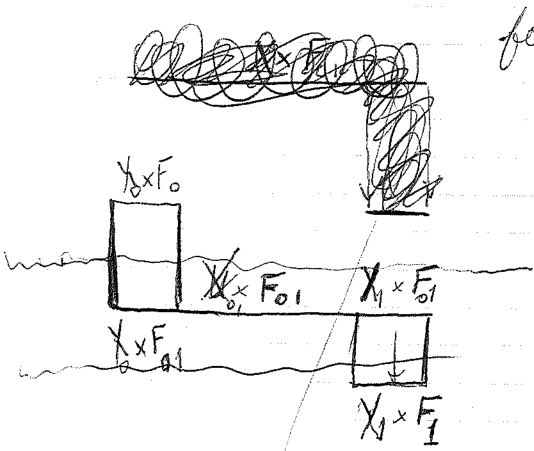
$\mathcal{L}$

$\uparrow$

E
$\downarrow$
P quad.

Two open sets $X = U_0 \cup U_1$

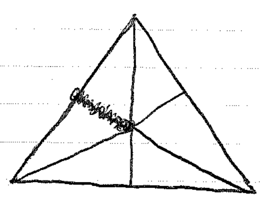
$F_0 \longleftarrow F_{01} \longrightarrow F_1$



form over X the space

$$\underbrace{(X-U_1)\times F_0}_{Y_0} \cup (U_0 \cap U_1)\times F_{01} \cup \underbrace{(X-U_0)\times F_1}_{Y_1}$$

So perhaps what I do is to form over K $\underrightarrow{\mathrm{holim}}\, F_\sigma$ which is $\bigcup \sigma \times F_\sigma$. Then over $X \times \left(\bigcup \sigma \times F_\sigma\right)$ I will be interested in a certain subspace to be described. So first I have to describe what should appear over C_σ



$C_\sigma =$ that part of the bary. subd. cons. of chains $\sigma_0 < \sigma_1 < \cdots < \sigma_q$ with $\sigma \leq \sigma_0$.

Simplex $\sigma = v_0, \ldots, v_q$ Bary. subd.

$\sum t_i v_i$

So arrange the t_i in order and you get a ~~[illegible]~~

$t_1 = \cdots$.

$$SP(T) \xrightarrow{f} SP(T/A).$$

$$\{t_1, t_2, \ldots, t_n, *, *, *, \ldots\}$$

$$SP(T/A) = \coprod_{k=0}^{\infty} SP^k(T-A)$$

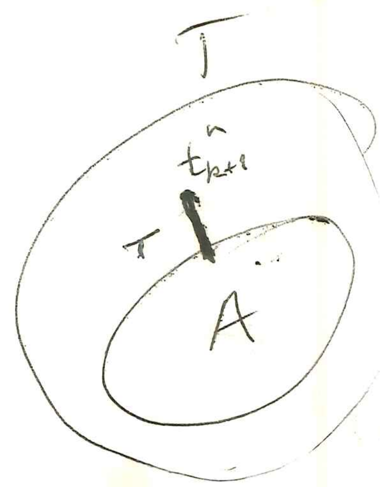
276
~~138~~
138
26
112

340

$$F_k\, SP(T/A) = \text{those } \{t_1, t_2, \ldots\} \text{ having } t_{k+1} = t_{k+2} = \cdots *$$

$$* = F_0 \subset F_1 \subset F_2 \subset \cdots\cdots$$

$$F_k - F_{k-1} = SP^k(T-A).$$

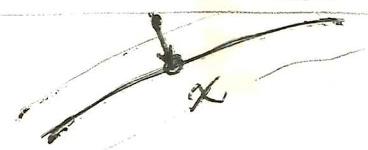


if

$$x = \{t_1, \ldots, t_k, *, *, \ldots\} \in SP^k(T-A)$$

$$f^{-1}(x) = \{t_1, \ldots, t_k, \underbrace{\qquad\qquad}_{SP^\infty(A)}\}$$

$$\cong SP^\infty(A)$$



$$x = \{t_1, \ldots, t_k\} \in SP^k(T-A)$$

$$x^n = \{t^n_1, \ldots, t^n_k, \underline{t^n_{k+1}, \ldots, t^n_\ell}, *, *, \} \in SP^\ell(T-A)$$

$$f^{-1}(x^n) = \{t_1, \ldots, t_k, t^n_{k+1}, \ldots, t^n_\ell\} \times SP^\infty(A)$$

$$\{a_{k+1}, \ldots, a_\ell\}$$

$$f^{-1}(x) = \{t_1, \ldots, t_k\} \times SP^\infty(A)$$

Specializing from $SP^{l}(T-A)$ to $SP^{k}(T-A)$ multiplies the fibre by elements of $SP^{l-k}(A)$

So if I were to given the normal tube around $SP^{k}(T-A)$

for an element of $K(M \times T)$ which comes from a family of unitary flat bundles, one has the analytical side. Want to know for the universal element of

$$K(B\Gamma \wedge D(B\Gamma))$$

For given any coh. class of $B\Gamma$ it appears in this family

---

Without coh: For any $\xi \in K(B\Gamma)$, one knows

$$\mathrm{sgn}(M, f^*\xi) = \int_M L(\tau_M) \cdot ch(f^*\xi)$$

is a homotopy invariant.

index of signat. op twisted by $f^*\xi$

where ξ is flat this is clear, because one has a specific global operator. Presumably for elements of $K(B\Gamma)$ admitting a more general presentation one wins. quasi-flat

---

$$\xi \in K(B\Gamma \wedge T) = H^*(B\Gamma) \otimes H^*(T)$$
$$= \mathrm{Hom}(H_*(T), H^*(B\Gamma))$$

universal element $T = D(B\Gamma)$

$$X \wedge DX \longrightarrow S \qquad S \longrightarrow X \wedge DX$$

$$X \wedge T \longrightarrow K$$

$$T \longrightarrow \mathrm{Hom}(X, K) = DX \wedge K$$

Therefore it would appear that fixed, among families $\xi \in K(X \wedge T)$, there is a universal one namely with $T = DX \wedge K$

$$K_*(A[G]) \longrightarrow h(BG, \underline{\underline{K}}_A)$$

Because maybe I can define

$$R(X, A[G]) \longrightarrow [X \times BG, \underline{\underline{K}}_A]$$

$$\downarrow$$

(formal ~~structure~~:) Suppose one has a family E of unitary rep of Γ indexed by T. Then using the ~~Riemannian structure~~ cochains of M, $C(M, E)$, with its cup product we get

$$\boxed{\mathrm{sign}(M, E) \in K(T)}$$

{evidently a homotopy invariant.}

Want a [signature formula] which would say

$$ch[\mathrm{sgn}(M,E)] \in H^*(T)$$
$$\| $$
$$\int_M L(\tau_M)\, \underbrace{ch(E)}$$
$$H^*(M), \; H^*(M \times T)$$

defined analytically using the flat structure of E.

~~Next point: Take sign(M, E) ∈ K(T)~~

$E \in K(M \times T)$

and on M you have signature operator which gives you (over $\frac{1}{2}$) an integration map $K(M\times T) \to K(T)$.
$\downarrow$

$$\begin{array}{ccc} K(M \times T) & \longleftarrow & K(B\Gamma \times T) \\ \downarrow & & \downarrow \\ K(\pi) & \longleftarrow & K(T) \end{array}$$

$K(M \times T)$
$f_* \downarrow$ — given by the signature orientation of $K[\frac{1}{2}]$
$K(T)$ — better given by the signature diff op on M Hirz. operator fibrewise

$\ni \quad ch(f_* \alpha) = f_*(L(\tau_M)\, ch\, \alpha)$

But it is also analytical.

Novikov signature problem.

M manifold, $\pi_1 M = \Gamma$

$u \in H^*(\Gamma)$, $f: M \to B\Gamma$

$$\int_M L(\tau_M) f^* u \quad \text{is a homotopy inv. of } M?$$

Lusztig method: Let E_t be a family of unitary reps of Γ parameterized by T. Form over T the complex $C^\cdot(M, E_t)$ of chains on M with coeff. in E_t. Then this is a hermitian complex of vector bundles over T, hence it det. a sign. which belongs to $K(T)$. Then one has to show that this element ~~depends only~~ determines the signatures for certain u

$$\{E_t\} \in K(B\Gamma \times T)$$

$$ch(\{E_t\}) \in H^*(B\Gamma \times T) = \mathrm{Hom}(H_*(T), H^*(B\Gamma))$$

~~Signature K(BΓ×T)~~ want $H_*(B\Gamma)$ $\quad f_*[L(\tau_M)]$

So one wants the dual of $B\Gamma = X$

$$X \wedge X^\vee \longrightarrow S$$

Idea: Given E_t ~~those~~ unitary bundles one can pull back f^*E_t family over T, form

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ M \times T & \xrightarrow{f} & B\Gamma \times T \\ \downarrow & & \\ T & & \end{array}$$

$$\int_M L(\tau_M)\, ch(f^*E_t)$$

which is a cohomology class on T, which one has hopes of interpreting in terms of the signature of operators

$$\bigcup \sigma \times F_\sigma$$

subdivide σ by considering chains which end with σ. Thus

$$\sigma = \bigcup_{\tau_0 < \cdots < \tau_k \leq \sigma} \Delta(k)$$

$$\bigcup_{\tau_0 < \cdots < \tau_k} \Delta(k)^\circ \times F_{\tau_k} \qquad \bigcup \sigma \times F_\sigma$$

and the part I am interested in what sits over

$$C_\sigma = \bigcup_{\sigma \leq \tau_0 < \cdots < \tau_k} \Delta(k)^\circ$$

which is simply

$$\bigcup_{\sigma \leq \tau_0 < \cdots < \tau_k} \Delta(k)^\circ \times F_{\tau_k} = \underset{\sigma \backslash J}{\underrightarrow{\text{holim}}}\ F$$

Therefore if I put

$$X_\sigma = U_\sigma - \bigcup U_\tau$$

the space I am interested in is

$$\bigcup X_\sigma \times \bigcup_{\sigma \leq \tau_0 < \cdots < \tau_k} \Delta(k)^\circ \times F_{\tau_k}$$ and this doesn't make very much sense.

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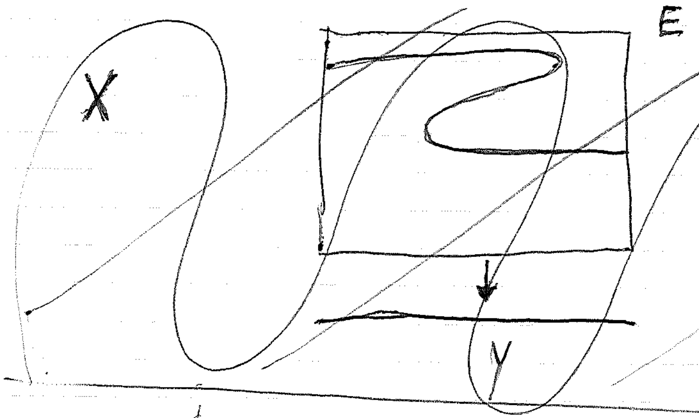


$X \overset{i}{\hookrightarrow} Y$

$$i^* c(\tau_Y) = c(\tau_X) \cdot c(\nu_i)$$

$$i_* c(\tau_X)$$

E vector bundle over Y, $X =$ zeroes of s trans. to 0.

$$\nu_i = i^* E$$

$$c(\tau_X) = \frac{i^* c(\tau_Y)}{i^* c(E)} = i^*\left(\frac{c(\tau_Y)}{c(E)}\right)$$

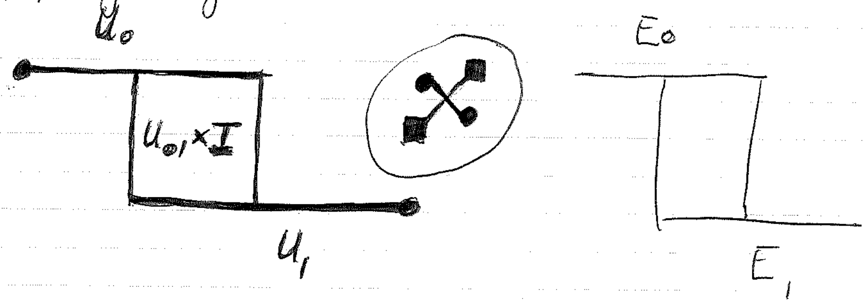
$$i_* c(\tau_X) = i_* 1 \cdot \frac{c(\tau_Y)}{c(E)}$$

$$= c_d(E) \frac{c(\tau_Y)}{c(E)}$$

So assume that $E_{01} \longrightarrow U_{01} \times_{U_0} E_0$ is an ~~homotopy~~ equivalence, in fact a universal homotopy equivalence

---

Start again: $E \xrightarrow{f} X = U_0 \cup U_1$ such that over $U_0, U_1, U_0 \cap U_1$ the actual and homot. fibres are equivalent. Show true for E itself. Replace $E_0 \to U_0$, $E_1 \to U_1$, $E_{01} \to U_{01}$ by fibrations and form the double mapping cylinder



Then I have arranged that $E_0 \to U_0$ is good for homotopy pull-back. In addition it should be ~~clear~~ that

$$\begin{array}{ccc} E_{01} & \longrightarrow & E_0 \\ \downarrow & & \downarrow \\ U_{01} & \longrightarrow & U_0 \end{array}$$

is homotopy-cartesian

One might better start with the two h-cartesian squares

$$\begin{array}{ccccc} E_0 & \longleftarrow & E_{01} & \longrightarrow & E_1 \\ \downarrow & & \downarrow & & \downarrow \\ U_0 & \longleftarrow & U_{01} & \longrightarrow & U_1 \end{array}$$

and form the double mapping cylinders. Now you want to show that the double map cylinders are OKAY

$f : X \longrightarrow Y$ map of cx. manifolds

Suppose f is an embedding. Then one has

$$0 \longrightarrow \tau_X \longrightarrow f^*\tau_Y \longrightarrow \nu_f \longrightarrow 0$$

so

$$f^* c_t(\tau_Y) = c_t(\tau_X)\, c_t(\nu_f)$$

$f : X \to Y$ proper smooth map.

$$0 \longrightarrow \tau_f \longrightarrow \tau_X \longrightarrow f^*\tau_Y \longrightarrow 0$$

$$c_t(\tau_X) = c_t(\tau_f)\, f^* c_t(\tau_Y)$$

$$f_* c_t(\tau_X) = f_* c_t(\tau_f) \cdot c_t(\tau_Y)$$

But $\quad c_t(\tau_f) = 1 + t c_1(\tau_f) + \cdots + t^d c_d(\tau_f)$

$$f_* c_t(\tau_f) = t^d \underbrace{f_* c_d(\tau_f)}_{\chi(\text{fibre})}.$$

[illegible]

$$\nu_i = i^* E$$

$$0 \to \tau_X \to i^*(\tau_Z) \longrightarrow i^*(E) \to 0$$

$$c(\tau_X) = i^*\left(\frac{c(\tau_Z)}{c(E)}\right)$$

$$i_*\left(c(\tau_X)\right) = i_* 1 \cdot \frac{c(\tau_Z)}{c(E)}$$

$$f_*\left(c(\tau_X)\right) = p_*\left(i_* 1 \cdot \frac{c(\tau_Z)}{c(E)}\right)$$

situation to understand very well: Let K be a finite simplicial complex, and $\sigma \mapsto F_\sigma$ a contravariant functor to spaces. Then I can form the thickening

$$F'_\sigma = \underset{\sigma \leq \tau}{\underrightarrow{holim}}\ F_\tau \simeq \bigcup_{\sigma \leq \sigma_0 < \cdots < \sigma_q} \Delta(q)^\circ \times F_{\sigma_q}$$

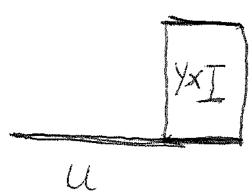
Example: Remember the case $U \subset X$, $E \to F$, $Y = X - U$

and you form the space

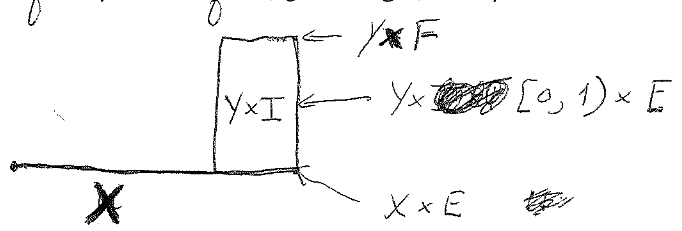
$$U \times E \cup Y \times F \simeq X \times E \underset{Y \times E}{\cup} Y \times F$$

but thickened you ~~[illegible]~~. replace F by the map cyl.

$$X \times E \underset{Y \times E}{\cup} Y \times ([0,1] \cup F) \quad (E \times)$$



Thus what I need to make things works is a nbd. V of Y of which Y is a strong deformation retract.



And over this one has the fibres as I have indicated. Suppose now I try to explain pull-backs with respect to a map $X \to X'$ ~~Suppose instead I~~

$U \subset X \qquad E \longrightarrow F$

I then form the space $U \times E \cup Y \times F$ topologized coarsely. Now I want to understand when the functor

$$(E \to F) \longmapsto (U \times E) \cup (Y \times F)$$

preserves equivalences. Sufficient condition is that around Y I find a nbd V of which Y is a s.d.r. Can always arrange this by replacing $U \subset X$ by

$$Y \times I \cup X \supset Y \times (0,1] \cup X.$$

~~Bec~~ Condition is sufficient because I can use Mayer-Vietoris. The point is that T rest. to V is heq to $Y \times F$ because of the deformation.

Suppose then that one has ~~[illegible]~~ $X = U_0 \cup U_1$ a torsor for the 1-simplex. Then one has the strata

$$X_0 = U_0 - U_{01}$$
$$X_{01} = U_{01}$$
$$X_1 = U_1 - U_{01}$$

and one wants to be able to deform a tub. nbd of a stratum down to that stratum

---

$0 < 1 < 2$

$X = U_0 \supset U_1 \supset U_2$

$F_0 \leftrightarrows F_1 \leftrightarrows F_2$

$X_0 = U_0 - U_1$

$X_1 = U_1 - U_2$

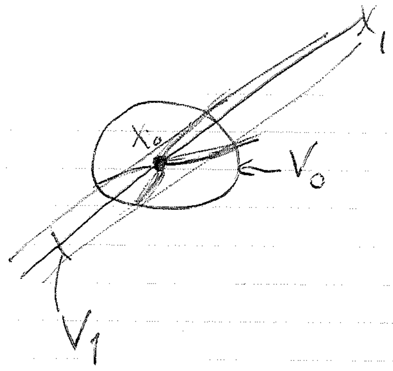
$X_2 = U_2$

three strata

~~[illegible]~~ have T over X

$$T|X_i = X_i \times F_i$$

So it seems I might want to thicken X_i to a tubular nbd. V_i and look at the associated covering V_0, V_1, V_2.

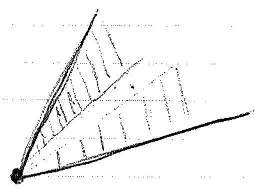


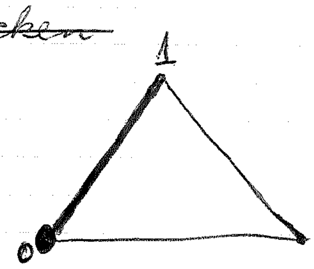
So then what would we have?

$T|V_0 \sim X_0 \times F_0$

$T|V_0 \cap V_1$

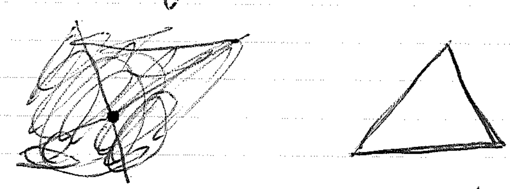
~~Might want to thicken~~





So therefore the first case to understand is that of $\underrightarrow{\text{holim}}$ itself. Thus over a simplicial complex K we form the space $T = \bigcup \sigma \times F_\sigma$ and we want to know why equiv. are preserved by this construction.

inductive proof.— consider adding one simplex σ to K



covering proof.— have $T|U_\sigma$ deforms to $\sigma \times F_\sigma$ and now you have to prove separately that if one has heq's "over" a numerable covering then one wins.

equivalence.

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \downarrow & \swarrow & \\ X & & \end{array}$$

$f_U, f_{U\cap V}, f_V$ heq's $\overset{?}{\Longrightarrow}$ f is?

$g_U|U\cap V$

$g_V|U\cap V$ both h-inv. for $f_{U\cap V}$ hence homotopic, i.e. one gets h. Then choose $\lambda : X \to [0,1]$

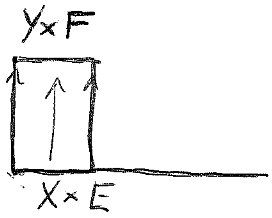
$$x \longmapsto \begin{cases} g_U(x) & \text{if } \lambda(x)=0 \\ h(x,\lambda(x)) & \text{if } 0 \le \lambda(x) \le 1 \end{cases} \qquad x \in U_\alpha \cap V_\beta$$

Still I don't understand the mechanism. I have the map $F'_\tau \longrightarrow F'_\sigma$ for $\sigma \le \tau$ and I want to show that because this is an equivalence for each τ, then the glueing should be an equivalence. ~~So~~ consider this part first. Assume I know that I have a ~~map of equivalences~~ map

---

So the problem is to show that if $F_\sigma \to G_\sigma$ is an equivalence, so is the twisting of $F'_\sigma \to G'_\sigma$

So suppose one tries to understand the $U \subset X$ case.

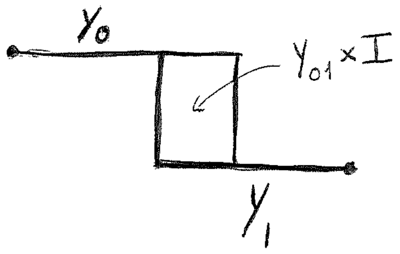


$U \times E \cup Y \times F$

and when you thicken it you replace F by the mapp. cyl. of $E \longrightarrow F$. The point then is that one ~~replaces the~~ makes the stratum where E changes to F have a nice collar

$$E_0 \longleftarrow E_{01} \longrightarrow E_1$$

so it is probably desirable to replace U_i by a shrinking



But now it is clear how to take pull-backs over X at least. ~~And the next thing is the following~~

$$f : E \longrightarrow X = U_0 \cup U_1 .$$ ~~Not further clear!~~

I can form a space

$$(E_0 | U_0 - U_{01}) \cup E_{01} \cup (E_1 | U_1 - U_{01})$$

which would be OK as we have $E_0 \longleftarrow E_{01} \longrightarrow E_1$. ~~And it seems clear that one has a map~~ This sits over $X = (U_0 - U_{01}) \cup U_{01} \cup (U_1 - U_{01})$ and seems to behave nicely for pull-backs. To check right homotopy property, one ~~takes the following~~ can suppose $U_0 = X$ in which case one wants to know that $(E_0 | U_0 - U_{01}) \cup E_{01} \longrightarrow E_0$ is an equivalence, which should be clear as $E_0 | U_{01} \longleftarrow E_{01}$ is ~~a~~ universal equivalence

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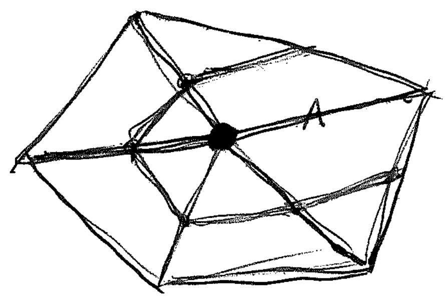
So thus it is clear that to describe the system of fibres of the map $SP^n(T) \to SP^n(T/A)$ I must use the fact that the normal directions at * in T/A determine points of A.



$$\begin{pmatrix} 1_r & * \\ & GL_n \end{pmatrix} \underset{s}{\overset{p}{\rightleftarrows}} GL_n$$

$ps = id.$

$$H_*\begin{pmatrix} 1_r & * \\ & GL \end{pmatrix} \underset{s_*}{\overset{p_*}{\rightleftarrows}} H_*(GL)$$

$$\begin{pmatrix} 1_r & * \\ & GL_p \end{pmatrix} \times \begin{pmatrix} 1_r & * \\ & GL_q \end{pmatrix} \xrightarrow{\perp} \begin{pmatrix} 1_r & * \\ & GL_{p+q} \end{pmatrix}$$

induces a ring structure on

$$(\perp)_* : \quad H_*\begin{pmatrix} 1_r & * \\ & GL \end{pmatrix} \otimes H_*\begin{pmatrix} 1_r & * \\ & GL \end{pmatrix} \longrightarrow H_*\begin{pmatrix} 1_r & * \\ & GL \end{pmatrix}$$

if field coefficients $\Rightarrow$ $H_*\begin{pmatrix} 1 & * \\ & GL \end{pmatrix}$ Hopf alg.

so now identity gives

$$\begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \perp \begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} = \begin{pmatrix} 1 & u & u \\ & \alpha & \\ & & \alpha \end{pmatrix} \sim \begin{pmatrix} 1 & u & \\ & \alpha & \\ & & \alpha \end{pmatrix}$$

$$\begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \perp sp\begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix}$$

Implies for $R = H_*\begin{pmatrix} 1 & * \\ & GL \end{pmatrix}$ that

$$R \xrightarrow{\Delta_*} R \otimes R \underset{id \otimes s_* p_*}{\overset{id \otimes id}{\rightrightarrows}} R \otimes R \xrightarrow{\perp_*} R$$

are the same. Done by induction. Suppose then we have $s_* p_*(x) = x$ if $\deg(x) < n$. Then for $x \in H_n^*\begin{pmatrix} 1 & * \\ & GL \end{pmatrix}$

$$\Delta(x) = 1 \otimes x + \sum_{\deg(x_i'') < n} x_i' \otimes x_i''$$

$$\perp_* \Delta_*(x) = x + \sum x_i' x_i''$$

$$\perp_* s_* p_* \Delta_*(x) = s_* p_*(x) + \sum x_i' \, s_* p_*(x_i'')$$

~~k field of char~~ $k = \mathbb{F}_q \quad q = p^d$

Thm: $H_*^i(GL(k), \mathbb{F}_p) = 0 \quad i > 0.$

Suffices to show

$$U_n = \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \\ & & & 1 \end{pmatrix} \hookrightarrow GL_n(k) \hookrightarrow GL(k)$$

induces zero map on homology. Use induction on n

$$H_*\begin{pmatrix}1_r & 0\\ 0 & GL\end{pmatrix} \xrightarrow{\sim} H_*\begin{pmatrix}1_r & *\\ & GL\end{pmatrix}$$

Denote by
$$\begin{pmatrix}1_r & * \\ & GL\end{pmatrix} \overset{i}{\underset{j}{\rightrightarrows}} GL$$
$$i\begin{pmatrix}1 & u \\ & \alpha\end{pmatrix} = \begin{pmatrix}1 & u \\ & \alpha\end{pmatrix}$$
$$j\begin{pmatrix} & '' & \end{pmatrix} = \begin{pmatrix}1 & 0 \\ & \alpha\end{pmatrix}$$

Then

<u>Cor</u>: $i_* = j_* : H_*\begin{pmatrix}1_r & * \\ & GL\end{pmatrix} \longrightarrow H_*(GL)$

$$\begin{array}{ccc} U_n = \begin{pmatrix}1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1\end{pmatrix} & \subset & GL_n(k) \\ \downarrow & & \Big\downarrow \text{inc.} \\ \begin{pmatrix}1 & * \\ & GL\end{pmatrix} & \overset{i}{\hookrightarrow} & GL \end{array}$$

The problem: I want nice formulas to describe a $M(T, t_0)$ - bundle over a space Y. My example is that of the symmetric product. Thus $SP_n(T)$ is stratified

$$SP_0 \subset SP_1 \subset SP_2 \subset \cdots\cdots$$

so it maps to the space

$$0 < 1 < 2 < \cdots\cdots$$

where the sets $\leq p$ are closed.

Now to finish things I only have to understand how to put a tube around

---

X collapse A to a point: can do by attaching a cone on A. ~~So therefore, so~~

$$Y \xrightarrow{f} X/A$$

$$f^{-1}(X/A - pt) = U$$

$$f^{-1}(\text{Cone}) = V$$

on $U \cap V$ I have a map to A and a map to X and a homotopy

---

on U I have a map to X

on V I have a map to a point

on $U \cap V$ I have a map to A

$Y = U_0 \cup U_1$ open sets say

$$\begin{array}{ccccc} U_0 & \longleftarrow & U_{01} & \longrightarrow & U_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \end{array}$$

given functor

then I can form

---

want to form a class. space

---

what is it I need at this point? ~~I need a sheaf of surf~~ I want to describe a map into $Y \to SP_2(T)$. Now I have

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ SP_1(T) & \subset & SP_2(T) \end{array}$$

Idea is that there should be a nbd of $SP_1(T)$ in $SP_2(T)$ which deforms down to $SP_1(T)$. So $SP_2(T)$ consists of set $\{t_1, t_2\}$. And $SP_1(T)$ is the subspace where one of the t_i is t_0. So a nbd. would consist of pairs t_1, t_2 where one is inside of a nbd U of t_0.

Memory: Suppose given $X_0 \leftarrow X_{01} \rightarrow X_1$, then one forms

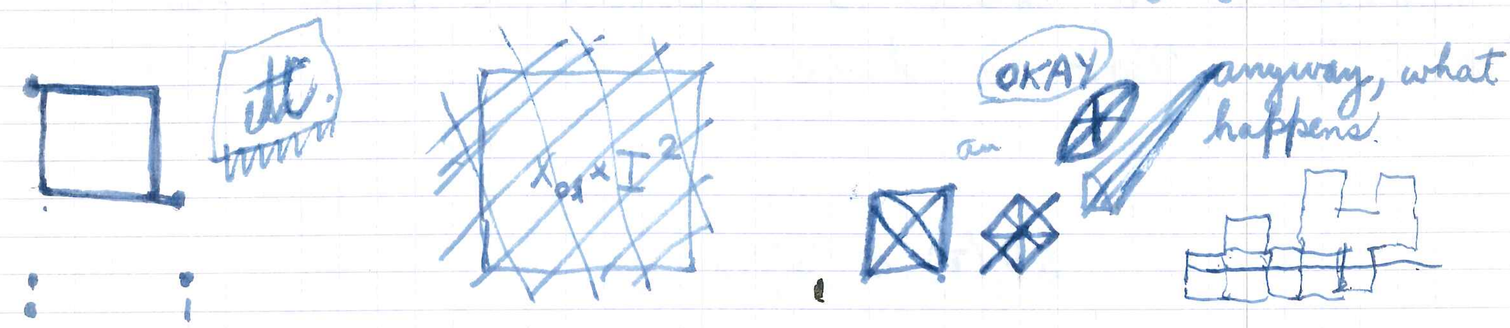
$$X_0 \cup_{X_{01}\times 0} X_{01}\times I \cup_{X_{01}\times 1} X_1.$$

double mapping cylinder. Regard as over and pull back to

$$\begin{array}{ccc} & \longrightarrow & Cyl(X_0 \leftarrow X_{01} \rightarrow X_1) \\ \downarrow & & \downarrow \\ [0,1] & \longrightarrow & \{0 > (01) < 1\} \end{array}$$

OKAY.

NO. The space which I recall as being good is.



map $Z \to [0,1]$. gives us the following

$$X_0 \times [0,1) \cup X_{01}\times (0,1)\times[0,1] \cup X_1 \times (0,1].$$

$Z \to 0$, $Z \to 1$.

$$U_0\times X_0 \cup U_{01}\times X_{01}\times[0,1] \cup U_1\times X_1$$

i.e. one has what

$$\begin{array}{c} Cyl\{U_0\times X_0 \leftarrow U_{01}\times X_{01} \rightarrow U_1 \times X_1\}. \\ \downarrow \\ Cyl\{U_0 \leftarrow U_{01} \rightarrow U_1\} \\ \downarrow \\ Y \end{array}$$

Formula for $D(V;T)$

If S is a finite set, we have

$$D(V;S) = \coprod_{\substack{f:S\to\mathbb{N} \\ \Sigma f(s)=n}} D_f(V) \qquad n = \dim V$$

where $D_f(V)$ is the space of flags of type $f(s_1), \dots, f(s_k)$ $\quad S = \{s_1, \dots, s_k\}$ $\quad s_i$ dist.

Then $D(V;S)$ is a covariant functor of S. If T is a space, then T^S is a contrav. functor of the finite set S, hence we can form the contraction: $D(V;\cdot) \underset{\times}{\otimes}{}_{\Gamma'} T^{\cdot}$. which set-theoretically is

$$\varinjlim_{S/T} D(V;S) \qquad \Gamma' = \text{finite sets}$$

the limit being taken over the cat of finite sets $/T$. One has a pairing

$$D(V;S) \times T^S \longrightarrow D(V;T)$$

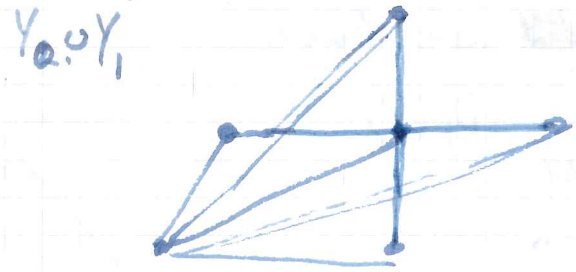
compatible with maps in S, hence we get a map

$$\boxed{D(V;\cdot) \underset{\times}{\otimes}{}_{\Gamma'} T^{\cdot} \longrightarrow D(V;T)}$$

which is clearly bijective, since set-theoretically $D(V;T) = \bigcup_{S\subset T} D(V,S)$, where S runs over finite subsets of T. On the other hand, both sides are compact when T is. Thus one has

$$D(V;\cdot) \times_{\Gamma'} T^{\cdot} = D(V;T)$$

which are forms the cone.



So how to understand iterated cones:

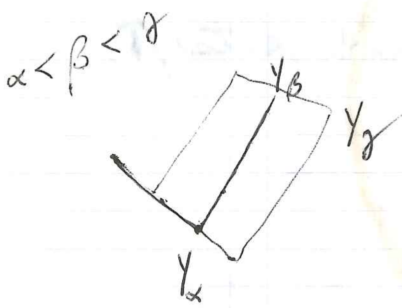
~~$Y_0 \cup Y_1 \cup Y_2 \cup \ldots$~~

~~Stratifica~~ $Y_0 \cup Y_1 = Y_0 \cup_{\partial \bar{Y}_1} \bar{Y}_1$

$= Y_0 \cup_{\partial \bar{Y}_1} \partial \bar{Y}_1 \times I \cup_{\partial \bar{Y}_1} \bar{Y}_1$

take cone on this

$pt. \cup I \times (\quad)$
$(\quad)$



U_α nbd of Y_α

In add. to Y_α you give $Y_{\alpha\beta}$

Guess. Take case of a simp. complex K where the strata are open simplices. Now let ~~us~~ us be given a map $Y \xrightarrow{f} K$

Karoubi tells me given D fredholm I consider the path
$$(\cos\theta)\, j + (\sin\theta)\Delta j$$
$$j\Delta = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$$

which is self-adjoint. Spectrum of j is ± 1 and of

$$(\Delta j) = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}$$

$$(\cos\theta) + (\sin\theta)\begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix} \qquad 0 \leq \theta \leq \pi$$

joins I to $-I$. Thus operators which are invertible except at $\theta = \pi/2$. Yes.

what happens to 0 eigenspace at $\theta = \pi/2$. Seems that $\begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}$ ~~calculates more simply.~~

Thus at $\theta = \pi/2$ one has $\begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}$ which has kernel $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$. Now as one runs ~~a tangent vector?~~ the tangent vector along this path one gets I

~~space Seems that what hap~~

consider projectors E such that $E - E_0$ compact.

Thus one handles things as follows.

wanted to consider A self-adjoint with ess. spectrum $\{0, +1\}$ such that $A - E_0$ is compact. Then $\exp(2\pi i A)$ is a unitary $\equiv 1$ mod compacts.

Now the point is to look at the fibre over I, i.e. at all $E \ni E - E_0$ is compact

The problem is to ~~define and~~ understand

$K(Y;T,t_0)$.

It would seem this is represented by $M(T;t_0)$ the classifying space of a top. category. Define $M(T;A)$ to be the top cat. of T-spaces in which the maps are unitary embeddings with complement having support in A. Can you describe now the way one might think of bundles for this top. cat?

~~[illegible]~~ <u>Idea</u>: Y should be stratified with $(T-t_0)$-bundles over each strata. $Y = \coprod_{n\geq 0} Y_n$

where ~~[illegible]~~ over Y_n we have a $(T-t_0)$-bundle E_n of rank n. Take a sequence of points y_α in Y_n such that ~~[illegible]~~ part of the support of E_n over y_α approaches t_0. ~~Then we want $E_n(y_\alpha) \to E_n$~~. Say that as $y_\alpha \to y_0$, a mult. k-piece of $E_n(y_\alpha)$ goes toward the basepoint. Then I want $E_n(y_\alpha)$ to converge to $E_{n-k}(y_0)$.

~~[illegible]~~ Seems then that Y_0, $Y_0 \cup Y_1$, $Y_0 \cup Y_1 \cup Y_2$, are closed.

Thus it would appear that Y_0 is closed, Y_1 is attached to Y_0, Y_2 attached ~~to~~ $Y_0 \cup Y_1$, etc.

~~I need this to see what is going on~~ How do I attach Y_1 to Y_0. You give the link of Y_0 in Y_1 call it Y_{01} and you form a ~~mapping cone~~. pushout

$$\begin{array}{ccc} Y_{01} & \longrightarrow & Y_1 \\ \downarrow & & \\ Y_0 & & \end{array}$$

can think of Y_{01} as a tubular nbd.

The point is that we can always form quotient spaces. Thus I can form the ~~appropriate s~~ monoid $M(T-pt)$ and ~~stratify~~ topologize it as a quotient of $M(T)$

$$M(T) = \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n; T) = M(\cdot) \times^{\Gamma'} T^{(\cdot)}$$

and $M(T; t_0) = M(T - t_0)$ topologized so as to be a quotient of $M(T)$.

$$M(\cdot) \times^{\Gamma} (T, t_0)^{(\cdot)}$$

where $M(S, s_0)$ for a pointed set is what?

T pointed space. Any idea of how to define (T, t_0) structures. We ~~need someway of~~ start with T-structures on V, $D(V; T)$ and stratify this according to the ~~dimension away from~~ mult of basept.

$$pt = D(V; T)_n \subset D(V; T)_{n-1} \subset \cdots\cdots \subset D(V; T)_0$$

~~Then~~

$$D(V; T)_p = \{\theta \mid t_0 \text{ has mult.} \geq p\}.$$

$\theta : \mathbb{C}^T \to \mathrm{End}(V)$

~~this~~ $V_{t_0} = \{v \mid \theta(f) v = f(t_0) v\}$ has dim $\geq p$.

Then $D(V; T)_p - D(V; T)_{p+1}$ fibres over $G_p(V)$ with fibre $D(V/W; T)$ over W.

$M(T, t_0)$ is stratified. $M(T - t_0) = \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n; T - t_0)$ groupoid

Thus the n.th stratum is a vector bundle of rank n with $T - t_0$ splitting. How to attach n onto $< n$ is becoming clear.

-quiring of operators compact. ~~It goes like~~

maybe there is some version mod n without the unit interval.

$C(A) \oplus C(A) \simeq C(A)$.

$C(A) \to S(A)$

A

A_I $\qquad$ A_{S^1}.

A

If I had a true functor I understood, then I should get a cosimplicial ring

$$A_{\Delta(0)} \rightrightarrows A_{\Delta(1)} \rightrightarrows A_{\Delta(2)} \rightrightarrows A_{\Delta(3)}$$

And if X is a simplicial set, then taking

$$X^{\cdot} \otimes A_{\Delta(\cdot)} = \varinjlim_{\Delta(\cdot) \to X} A_{\Delta(\cdot)}$$

would give me A_X. Suppose one wins these these

$A \qquad M_n(A)$

$$A \quad \mathcal{P}_A \underset{\text{rest.}}{\overset{\text{base.}}{\rightleftarrows}} \mathcal{P}_{M_n(A)} \qquad A^n$$

~~$Sp^k(\ldots)$~~

$$|p \longmapsto Sp^k(K_p)|$$

$$= |p \longmapsto (K_p{}^k)/\Sigma_k|$$

$$= |p \longmapsto K_p{}^k| / \Sigma_k \qquad \text{two left adj.}$$

$$= |K|^k/\Sigma_k \qquad \text{comm. prod.}$$

$$= Sp^k(|K|).$$

---

when you decompose V, someone gives you what sort of structure?

$$D(V;T) \longrightarrow T^k/\Sigma_k \qquad \dim V = k$$

$V = \oplus V_t \qquad d(V) = \sum a_t t \qquad a_t = \dim(V_t)$

$\alpha_1 =$ no $a_t = 1$

$\alpha_2 =$ no $a_t = 2$

$\vdots$

$$d(V) = \sum_{a_t=1} t + 2\sum_{a_t=2} t + \cdots$$

and ~~suppose~~ the decompositions corresp to $V = \oplus V_t$

$$D_{\underbrace{1\cdots1}_{\alpha_1}\underbrace{2\cdots2}_{\alpha_2}\cdots}(V) = D_\alpha(V)$$

and then one has decomp. $T^{|\alpha|} \qquad |\alpha| = \sum \alpha_i$

$$(T^{\alpha_1+\alpha_2+\cdots} - \text{diag}) \times^{\Sigma_\alpha} (D_\alpha(V)).$$

T space, V unitary vector space, $D(V;T)$ = space of T-decompositions of V.

unitary vector bundle of Y with T-decomposition

$\mathrm{Vect}(Y;T)$ = homotopy classes of T-bundles over Y. $= [Y, \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T)]$

$K(Y;T)$ assoc. abelian group.

If t_0 is a basepoint of T + T is conn. one has

$$\mathrm{Vect}(pt;pt) \longrightarrow \mathrm{Vect}(Y;T)$$
$$\overset{\shortparallel}{\mathbb{N}}$$

claim cofinal. In effect ~~Vect(pt Vect Vect~~ Vect(pt;T): $\mathrm{Vect}(pt;T) = \mathbb{N}$, since any T-dec. of V is supp. on a finite subset of T which contracts to the basepoint. Thus $\mathrm{Vect}(pt;pt)^{-1}\mathrm{Vect}(Y;T) = F(Y;T)$ is rep. monoid functor of Y which is a group for $Y = pt$. $\therefore$ ....

Things to understand – relation of

$$\coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T)$$

to the Anderson-Segal space. Suppose $T = |K|$ where K is a simplicial ~~set~~ One can form the simplicial space

$$p \longmapsto D(V;K_p)$$

and

Question: $D(V, |p \mapsto K_p|) = |p \mapsto D(V, K_p)|$?

Example: Instead ~~use~~ of $D(V;-)$ use $SP^k(-) = X^k/\Sigma_k$. Then one sees that

I need a formula for $D(V;T)$:

If $\dim(V) = n$ we want to have a space with strata indexed by partitions $\alpha \vdash n$.

$$n = \alpha_1 + 2\alpha_2 + \cdots$$

To this strata one will have attached the space $T^{\alpha_1 + \cdots}$ − big diag., and the specialization maps will allow one to go to another partition which is coarser, i.e. the indices

$$2\alpha_2 + \cdots, \quad 3\alpha_3 + \cdots$$

will increase.

What is the basic category here?

Now a decomposition $D_\alpha(V)$ is indexed by $\alpha \vdash n$ and one is allowed to permute the different things of the same level and to coallesce eigenspaces.

Forgetting N what we have is $\alpha_1, \alpha_2, \cdots$ and the group $\Sigma_{\alpha_1} \times \cdots$ and we are allowed to coallesce a point of α_i and α_j to get a point of α_{i+j}

S finite set with basepoint s_0.

$$M(S) = \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n; S) \sim \prod_S \left[\coprod_n BU_n\right].$$

~~Let $s_0, s_1, s_2, \ldots, s_{15}$ determine what happens.~~

~~$M(S)$ is the classifying space for the ...~~

$$M(T) = \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n; T) = M(S) \otimes^{\Gamma'} T^S$$

classifies bundles with T-structure up to homotopy. It is a disconnected monoid whose group-completion is what ~~conc~~ concerns me.

---

can formulate things without basepoint I think.

$$\begin{array}{ccccc} M(A \cap B) & \longrightarrow & M(B) \times M(A) & \longrightarrow & M(A \cup B) \\ & & \uparrow \text{heq} & \nearrow & \\ & & M(A \amalg B) & & \end{array}$$

~~[illegible]~~

$$M(S)/M(pt) \longleftarrow M(S - pt) \qquad \text{up to topology.}$$

~~Suppose now I can find things so~~

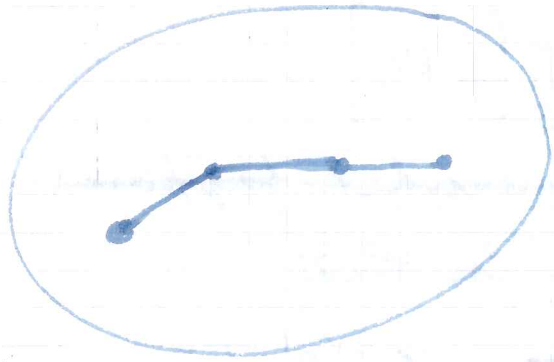
Ex: $M(S) = \coprod_n S^n$

$$M(T) = \coprod_n T^n$$

Then $M(T)/M(pt) \longleftarrow M(T-pt)$ is bijective but not topological isom.

So if T is a pointed space, I want $M(T; pt)$ ~~to~~ be defined as a suitably stratified thing with strata $M(T-pt)$.

X closed in a s.comp. Y.



Get a nbd of X by considering the open star i.e. all points y ~~such that their coords~~ whose support meets X, i.e. $\exists$ for some vertex x of X $\rho_x(y) > 0$. $U = \{y \mid \sum_{x \in X_{vert.}} \rho_x(y) > 0\}$.

Have a retraction of U back to X. Linearly slide

clear

~~Linearly slide U into X.~~

Next point will be more refined.

~~It would seem then that~~ I want to give carefully the structure over Y with thickenings if necessary. ~~[illegible]~~ Think of Y_n as a manifold

Y_0 manifold whose normal ~~structure~~ tube is a cone.

Y_1 open manifold. $\overline{Y}_1$ = manifold with boundary, $\partial \overline{Y}_1$ = smooth thing ~~[illegible]~~ over Y_1 of or better $D(Y;T)$ is the inductive limit taken suitably wrt topology of $D(V;S)$ S = finite sets over T.

The class. space for rank n bdles with T-structure is therefore

$$PU_n \times^{U_n} D(\mathbb{C}^n;T) = \varinjlim_{S/T} \coprod_{\substack{f:S\to\mathbb{N} \\ \Sigma f(s)=n}} PU_n \times^{U_n} D_f(\mathbb{C}^n)$$

so

$$\coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T) = \varinjlim_{S/T} \coprod_{f:S\to\mathbb{N}} PU_{|f|} \times^{U_{|f|}} D_f(\mathbb{C}^{|f|})$$

$$\int$$

$$\prod_S \left(\coprod_n BU_n\right) = \coprod_{f:S\to\mathbb{N}} \left\{\prod_{s\in S} BU_{f(s)}\right\}$$

This perhaps won't be helpful as it is not clear how to organize the limit which allows for the variations of eigenvalues.

Role of basepoint: We understand

$$\coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T)$$

which represents $Vect(Y;T)$. Suppose then that T is connected with basepoint, and I want to represent $Vect(\ ;T)/Vect(\ ;pt)$.

Thus I have to kill the basepoint t_0 of T somehow.

Idea take realization of $\coprod_n SP_n(T)$ acted on by $\mathbb{N}$ and to use standard

---

$\mathbb{N}$ poset.

$Z_0 \subset Z_1 \subset Z_2 \subset \cdots\cdots \subset Z$ closed sets

define $f : Z \longrightarrow \mathbb{N}$

by putting $f(z) = \text{least } n \ni z \in Z_n$.

Then ~~$f^{-1}\{p \geq n\} = \{z \mid f(z) \geq$~~

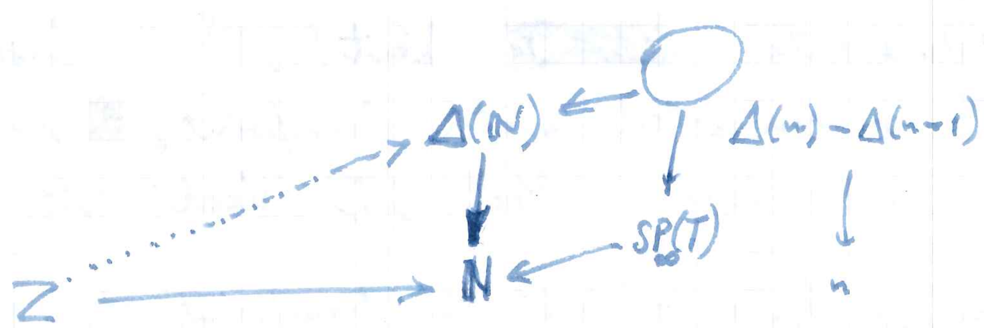
$f(z) \leq n \iff z \in Z_n$

$$\boxed{f^{-1}\{p \leq n\} = Z_n}$$

$\therefore f$ continuous.

$f(z) =$ ~~[illegible]~~ rank (Θ_z)

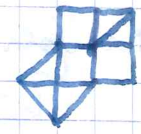
For homotopy purposes I need something more that just $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$, I need to give ~~[illegible]~~ open nbds. + normal bundles.



Problem: Given a Banach alg. A and a space X compact, define a Banach alg A_X.

Property of A_X: It's homotopy groups should be the homology of X with coeff. in the K-spectrum $\underline{\underline{K}}_A$.

$$\{Y, \underline{\underline{K}}_{A_X}\} = \{Y, X \wedge \underline{\underline{K}}_A\}$$



Possible approaches to the problems:

1) ~~[illegible]~~ Anderson's - Given a ~~connected~~ simplicial X and a permututative category $\mathcal{P}$, one has $C(X, \mathcal{P})$ the simp. perm. cat of chains on X coeffs. in $\mathcal{P}$.

2) Segal's version of configurations. Given a manifold M, one can consider configurations in M indexed by X, i.e. finite subsets of M over X. In particular if $M = R^n$, one gets a space $\Omega^n S^n X$

3) Atiyahs operators. If X is a compact manifold, one can consider the algebra of pseudo diff. operators of degree ≤ 0 on $L^2(X)$. ~~[illegible]~~ One gets a map $C(ST_X) \longrightarrow$ [bdd mod compact operators on $L^2(X)$]

which is norm-preserving. Hence one gets

$$C(ST_X)^* \longrightarrow \text{Fredholm ops on } L^2(X)$$

which gives one the index.

Karoubi's flask idea.

$\mathcal{P}_A \hookrightarrow \mathcal{F}$ $\qquad$ $\mathcal{F}$ flask category.

e.g. $\mathcal{P}_{CA}$.

X space. Consider a Hilbert space bundle over X; by Kuiper it is ~~contractible~~ (trivial); First produce a section. This is trivial.

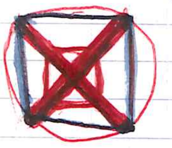
~~A~~ X space. ~~Then X is trivial.~~ try to understand the oper over X.

X space. All Hilbert space bundles over X are trivial. Let R be the ring of endos. describing these bundles. ~~Let~~ Clearly R = ~~[illegible]~~ $\mathrm{End}(H)^X$. Next ~~let R be~~ ~~[illegible] form~~.

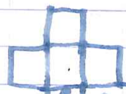
Thus I start from the category of vector bundles on X, embed into the cat of Hilbert bundles, and form the quotient category, ~~which is~~ whose objects are Hilbert bundles and Fredholm bundle maps. This quotient category is described by the ring of endos. of ~~End(H) × H → X~~ H_X in the quotient category, i.e. $\mathrm{End}(H)^X$ divided out by the compact operators. = uniform closure of operators with finite diml. image.

$$\mathrm{Comp}(H)^X \longrightarrow \mathrm{End}(H)^X \longrightarrow \mathrm{End}(H)^X / C(H)^X$$

ideal $\qquad$ ring



example: [150] $X = R$ or I



$\mathfrak{C}_X$ = operators on Hilbert space

whereas $X = S^1$. $\mathfrak{C}_X$ = bdd mod. comp. ops.

$X \longmapsto A_X$ should be a covariant functor, so that $K(A_X) = h_0(X, \underline{K}_A)$ is covariant. If Y is then a subspace of X we have

$$\begin{array}{ccc} A_Y & \longrightarrow & A_X \\ \downarrow & & \downarrow \\ A_{pt} & \longrightarrow & A_{X/Y} \end{array} \qquad A_{pt} = A.$$

X compact manifold

X compact almost complex manifold, then

$$K_0(X) = K^0(X)$$

is covariant in X.

Think of X as a basespace. $X \xrightarrow{f} pt$ I want $f_! f^* \underline{K}_A$ Yes. Would be such that

$$[f_! f^* \underline{K}_A, \Gamma] = [f^* K_A, f^* \Gamma]$$

but a hmo. of $f^* K_A$ to $f^* \Gamma$ over X?

$f^* K_A$ $\qquad X \xrightarrow{f} pt$ clear.

If X is a space, what is A_X??

Can you describe $\mathrm{End}(H)^X$ starting from $\mathbb{C}^X$. Roughly

$$\mathrm{End}(H) = \mathrm{End}(\underbrace{\mathbb{C}\oplus\cdots\oplus\mathbb{C}}_{\mathbb{N}\text{ times}})$$

$$\mathrm{End}(\mathbb{C}^n)^X = \mathrm{End}_{\mathbb{C}^X}(\mathbb{C}^{X\,M})$$

Guess. Take a Hilbert space on which $\mathbb{C}^X$ acts with infinite multiplicity.

Let H be a Hilbert space on which $\mathbb{C}^X$ acts by self adjoint operator $f^* = \bar{f}$. Not unique

---

e.g. Assume $\exists$ cyclic vector v. This means that $\mathbb{C}^X v$ is dense in H. So one gets a measure: $f \mapsto (fv, v)$ on X. Note that

$$|f|^2 \longmapsto (\bar{f}fv, v) = (fv, fv) \geq 0.$$

Thus one has $H \simeq L^2(X, \mu)$ with $\mathbb{C}$ acting via mult.

Other possibility: Use a smoothness structure on X which singles out a collection of measure. Thus one would find essentially one Hilbert space V on which $\mathbb{C}^X$ acts cyclically. Take this and define $\mathrm{End}(H)^X = \mathrm{End}_{\mathbb{C}^X}(V)$

Comp.

Consider an exact cat $\mathcal{P}$ and let $\mathcal{A}$ = all left exact contravariant functors from $\mathcal{P}$ to $\mathcal{A}b$. $\mathcal{A} = \mathrm{Lex}(\mathcal{P}^\circ, \mathcal{A}b)$.

$$\mathcal{P} \xrightarrow{h} \mathcal{A}$$

$$P \longmapsto h_P = \mathrm{Hom}(?, P)$$

Review why $\mathcal{A}$ is an abelian cat. Because,

$$\mathcal{A} \qquad \mathrm{Hom}(\mathcal{P}^\circ, \mathcal{A}b)$$

Inside $\mathrm{Hom}(\mathcal{P}^\circ, \mathcal{A}b)$ we have the subcategory of functors which will go to zero in $\mathcal{A}$. $\mathcal{S} = \{F : \mathcal{P}^\circ \to \mathcal{A}b \mid F \text{ eff.}\}$

F eff. means $\forall P,\ \xi \in F(P)\ \exists\ P' \overset{u}{\twoheadrightarrow} P \ni u^*(\xi) = 0$.

First must show $\mathcal{S}$ is a Serre subcategory of $\mathrm{Hom}(\mathcal{P}^\circ, \mathcal{A}b)$, i.e. given

$$0 \to F'(P) \to F(P) \to F''(P) \to 0 \qquad F \in \mathcal{S} \Rightarrow F', F'' \in \mathcal{S}$$

$$\xi$$

$$F(P_1) \qquad F''(P_1) \quad (\text{with } F''(P) \downarrow F''(P_1))$$

$$F(P_2)$$

$\therefore$ closed under sub. quot. ext.

$\mathcal{P} = \mathcal{P}_A$. Then $\mathrm{Lex}(\mathcal{P}^\circ, \mathcal{A}b) = \mathrm{Hom}(\mathcal{P}^\circ, \mathcal{A}b) = \mathrm{Mod}(A)$.

$\mathcal{P}_A \subset \mathrm{Mod}_A$ and $\mathrm{Mod}_A / \mathcal{P}_A$ is some triangulated gadget.

It would appear that if I want to continue ~~in this vein~~ with this idea of def

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GUESS: THERE SHOULD BE A REALLY GOOD DEFINITION OF A_X FOR X CONNECTED WITH BASEPOINT. IT SHOULD AGREE WITH WHAT KAROUBI HAS FOR $X = S^n$: $A_{S^n} = S^n(A)$.

Question: What should be the pair construction for $S(A)$? A map should be a pair of projectors e_1, e_2 in $S(A)$, together with an isomorphism of e_1, e

---

$$f : X \longrightarrow Y$$

$$f^* : \mathbb{C}^X \longleftarrow \mathbb{C}^Y$$

so if $\mathbb{C}^X$ operates somewhere, then now $\mathbb{C}(Y)$ does.

---

e.g. I want a ring A_X covariant in X.

A_X covariant ???

$A_{\mathbb{R}}$ = bdd. operators.

A_I = bdd. operators.

In top K-theory I expect two operations

$A \longmapsto A^X$

$A \longmapsto A_X$

and one point is that these two operations are inverse to each other.

~~K(A^X)~~ $K(Y, A^X) = K(Y \times X, A)$.

Thus

$K(A^{S^1}) = K(S^1, A) = K(A) \oplus K^{-1}(A)$

whereas A_X should involve K-homology of X.

$A_{S^1} = S(A)$ in Karoubi's sense

$K(S(A)) = K^1(A)$

ΩA	$A^I \longrightarrow A \times A$	$K\ A^{I,I}$

$K(Y, A_X) = [Y, X \wedge \underline{\underline{K}}_A]$

X simplicial set, then $(X \wedge \underline{\underline{K}}_A)_0$ is probably chains on X with values in P_A. This is a simplicial monoid.

M is a monoid
Adjoin to M the projective f.t. M-sets.
Adjoin to M to set of projective f.t. M-sets.
get a category with the same topos?
small proj. gen.

suspension of a ring A. $K_1(SA) = K_0 A$

$$I \longrightarrow CA \longrightarrow SA$$

$$\underset{\substack{\| \\ 0}}{K_1 CA} \longrightarrow K_1(SA) \longrightarrow \underset{\substack{\text{is} \\ K_0 A}}{K_0(I)} \longrightarrow \underset{\substack{\| \\ 0}}{K_0 CA}$$

Thus I want to understand the significance of the suspension of A.

$A = B \times B^0$ Here $B\underline{V}$ = my pair cat.

$\underline{\underline{U}}$ consists of ~~[illegible]~~ $(P \supset M, N)$ modulo pairs with complements.

and the function sends $V \mapsto H(V), V, V^*$

$(P, P') \longmapsto (P \oplus P', P, P')$

---

objects $(P; M, N)$ M, N are admiss. subobjects of P modulo action of ~~$P \oplus P$~~ $(M \oplus N, M, N)$

~~[illegible]~~ In the case of a field given

$(P; M, N)$ one forms $0 \subset \underline{M \cap N} \begin{smallmatrix} \subset M \supset \\ \subset N \supset \end{smallmatrix} M + N - P$

and splits this so that every

$$(P; M, N) = (M \oplus N, M, N) \oplus (P; P, P) \oplus (Q, 0, 0)$$

B = field = k. ~~so take the category of~~ Basically an excellent construction this $\underline{\underline{U}}$ gadget

---

~~[illegible]~~ suspension $S(A)$.

X space, A ring, then I want to describe the theory $X \wedge \underline{\underline{K}}_A$ intelligently so that

If $X = S^1$, then there should be a map of

$$0 \leftarrow S^1 \wedge \underline{\underline{K}}_A \leftarrow \underline{\underline{K}}_{A[t, t^{-1}]} \leftarrow \underline{\underline{K}}_{A[t]} \oplus \underline{\underline{K}}_{A[t^{-1}]} \leftarrow \underline{\underline{K}}_A \leftarrow 0.$$

where the fibre is $\underline{\underline{K}}_A$

$$\frac{2}{7} \quad \frac{4}{14} \quad \frac{6}{21}$$

$$\frac{2}{7} = \frac{4}{14} = \frac{6}{21} = \frac{8}{28} =$$

Karoubi periodicity thm. ${}_{-1}\underline{V} = \Omega\, {}_1\underline{U}$.

$${}_{-1}Q \xrightarrow{F} \mathcal{P}_A \underset{*}{\overset{1}{\rightrightarrows}} \mathcal{P}_A \xrightarrow{H} {}_1 Q$$

For $B\underline{V}$, I have the candidate of ~~quadratic~~ P mod action of Q. And for $\underline{U}$ I have the candidate of formations $\mathcal{F}$: (Q, F, G) modulo action of trivialized formations (Q, F, G, H) and I have this functor which goes from

$$B\underline{V} \longrightarrow \underline{U}$$

~~so it would seem that this category $B\underline{V}$ might be realized Q-style. Thus a map is a $V' \oplus Q \xrightarrow{\sim} V$ pairs~~

$$B\underline{V} \xrightarrow{f} \underline{U} \qquad V \longmapsto (H(V), V, V^*)$$

so now the problem is to show this functor is a homotopy equivalence. ~~does succeed at all.~~ So I could consider things of the form $f/(Q,F,G)$. meaning we could look at direct summands of Q with the following properties: Assume that

$$\pi_0(B\underline{V}) = K_0 A / F_{-1}L_0(A) = {}_{-1}W_0(A) \subset {}_{-1}V_1(A)$$
$$\pi_0(\underline{U}) = {}_1U_0(A)$$

$A = \mathbb{C}$. X space — to define A_X roughly by means of Segal's theory so that A_X gives the K-homology of X with coeffs. in A.

Now Atiyah has suggested one define A_X using a space of operators containing $C(X, A_X)$ and commuting mod lower order.

X = closed manifold

T_X = tangent bundle of X

Then duality says what. If X gets embedded in Euclidean space E with normal bundle ν, then

$$H_{N-i}(E, E-X) \xleftarrow{\sim} H^i(X)$$

$\nu, S\nu$

where $N = \dim(E)$. Thus it would appear that

$E^+ \longrightarrow \nu^+$ $\qquad \nu = E - \tau$

thus the dual of X is $X^{-\tau}$.

Index: $K_c(\tau_X) \longrightarrow \mathbb{Z}$. Question: Is $K_c(\tau_X)$ the K-homology of X in degree 0?

~~If X is almost complex, then~~ X manifold N even.

$$K_0(X) = K^0(X^{-\tau}) = K^0(S^{-N}X^{\nu}) = K^0(X^{\nu})$$

Is $K^0(X^{\tau})$ the K-homology of X in degree 0?

so how are $K^0(X^{\nu})$ and $K^0(X^{\tau})$ related?

A Banach alg. e.g. cont. functions on a space Z
Two operations:

1) $A \longmapsto A^X$. Here

$$K(Y, A^X) = K(Y \times X, A)$$

so that one has

$$(\underline{\underline{K}}_A)^X = \underline{\underline{K}}_{(A^X)}$$

[illegible]

2) $A \longmapsto A_X$ This is to be defined à la Segal so that

$$K(Y, A_X) = h(Y, X \wedge \underline{\underline{K}}_A).$$

In other words once the K-spectrum of A is appropriately defined, then A_X should be related to homology. e.g.

~~E E C~~

[illegible] $$K(A_{S^n}) = h(S^0, S^n \wedge \underline{\underline{K}}_A)$$

~~Suppose now that one understands all this~~

Now using indexes I will be able to define $S^n \wedge \underline{\underline{K}}_A$ hopefully.

~~So let Γ be a sheaf of differentiable functions~~

Suppose then that $X = \mathbb{R}_c$ = line with compact support. Then I will define a ring of operators A_X which hopefully ~~[illegible]~~ will classify $B\,GL(A)$. Then skew-adjoint Fred. operators ~ skew adjoint Fred ops with essential spectrum $\{\pm i\}$.

$\downarrow \exp \pi(?)$

unitary of form $-1 + C$ ~ $\mathcal{U}$

is a classifying space for K^{-1}.

---

self-adjoint Fred. operators ~ self-adjoint ones with essential spectrum $\{\pm 1\}$

~ projectors in <u>bdd mod compact ops.</u>

suspension

is a classifying space for K^1

---

periodicity thm: $K^{-2}(X) = \tilde{K}(S^2 \wedge X) =$

$= K^{-1}(S^1 \times X, X)$

387

Europe 41 50 ∞ [1974] [redacted]

$K^{-1}(S^1 \times X, X)$ or so

$\parallel$

$K_1(A\langle z, z^{-1}\rangle, A)$.

$\downarrow \cong$ ← this part here is general in some sense, works over $\mathbb{R}$.

$K_1(SA) \cong K_0(A)$.

A Banach algebra. Have two operations

1) $A \longmapsto A^X$ such that $K(Y, A^X) = K(X \times Y, A)$

cohomology sides.

2) $A \longmapsto A_X$ such that $K(Y, A_X) = h(Y, X \wedge k_A)$

2) has something to do with the K-homology of X. So now I ought to take up K-homology maybe in the way described by Atiyah i.e. one considers operators partially commuting with those of A.

~~Thus if $X \simeq S^1$, we win the resultant.~~

But more: one should be able to understand K_*A from a good description of $X \wedge \underline{B}_A$.

e.g. $(X \wedge P_A)$. Suppose X is a simp. set. Then I can get a simp. category monoidal, which is $\bigoplus_{x \in X} P_A$. etc.

One proves one has a homology theory somehow!!!

exponential map is algebraic modulo m.

M monoid. $\mathcal{C}$ = category of M-sets

Let $\mathcal{P}$ = projective M-sets i.e.

A ring. Mod(A) category of A-modules

abelian with a small projective generator

M monoid $\mathcal{C}$ = topos of M-sets

Consider now small projective objects of $\mathcal{C}$. Projective clear, small also. Thus we get an idempotent operator in $M \amalg M \amalg \cdots \amalg M$, whose image is what we are after.

$M = \mathbb{N}$. $\quad \mathbb{N} \amalg \mathbb{N} \amalg \cdots \amalg \mathbb{N} \underset{p}{\overset{s}{\rightleftarrows}} S \qquad \mathbb{N} \times I \overset{s}{\rightleftarrows} S$

S is an $\mathbb{N}$-set; ~~the structure~~

S must split into pieces according to s

$s(x) \in i$th orbit.

$S_i = s^{-1}(\mathbb{N}_i)$. Then

$$\begin{array}{ccc} \Delta S_i & \xleftarrow{\ \sim\ } & S_i \\ \cap & & \cap \\ \mathbb{N} \times i & \longrightarrow & S \end{array}$$

2:~~00~~15 – prove periodicity

I have seen that topologically I get a facinating thing ~~[illegible]~~ by taking the Γ-space in which

$S \longmapsto$ space of fin. diml. orth. subspaces indexed by $s \neq *$

~~[illegible]~~ This is a nice explicit point of view

Suppose now I try to understand $B\mathcal{P}_S$ – where we find ourselves interested in ~~[illegible]~~ the cat consisting of an object of $\mathcal{P}$, V divided up according to S

Let H be an infinite dimensional space. Consider all projections e in H whose image is of rank k say. Can you make sense ~~[illegible]~~ out of the set of these projections as a category.

$\{V \subset H\}$

References

[1] M. Artin, A. Grothendieck, and J. L. Verdier, Theorie des Topos et Cohomologie Etale des Schemas (SGA4), v. 269, 270, 305, Lecture Notes in Mathematics, Springer-Verlag.

[2] C. Chevalley, "Une demonstration d'un théoreme sur les groupes algebriques," J. de Math. Pures et Appl., v. 39(1960), 307-317.

[3] M. Demazure, "Schemas en Groupes Reductifs," Bull. Soc. Math. France, v. 93(1965), 369-413.

[4] M. Demazure and A. Grothendieck, Schemas en Groupes (SGA3), v. 151, 152, 153, Lecture Notes in Mathematics, Springer-Verlag.

[5] D. Gieseker, "Flat Bundles and the Fundamental Group," to appear.

[6] R. Hoobler and D. Rector, "Arithmetic K-theory," Proceedings of the Seattle Conference on Localization and Completion in Homotopy Theory, to appear in Lecture Notes in Mathematics.

[7] A. Magid, "Left Algebraic Groups," to appear.

[8] M. Miyanishi, "On the algebraic fundamental group of an algebraic group," J. of Math. of Kyoto Univ., v. 12(1972), 351-367.

[9] J. P. Murre, On an Introduction to Grothendieck's Theory of the Fundamental Group, Tata Lecture Notes in Mathematics, v. 40(1967), Tata Institute of Fundamental Research, Bombay.

σ simplices of a simplicial complex (finite) K.

$\sigma \mapsto F_\sigma$

I want now to know that for each $\sigma \subset \tau$ the map $F_\tau \longrightarrow F_\sigma$ is to be a cofibration.

So I should check that if U_α $\alpha \in I$ is an I-torsor over X, $\ni$ $U_{\alpha_0} = X$, and if $\alpha \mapsto F_\alpha$ $\in$ Funct(I°, Sp), then

$$R(I, U_\alpha, F_\alpha) = R(\alpha_0 \backslash I, U_\alpha, F_\alpha)$$

So $$\mathrm{Hom}_{/X}(T, R(I, U_\alpha, F_\alpha)) = \mathrm{Ker}\left\{\prod_\alpha \mathrm{Hom}(U_\alpha \times_X T, F_\alpha) \rightrightarrows \prod_{\alpha \leq \beta} \mathrm{Hom}(U_\beta \times_X T, F_\alpha)\right\}$$

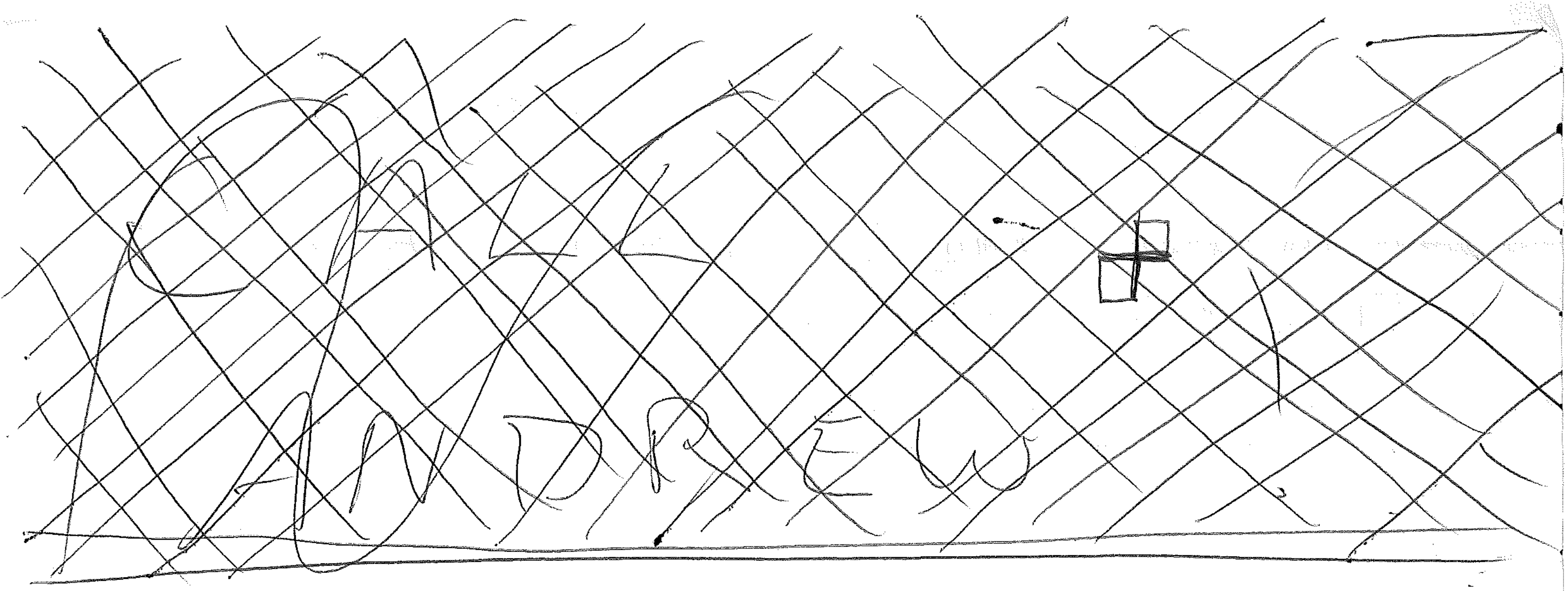
Thus if $p_\alpha : R(I, U_\alpha, F_\alpha) \times_X U_\alpha \longrightarrow F_\alpha$ are the canonical proj. then restricting these to the $\alpha \geq \alpha_0$ we get a map

$$R(I, U_\alpha, F_\alpha) \longrightarrow R(\alpha_0 \backslash I, U_\alpha, F_\alpha)$$

which I want to show is a homeo. Suppose then I have I have a map of T to the latter, i.e. have $p_\beta : T \times_X U_\beta \to F_\beta$ for $\beta \geq \alpha_0$. To define $p_\alpha : T \times_X U_\alpha \to F_\alpha$. Take a point t, then $\mathrm{im}(t)$ in X is contained in U_α and U_{α_0} hence one has a $\beta \geq \alpha, \alpha_0$ $\ni$ $\mathrm{im}(t) \in U_\beta$. Then I can define $p_\alpha(t)$ to be $p_\beta(t)$ etc.

T_{U_α} is covered by T_{U_β} $\beta \geq \alpha_0, \alpha$

thus p_α can be recovered from p_β for $\beta \geq \alpha_0, \alpha$



So suppose I assume $E \longrightarrow F$ is a nice embedding — meaning E is a strong defm. retract of a nbd V in F. Thus we have $F = (F-E) \cup V$ open covering, and since we have

$$Z \longrightarrow X \times F \qquad U \times E \cup Y \times F = X \times E \underset{Y \times E}{\cup} Y \times F$$

we get an induced open covering of Z. Inverse image of $F-E$ is the complement of $X \times E$, namely $Z-(X \times E) = Y \times F - Y \times E$. Thus one has

$$Z = \Big(X \times E \underset{Y \times E}{\cup} Y \times V \Big) \quad \underset{(Y \times V - Y \times E)}{\cup} \quad \big(Y \times F - Y \times E \big)$$

$$\int$$

$$X \times E$$

so one might compare $Y \times (V-E) \subset Y \times (F-E)$. So perhaps it appears desirable ~~to~~ have V of the form $E \times [0,1)$ In which case $V-E \sim E$, $F-E \sim F$.

$\mathcal{U}_\alpha$ is contractible one has a classifying space

---

when one forms $\underrightarrow{\text{holim}}\, F_\alpha$ one replaces F_α by something thickened and takes

So I have a simplicial complex K and a ~~covariant~~ contravariant functor $\sigma \mapsto X_\sigma$ to spaces and I form the

Contraction $\underset{\sigma \in K}{\underrightarrow{\text{holim}}}\, X_\sigma$ = contraction of $\sigma \mapsto \bar{\sigma}$, $\sigma \mapsto X_\sigma$

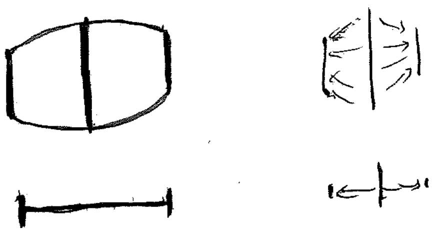
Since each point of $|K| = \underset{\sigma}{\underrightarrow{\lim}} \{\sigma \mapsto \bar{\sigma}\}$ lies in a smallest simplex, I find that a point of $\underrightarrow{\text{holim}}\, X_\sigma$ is simply?? Better

$$\underrightarrow{\text{holim}}\, X_\sigma = \bigcup \sigma \times X_\sigma$$

But now I want to topologize $\underrightarrow{\text{holim}}\, X_\sigma$ so that I can see easily what are the maps $Z \xrightarrow{f} \underrightarrow{\text{holim}}\, X_\sigma$. Thus f gives rise ~~to~~ an open covering with partition $\{f^{-1}(U_\sigma),\ \lambda_\sigma f\}$

and one finds over $f^{-1}(U_\sigma)$ a map to X_σ. Thus with the weak topology on $f^{-1}(U_\sigma)$.

<u>Assertion</u>: If I is a poset, $i \mapsto X_i$ a cont. functor to spaces, then can topologize $\bigcup X_i$ so that the functor assoc. to Z a covering $\mathcal{U}_i$ indexed by I +



$$\begin{array}{ccc} \bigcup \sigma \times X_\sigma & \longrightarrow & \bigcup X_\sigma \\ \downarrow & & \downarrow \\ \bigcup \sigma & \longrightarrow & \bigcup_\sigma pt \end{array}$$

$\mathbb{Z}$ comp

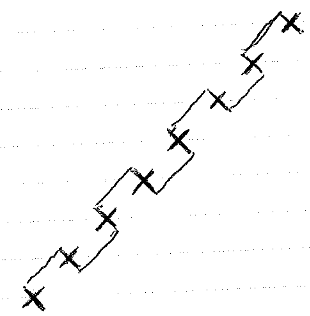
$U\times E$ $\qquad$ $Y\times F$

$$U \subset X \supset Y$$

So if one works with the inductive limit

$$X\times E \underset{Y\times E}{\cup} Y\times F$$

So if I could compute this as a cofibration square

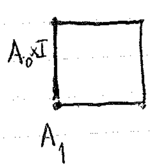


$$\begin{array}{ccc} X\times E & \longrightarrow & Y\times F \\ \downarrow & & \downarrow \\ X\times E & \longrightarrow & (X\times E)\underset{(Y\times E)}{\cup}(Y\times F) \end{array}$$

Then I would have that as $E\longrightarrow F$ is a homology isom. so then same holds for $Y\times E\longrightarrow Y\times F$. So one should start with assuming that $E\longrightarrow F$ is nice, i.e. replacing F by $E\times I \underset{E}{\cup} F$. Then the end result is

$$X\times E \underset{Y\times E}{\cup} Y\times E\times I \underset{Y\times E}{\cup} Y\times F$$

$$A_0 \longrightarrow A_1 \longrightarrow A_2$$



$$A_0 \longrightarrow A_0\times[0,1]\cup A_1$$

So now it is necessary to consider the problem of replacing a functor $\alpha \longmapsto F_\alpha$ by one which is excellent for homotopy glueing considerations. Thus I would want to know that

Let $Y \xrightarrow{f} X$ be etale. Then can I represent

$$T \longmapsto \mathrm{Hom}(T \times_X Y, F)$$

on spaces over X? Let U_i be an open covering of Y such that $U_i \longrightarrow X$ is an open immersion. Then

$$\mathrm{Hom}(T \times_X Y, F) = \mathrm{Ker}\left\{\prod_i \mathrm{Hom}(T \times_X U_i, F) \rightrightarrows \prod_{i,j} \mathrm{Hom}(T \times_X (U_i \cap U_j), F)\right\}$$

$$= \mathrm{Ker}\left\{\prod_i \mathrm{Hom}_{/X}(T, {}_{U_i}F) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{/X}(T, {}_{U_i \cap U_j}F)\right\}$$

$$= \mathrm{Hom}_{/X}\left(\mathrm{Ker}\left\{\prod_{i \, /X} {}_{U_i}F \rightrightarrows \prod_{i,j \, /X} {}_{U_i \cap U_j}F\right\}\right).$$

so it seems to be OKAY. The fibre of ${}_Y F$ over a point x is $\mathrm{Hom}(\{x\} \times_X Y, F)$ i.e. it is the product of F for each point in the fibre over x.

Next to do homotopy theory.

$$\mathrm{Hom}_{/X}(T, R(I, U_\alpha, F_\alpha \times Z_\alpha)) = \mathrm{Hom}_{\mathrm{Funct}(I, Sp)}(T \times_X U_\alpha, F_\alpha \times Z_\alpha)$$

$$= \mathrm{Hom}_{\mathrm{Funct}(I, Sp)}(T \times_X U_\alpha, F_\alpha)$$

$$\times \mathrm{Hom}_{\mathrm{Funct}(I, Sp)}(T \times_X U_\alpha, Z_\alpha)$$

$$R(I, U_\alpha, F_\alpha \times Z_\alpha) = R(I, U_\alpha, F_\alpha) \times_X R(I, U_\alpha, Z_\alpha).$$

If Z_α constant functor Z, then to give $T \times_X U_\alpha \longrightarrow Z$ $\forall \alpha$ is same as giving $\varinjlim_\alpha T \times_X U_\alpha \longrightarrow Z$

$$\overset{\shortparallel}{T}$$

so $$R(I, U_\alpha, F_\alpha \times I) = R(I, U_\alpha, F_\alpha) \times I$$

So one sees that if one has a homot. of morphisms $F_\alpha \times I \longrightarrow F'_\alpha$ one gets a homotopy of $^{\text{Imp}} R(I, U_\alpha, F_\alpha) \to$ over X.

Assume $U_{\alpha_0} = X$ for some α_0. Then

$$R(I, U_\alpha, F_\alpha) = R(\alpha_0 \backslash I, U_\alpha; F_\alpha)$$

$$\begin{array}{ccccc} U_0 \times F_0 & \longleftarrow & U_{01} \times F_{01} & \longrightarrow & U_1 \times F_1 \\ \| & & \| & & \| \\ X \times F_0 & \longleftarrow & U_1 \times F_{01} & \longrightarrow & U_1 \times F_1 \end{array}$$

~~$X \times F_{01}$ $\quad X \times F_{01}$ $\quad X \times F_{01}$~~

$\sigma \subset \{0, \dots, n\}$

Cyl $(U_\sigma \times F_\sigma)$

assume $\exists\, i$ such that $U_i = X$ whence

$$U_{\sigma \cup \{i\}} = U_\sigma$$

hence

Cyl $(U$

---

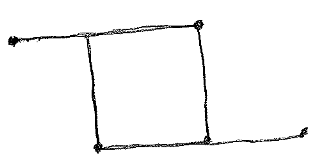
wrong philosophy.

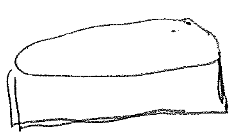
have functor $\sigma \longmapsto F_\sigma$

and I want to form $\underrightarrow{\mathrm{holim}}_{\sigma}\, F_\sigma$

over $\underrightarrow{\mathrm{holim}}_{\sigma}\, pt = K.$

Now I want to see why $\underrightarrow{\mathrm{holim}}_{\sigma}\, F_\sigma$ over K in the nbd of σ retracts to F_σ, which is clear.





$$\begin{array}{ccc} \coprod \sigma \times U_\sigma \times F_\sigma & \longrightarrow & \coprod \sigma \times F_\sigma \\ \big\downarrow & & \big\downarrow \\ \coprod U_\sigma & \longrightarrow & \coprod pt \end{array}$$

~~So suppose then I give a poset J and a family $\sigma \mapsto U_\sigma$ of open sets of X $\ni$~~
~~$\sigma \subset \tau \Rightarrow U_\sigma \supset U_\tau$~~
~~$\forall x$ $\{\sigma \mid x \in U_\sigma\}$ has a largest element.~~

~~Let me give you an map of~~

Let K be a simplicial complex with vertices I simplices J, let $\sigma \longmapsto F_\sigma$ be a contrav. functor from J to spaces. I want then to form over K the space E with

$$\sigma \subset \tau \qquad \begin{array}{ccc} E|_{U_\sigma} & \xrightarrow{p_\sigma} & U_\sigma \times F_\sigma \\ \cup & & \uparrow \\ E|_{U_\tau} & \xrightarrow{p_\sigma} & U_\tau \times F_\tau \end{array}$$

and universal with this property. Thus

$$\left(U_\sigma \times U_\sigma \times F_\sigma\right)_!$$

'till more confusing than I want

Now the point is to thicken E up a bit. So I can form the ~~[illegible]~~ analogous space

$$\bigcup \sigma \times U_\sigma \times F_\sigma$$

which sits over $\bigcup U_\sigma = K$. One would hope that when the F_σ are all spaces over Y, then

$$\bigcup \sigma \times F_\sigma \quad \text{sits over } Y$$

and is compatible with base change in Y.

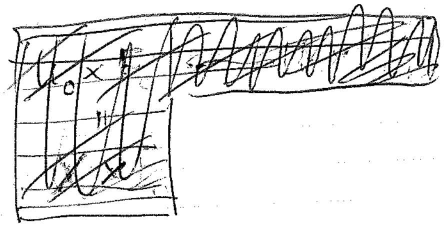
So the case to consider is two open sets. Given $F_0 \leftarrow F_{01} \rightarrow F_1$ and I form

$$Cyl(F_0 \leftarrow F_{01} \rightarrow F_1) = F_0 \cup F_{01} \times [0,1] \cup F_1$$

Now ~~we~~ given $X = U_0 \cup U_1$, I can form

$$\begin{array}{c} Cyl(U_0 \times F_0 \leftarrow U_{01} \times F_{01} \rightarrow U_1 \times F_1) \\ \downarrow \\ X \end{array}$$

sitting over X. Compatible with base change in X. Next I want to see that if $U_0 = X$, then this has the right homotopy-type. But



$$\begin{array}{ccccc} U_0 \times F_0 & \leftarrow & U_{01} \times F_{01} & \longrightarrow & U_1 \times F_1 \\ \| & & \| & & \| \\ X \times F_0 & \leftarrow & U_1 \times F_{01} & \longrightarrow & U_1 \times F_1 \end{array}$$

Given a space X with an open set U, I then get a functor

$$\xi : (E \longrightarrow F) \longmapsto (U \times E) \cup [(X-U) \times F]$$

which is twisting with respect to a torsor. I want to understand whether this preserves heq's. Thus if I have an arrow

$$\begin{array}{c} (E \longrightarrow F) \\ \downarrow \\ (E' \longrightarrow F') \end{array}$$

such that $E \to E'$ and $F \longrightarrow F'$ are heq's, does it follow that $\xi(E \longrightarrow F) \longrightarrow \xi(E' \longrightarrow F')$ is an heq.

So we have this way of going from an arrow $F' \to F$ to a space E over $X, \mathcal{U}$.

$\mathcal{U}, X$ are given and I have this functor

$$(F' \to F) \longmapsto E(F' \to F)$$

clear that $E(F'_{\times T} \to F_{\times T}) \leftarrow E(F' \to F) \times T$

so that I have a homotopy functor.

Also if $F' \to F$ is an isom

$$E(F \xrightarrow{id} F) = X \times F$$

Suppose now that $F' \xrightarrow{f} F$ is a h.e. with h. inverse g.

then we get

$$F' \xrightarrow{f} F$$

First case. Suppose that $F' \hookrightarrow F$ is a strong def. retract situation. So that we have

$$\begin{array}{ccc} F' & \xrightarrow{id} & F' \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle i} \\ F' & \xrightarrow{i} & F \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle r} \\ F' & \hookrightarrow & F' \end{array}$$

$$(F' \xhookrightarrow{i} F) \times I$$
$$\downarrow (pr_1, i)$$
$$F' \xhookrightarrow{i} F$$

$h_0(f) = f$

$h_1(f) = ir$

$h_t i = i$

$F' \xrightarrow{u_t} F$ family of maps

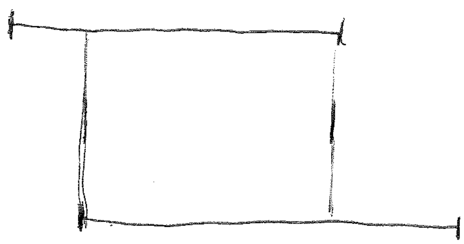
$F' \times I \longrightarrow F \times I \qquad y, t \mapsto (u_t(y), t)$

$$\begin{array}{ccc} & & SP(T) \\ & & \downarrow f \\ [0,1] & \xrightarrow{\lambda} & SP(T/A) \end{array}$$

I want to form the pull-back, except that this perhaps is too rigid. Why? Because the fibre of f over a point x on the k-th stratum is determined by $p_k : f^{-1}(U_k) \longrightarrow SP(A)$, and p_k is not an isom. on the fibres on all of U_k.

~~Suppose then that~~

Psychologically what I need is to have over each U_k a coordinate function, and a way of passing between.



~~so for each open set U~~

$U_i \quad i \in I$ open covering

form $\coprod U_\sigma \times \sigma$ = contraction of $\sigma \longmapsto U_\sigma$ contrav. $\sigma \longmapsto \bar{\sigma}$ cov.

over this I would have

$$\begin{array}{ccc} \bigcup \sigma \times U_\sigma \times F_\sigma & & \\ \downarrow & & \\ \bigcup \sigma \times U_\sigma & \longrightarrow & X \end{array}$$

[illegible] In general suppose I take ~~a vertex t_0 and~~ a simplex σ_0 and consider $E|_{U_{\sigma_0}}$

Question: Let V_i, $i \in I$, be a locally finite open covering of X and F_σ a space for each $\sigma \subset I$, $V_\sigma \neq \phi$; F_σ contrac. in σ. Form then the glue space E over X ~~[illegible]~~. Assuming that $F_\tau \to F_\sigma$ is an heq $\forall \sigma \subset \tau$, does it follow that

$$E_{U_\sigma} \longrightarrow U_\sigma \times F_\sigma$$

is a heq.

~~Psychology~~

~~Example~~ ~~[illegible]~~

E_U

Example:

$$\begin{array}{ccc} U & \subset & X \\ \downarrow & & \downarrow \\ F' & \to & F \end{array}$$

Thus

$$\begin{array}{ccccc} & & X \times F & & \\ & \nearrow & & \searrow & \\ E & & & & (U \times F)_* \\ & \searrow & & \nearrow & \\ & & (U \times F')_* & & \end{array}$$

is cartesian.

the fibre of E over a point of U is F'

the ——————————— $X-U$ is F

and specializing from a point of U to a point of $X-U$ corresponds to the given map of F' to F.

Question: If $F' \to F$ is a homotopy equivalence, is it true that $E \to X \times F$ is a homotopy equivalence

Thus I want to know when I might expect

$(X \times F) \longrightarrow (U \times F)_*$ to be a

So over $[0,1]$ I form $Cyl(F_0 \leftarrow F_{01} \rightarrow F_1)$.

More generally over K I can form $\underrightarrow{holim}(\sigma \mapsto F_\sigma)$ which I can think of as a space E over K together with a morp. of functors

$$\begin{array}{lccc} & E|_{U_\sigma} & \longrightarrow & F_\sigma \\ & \cup & & \uparrow \\ \sigma \subset \tau & & & \\ U_\sigma \supset U_\tau & E|_{U_\tau} & \longrightarrow & F_\tau \end{array}$$

Now what I want to prove is that if $F_\tau \to F_\sigma$ is a heq for every $\sigma \subset \tau$, then $E \to K$ is good for homotopy base change.

So for $Cyl(F_1 \leftarrow F_{01} \rightarrow F_1)$ one has

$$\begin{array}{ccc} & & F_0 \cup F_{01} \times [0,1] \cup F_1 \\ & & \downarrow \\ X & \xrightarrow{\lambda} & [0,1] \end{array}$$

now the map λ gives me two open sets $V_0 = \lambda^{-1}[0,1)$ $V_1 = \lambda^{-1}(0,1]$ and the pull-back of $Cyl(F_0 \leftarrow F_{01} \rightarrow F_1)$ via λ is $\lambda^* E$ universal for

$$\begin{array}{ccc} V_0 & \longrightarrow & F_0 \\ \cup & & \uparrow \\ V_{01} & \longrightarrow & F_{01} \\ \cap & & \downarrow \\ V_1 & \longrightarrow & F_1 \end{array}$$

and so what I want to show is that when the arrows $F_0 \leftarrow F_{01} \rightarrow F_1$ are heqs then $\lambda^* E \longrightarrow X \times E$ is a heq.

Summary of the position

If T is a ~~compact metr~~ finite polyhedron, with basepoint t_0, I want to define ~~[illegible]~~

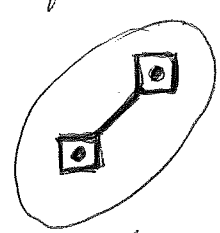
$$k(X; T, t_0)$$

which is the K-theory of unitary $\mathbb{C}^T$-bundles on X modulo unitary $\mathbb{C}^{t_0}$-bundles. Recall that

$$k(X;T,t_0) = \mathrm{Vect}(X;T) \big/ \mathrm{Vect}(X, t_0)$$

by definition, and that this is ~~[illegible]~~ probably a representable functor.

Recall that $\mathrm{Vect}(X;T)$ is the set of homotopy classes of unitary $\mathbb{C}^T$-bundles on X. This ~~[illegible]~~ functor of X is represented by



$$\coprod_{n\geq 0} PU_n \times^{U_n} D(\mathbb{C}^n;T)$$

where $D(\mathbb{C}^n;T) =$ ~~[illegible]~~ $*$ homos. of $\mathbb{C}^T$ into $\mathrm{End}(\mathbb{C}^n)$.

~~Another way is to say $\mathrm{Vect}(X;T)$ is represented by the~~ ~~[illegible]~~ This space is the class. space of the top. groupoid ~~[illegible]~~ consisting of unitary v. spaces ~~[illegible]~~ with $\mathbb{C}^T$-structure and their isos. ~~[illegible]~~

Now $k(X;T,t_0)$ should be represented by the classifying space of the top. cat. whose objects are unit. v.s. with $\mathbb{C}^T$-action, and whose maps are unitary embeddings with quotient given the basepoint structure

So suppose I is a poset and $\sigma \mapsto F_\sigma$ is a contravariant functor from simplices to spaces, then I can contract

$$\sigma \longmapsto \bar{\sigma} = \text{geometric simplex}$$

$$\sigma \longmapsto$$

Let I be a category. Then given a functor $i \mapsto F_i$ covariant I can ask for $\underrightarrow{\text{holim}}\, F_i$. In practice this means I **thicken** F_i **to** $V_i \times F_i$ where V_i is contractible and then take the usual ind.

$$E_{U_\sigma} \longrightarrow U_\sigma \times F_\sigma$$

$$\text{Hom}_{/X}\left(T, \text{Glue}(V_\sigma \times F_\sigma)\right) = \left\{ f_\sigma : T_{V_\sigma} \longrightarrow F_\sigma \;\middle|\; \forall \sigma \subset \tau \quad \begin{matrix} T_{V_\sigma} & \longrightarrow & F_\sigma \\ \cup & & \uparrow \\ T_{V_\tau} & \longrightarrow & F_\tau \end{matrix} \right\}$$

If T sits over V_{σ_0}, then can we recover $\{f_\sigma\}$ from the subfamily of $\{f_\sigma : \sigma \geq \sigma_0\}$. So fix a σ_1 and a $t \in T_{V_{\sigma_1}}$. Then because $\text{im}(t) \in X$ belongs to V_{σ_1} and V_{σ_0} $\exists$ $\tau \geq \sigma_1, \sigma_0$ such that $x \in V_\tau$. Thus f_{σ_1} in a nbd. of t is determined by f_τ where $\tau \geq \sigma_0$.

~~[illegible]~~ given $y \in SP(T/A)$ on the k-th stratum $s_k(y) < s_{k+1}(y) = 1$. Then p_k is the identity on the fibre over y. Thus $SP(T)$ is the full glueing.

Is it possible that $SP(T \cup A\times I)$ is the layered glueing? ~~In either case one finds no difference here~~ Thus let me ~~[illegible]~~ try to understand the fibre of $SP(T \cup A\times I)$ over a point $y \in SP(T/A)$ on the k th stratum, i.e. in $SP^k(T-A)$. Observe $T \cup A\times I / A \times I = T/A$, ~~and that~~ hence all fibres are $\sim SP(A\times I)$. Problem – no basepoint. ~~Therefore~~ Thus ~~[illegible]~~ I can't form $SP(T \cup A \times I)$. ~~one has the following idea~~

The point is that over $U_k \subset SP(T/A)$ one puts $U_k \times SP(A)$, and over $U_k \cap U_\ell$ one puts $U_k \cap U_\ell \times [0,1] \times SP(A)$ attached vertically

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<u>small open sets:</u> Let $\mathcal{U} = \{U_i,\ i \in I\}$ be a locally-finite open covering of a space X. $\mathcal{N}(\mathcal{U}) = \{\sigma \subset I \mid \sigma \text{ finite},\ U_\sigma \neq \phi\}$ ordered by $\sigma \leq \tau \iff \sigma \supset \tau$. $Op(\mathcal{U}) = \{U \text{ open in } X \mid U \subset U_i \text{ some } i\}$.

$\sigma \leq \tau \iff \sigma \supset \tau \Rightarrow U_\sigma \subset U_\tau$.

Then have functors

$$\sigma \longmapsto U_\sigma$$
$$\mathcal{N}(\mathcal{U}) \rightleftarrows Op(\mathcal{U})$$
$$f(u) = \{i \mid U \subset U_i\} \longleftarrow U$$

Then one has $U \subset U_i \iff i \in f(u)$

$U \subset U_\sigma \iff \sigma \subset f(u) \iff \sigma \geq f(u)$

$\mathrm{Hom}(U, U_\sigma) = \mathrm{Hom}(f(u), \sigma)$. So the functors are adjoint

Geometry of stratification: Open covering U_k of normal tubes around strata X_k. When one is in U_k one thinks of being able to retract down to X_k.

Suppose then we have a path $\lambda : [0,1] \to SP(T/A)$ and I want to understand the pull-back of the bundle $SP(T)$ over $SP(T/A)$ with respect to λ. And in particular I want to be able to lift λ. ~~[illegible] I don't proceed~~

$SP(T)$
$\downarrow f$
$SP(T/A)$

Lifting of paths to be understood first.

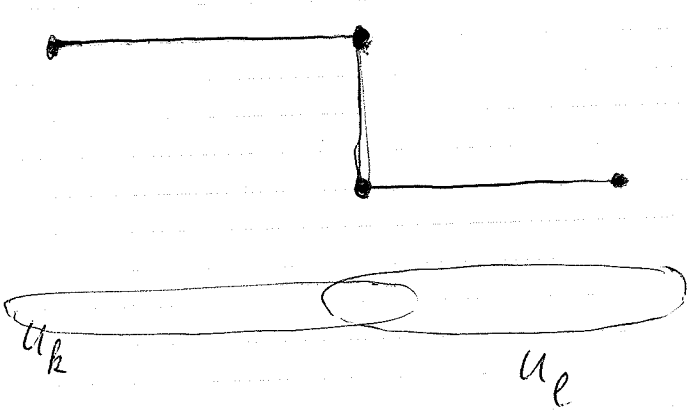
$$\lambda : [0,1] \longrightarrow SP(T/A)$$

Now $SP(T/A)$ has the covering $U_k = \{y \mid s_k(y) < s_{k+1}(y)\}$, and over U_k one has the fibre projection

$$p_k : f^{-1}(U_k) \longrightarrow SP(A)$$
$$\{t_1, \dots\} \longmapsto \{\pi t_{k+1}, \dots\}$$

which gives the fibre over the stratum $SP^k(T-A) \subset SP(T/A)$. ($U_k$)

I have the induced covering λ^{-1} of the unit interval, which I can refine into intervals in the usual way. So the critical case is where $[0,1] = \lambda^{-1} U_k$ for some k and we have constructed something over 0. This means that we have for each $0 \leq z \leq 1$ a point $\lambda(z) = \{t_1(z), t_2(z), \dots\}$ in $SP(T/A)$ such that $s_k(\lambda(z)) < s_{k+1}(\lambda(z))$, and a lifting of this point at $z=0$. Now because $s_k(\lambda(z))$ is always < 1, there is only one way of lifting the set of points $\{t_1(z), \dots, t_k(z)\}$ up to a set in T. Now the given lifting I have for $\lambda(0)$



Reason: I am looking at all of this carefully is that I want to have an example of glueing a cocycle.

Thus let me be given on a space X a covering $U_0, U_1, U_2, U_3 \ldots\ldots$ and cocycles

$$m_{kl} : U_k \cap U_l \longrightarrow M$$

so that I can on one hand glue so as to get an E/X with map

$$p_k : E_{U_k} \longrightarrow U_k \times M$$

such that $\quad p_j = m_{jk} \, p_k \quad$ for $\quad j < k.$

On the other hand there should be some way of working in the unit intervals. I have seen how over $SP(T/A)$ I have an open covering and a cocycle. I hope that $SP(T \cup A \times I)$ is the thing I had in mind. Better, what about $SP(T \cup CA)$ over $SP(T/A)$.

$$SP(T)_{U_k} \longrightarrow SP(A)$$

$$\{t_1, \ldots\} \longmapsto \{\pi t_{k+1}, \ldots\}$$

(Yes)

$$U_k = \{x \mid s_k(x) < s_{k+1}(x)\}$$

$$U_0 = \{x \mid s_1(x) > 0\} = SP(V)$$

Suppose one has a functor from small open sets to a monoid M. — small with respect to a covering $\{U_i\} = \mathfrak{U}$. Meaning that if $U \subset V$ are small open sets, then one has $c_{UV} : U \longrightarrow M$ satisfying transitivity, etc. Suppose then I try to glue wrt this cocycle. So over X I seek a space E with projections

$$p_U : E_U \longrightarrow U \times M$$

such that for $U \subset V$ one has

$$\begin{array}{ccc} E_U & \longrightarrow & U \times M \\ \cap & & \downarrow c_{UV} \\ E_V & \longrightarrow & V \times M \end{array}$$

commutes. Then universally it would seem that one has

$$E = \varprojlim (U \times M)!$$

and in particular given x

$$E_x = \varprojlim_{U \ni x} (U \times M)!$$

Examples I examined

$$c_{kl} : U_k \cap U_l \longrightarrow M$$

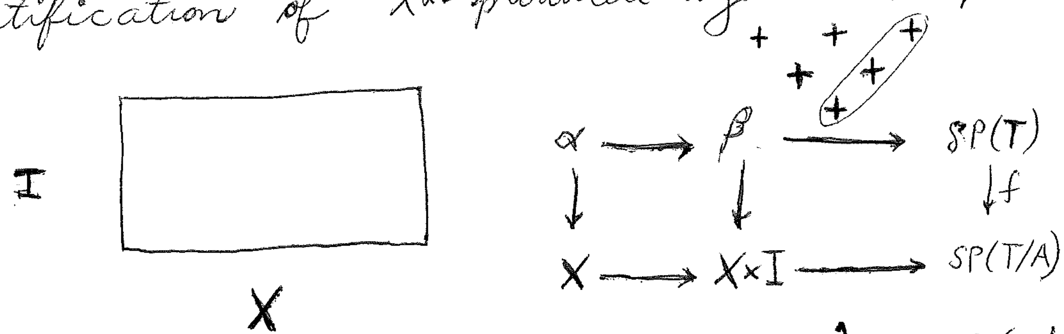
and $p_k = c_{kl}\, p_l$ and I want

$$\begin{array}{ccc} E|_{U_k \cap U_l} & \xrightarrow{p_k} & U_{kl} \times M \\ & \searrow^{p_l} & \uparrow c_{kl} \\ & & U_{kl} \times M \end{array}$$

example: Suppose I were directly to prove the result on the symm. product.

$$T \xrightarrow{f} T/A$$

Thus suppose I have a map $\xi : X \to SP^n(T)$ such that $f(\xi) : X \to SP^n(T/A)$ is homotopic to zero. I want to lift this homotopy back to ξ itself. So I consider the stratification of $X \times I$ produced by the homotopy of $f \cdot \xi$.



So what happens. X pt — have a path λ in $SP(T/A)$ which I want to lift. Cases: Suppose the path λ starts out at a point $\lambda(0) \in$ kth stratum and upstairs one gives the unique minimal rep. Now one can move to the jth stratum $j \leq k$ by letting part go to the basepoint — this is no trouble. But also one can move to the l th stratum $l > k$ by moving normally to the stratum — i.e. creating new particles away from the basepoint. This requires me to add new things upstairs.

Essential idea: ∃ a unique way of embedding $\mathbb{C}^T$ into Calkin alg^A which lifts to Bdd ops. Thus given a $\mathbb{C}^T$ vector space E, I should be able to add E up an inf. no. of times

$$E \oplus \cdots$$

and then add in the dense Hilbert representations whence I gets two Hilbert bundles on which $\mathbb{C}^T$ acts faithfully mod compacts, and a commuting Fredholm operators. Now trivializing these $\mathbb{C}^T$-Hilbert bundles, which should be possible mod compacts, one gets a family of units in the $\mathbb{C}^T$-Calkin algebra.

Essential idea: T compact metric space — there is distinguished embedding of $\mathbb{C}^T$ in A, hence a centralizer which one might call the $\mathbb{C}^T$ Calkin alg.

New version of Fredholmization — given a bundle E consider the Hilbert bundle map

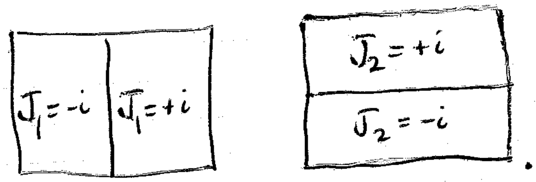
$$E \oplus H \xrightarrow{pr_2} H$$

and trivialize.

Fredholmization in $\mathbb{C}^T$-sense. Given a $\mathbb{C}^T$-bundle

Idea: Take Clifford alg. with gen. $J_1, \dots, J_N$ and relations
$$\begin{cases} J_i J_j = -J_j J_i & i \neq j \\ J_i^2 = -1 \end{cases}$$

Then take a Hilbert space H which is a C_N module of inf. mult. Action of J_1 given by decomposing H into $\pm i$ eigenspaces. These get interchanged by J_2



A = Calkin algebra.

$$\Omega(A^{u}) \sim \{ a \in A^{u} \mid a^2 = -1 \quad a^* = -a \}$$

$$\Omega \left\{ a \in A^* \,\middle|\, \begin{array}{l} a^2 = -1 \\ a^* = -a \end{array} \right\} = u_n(A)/\{$$

---

formula $\quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

~~essentially self~~

Fredholmization process.

$$\begin{array}{ccc} E & & E \oplus \cdots \\ \downarrow & & \downarrow \\ X & & X \end{array} \qquad \xrightarrow[\text{shift op}]{F}$$

{Fredholm bundle map}

Assume now that I have a space T around. Here the basic idea is that modulo compact operators there is exactly one embedding of $\mathbb{C}^T$ into bounded operators on H. Thus

So geometrically at least we see two different types of stratifications.

Glueing process.

~~Let π~~

$A =$ Calkin alg

$\mathcal{F} = \{a \in A \mid aa^* = a^*a = 1\}$

$\mathcal{F}_1 = \{a \in \mathcal{F} \mid a^2 = -1,\ a = -a^*\}$

$\mathcal{F}_2 = \left\{a \in \mathcal{F} \;\middle|\; \begin{array}{l} a^2 = -1, \quad a = -a^* \\ aJ_1a^{-1} = -J_1 \end{array}\right\}$

$$\mathcal{F}_1 \longrightarrow \Omega(\mathcal{F}; 1, -1)$$

$$J \longmapsto (\cos\theta + J\sin\theta) \qquad 0 \leq \theta \leq \pi$$

$$\mathcal{F}_2 \longrightarrow \Omega(\mathcal{F}_1; J_1, -J_1)$$

$$J \longmapsto J_1\cos\theta + J\sin\theta$$

Analysis of $\mathcal{F}_2$. From J_1 we get a projector E onto i-eigenspace, $1-E$ proj. onto $-i$ eigenspace. Then $aEa^{-1} = 1-E$, so a splits into 2 pieces ~~according to~~ $a = (1-E)aE + Ea(1-E)$ so $(1-E)aE$ should be any unitary thing between etc. $\therefore$ $\mathcal{F}_2 \cong \mathcal{F}$.

so suppose in a Hilbert space H I have a J_0 such that $J_0^2 = +1$ and the two eigenspaces are of infinite multiplicity. Now consider all J such that $J^2 = 1$ and such that $J \equiv J_0$ mod compacts. Hopefully the group $\mathcal{U}$ of unitaries $\equiv I$ mod compacts acts transitively on this J in which case it would give us $\mathcal{U}/\mathcal{U}\times\mathcal{U} \sim \mathbb{Z}\times BU$

?

Prob: Why is Ω(self-adj Fred.) $\sim$ Fred. ?

Idea: Is that we have

$$\text{self-adj. Fred} \sim \Omega(\text{Fred})$$

using the maps.

$$\begin{matrix} \left\{\begin{matrix}\text{unitaries} \\ \text{ess. spec } 1\end{matrix}\right\} & \longrightarrow & \begin{matrix}\text{linear} \\ \text{gp of } H\end{matrix} & \longrightarrow & \text{Fred.} \\ \uparrow & & & & \\ \left\{\begin{matrix}\text{self-adj} \\ \text{Fred. op} \\ \text{spec } \pm 1\end{matrix}\right\} & & & & \end{matrix}$$

~~Next part: Take Fredholm operators~~

Work in J. Take a skew-adj. F op. essential spec $\pm i$ call it F so that $F^2 = -I$ mod compacts. could try covering this by the contr. space of $F \ni F^2 = -I$, and the fibre over J_0 is the space

$$\{ J \mid J^2 = 1,\ J \equiv J_0 \text{ mod comp.} \}$$

I looked at above.

problem: Map Fredholm ops into loops in self-adjoint Fred. ops.

Given F a Fredholm op. I need a loop, thus

Start with a Hilbert space H on which one has the operators $J_1, J_2, \ldots, J_n$.

$$J_i^2 = -1 \qquad J_i J_j + J_j J_i = 0.$$

with infinite multiplicities.

Then $\{J \mid J^2 = -I\}$, $\pm i$ eigenspaces of inf. mult.$\}$?

---

work in a fin. diml. space $V = \mathbb{C}^N$ N large.

Suppose you consider $\{J \mid J^2 = +I\}$ so this space is the Grassmannian of V. $\coprod_k G_k(V)$.

So fix such an operator J and consider things commuting with J, better anti-commuting with J i.e. if $AJA^{-1} = -J$, then A must map the $+1$ eigenspace to the -1 eigenspace

$$J(Av) = -AJv = -Av$$

and the -1 to the $+1$ eigenspace. Thus if

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 0 & \beta^{-1} \\ \alpha^{-1} & 0 \end{pmatrix}$$

~~A unitary $\Rightarrow A^* = A^{-1}$~~

~~$\Rightarrow \beta^{-1} = \alpha^*$~~ $\beta^* = \alpha^{-1}$

~~$\beta = (\alpha^*)^{-1}$~~

$$A^* = \begin{pmatrix} 0 & \beta^* \\ \alpha^* & 0 \end{pmatrix}$$

$V \oplus W$

$\downarrow$

$V \oplus W$

Example: $\mathcal{F}$ = self. adj. Fredholms ess. spec. ±1.

Then ~~[illegible]~~ define

$$s_k(A) = |\lambda_k| \quad \text{where}$$

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots\cdots$$

Here one has

$$\mathcal{U}_k = \{A \mid s_k(A) < s_{k+1}(A)\}$$

$$X_k = \{A \mid 0 = s_1(A) = \cdots = s_k(A) < s_{k+1}(A)\}$$

$$\sim \quad B\mathcal{U}_k$$

~~Then $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots$~~

$\mathcal{U}_k$ is a normal tube around X_k.

$$X_0 = \mathcal{U}_0$$

$$X_1 = \mathcal{U}_1 - \mathcal{U}_0$$

---

do I get a cocycle here. On $\mathcal{U}_k \cap \mathcal{U}_\ell$ I have

$$s_k < s_{k+1} \leq \cdots \leq s_\ell < s_{\ell+1}$$

thus I have the ~~[illegible]~~ eigenvalues ~~[illegible]~~ and eigenspaces with $s_{k+1} \leq \cdots \leq s_\ell$ which I can ~~project onto~~ split up into

$$\coprod_{i+j=k-\ell} F_{ij}(H)$$

$$\wr$$

$$B\mathcal{U}_i \times B\mathcal{U}_j$$

It would seem one might have a ~~sheaf~~ cocycle with values in the monoid $(\coprod B\mathcal{U}_n)^2$

The basic method to think is in terms of the stratification of X defined as follows:

$$Z_0 = \{x \mid s_1(x) = 1\}$$

$$Z_k = \{x \mid s_{k+1}(x) = 1\}$$

so that one thinks of U_k as a normal tube around $X_k = Z_k - Z_{k-1} = \{x \mid s_k(x) < s_{k+1}(x) = 1\}$. The point is that over X_k one has p_k which determines the others p_j. Thus if

$$x \in X_k \cap U_j \qquad j < k$$

then $\quad p_j(x) = c_{jk}(x)\, p_k(x)$,

hence over X_k one has that $p_k : \quad X_k \times_{BM} PM \xrightarrow{p_k} X_k \times M$ is an isomorphism. YES.

The other possibility is to have

$$p_k m_{kl} = p_l$$

so that over $U_k \cap U_l$ p_k det. p_l

Here one would be inclined to have the reverse strat.

$$X_0 = U_0 \qquad f^{-1}(U_0) \xrightarrow[\cong]{p_0} U_0 \times M$$

$$X_1 = U_1 - U_0$$

$$X_2 = U_2 - (U_1 \cup U_0)$$

So this model of BM sits over simplex with vertices $k = 1, 2, \ldots$ etc. With the coarse topology, a map $X \longrightarrow BM$ can be identified with a partition, or better, a family of functions

$$0 \le s_1 \le s_2 \le \cdots\cdots \le 1.$$

together with for every $x \in X$, $s_k(x) < s_{k+1}(x)$, $s_\ell(x) < s_{\ell+1}(x)$ an element $m_{k\ell}(x)$,

$$m_{k\ell} : U_k \cap U_\ell \longrightarrow M$$

satisfying the cocycle condition.

If now I form the class space of the cat of M acting on itself the objects are (k, m) etc. so that the non-deg. simp. are

$$(k_0, m_0) \xrightarrow{m_{01}} (k_1, m_1) \longrightarrow \cdots \longrightarrow (k_p, m_p)$$

better $m_i = m_{ij} m_j$

Thus a map $X \longrightarrow PM$ can be ident. with a ~~covering U_k~~ family $s_1 \le \cdots$ and a functor

$$U_k \xrightarrow{p_k} M$$

$$U_k \cap U_\ell \xrightarrow{c_{k\ell}} M$$

such that on $U_k \cap U_\ell$.

$$p_k = c_{k\ell}\, p_\ell$$

$$x \in f^{-1}(U_k) \cap f^{-1}(U_\ell) \xrightarrow{p_k} SP(A)$$
$$\xrightarrow{p_\ell} SP(A)$$
$$\downarrow$$
$$U_k \cap U_\ell \xrightarrow{c_{k\ell}}$$

$x = \{t_1, t_2, \dots\}$ $\qquad$ $p_k(x) = \{\pi t_{k+1}, \dots\}$

$p_\ell(x) = \{\pi t_{\ell+1}, \dots\}$

$c_{k\ell}(f(x)) = \{\pi t_{k+1}, \dots, \pi t_\ell\}$

so $\boxed{p_k = (c_{k\ell} \circ f)\, p_\ell \quad \text{on } f^{-1}(U_k \cap U_\ell)}$

Review BM. Milnor model.

Replace M by the cat whose objects are pairs $(k, \cdot)$ $k = 1, 2, \dots$ in which maps are of form

$$(k, \cdot) \xrightarrow{m} (\ell, \cdot) \qquad \text{if } k < \ell$$

and identities.

So non-deg. simp. of form

$$(k_0, \cdot) \xrightarrow{m_{10}} (k_1, \cdot) \xrightarrow{m_{21}} (k_2, \cdot) \xrightarrow{m_{32}} \cdots\cdots$$

Good formulas:

$$(k_0, \cdot) \xrightarrow{m_{01}} (k_1, \cdot) \xrightarrow{m_{12}}$$

and so $\quad m_{j\ell} = m_{jk}\, m_{k\ell}$.

and that the points $t_{k+1}, \dots, t_\ell$ lie in $V/A - *$
because $0 < \rho(t_{k+1}) \leq \dots \leq \rho(t_\ell) < 1$.

Thus if I apply the ~~retr~~ retraction I get

$$\{\pi(t_{k+1}), \dots, \pi(t_\ell)\} \in SP^{\ell-k}(A)$$

so I define

$$c_{k\ell} : \mathcal{U}_k \cap \mathcal{U}_\ell \longrightarrow SP^{\ell-k}(A)$$

in this manner. $c_{k\ell}$ applied to $x = \{t_1, \dots\}$
is the image under retraction π of the seq. $t_{k+1}, \dots, t_\ell$.

Note the cocycle condition

$$\mathcal{U}_k \cap \mathcal{U}_\ell \cap \mathcal{U}_m \xrightarrow{(c_{k\ell},\, c_{\ell m})} SP^{\ell-k}(A) \times SP^{m-\ell}(A)$$

$$\{t_1, \dots\} \longmapsto \{\pi t_{k+1}, \dots, \pi t_\ell\},\ \{\pi t_{\ell+1}, \dots, \pi t_m\}$$

$$\mathcal{U}_k \cap \mathcal{U}_\ell \cap \mathcal{U}_m \xrightarrow{c_{km}} SP^{m-\ell}(A), \qquad SP^{\ell-k}(A) \times SP^{m-\ell}(A) \longrightarrow SP^{m-\ell}(A)$$

Next define fibre projections

$$p_k : f^{-1}(\mathcal{U}_k) \longrightarrow SP(A)$$

as follows. If $x \in SP(T)$ is in $f^{-1}(\mathcal{U}_k)$, so that

$$x = \{t_1, t_2, \dots\} \quad \text{with } \rho(t_k) < \rho(t_{k+1})$$

then $p_k(x) = \{\pi t_{k+1}, \pi t_{k+2}, \dots\}$.

Note that the p_k and the cocycle are related as follows:

symmetric products

$$\begin{array}{c} SP(T) \\ \downarrow f \\ SP(T/A) \end{array}$$

choose $\rho : T/A \longrightarrow [0,1]$

$\rho^{-1}(1) = A/A$

[illegible]

Given $x \in SP(T/A)$, $x = \{t_1, t_2, \cdots\}$ and can arrange

$$\rho(t_1) \leq \rho(t_2) \leq \cdots$$

then define $s_k(x) = \rho(t_k)$.

This way I get functions cont.

$$s_1 \leq s_2 \leq \cdots\cdots \quad \text{on } SP(T/A)$$

such that $s_k(x) = 1$ [illegible] k large.

Now put

$$U_k = \{ x \in SP(T/A) \mid s_k(x) < s_{k+1}(x) \} \qquad k = 1, 2, \ldots$$

and define for $k < l$

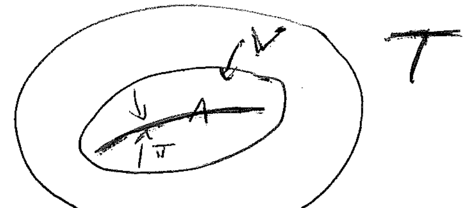
$$c_{kl} : U_k \cap U_l \longrightarrow SP^{l-k}(A)$$

as follows: ~~Let [illegible] me put~~ Given

$x \in U_k \cap U_l$ $\qquad s_k(x) < s_{k+1}(x) \leq \cdots \leq s_l(x) < s_{l+1}(x)$

$$x = \{t_1, t_2, \cdots t_k, t_{k+1}, \ldots, t_l, t_{l+1}, \cdots\}$$

the points $\{t_{k+1}, \ldots, t_l\}$ lie in a nbd. of A?



Have $\rho : T \longrightarrow [0,1]$

$\rho^{-1}(1) = A$.

$\rho^{-1}(0,1] = V$ retracts down to A.

Call $\pi : V \longrightarrow A$ the retraction. Note that

$$V - A = V/A - *$$

I have the feeling that over ~~[crossed out]~~ U, V one should give objects f_U, f_V and a transition cocycle over $U \cap V$, and that one should form over $U \cup V$. ~~[crossed out]~~ When one glues over $U \cup V$ one puts in an interval over $U \cap V$ to do the good thing.

Example: ~~[crossed out]~~ Compare $SP(T/A)$ with $SP(T \cup_A CA)$. There is a map

$$SP(T \cup CA) \longrightarrow SP(T/A)$$

and also one has the realization of

~~SP(T)~~

$$\coprod SP^n(T) \times SP(A) \rightrightarrows \coprod_n SP^n(T)$$

So one might hope that $SP(T \cup CA)$ might just be the realization of $\coprod SP^n(A)$ acting on $\coprod SP^n(T)$. What is a point of the realization? Non-degenerate k simplex: ~~[crossed out]~~

$$\coprod SP^\cdot(T) \times \left(\coprod SP^\cdot(A)\right)^k$$

$\tau, \alpha_1, \ldots, \alpha_k$ ~~[crossed out]~~ no $\alpha_i = 1$

so one gets integers $n_0, n_1 - n_0, \ldots n_k - n_{k-1}$, ~~[crossed out]~~ $\alpha_i \in SP^{n_i - n_{i-1}}(A)$. $i=1$

$\tau \in SP^{n_0}(T)$

hence a point will be a partition

$$t_{n_0} + \cdots + t_{n_k} = 1$$

plus $\quad \tau \quad \tau\alpha_1 \quad \tau\alpha_1 \cdots \alpha_k$

in terms of the increasing sequence

$$s_m = \sum_{i \le m} t_i$$

this means we have

$$s_{n_0} < s_{n_0+1} \quad , \quad s_{n_1} < s_{n_1+1} \quad \cdots$$

etc.

Now a point of $SP(T \cup CA)$ ~~[illegible]~~ sits over a point of $SP([0,1])$ which is a sequence

$$s_1 \le s_2 \le \cdots$$

assuming that the jumps are

$$0 = s_1 = \cdots = s_{n_0} < s_{n_0+1} = \cdots = s_{n_1} < s_{n_1+1} = \cdots$$

the point that lies over $s_1 \cdots s_{n_0}$ will be a point of $SP^{n_0}(T)$ and the point that lies over $s_{n_0+1} = \cdots = s_{n_1}$ will be in $SP^{n_1 - n_0}(A)$ etc. Thus it seems immensely clear that the realization of $\coprod SP^n(A)$ acting on $\coprod SP^n(T)$ is exactly ~~[illegible]~~ $SP(T \cup CA)$.

~~On the other hand I think I understand $SP(T/A)$ in terms of a function $p: T/A \to pt$ and a retraction of $p^{-1}[0,1]$ to A~~

$$\begin{array}{ccc} SP(T \cup A \times I) & \xrightarrow{\sim} & SP(T) \\ \downarrow & & \downarrow \\ SP(T \cup CA) & \xrightarrow[\sim]{} & SP(T/A) \end{array}$$

Good statement might be that if one has $I' \to I$ cofinal at each point x, then

$$R(I', U, F) \xleftarrow{\sim} R(I, U, F)$$

Generalizations – M monoid, to U to be an M-torsor sheaf with right M action and nice condition on the stalks. F to be a **right M** space. **Form** $U \times F$, can push down to X and form the appropriate inverse limits.

Basic problem now is to take a functor $\alpha \mapsto F_\alpha$ and **to replace it** by something which is good for this glueing process

---

~~The problem~~ The situation now. Given the torsor $P : \alpha \mapsto U_\alpha$ for the ordered set I, over X, and $\alpha \mapsto F_\alpha$ a functor to spaces, I have the glueing process $R(I, U_\alpha, F_\alpha) = \mathbf{Twist}(P, F)$. Thus I get a functor of twisting via the torsor P

$$\mathrm{Funct}(I^\circ, \text{spaces}) \longrightarrow \text{spaces}/X.$$

Now what I need is the homotopy properties. So suppose that I has an initial element α_0 in which case $P * F$ maps to F_{α_0}. What I want to understand is when, from the fact that $F_\beta \to F_\alpha$ are equivalences, I can conclude that $P * F \to X \times F_{\alpha_0}$ is an equivalence.

Example: Given $E \to F$, $U \subset X$

one twists to get a space [redacted] $(U\times E)\cup(Y\times F)$.

What do I need to argue ~~[illegible]~~ that if $E \to F$ is a hom. isom., then

$$Z = (U\times E)\cup(Y\times F) \longrightarrow X\times F$$

is a homology isomorphism?

[redacted]

$$\begin{array}{ccc} Y\times E & \longrightarrow & Y\times F \\ \downarrow & & \downarrow \\ X\times E & \longrightarrow & Z \end{array}$$

$U\times E$

Thus one should really ask when [redacted] excision holds.

[redacted] The point is that we have the map $Z \to X\times F$ which when restricted to both strata U, Y is a homology isom.

$$\begin{array}{ccccc} U\times E & \hookrightarrow & Z & \hookleftarrow & Y\times F \\ \downarrow & & \downarrow & & \| \\ U\times F & \hookrightarrow & X\times F & \hookleftarrow & Y\times F \end{array}$$

One possible method to proceed is to find ~~[illegible]~~ nbd. V of Y ~~which retracts to Y.~~ of which Y is a strong defm. retract. Then Z_V would also have Z_Y as a strong defm. retract, so one could use Mayer-Vietoris.

$\sigma \longmapsto F_\sigma$ contravariant, σ ranges over K

Then I form E by glueing $U_\sigma \times F_\sigma$. Now I want to show that $E_{U_\sigma} \longrightarrow U_\sigma \times F_\sigma$ is a heq provided I know that $F_\tau \longrightarrow F_\sigma$ is a heq for every $\tau \geq \sigma$.

So when I pull back to U_σ what do I get? ~~Under what conditions would one know that~~

$$U_{\sigma_0} \times_X \text{Glue}\,(U_\sigma \times F_\sigma) = \text{Glue}\,(U_{\sigma_0 \cup \sigma} \times F_\sigma)$$

$$\uparrow$$

$$\text{Glue}\,(U_{\sigma_0 \cup \sigma} \times F_{\sigma_0 \cup \sigma}$$

V_σ open sets on X.

$$\text{Glue}\,(V_\sigma \times F_\sigma) = \varprojlim_{\sigma \subset \tau} (V_\sigma \times F_\tau)_*$$

$$(V_\sigma \times F_\sigma)_* \rightarrow (V_\tau \times F_\sigma)_* \leftarrow (V_\tau \times F_\tau)_*$$

so the fibre over a point x is the inverse limit of F_σ and σ runs over all $\ni x \in U_\sigma$

$\therefore$ is F_σ where σ largest $\ni x \in U_\sigma$.

So it would seem that if we took the subcategory of all things such that

~~$x \in V_\sigma$~~

$$\text{Glue}_{\sigma_0 \subset \sigma} (V_\sigma \times F_\sigma)$$

fibre over x is ~~the~~ F_σ where σ largest $\ni x \in U_\sigma$ in part.

we are assuming $\forall x$ there is a largest $\sigma \ni x \in U_\sigma$. Thus when forming

$$L(\sigma \mapsto U_\sigma \times F_\sigma)$$

the fibre over x is

$$L(\underset{U_\sigma \ni x}{\sigma} \mapsto F_\sigma)$$

---

generalization: functor F_σ determines what sort of result?

So now the problem is to generalize this somehow. So given $\sigma \mapsto F_\sigma$ I want to replace it by a real good thing such that I won't have problems with glueing. Possible start: Let me first try to understand the covering U_σ of a simplicial complex, which presumably has all of the collaring properties desired.

Possibility given U_α indexed by a poset I I can then define strata

$$X_\alpha = \{x \mid \alpha \text{ last index such that } x \in U_\alpha\}$$

$$U_\alpha - \bigcup_{\beta \underset{\neq}{>} \alpha} U_\beta$$

For a simplicial complex $X_\sigma = U_\sigma - \bigcup_{\tau > \sigma} U_\tau$ = open simplex σ itself. In some sense U_σ should be a collaring around the stratum X_σ.

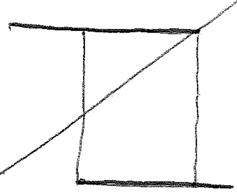
Example: Let K be a finite simplicial complex, and $\sigma \mapsto F_\sigma$ a contrav. functor ~~to~~ spaces. Then over K, I want to form the space

$$\bigcup \sigma \times U_\sigma \times F_\sigma$$

$$\downarrow$$

$$\bigcup U_\sigma = K$$

Definition of this space: It is a space E over K with maps $p_\sigma : E|_{U_\sigma} \longrightarrow U_\sigma \times F_\sigma$ such that



(i) Over

Example: K finite simplicial complex
U_σ = open star of σ
$\sigma \longmapsto F_\sigma$ contrav. functor

Define a space E over K with nat. transf

$$p_\sigma : E|_{U_\sigma} \longrightarrow F_\sigma$$

which is universal. Then one shows I think that

$$E = \bigcup \sigma \times F_\sigma$$

set-theoretically, and that except for topology E is the contraction of $\sigma \longmapsto \bar{\sigma}$ and $\sigma \longmapsto F_\sigma$.

Problem: Let $Vect(X;T)$ be the monoid of unitary v.b. over X with $\mathbb{C}^T$-action, and put

$$k(X;T,t_0) = Vect(X;T)/Vect(X,t_0).$$

Then is $k(X;T,t_0)$ represented by the classifying space of the top. cat consisting of unitary v.s. with $\mathbb{C}^T$-action and unitary embeddings with cokernel having the basepoint $\mathbb{C}^T$-action? Denote this cat by $\mathcal{V}(T,t_0)$.

I had some idea of what the classifying space of this category should represent. Namely over X one should have a bundle with fibres of different dimensions, and on a given fibre one has an action of $\mathbb{C}^{T-t_0}$. One has the possibilities of specializing — i.e. letting some eigenvalues go to the basepoint. This means intuitively that the places where $\leq k$ eigenvalues are not at the basepoint is a closed set.

~~idea: suppose~~

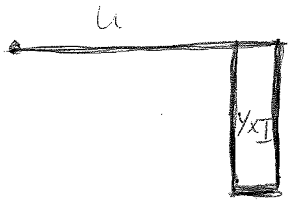
Precise idea is first that a T mod t_0-bundle over X has to induce a map $X \longrightarrow SP(T)$ which will then induce an open covering of X etc. and strata once I choose $\rho: T \longrightarrow [0,1]$ $\ni$ $\rho^{-1}(1) = t_0$, i.e. ρ is the barycentric coordinate of t_0. The rest of the data is <u>determined</u> in a way I understand. Except in a certain technical sense, where one thinks that to give the map $X \longrightarrow SP[0,1]$ in advance is a bit too strong.

Goal: To thicken F_α and/or U_α to get a good glueing. I can thicken F_α say by generalized mapping cylinder construction:

F generated by functors of the form

$$\alpha \longmapsto \operatorname{Hom}(\alpha, \alpha_0) \times T$$

Example. Thickening U, X



$$\left.\right\} \quad U' \times E \cup Y \times F$$

$$\quad\quad = \quad U \times E \cup Y \times [0,1) \times E \cup Y \times F$$

$$U \times E \cup Y \times (E \times [0,1] \cup F)$$

same set.

---

thickening process. First suppose we have a simp. cx. K. Then I can form over K the space $\bigcup \sigma \times F_\sigma = Z$ and now call the thickening maybe should by something like $\sigma \mapsto \bigcup_{\tau \geq \sigma} \tau \times F_\tau = Z_{U_\sigma}$

~~Suppose then that one has $\sigma \mapsto \bigcup_{\tau \geq \sigma} \tau \times F_\tau$~~

$$\alpha \longmapsto |\alpha \backslash I|$$

If I have a class. space then each U_α is contractible, and is what you get from the functor

$$\sigma \mapsto \operatorname{Hom}(\ ,\ )$$

Review the situation: Let J be a poset and X a space. Then a J-torsor over X may be identified with a contravariant functor $J \longrightarrow \mathrm{Open}(X)$, $\sigma \mapsto U_\sigma$ such that for every $x \in X$, $\{\sigma \in J \mid x \in U_\sigma\}$ is a directed set. (meaning $x \in U_\sigma,\ x \in U_\tau \Rightarrow \exists\ \rho \geq \sigma, \tau \ni x \in U_\rho$)
If $\sigma \mapsto F_\sigma$ is a contravariant functor from J to spaces I can twist this functor with the torsor $\sigma \mapsto U_\sigma$ to get a space $T(J, \sigma \mapsto U_\sigma, \sigma \mapsto F_\sigma)$ with the following universal property: A map $Y \longrightarrow T(J, U, F)$ of spaces over X is the same as a natural transf. of functors

$$U_\sigma \times_X Y \longrightarrow F_\sigma$$

(I recall that a covariant functor $\sigma \mapsto G_\sigma$ to sets can be twisted with $\sigma \mapsto U_\sigma$ so as to give a sheaf

$$U \overset{J}{\times} G = \text{contraction}(\sigma \mapsto U_\sigma)\text{with}(\sigma \mapsto G_\sigma)$$

$$= \varinjlim_{(\sigma, \xi) \in J_G} (\sigma \mapsto U_\sigma)$$

and that $G \mapsto U \times^J G$ is the base change functor for a morphism of topoi: (Sheaves$/X$) $\longrightarrow$ Funct(J, Sets).)

It following immediately from the definition that $\forall$ map $f : X' \longrightarrow X$ one has

$$f^* T(J, U, F) = T(J, f^* U, F),$$

hence taking $X' =$ point x of X, one sees the fibre of $T(J, U, F)$ over x is $\varprojlim (\sigma \mapsto F_\sigma)$. In the interesting case where $\sigma, x \in U_\sigma$ $\forall x$ there is a large σ such that $x \in U_\sigma$, this means the fibre over x of $T(J, U, F)$ is just the space F_σ.

Better: Make J into a space by declaring each of the sets $\{\tau \mid \tau \geq \sigma\}$ to be open. i.e. the open sets are the ones closed under generalizing. Then each point $\sigma \in J$ has a smallest nbd. $U_\sigma = \{\tau \geq \sigma\}$, and a map $f: X \to J$ gives

$$\sigma \mapsto f^{-1}(U_\sigma) \qquad \text{cont.} \qquad \sigma \leq \tau \quad U_\sigma \supset U_\tau$$

$\ni \forall x$ there is a least $\sigma \ni x \in U_\sigma$ namely $f(x)$.

$$x \in f^{-1}(U_\sigma) \iff f(x) \geq \sigma$$

Therefore what seems to happen is that a map $f: X \to J$ is the same thing as a J-torsor $\sigma \mapsto V_\sigma$ on X such that $\forall x\ \exists$ largest $\sigma \ni x \in V_\sigma$. Then $f(x) =$ this largest σ.

Now that I seem to understanding twisting I should like to understand what is a classif. space. Basically a J-torsor $\sigma \mapsto U_\sigma$ should be universal when each U_σ is contractible.

Suppose I have two spaces X, Y with J-torsois $\sigma \mapsto U_\sigma$, $\sigma \mapsto V_\sigma$ resp. Then one can form a space over X and a space over Y.

$$\begin{array}{c} T(J, V, \sigma \mapsto U_\sigma) \\ \downarrow \\ Y \end{array} \qquad\qquad \begin{array}{ccc} & & T \\ & & \downarrow \\ Y & \longrightarrow & J \end{array}$$

$T(V \times^J U) = Y \times$

$$V \times^{\sigma} (U) \qquad \overline{V} \times_{Y} V_\sigma \longrightarrow U_\sigma$$
$$\downarrow$$
$$Y$$

suppose Y is now

[illegible]

$$\begin{array}{ccc} T_{/X}(U_\sigma, V_\sigma) & \ni & V_{f(x)} \\ \downarrow & & \\ X & \ni & x \end{array}$$

$$T_{/X}(U_\sigma, V_\sigma) \hookrightarrow T_{/X}(U_\sigma, Y) = X \times Y$$

$$T_{/X}(U, V) = \bigcup_x x \times V_{f(x)} = \{(x,y) \mid f(y) \geq f(x)\}$$

$$T_{/Y}(V, U) = \bigcup_y U_{f(y)} \times y = \{(x,y) \mid f(x) \geq f(y)\}$$

The problem now is to understand [illegible] when this glueing process is reasonable.

What you would like to prove is that if U_σ is the standard open covering of a s. cx. X and if the maps $F_\sigma \to F_\tau$ are equiv. then F_σ = homotopy fibre.

Try some more. I start with X and $\sigma \mapsto U_\sigma$ and $\sigma \mapsto F_\sigma$ and I get the space

$$T(U \times^J F) \text{ over } X.$$

Now when J has an initial element σ_0 one has a map

$$T(U \times^J F) \longrightarrow X \times F$$

which one wants to show is an equivalence when each $F_\sigma \to F_\tau$ is. Thus one is reduced to the case of showing that $F_\sigma \to F'_\sigma$ equiv. $\forall\, \tau \le \sigma \Rightarrow T(U \times^J F) \to T(U \times^J F')$ is an equivalence. ~~Suppose instead that~~.

~~T(U×~~

Suppose X is a s. cx. + U_σ st. family. Then is $T(U \times^J F)\big|_{U_\sigma}$ hom. equiv. to F_σ?

Yes Thus if $F \to F'$ is an eq. pointwise $\Rightarrow T(U \times^J F) \longrightarrow T(U \times^J F')$ equiv. over each U_σ

Suppose you thicken F. ~~Then~~ and replace T by a LT. Is it clear, then?

---

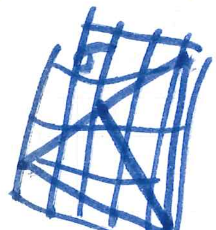
equiv. over each $U_\sigma \overset{?}{\Rightarrow}$ equiv.?

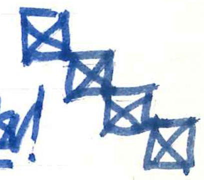
The idea now is to replace $\sigma \mapsto F_\sigma$ by 
its thickening which is

$$\sigma \longmapsto \underset{\tau \geq \sigma}{\underrightarrow{\text{holim}}}\ F_\tau$$

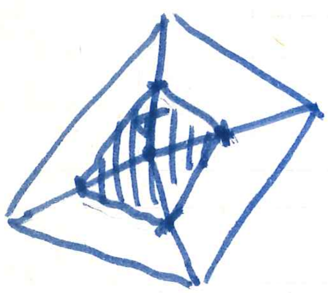
Somehow this is the link$_\wedge^{\text{cone}}$ of the simplex σ.
Thus U_σ Take U_σ = open star of σ

$U_\sigma \sim \sigma \times$ Cone Link(σ).

I can  identify Link(σ) with what?

$\sigma \longmapsto U_\sigma$ contravariant 

C_σ = cone on Link(σ) ~~[illegible]~~!



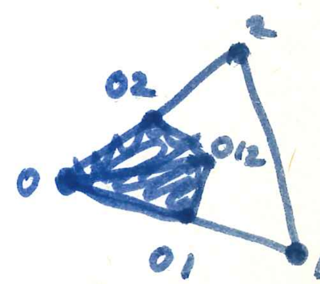
The simplicial complex consisting of
~~elements $\sigma = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_q$~~

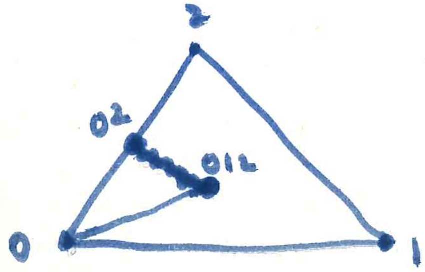
~~$C_\sigma = \{(\sigma_0 \leq \cdots \leq \sigma_q) \mid \sigma \leq \sigma_0\}$~~

~~= simp. cx. of $\{\sigma' \mid \sigma' \geq \sigma\}$~~

Link(σ) = simp. τ [redacted] disj. from σ ∋ $\sigma \cup \tau$ is a simp.

C_σ = [redacted] $\frac{1}{2}$ bary.(σ) $*$ $\frac{1}{2}$ Link(σ).





$(0) < (02) < (012)$.

would seem that C_σ = [redacted] $\{(\sigma_0 \leq \cdots \leq \sigma_q) \mid \sigma \leq \sigma_0\}$

$C_\sigma = \{(\sigma_0 \leq \cdots \leq \sigma_q)$

Geometrically: Take following.

C_σ = part of baryc. subd. cons. of $\sigma_0 < \cdots < \sigma_q$ with $\sigma \leq \sigma_0$

$C'_\sigma \equiv \text{Cone}\{\text{Link}(\sigma)\}$

Are these the same simplicial complex.

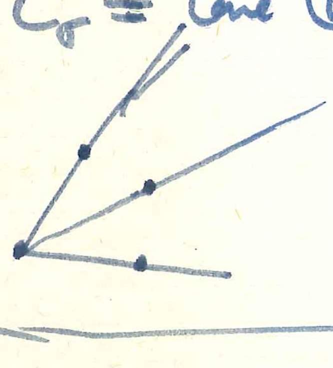
~~Given a vertex σ_0 of C_σ~~ Let $\sigma_0 = (v_0 \cdots v_k)$. Let $\tau = (v_0 \cdots v_k\ v_{k+1} \cdots v_p)$ be a vertex of C_σ. Then can assoc. to τ either

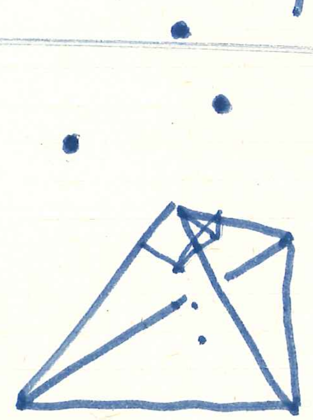
$$(v_{k+1} \cdots v_p) \text{ if } p > k$$

$$\underset{\text{Link}(\sigma)}{\wedge}$$

or the vertex of the cone. Thus can send τ to the barycenter of $v_{k+1} \cdots v_p$

$C'_\sigma = \text{Cone}(\text{Link})$





Conjecture: C_σ = barycentric subdivision of Cone $\{\text{Link}(\sigma)\}$. Such a simp. is a chain $\tau_0 < \cdots < \tau_q$ of simp. in the cone. Can be ident. with a chain $\sigma_0 < \cdots < \sigma_q$ in C_σ.

X space, I poset,

$\alpha \longmapsto U_\alpha \qquad \alpha \le \beta \Rightarrow U_\alpha \supset U_\beta$ contrav. functor $I \to Op(X)$.

such that : $\forall x \quad \{\alpha \mid x \in U_\alpha\}$ is directed.

$\alpha \longmapsto F_\alpha$ functor $I^0 \longrightarrow$ Spaces.

For each space T over X consider the set whose elements are natural transf $T_{U_\alpha} \longrightarrow F_\alpha$

$$T \longmapsto \mathrm{Hom}\big((\alpha \mapsto U_\alpha \times_X T), (\alpha \mapsto F_\alpha)\big)$$

I want to prove this functor of T is representable.

~~First consider~~ Assume $\forall U$ open in X, space F the functor $T \longmapsto \mathrm{Hom}(U \times_X T, F)$ is representable by a space ${}_U F$. Then

$$\begin{array}{ccc} U_\alpha \times_X T & \longrightarrow & F_\alpha \\ \cup & & \uparrow \\ U_\beta \times_X T & \longrightarrow & F_\beta \end{array}$$

$$\mathrm{Hom}\big((\alpha \mapsto U_\alpha \times_X T), (\alpha \mapsto F_\alpha)\big)$$

$$\|$$

$$\mathrm{Ker}\Big\{\prod_\alpha \mathrm{Hom}(U_\alpha \times_X T, F_\alpha) \rightrightarrows \prod_{\alpha \le \beta} \mathrm{Hom}(U_\beta \times_X T, F_\alpha)\Big\}$$

$$\|$$

$$\mathrm{Ker}\Big\{\prod_\alpha \mathrm{Hom}_{/X}(T, {}_{U_\alpha}F_\alpha) \rightrightarrows \prod_{\alpha \le \beta} \mathrm{Hom}_{/X}(T, {}_{U_\beta}F_\alpha)\Big\}$$

$$\|$$

$$\mathrm{Hom}_{/X}\Big(T, \mathrm{Ker}\Big\{\prod_{\alpha / X} {}_{U_\alpha}F_\alpha \rightrightarrows \prod_{\alpha \le \beta \, / X} {}_{U_\beta}F_\alpha\Big\}\Big)$$

To represent $\mathrm{Hom}(U \times_X T, F)$ ~~consider the space~~ define

$$R = (U \times F) \cup (X - U) \quad \text{as a set}$$

so that one has maps $\begin{cases} R \longrightarrow X \\ R \times_X U \longrightarrow F \end{cases}$

Put least topology on R such that these maps are cont. Then a function $Z \longrightarrow R$ is cont. $\Leftrightarrow$ $Z \longrightarrow X$ is and $Z \times_X U \longrightarrow F$ is.

In particular given T over X function $T \to R$ over X is cont. $\Leftrightarrow$ $U \times_X T \longrightarrow F$ is cont. So

$$\mathrm{Hom}_{/X}(T, R) \longrightarrow \mathrm{Hom}(U \times_X T, F)$$

Clearly injective. etc.

Notation $C(\alpha \mapsto U_\alpha \times F_\alpha)$ for this space.

$f : Y \longrightarrow X$

then $f^* R(\alpha \mapsto U_\alpha \times F_\alpha) = C(\alpha \mapsto f^* U_\alpha \times F_\alpha)$

Proof:

$$\mathrm{Hom}_{/Y}(T, f^* C(I, U, F))$$

$$\parallel$$

$$\mathrm{Hom}_{/X}(T, R(I, U, F))$$

$$\parallel$$

$$\mathrm{Hom}_{\mathrm{Funct}(I, Sp)}(U \times_X T, F)$$

$$\parallel$$

$$\mathrm{Hom}_{\mathrm{Funct}(I, Sp)}(\overset{f^* U}{\cancel{U \times_X Y}} \times_Y T, F)$$

$$\parallel$$

$$\mathrm{Hom}_{/Y}(T, R(I, f^* U, F)).$$

~~outline in time~~
~~15 min on $K_n = \pi_n BGL(A)^+$~~
~~15 min on Segal Anderson~~

Segal – Anderson theory: which I like to think of as an ext. of Dold-Thom theory of inf. symm. products.
Begin with an example:

Γ = cat whose objects are finite sets with basept whose morph of basepoint pres. maps.

or to more concrete, replace Γ by full subcat cons. of $\underline{n} = \{0, 1, \ldots, n\}$, basept 0, $\forall n \geq 0$.

X space with basepoint, and $S \in Ob\,\Gamma$ put

$$X^S = Hom^{pt}(S, X) = \prod_{S-*} X$$

$$S \longmapsto X^S \quad \text{contrav. from } \Gamma \text{ to spaces}$$

M top. ab. monoid. Put

$$M[S] = \prod_{S-*} M = \left\{\text{reduced chains on } S \text{ coeff in } M\right\}$$
$$\sum_{s \in S} m_s \cdot s \qquad m_* = 0$$

Given $S \xrightarrow{u} S'$, we have induced map

$$u_* : M[S] \longrightarrow M[S']$$

$$u_*\left(\sum m_s \cdot s\right) \longmapsto \sum_{s'} \left(\sum_{u(s)=s'} m_s\right) \cdot s'$$

Then $S \mapsto M[S]$ covariant from Γ to spaces.

Put $M[X]$ = contraction of $S \mapsto M[S]$ with $S \mapsto X^S$

$$= \coprod_n M[\underline{n}] \times X^{\underline{n}} \Big/ (\theta_* \alpha, \beta) = (\alpha, \theta^* \beta).$$

$M[X]$ is the universal gadget with
Thus canonical maps $M[S] \times X^S \longrightarrow M[X]$
comp. with morphisms in Γ.

Easily seen that a point of $M[X]$ is a
0-chain $\sum_i m_x \cdot x$ finite sum, $m_* = 0$. Ex. $M = \mathbb{N}$
$M[X] = SP(X)$. Key result of D-T theory is

$$Y \subset X \text{ finite complexes}$$
$$\Rightarrow \quad M(Y) \longrightarrow M(X) \longrightarrow M(X/Y) \quad \text{has h. type}$$
of a fibration (hence
$$X \longmapsto \pi_q M(X) \qquad q \geq 0$$
is a gen. hom. theory ;)

[Segal-And. gen. goes as follows: First I need to define the class. space of a category $\mathcal{C}$. Let Δ = cat consisting of the posets $[n] = \{0, 1, \ldots, n\}$ all $n \geq 0$ nat. ordering, + all weakly monotone maps.

$$[p] \longmapsto N_p\mathcal{C} = \text{diagrams}: \ X_0 \to \cdots \to X_p \quad \text{in } \mathcal{C}$$
contrav. from Δ to Sets. On the other hand
$$[p] \longmapsto \Delta[p] = \text{simplex with vertices } 0, 1, \ldots, n$$
is covariant from Δ to Sets. Put
$$B\mathcal{C} = \text{contraction} = \coprod_n \Delta(p) \times N_p\mathcal{C} \Big/ (\theta_* \alpha, \beta) = (\alpha, \theta^* \beta) \quad \text{all } \theta$$

<u>Mention</u>: G discr. gp. $\Rightarrow BG$ is usual class. space.

$\mathcal{P} = \mathcal{P}_A$. Given S in Γ, define $\mathcal{P}[S]$ = categ.

whose objects are obj. P of $\mathcal{P}_A$ equipped with a direct sum decomp $P = \bigoplus_{s \in S} P_s$, $P_* = 0$; morphisms are isos

$S \mapsto \mathcal{P}_A[S]$ ~~functor~~ cov. from Γ to Cats

$S \mapsto B\mathcal{P}_A[S]$ ——————————— Spaces.

So can put

$$B\mathcal{P}_A[X] = \text{contraction } S \mapsto B\mathcal{P}_A[S], \; S \mapsto X^S$$

Can think of a point of $B\mathcal{P}_A[X]$ as a chain $\sum P_x \cdot x$ $P_x \in \mathcal{P}_A$.

<u>Thm 1</u>: ~~Ca~~ For <u>conn.</u> X, $X \mapsto \pi_q(B\mathcal{P}_A[X])$ is a gen. homol. theory.

<u>Thm 2</u>: $\pi_{q+n}(B\mathcal{P}_A[S^n]) \cong K_q A$

25 min. $\Omega^n B\mathcal{P}_A[S^n] \simeq K_0 A \times BGL(A)^+$.

——

~~Problem: Suppose one were to understand the relations which~~

Problem: To understand the classifying space of a category in some suitable framework.

I begin with the case of $X_0 \leftarrow X_{01} \rightarrow X_1$ which has $Cyl(X_0 \leftarrow X_{01} \rightarrow X_1) = X_0 \cup X_{01} \times (0,1) \cup X_1$ as its classifying. One takes the weak topology on this space so that we know what are the maps. Thus a map ~~Cyl(X_0~~

$$Z \xrightarrow{f} Cyl(X_0 \leftarrow X_{01} \rightarrow X_1)$$

will consist first of all of a map $Z \xrightarrow{pf} [0,1]$ which I denote λ. ~~we put $Z_0 = \{z \mid \lambda$~~

Better to think of $\lambda : Z \rightarrow \Delta(1)$ $\quad \lambda_i = i$-th coord.

$$\lambda_i(z) = \begin{cases} pf(z) & i=0 \\ 1-pf(z) & i=1 \end{cases}$$

Then over Z_0 I have $Z_0 \longrightarrow X_0$

Z_{01} $\qquad Z_{01} \longrightarrow X_{01}$

$\qquad Z_1 \longrightarrow X_1$

Would seem that given $\lambda : Z \longrightarrow \Delta(1)$

$$Hom(Z, Cyl(X_0 \leftarrow X_{01} \rightarrow X_1))_\lambda = Hom(Z_0 \leftarrow Z_{01} \rightarrow Z_1 \, , X_0 \leftarrow X_{01} \rightarrow X_1)$$

Therefore the thing to notice is that $Cyl(X_0 \leftarrow X_{01} \rightarrow X_1)$ comes first of all with a canonical λ and functor.

Generalization: Let K be a simplicial complex, let $\sigma \longmapsto X_\sigma$ be a functor from simplices of K to spaces such that $\sigma \subset \tau$ (σ face of τ) $\Longrightarrow X_\sigma \leftarrow X_\tau$ i.e. $\sigma \mapsto X_\sigma$ is contravariant. Then one can form a quotient space over K by contracting the functors $\sigma \longmapsto \bar{\sigma}$ closed simplex

$\sigma \longmapsto X_\sigma$

but given the weak topology, meaning this.

Set theoretically a map $Z \to \underrightarrow{\mathrm{holim}}(\sigma \mapsto X_\sigma)$

It would seem that in good conditions one gets something reasonable.

unitary vector spaces of finite dim.

For any finite set with basepoint S consider orthogonal decompositions of Hilbert space H indexed by $s \in S$ i.e. $\forall s$ we give s.a. proj. E_s such that $E_s E_t = E_t E_s = 0$ $s \neq t$ and such that rank E_s finite for $s \neq$ basept.

These form a space in an obvious way with components indexed by functions $d : S - * \longrightarrow \mathbb{N}$

$S = \{*, 1, \ldots, n\}$, then the component of deg. d is the space of flags in H $\quad 0 \subset V_1 \subset \cdots \subset V_n$ with V_i / V_{i-1} of dim d_i.

This space has the homotopy type of $BU_{d_1} \times \cdots \times BU_{d_n}$.

Next – one has the following: If D_S is as above and one has $S \longrightarrow S'$ then one has $D_S \longrightarrow D_{S'}$.

Therefore when I mix $S \mapsto X^S$ and $S \longmapsto D_S$ what I seem to get is a space consisting of $\mathbb{C}^X$- actions on H such that the multiplicity is finite outside of the basept.

~~So the next point is that one wins certain things~~

Compare with $\mathbb{C}^X$-actions such that mult. by f is compact if f vanishes at basepoint.

$U_\sigma = \bigcap U_i$ etc. then one forms.

$$\underrightarrow{\text{holim}}\, U_\sigma \times F_\sigma \longleftarrow \underleftarrow{\text{holim}}\, U_\sigma \times F_{\sigma \cup i_0} \longrightarrow \underleftarrow{\text{holim}}\, U_\sigma \times F_{i_0}$$

$$\underleftarrow{\text{holim}}\, U_\sigma \times F_{i_0} \downarrow \left(\underleftarrow{\text{holim}}\, U_\sigma\right) \times F_{i_0} \downarrow X \times F_{i_0}$$

This is a heq and compatible with base change over X.

Let

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

be a universal heq/X. Does this imply u is a fheq?

---

formula for $\underleftarrow{\text{holim}}\, F_\sigma$ σ simplices in K which has the good properties.

Take the set $\bigcup \sigma \times F_\sigma$ σ open simplex

which sits over $K = \bigcup \sigma$, let U_σ = open star of $\sigma = \bigcup_{\tau \geq \sigma} \tau$, so that we have

$$U_\sigma \quad + \quad \text{maps } p^{-1}U_\sigma = \bigcup_{\tau \geq \sigma} \tau \times F_\tau \longrightarrow F_\sigma$$

coordinate projections. Now define the top on $\bigcup \sigma \times F_\sigma$ so as to have fewest open sets $\ni$ $p^{-1}U_\sigma$ is open + $p_\sigma : p^{-1}U_\sigma \longrightarrow F_\sigma$ is continuous.

$$\underrightarrow{\text{holim}}\ U_\sigma \times F_\sigma \longleftarrow \underrightarrow{\text{holim}}\ U_\sigma \times F_{\sigma \cup i_0} \longrightarrow \underrightarrow{\text{holim}}\ (U_\sigma \times F_{i_0})$$

$$\downarrow$$

$$X \times F_{i_0}$$

this is a heq as the covering is numerable

which will be natural over each open set $U \subset X$.

What have I used in this proof.

$\sigma \longmapsto \underrightarrow{\text{holim}}_\sigma\ F_\sigma$ has certain formal properties.

1) preserves heqs

2) Base change - If all the F_σ are over a fixed space X, and one has $Y \xrightarrow{f} X$, then

$$\underrightarrow{\text{holim}}\ Y \times_X F_\sigma = Y \times_X \underrightarrow{\text{holim}}\ F_\sigma$$

If this is true, then for U_σ over X ∋ $\underrightarrow{\text{holim}}\ U_\sigma \longrightarrow X$ is a fheq, we have for all Y over X, that holim

$$\underrightarrow{\text{holim}}\ Y \times_X U_\sigma \longrightarrow Y$$

$$\| \qquad \nearrow$$

$$Y \times_X \underrightarrow{\text{holim}}\ U_\sigma$$

is a heq.

~~[illegible]~~

$\underrightarrow{\text{holim}}\ U_\sigma \longrightarrow X$ "shrinkable" then one has a ~~cell like map~~ map $\underrightarrow{\text{holim}}\ U_\sigma$

classifying space of a top category. $\mathcal{F}$

quasi-fibrations.

Example: Let $\{U_i, i \in I\}$ be a numerable open covering of X, put

$$U_\sigma = \bigcap_{i \in \sigma} U_i$$

for each finite set $\sigma \subset I$, and let for each $\sigma \ni U_\sigma \neq \phi$ there be given a functor $\sigma \longmapsto F_\sigma$ contravariant.

Form the space E over X by contracting

$$\left| \begin{array}{lll} \sigma \longmapsto U_\sigma \times F_\sigma & & \text{contravariant} \\ \sigma \longmapsto \Delta(\sigma) & & \text{covariant} \end{array} \right.$$

Claim $E \longrightarrow X$ is "good" for homotopy base-change if $\forall \tau \subset \sigma$ with $U_\sigma \neq \phi$ we have $F_\sigma \to F_\tau$ is a homotopy equivalence.

Proof: The conclusion is local over numerable coverings. Also if one has an equiv. relation R on a Y over X, then forming Y/R is local on X, i.e.

$$\begin{array}{ccc} Y_U/R_U & \subset & Y/R \\ \downarrow & & \downarrow \\ U & \underset{\text{open}}{\hookrightarrow} & X \end{array}$$

is cartesian.

Thus formally one can suppose that $U = U_{i_0}$ for some i_0. Here one has $\sigma \cup i_0$ is in the nerve for any σ. So one has maps of contra. functors

$$U_\sigma \times F_\sigma \longleftarrow U_\sigma \times F_{\sigma \cup i_0} \longrightarrow U_\sigma \times F_{i_0}$$

which are heqs. Thus get over X a heq

~~Suppose I is a poset, [illegible] then might a classifying space for I consist of~~

~~$i \longmapsto U_i$ contravariant~~

~~I poset, $i \mapsto U_i$ contra functor to spaces over X~~

~~$i \longmapsto F_i$ a ϕ contra functor to spaces. [illegible]~~

~~Assume holim~~

K simplicial complex, $\sigma \longmapsto U_\sigma$ = open star of σ,

$F \longmapsto F_\sigma$ contravariant. Then

$$\underrightarrow{\mathrm{holim}}\, F_\sigma = \bigcup \sigma \times F_\sigma$$

$$= \text{contraction of } \sigma \longmapsto \sigma \text{ and } \sigma \longmapsto F_\sigma$$

has open covering $p^{-1}(U_\sigma) = \bigcup_{\tau \geq \sigma} \tau \times F_\tau$

and coordinate proj.

$$\gamma_\sigma : p^{-1}(U_\sigma) \longrightarrow F_\sigma$$

~~And in [illegible] some sense $\underrightarrow{\mathrm{holim}}\, F_\sigma$ is the result of glueing~~ Then a map

$$T \longrightarrow \underrightarrow{\mathrm{holim}}\, F_\sigma$$

is a map $T \xrightarrow{f} K$, ~~[illegible]~~ and a ~~[illegible]~~ morphism of functions

$$f^{-1}(U_\sigma) \longrightarrow F_\sigma$$

Thus it would appear that $\underrightarrow{\mathrm{holim}}$ is an inverse limit taken over spaces over K.

Define $Y(V)$ to consist of lattices Λ A A S
whose simplices are chains $\Lambda_0 < \cdots < \Lambda_n \ni \pi\Lambda_n \subset \Lambda_0$

Contract $Y(V)$ by fixing Λ_0 and mapping

$$T_n(\Lambda) = \Lambda + \pi^{-n}\Lambda_0$$

Then $T_n(\Lambda) \longrightarrow T_{n+1}(\Lambda)$ are homotopic.

Next define $X(V)$ as a quotient of $Y(V)$.

$$\begin{array}{ccc} Y(V) & \longrightarrow & Y(V/H) \\ \cup & & \psi \\ X(V) & \longrightarrow & Ae \end{array}$$

In general fibres of

$$Y(V) \xrightarrow{\varphi} Y(V/W)$$

are contractible. In general suppose Z is a full subcomplex such that $\Lambda \in Z \Longrightarrow T_W(\Lambda) \in Z$, then Z contracts to its image in $Y(V/W)$.
More precisely define s to be the section of φ given by $V/W \longrightarrow V$ and then adding a fixed lattice of W. Then $Z \sim s\varphi(Z)$

$$\begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & u & \\ & \alpha & \\ & & \alpha \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & u & u \\ & \alpha & \\ & & \alpha \end{pmatrix}$$

$$\begin{array}{ccccc} X(V) & \longrightarrow & X(V/W) & \longrightarrow & X(V/H) \\ \| & & \| & & \\ Y(H) & \longrightarrow & Y(H/W) & & \end{array}$$

$W \subset H$ hyp

$X(V)$

$Y(V)$ $\quad L_0 < \cdots < L_n$

work mod homothety.

$$\begin{array}{ccc} X(V) & \longleftarrow & Y(H) \\ \uparrow & & \\ Y(V) & & \end{array}$$

no — one has $X(V) \longrightarrow Y(H)$

Identify $X(V)$ with subcomplex of

$Y(V)$ = simp. cx. of lattices Λ

$X(V)$ quotient by homothety.

$W^1 < V$ line

$X(V) \subset Y(V)$

complex of [illegible] $\Lambda \ni \Lambda \cap W^1 = \Lambda^1$.

contraction.

Better choose a hyperplane $H \subset V$ and [illegible] [illegible] under projection $V \longrightarrow V/H$ one gets a fixed lattice.

$$\begin{array}{ccc} Y(V) & \longrightarrow & Y(V/H) \\ \cup & & \cup \\ X(V) & \longrightarrow & \Lambda_0 \end{array}$$

$V \qquad F \leftarrow A$

$Y(V)$ building of lattices in V.

Assertion: Given a flag $0 < W_0 < \cdots < W_p < V$, and a ~~subset~~ subcomplex stable under geodesic

---

W subspace of V $\qquad 0 < W < V$.

Then have ~~projection~~

$$X(V) \xrightarrow{\varphi} X(V/W)$$

and one has ~~[illegible]~~ "geodesics" flow

$$\begin{array}{ccccccccc} 0 \to & L \cap W & \to & L & \to & L+W/W & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & \pi^{-1}(L \cap W) & \to & T_W(L) & \to & L+W/W & \to 0 \end{array}$$

<u>Lemma</u>: Let $Z \subset X(V)$ be a subcomplex ~~closed~~ ~~such that~~ stable under geodesic flow whose image in $X(V/W)$ is contractible. Then Z is contractible. | bad.

<u>Proof</u>: Enough to take a finite complex in Z and contract it to a point. Take a section

---

$$\begin{array}{ccc} Z \subset X(V) & \to & X(V/W) \\ \uparrow & & \uparrow \\ Y(V) & \to & Y(V/W) \end{array} \qquad T_V(L) = \pi^{-1} L$$

horizontal maps have same fibres ~~maps~~ over vertices

square is cartesian

$$E(T) = \left\{ (f_\sigma) \;\middle|\; \begin{matrix} f_\sigma : \mathcal{U}_\sigma \times_K T \longrightarrow F_\sigma \\ \forall \sigma \subset \tau \qquad \cup \qquad\qquad \uparrow \\ \mathcal{U}_\tau \times_K T \longrightarrow F_\tau \end{matrix} \text{ commutes} \right\}.$$

if T is a point x of K, then $x \in \mathcal{U}_\tau$ $\sigma \subset \tau \Longrightarrow$ f_τ determines f_σ. Thus the fibre over x is F_τ where τ largest $\ni x \in \mathcal{U}_\tau$ i.e. τ is the open simplex containing x

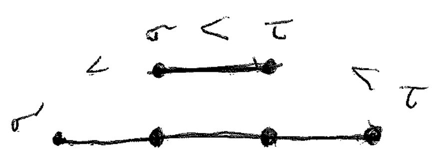
Thus set-theoretically $E = \coprod_\tau \tau \times F_\tau$

Two descriptions

$$\bar{\sigma} \times F_\sigma \longrightarrow E \longrightarrow (j_{\mathcal{U}_\sigma})_* (\mathcal{U}_\sigma \times F_\sigma)$$

which give somehow the same point set with two topologies.

is a functor on the box category



$\sigma \subset \tau$

$F_\sigma \times \bar{\sigma}$ ⟵ $F_\tau \times \bar{\sigma}$ ⟶ $F_\tau \times \bar{\tau}$

$F_\sigma \times \bar{\tau}$ ⟶ $F_{\sigma'}$; $\bar{\tau} \cap \bar{\tau}'$

$F_{\sigma'} \quad \times \quad \bar{\tau}'$

So at the moment I can handle the following. Let K be a simp. complex, $\sigma \mapsto F_\sigma$ a contravariant functor from simp. to spaces. $\sigma \mapsto U_\sigma$ = open star. Then

$$\mathrm{Hom}_{sp/K}\left(T, (j_{U_\sigma})_*(U_\sigma \times F_\sigma)\right)$$

$$\|$$

$$\mathrm{Hom}_{sp/U_\sigma}\left(U_\sigma \times_K T,\ U_\sigma \times F_\sigma\right) = \mathrm{Hom}_{sp}\left(U_\sigma \times_K T, F_\sigma\right)$$

and since this is contravariant in σ I can take

$$\mathrm{Hom}_{sp/K}\left(T, \varprojlim_\sigma (j_{U_\sigma})_*(U_\sigma \times F_\sigma)\right)$$

$$\|$$

$$\varprojlim_\sigma \mathrm{Hom}_{sp}\left(U_\sigma \times_K T, F_\sigma\right)$$

An elt. of this is a family $\rho_\sigma : U_\sigma \times_K T \longrightarrow F_\sigma$ such that $\forall\ \sigma \subset \tau$ the diag

$U_\sigma \supset U_\tau$

$F_\tau \to F_\sigma$

$$\begin{array}{ccc} U_\sigma \times_K T & \longrightarrow & F_\sigma \\ \cup & & \uparrow \\ U_\tau \times_K T & \longrightarrow & F_\tau \end{array}$$

$$\begin{array}{c} E \\ \downarrow \\ U \subset V \end{array}$$

$(j_U)_*(E|U) \leftarrow (j_V)_*(E)$

$\sigma \subset \tau$

$U_\sigma \supset U_\tau$

$$\begin{array}{ccc} (j_{U_\sigma})_*(U_\sigma \times F_\sigma) & \longrightarrow & (j_{U_\tau})_*(U_\tau \times F_\sigma) \\ & & \uparrow \\ & & (j_{U_\tau})_*(U_\tau \times F_\tau) \end{array}$$

$$\overset{E}{\underset{}{U \subset V}}$$

$$\Gamma(T, (j_V)_*(E)) \overset{\text{defn}}{=} \Gamma(V \times_K T, E)$$

$$\downarrow \text{rest}$$

$$\Gamma(T, (j_U)_*(E|U)) = \Gamma(U \times_K T, E|U)$$

∴ There is a canonical map

$$(j_V)_*(E) \longrightarrow (j_U)_*(E|U)$$

so that this depends covariantly on the open set U. One ought to think of $(j_U)_*(E)$ as the result of extending E by 0 outside of U.

---

so when I have $\sigma \mapsto U_\sigma$ $\quad \sigma \mapsto F_\sigma$ then

$\sigma \subset \tau$

$U_\tau \subset U_\sigma$

$F_\sigma \leftarrow F_\tau$

$$(j_{U_\sigma})_*(U_\sigma \times F_\sigma) \longrightarrow (j_{U_\tau})_*(U_\tau \times F_\sigma)$$

$$\uparrow$$

$$(j_{U_\tau})_*(U_\tau \times F_\tau).$$

~~Sh...~~ so I have to take this end - -

~~Thus if one has a sheaf one~~

Take a point $x \in K$. Then

$$\Gamma(x, (j_{U_\sigma})_*(U_\sigma \times F_\tau)) = \begin{cases} F_\tau & x \in U_\sigma \\ 0 & x \notin U_\sigma. \end{cases}$$

$$\begin{array}{ccc} A & \longrightarrow & A[S^{-1}] = B \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{A}[S^{-1}] = \hat{B} \end{array}$$

Claim $\quad E_n(\hat{B}) = \underbrace{E_n(\hat{A})}_{X}\,\underbrace{E_n(B)}_{Y} \qquad n \geq 3$

start with $\quad \varepsilon \in E_n(\hat{B})$

$$\varepsilon = e_{i_r j_r}(\beta_r) \cdots \cdots e_{i_1 j_1}(\beta_1)$$

So the point to prove is that XY is ~~[illegible]~~ stable by multiplication on the left by $e_{ij}(b)$. Now take $x = x_1 \cdots\cdots x_m$ and suppose I ~~[illegible]~~ multiply by $e_{ij}(b)$.

$$\boxed{e_{ij}(b)\, x_1} \cdots x_m \in X\, e_{ij}(b')\, x_2 \cdots x_m$$

$$e_{ij}(b)\, e_{jk}(\alpha)$$

What is the way to think of a map
$X \longrightarrow SP^n(T)$

$$SP^n(A \amalg B) = \coprod_{i+j=n} SP^i(A) \times SP^j(B)$$

Hence if we look at $f: SP^n(T) \longrightarrow SP^n(T/A)$ the fibre over the stratum $SP^k(T-A)$ is $SP^{n-k}(A)$.
Thus

$$SP^n(T) = \coprod SP^{n-k}(A) \times SP^k(T-A)$$

set-theoretically. Now what remains somehow is to describe the ~~precise~~ way these pieces are glued together. i.e. take a ~~sequence~~ point $x \in SP^k(T-A)$ $x = \{t_1 \cdots t_k\}$ where again using $\rho: T \longrightarrow [0,1]$, $\rho^{-1}(1) = A$, I arrange $\rho(t_1) \leq \cdots \leq \rho(t_k) < 1$.
Then ~~any nearby norm~~ the normal bundle to the stratum $SP^k(T-A)$ at x consists of all sequences $t_i \ni \rho(t_k) < \rho(t_{k+1})$ and such that the first k terms is x.

So let $y^\lambda = \{t_1 \cdots t_k, t_{k+1}^\lambda \cdots t_n^\lambda\}$ $t_i \in T/A$
converge to $x = \{t_1, \ldots, t_k\}$ as $\lambda \longrightarrow \infty$.

Better I want $y^\lambda \in SP^{k+1}(T-A)$

$$y^\lambda = \{t_1, \ldots, t_k, t_{k+1}^\lambda)$$

and I let $t_{k+1}^\lambda \longrightarrow A/A$. Then $f^{-1}(y^\lambda) = SP^{n-k-1}(A)$
$f^{-1}(x) = SP^{n-k}(A)$ and to really have a specialization map $f^{-1}(y^\lambda) \longrightarrow f^{-1}(x)$ one must have ~~a home~~ that $t_{k+1}^\lambda \in T-A$ converges to a pt. in A.

So over each U_σ I have a space F_σ $\ni$

if $\sigma \subset \tau$ then
$$\begin{array}{ccc} F_\sigma & \supset & F_\tau \\ \downarrow & & \downarrow \\ U_\sigma & \supset & U_\tau \end{array}$$
not cartesian

$$\sigma \subset \tau \quad \prod_\sigma (j_{U_\sigma})_*(F_\sigma) \rightrightarrows \prod_{\sigma\subset\tau} (j_{U_\tau})_*(U_\tau \times_{U_\sigma} F_\sigma)$$

so if there is a ~~least~~ largest $\sigma \ni x \in U_\sigma$ then the fibre of this space over x is precisely F_σ. So this means that over $x \in SP^k(T-A)$, so if $x \in U_\sigma = \cdots \cap U_k$ one has $F_\sigma = SP(\rho^{-1}[s_{k+1}(x), 1]) = SP(A)$.

Something is wrong because of the transitions.

So can I map
$$SP(T) \longrightarrow \prod_\sigma \cdots$$

thus over U_σ what is $SP(T)$ like. So if we have a point $x \in U_\sigma$ ~~say~~ then is there a map
$$SP(T)_x \longrightarrow SP(\rho^{-1}[s_{k+1}(x), 1])$$
throw away the ~~[illegible]~~ first k points

$s_k(x) < s_{k+1}(x)$

But suppose we pass from U_ℓ to U_k

Given a point x in $U_k \cap U_\ell$, then we have the fibre

$$SP(T)_x \to SP(\rho^{-1}[s_{\ell+1}(x), 1]) \times SP^{\ell-k}([s\ , s_\ell(x)]$$
throw away ℓ eigenvalues

$$SP(T)_x \searrow SP(\rho^{-1}[s_{k+1}(x), 1])$$

~~$SP^{\ell-k}$~~

Now then given $U_k \cap U_\ell \cap \cdots \cap U_m$ we assoc. $SP(\rho^{-1}(s_m, 1])$ as the fibre. So that

$$\sigma \subset \tau$$

$$\Rightarrow \quad U_\sigma \supset U_\tau$$

$$\text{and} \quad lv(\sigma) \leq lv(\tau)$$

$$\text{so} \quad SP(\rho^{-1}(s_{lv(\sigma)}, 1]) \supset SP(\rho^{-1}(s_{lv(\tau)}, 1])$$

so it seems clear that one has a contravariant functor of σ.

Now given a point [redacted] x has

$$s_k(x) < s_{k+1}(x) = 1$$

thus k is the last $\ni x \in U_k$.

So if x lies on $SP^k(T-A)$. Then $s_k(x) < s_{k+1}(x) = 1$. and on this stratum; which is in U_k.

Other possibility is to define the fibre to consist of $SP(\rho^{-1}[s_{k+1}(x), 1])$

Try to put over U_k the bundle

$$x \longmapsto SP(\rho^{-1}[s_{k+1}(x), 1])$$

then over $U_k \cap U_\ell \cap \cdots \cap U_m$ $\quad k < \ell < m$

one has $SP(\rho^{-1}[s_{k+1}(x), 1]) \supset SP(\rho^{-1}[s_{m+1}(x), 1])$

and this is a fibre-homotopically trivial thing over U_σ and the maps.

Back to ~~~~ symmetric product in particular to the map

$$SP(T) \xrightarrow{f} SP(T/A).$$

now if I pick a point $z \in SP(T/A)$ say $z = \{t_1, \ldots, t_k\} \in SP^k(T-A)$, then $f^{-1}(z) \simeq SP^\infty(A)$.

Now I was going ~~to~~ try to ~~~~ introduce a tubular nbd U_k of ~~~~ the stratum $SP^k(T-A)$ by using a function $\rho: T \to [0,1]$ such that $\rho^{-1}(1) = A$. Then I define

$$U_k = \left\{ z \in SP(T/A) \;\middle|\; \begin{array}{l} \text{if } z = \{t_1, \ldots\} \text{ and the } t_i \text{ are arranged} \\ \text{with } \rho(t_1) \le \rho(t_2) \le \cdots \\ \text{then } \rho(t_k) < \rho(t_{k+1}) \end{array} \right\}$$

Now my idea was to ~~~~ describe $SP(T)$ efficiently ~~~~ over the open sets U_k and more gen. $U_\sigma = U_{i_0} \cap \cdots \cap U_{i_q}$ if $\sigma = \{i_0, \ldots, i_q\}$.

The first thing I tried was to ~~~~ look at $f^{-1}(U_k)$, but this didn't seem reasonable. Instead it seems that given a point $z = \{t_1, \ldots\}$ of U_k I want the fibre over z to be $SP(V_\varepsilon)$ where $V_\varepsilon = \rho^{-1}(\varepsilon, 1]$ $\varepsilon = s_k(z)$. This works. Thus it seems I can find over U_k a bundle with fibre $SP(\rho^{-1}(s_k(x), 1])$ over $x \in U_k$. Now over ~~~~ $U_k \cap U_\ell$ $k < \ell$ one has $s_k < s_{k+1}$ $s_\ell < s_{\ell+1}$ so in part. $s_k < s_\ell$ and hence we have a map

$$SP(\rho^{-1}(s_k, 1]) \supset SP(\rho^{-1}(s_\ell, 1])$$

On $U_k \cap U_\ell$ you have first a splitting of the map $U_k \cap U_\ell \hookrightarrow U_\ell \longrightarrow SP^\ell(T-t_0)$ into

$$\begin{array}{ccc} U_k \cap U_\ell \hookrightarrow U_\ell & \longrightarrow & SP^\ell(T-t_0) \\ \searrow & & \uparrow \\ & & SP^k(T-t_0) \times SP^{\ell-k}(U_{t_0}-t_0) \end{array}$$

where U_{t_0} is the open star of t_0.

and relative to this splitting one has a canon. splitting of the bundle ξ_ℓ into ξ_k and a bundle supported in $U_{t_0}-t_0$.

K simplicial complex

$\sigma \longmapsto U_\sigma$ open star of σ

$\sigma \longmapsto F_\sigma$ contravariant functor

Then if $j_{U_\sigma}: U_\sigma \hookrightarrow K$ is the inclusion, I will put

$$(j_{U_\sigma})_*(U_\sigma \times F_\sigma)$$

for the space over X which is $U_\sigma \times F_\sigma$ over U_σ and $\emptyset$ elsewhere. Now form the inverse limit of the diagram

$$\prod_\sigma (j_{U_\sigma})_*(U_\sigma \times F_\sigma) \rightrightarrows \prod_{\sigma \subset \tau} (j_{U_\tau})_*(U_\tau \times F_\sigma)$$

it gives me the universal property I want.

$$I \longrightarrow \prod_{\sigma} (j_{U_\sigma})_* (U_\sigma \times F_\sigma) \rightrightarrows \prod_{\sigma \subset \tau} (j_{U_\tau})_* (U_\tau \times F_\sigma)$$

Then for any space X over K, a map $X \to I$ over K may be identified with a family of maps

$$\begin{array}{ccc} X \times_K U_\sigma & \longrightarrow & F_\sigma \\ \cup & & \uparrow \\ X \times_K U_\tau & \longrightarrow & F_\tau \end{array}$$

, $\sigma \subset \tau$ commutes.

It should be clear that the construction ~~should~~ co be

essential problem now is to prove an exactness result:

$$\boxed{M(A, t_0) \longrightarrow M(T, t_0) \longrightarrow M(T, A)}$$

~~Suppose that one understands what happens when one~~ ~~[illegible]~~

$$M(A, t_0) \longrightarrow M(T, t_0)$$

Understand case of symm. product.

$$\begin{array}{ccc} SP(T, t_0) & \longrightarrow & SP(T/A) \\ \cup & & \cup \\ SP(T/t_0)_n & \longrightarrow & SP^n(T/A) \\ & & \cup \\ & & SP^{n-1}(T/A) \end{array}$$

and if it is indifferent

$$\begin{array}{ccc} SP(T) & \supset & E_n \\ f\downarrow & & \downarrow \\ SP(T/A) & \supset & SP^n(T/A) \end{array}$$

now take U_k = open set in $SP(T/A)$ where function $s_k < s_{k+1}$

[illegible] It is a tube around $SP^k(T-A) = SP^k(T/A) - SP^{k-1}(T/A)$

Precisely: Given a point in $SP^k(T-A)$ its image in $SP^k[0,1]$ gives a sequence $s_1 \leq \cdots \leq s_k < 1$ and then a normal motion from this point consists of ~~adding~~

$$SP^\infty(V_\varepsilon) \qquad \varepsilon = 1 - s_k$$

Over U_k = open set in $SP(T/A)$ where ~~$SP(T/A)$~~

$$s_k < s_{k+1}$$

~~[illegible]~~

$$U_k \sim SP^k(T-A)$$

and $f^{-1}(U_k) = ?$ Fix a point in U_k

i.e. $(t_1 \cdot \text{———})$ where $\rho(t_k) < \rho(t_{k+1})$

and what is the fibre? ~~assuming that one has~~

[illegible] So once I fix the fibres it is clear what happens

---

As U_k contracts to $SP^k(T-A)$ ~~[illegible]~~ $f^{-1}(U_k)$ deforms to $SP^k(T-A) \times SP^\infty(A)$.

$E \overset{f}{\downarrow} B$ Suppose I can cover B by an open covering U_σ such that $U_\sigma \times_B E \longrightarrow U_\sigma \times F$ is a heq.

$\begin{matrix} E \\ \downarrow \\ K \end{matrix}$ Suppose that

Then

$$\begin{array}{ccc} U_\sigma \times_K E & \xrightarrow{heq} & F_\sigma \\ \cup & & \uparrow \\ U_\tau \times_K E & \xrightarrow[heq]{} & F_\tau \end{array}$$

So if $F_\tau \to F_\sigma$ is a heq. one knows that E/K has the homot. fibre F.

So for $\begin{matrix} SP(T) \\ f\downarrow \\ SP(T/A) \end{matrix}$

and the open covering U_k of $SP(T/A)$. Now over U_k I have a fibre-wise retraction of

$$\begin{array}{ccc} f^{-1}(U_k) & \longleftarrow & SP^k(T-A) \times SP(A) \\ \downarrow & & \downarrow \\ U_k & \longleftarrow & SP^k(T-A) \end{array}$$

But what was Dold's argument

$\begin{matrix} SP(T) \\ f\downarrow \\ SP(T/A) \end{matrix}$ point was to show $f \mid SP^k(T/A)$ is a quasi-fibration. So the lemma is tubular nbd.

So $\begin{matrix} & E & \\ & \downarrow & \\ Z \underset{i}{\subset} & B & \underset{j}{\supset} U \end{matrix}$ assume $\begin{matrix} E_U \\ \downarrow \\ U \end{matrix}$ is a quasi-fibration

$\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ quasi-fibration.

Finally I want a tube V around $Z \ni$ $\begin{array}{ccc} E_V & \leftarrow & E_Z \\ \downarrow & & \downarrow \\ V & \leftarrow & Z \end{array}$ homotopy cartesian

NOT ENOUGH

ha

$$\begin{array}{lccc} D: & E_V \times I & \longrightarrow & E_V \\ & \downarrow & & \downarrow \\ d: & V \times I & \longrightarrow & V \end{array}$$

$D_0 = id$

$D_1 =$ retraction onto E_Z

So it would seem ~~one wants to know that~~

$$\begin{array}{lccc} D_1: & E_V & \longrightarrow & E_Z \\ & \downarrow & & \downarrow \\ d_1 & V & \longrightarrow & Z \end{array}$$

~~and~~ one wants to show this is <u>cartesian</u>

Thus if $Z = SP^{k-1}(T/A) \subset SP^k(T/A)$.

<u>Let</u> V be a small nbd of $SP^{k-1}(T/A)$. Thus it a

~~Suppose one finds the differences~~

fibre over $SP^k(T-A)$ is $SP^\infty(A)$ in a uniform way.

But I have to understand what happens as we go to $SP^{k-1}(T/A)$. Thus if I have $\{t_1, \dots, t_k\} \subset T-A$ and I let ~~the t_i converge~~ this point go to $SP^{k-1}(T/A)$. Say precisely we end in $SP^j(T-A)$ say

$s_1 \cdots\cdots s_\ell$

nearby $\underbrace{t_1 \text{——} t_\ell}$ $\underbrace{t_{\ell+1} \cdots\cdots t_k}_{\text{near to } A}$ fibre over

~~Suppose that $t_1, \dots, t_\ell$ ~~ $\to 0$.

$t_1 \cdots t_j$ $\underbrace{t_{j+1} \cdots t_k}$ ~~are~~ in V_ε a nbd of A

$\downarrow$

$t_1 \cdots t_j$

thus they have images in A which give the multip as the fibre.

The principle: If I have a point

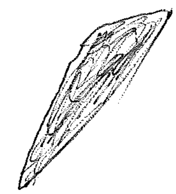
$t_1 \cdots t_\ell \in SP^\ell(T-A)$

converging to $t_1 \cdots t_k \in SP^k(T-A)$

then $t_{k+1} \cdots t_\ell$ are approaching A, hence if V is a tube around A, then t

$U_k' = \{ (t_i) \mid \rho(t_k) < \rho(t_{k+1}) \}$ is a tube around $\underbrace{SP^k(T-t_0)}_{k\text{-th st.}}$

So over $U_k' \subset SP^\infty(T)$ I give a rank k bundle ξ_k



$$\begin{array}{ccccc} X \supset U_k & \dashrightarrow & F_k & \longrightarrow & PU_k \times^{U_k} D(\mathbb{C}^k, T-t_0) \\ & \searrow & \downarrow & & \downarrow \\ & & U_k' & \longrightarrow & SP^k(T-t_0) \end{array}$$

Over $U_k \cap U_\ell$ I have to embed ξ_k into ξ_ℓ

$$\begin{array}{ccc} & & SP^\ell(T-t_0) \\ & \nearrow & \uparrow \\ U_k' \cap U_\ell' & & \\ & \searrow & \\ & & SP^k(T-t_0) \times SP^{\ell-k}(U_{t_0}-t_0) \end{array}$$

$U_k \cap U_\ell \qquad \xi_k \hookrightarrow \xi_\ell$

$$U_k' \cap U_\ell' \cap \dots \cap U_m' \longrightarrow SP^k(T-t_0) \times SP^{\ell-k}(U_{t_0}-t_0) \times \dots \times SP^{m-\ell}(U_{t_0}-t_0)$$

$PU_k \times^{U_k} D(\mathbb{C}^k, T-t_0) \times$

$PU_{k,\ell-k,\dots,m-\ell} \times^{U_{k,\ell-k,\dots,m-\ell}} D(\mathbb{C}$

etc.

~~So recall what should be a T mod t_0 bundle ξ~~

~~I am given $T \xrightarrow{\rho} [0,1]$ with $\rho^{-1}(1) = t_0$.~~

~~First part of ξ is a map~~

~~$$X \longrightarrow$$~~

~~$$SP^\infty(T) = \bigcup_n SP^n(T)$$~~

~~Because of $SP^n(T)$~~

I want to recall my definition of a T mod t_0 bundle. ξ over X. First thing ξ determines is a map

$$X \longrightarrow SP^\infty(T) = \bigcup SP^n(T).$$

Having chosen $\rho : T \to [0,1]$ such that $\rho^{-1}(1) = t_0$ I get a map

$$SP^\infty(T) \longrightarrow SP^\infty([0,1])$$

$$\|$$

$$\{ (s_1 \le s_2 \le \cdots) \mid 0 \le s_1, \; s_n = 1 \quad n \text{ large} \}$$

So I get cont. functions $0 \le s_1(x) \le s_2(x) \le \cdots \le s_n(x) = 1$ on X. Now take on X the open covering which is the inverse image of the standard on

$$SP^n [0,1] = \Delta(n)$$

i.e. put $U_k = \{ x \mid s_k(x) < s_{k+1}(x) \}$. Then ξ should give a bundle ξ_k on U_k sitting over the appropriate part of $SP^k(T)$.

The effect of ρ is to make explicit the stratification on $SP^n(T)$. Thus given a point $\{t_1, \ldots, t_n\}$ in $SP^n(T)$ we put all the points at the basepoint at the end — in fact we arrange

$$\rho(t_1) \le \cdots \le \rho(t_n)$$

and then we define the stratification analogously.

$M(T) \qquad M(T) \times M(t_0)$

Thus I get something like

$$\coprod M(T) \times M(t_0)^p \times \overset{\text{int}}{\Delta}(p)$$

Next ingredient — form sheaves of finite type over $T - t_0$.

So over X I want to consider

Now why might you get a fibration in T.
Thus I want to define

Another variation: Suppose over X we give the covering $\mathcal{U}_k$ the bundles ξ_k and the embeddings

$$\xi_k | \mathcal{U}_k \cap \mathcal{U}_\ell \hookrightarrow \xi_\ell | \mathcal{U}_k \cap \mathcal{U}_\ell$$

Here try this: Given the strata X_k, the bundle ξ_k over X_k, for $T - t_0$. Then I have to show how ξ_ℓ on X_ℓ spec. to ξ_k on X_k for $k < \ell$. Thus I give a small tubular nbd $\mathcal{U}_k$ of X_k and give embedding of $\xi_k | \mathcal{U}_k \cap \mathcal{U}_\ell \hookrightarrow \xi_\ell | \mathcal{U}_k \cap \mathcal{U}_\ell$ $\ni$ the [redacted] support of the cokernel tends to the basepoint at we go from a point of $\mathcal{U}_k \cap \mathcal{U}_\ell$ to a point of [redacted] X_k

~~Assume maps that~~ [illegible]

OK

$$\begin{array}{ccccc} U_k & \longrightarrow & PU_k \times^{U_k} D(\mathbb{C}^k, T - t_0) & \longrightarrow & BU_k \\ \downarrow & & \downarrow & & \\ V_k & \longrightarrow & SP^k[0,1) & & \end{array}$$

~~This the map~~

$$M(T) = \coprod_n PU_n \times^{U_n} D(U_n; T)$$

$$M(t_0) = \coprod_n BU_n$$

and we were to form [illegible] $M(T)/M(t_0)$.

This means something stratified

It [illegible] means something like a stratified set?

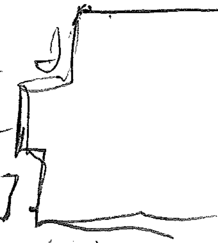
Idea: classifying space of a ~~monoid~~ category.

Objects were to be [illegible] $\mathbb{C}^n$, $\xi \in D(\mathbb{C}^n, T)$ $n \geq 0$.

maps spaces of unit embed. $\mathbb{C}^m \hookrightarrow \mathbb{C}^n$ $\ni$ complement is supported at the basepoint.

~~So therefore what happens is that one can determine the following. Let T be a sheaf~~

Suppose I give a subset A of T. Then I can define the analogous type of bundles but where I use a function $\rho : T/A \longrightarrow [0,1]$



that is, a function on $\rho : T \to [0,1]$ $\ni$ $\rho^{-1}(1) = A$.

It is clear that the definition of bundles I take then is the same as for T/A, since ξ_k on U_k makes use only of the eigenvalues $s_1(x) \leq \cdots \leq s_k(x) < 1$.

Classifying space for the type of bundles considered.

So start with $SP^n(T) \longrightarrow SP^n([0,1]) = \Delta(n)$

and now define V_k to be the place where the sequence jumps at the kth spot. Over V_k I have then the sequence

$$s_1(t) \leq \cdots \leq s_k(t) < s_{k+1}(t) \cdots$$

so I can consider possible decompositions of a unitary v.s. wrt this.

$$D(\mathbb{C}^k, T) \longrightarrow D(\mathbb{C}^k, [0,1])$$

$$\downarrow$$

$$SP^k([0,1])$$

$$PU_k \times^{U_k} D(\mathbb{C}^k, T)$$

$$\downarrow$$

$$BU_k \times SP^k[0,1] \longleftarrow$$

Suppose then I have a space X with a map $X \longrightarrow SP^n([0,1])$ which I think of as n functions

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq s_n(x) \leq s_{n+1}(x) \equiv 1.$$

Then put $U_k = \{x \mid s_k(x) < s_{k+1}(x)\}$, and suppose given a bundle ξ_k of rank k over U_k decomposed according to T such that on mapping the eigenvalue sequence to $SP^n([0,1])$ we get $s_1, \ldots, s_k$ on U_k. One $U_k \cap U_\ell$ I want

$$\xi_k | U_k \cap U_\ell \hookrightarrow \xi_\ell | U_k \cap U_\ell$$

with image the part belonging to the appropriate eigenvalues. $k < l$

Thus on $U_k \cap U_\ell$, ξ_ℓ completely determined ξ_k except the isomorphism has to be given.

Thus finally I want to know that given x, that the last $k \ni s_k(x) < s_{k+1}(x)$ matters. But this implies $s_{k+1}(x) = 1$. Seems fairly natural what I have done since if x approaches a point of X_k the eigenvalues have to move to the basepoint.

So I am trying to describe what is a bundle decomposed according to T mod t_0. Now using the barycentric coordinate of t_0 I get these functions

$$s_1(x) \le s_2(x) \le s_3(x) \le \cdots \le s_n(x) = 1.$$

Then I ~~[illegible]~~ try to define the basic sets of interest which are:

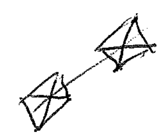
X_k is where $s_k(x) < s_{k+1}(x) = 1$

so that we have ξ_k over X_k parameterized by $T - t_0$. Partial approximation: ~~If I am given a~~

I can define U_k by the condition $s_k(x) < s_{k+1}(x)$. This is an open set which contains X_k and which ~~[illegible]~~ retracts onto X_k because I can push $s_{k+1}(x) \le \cdots$ to 1. ~~But now these sets~~ Over U_k I have a well defined bundle ξ_k of rank k.

Now the U_k do not increase. **Observe** that

$$U_k \cup U_{k+1} \cup \cdots = \{x \mid s_k(x) < 1\}.$$



is our decreasing family so that

$$X_k = [U_k \cup U_{k+1} \cup \cdots] - [U_{k+1} \cup U_{k+2} \cup \cdots]$$

~~[illegible] I am given a family of [illegible]~~ Now over $U_k \cap U_l$ etc. I know the transitions I have to have

So it seems coherent.

This means that I have $X \xrightarrow{f} SP^n([0,1])$

$$= \{(t_0, \ldots, t_n) \mid 0 \le t_0 \le t_1 \le \cdots \le t_n \le 1\}$$

and hence I get open sets $V_k = \{x \mid t_k f(x) < t_{k+1} f(x)\}$.

or even better a partition of unity

$$\rho_k f(x) = t_{k+1} f(x) - t_k f(x)$$

~~Suppose instead that~~ Suppose now I have a bundle ξ over X, then because of the function

$$\rho : T \to [0,1] \qquad \rho^{-1}(0) = 0$$

Then I get a map $X \to SP^n(T) \to SP^n([0,1])$

by taking the determinant, i.e. the divisor of eigenvalues

Now $SP^n([0,1]) = \{0 \le s_1 \le \cdots \le s_n \le 1\}$

Now I am interested in the number of eigenvalues at the basepoint t_0; which corresponds to the number of $s_i = 0$.

So over X I have this map $X \to SP([0,1])^n = \Delta(n)$

whose barycentric coords I call $\lambda_0, \ldots, \lambda_n$

$$\lambda_i = s_{i+1} - s_i \qquad i = 0, \ldots, n$$

~~and I define V_k = open set where~~

I am going to define a T mod t_0 bundle over a space X (assume X compact). First I will have a map

$$X \longrightarrow SP^{\infty}([0,1]) \qquad 1 \text{ basepoint}$$

which I will think of as a sequence of cont. fns. $s_1 : X \longrightarrow [0,1]$ which is monotone

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots$$

such that $s_n(x) \equiv 1$ for n large. (X compact & SP^∞ has the ind. lim. topology).

Define

$$V_n = \{ x \mid s_n(x) < 1 \} \quad \text{open}$$

$X - V_n$ is where $S_n = 1$ i.e. stratum $\overline{X_n}$

so that

$$X = V_0 \supset V_1 \supset V_2 \cdots \supset V_n = \phi$$

On V_n I propose to give a unit. vector bundle ξ_n of rank n with eigenvalue sequence

$$s_1(x) \;\cdots\; s_n(x)$$

over x. Moreover I will relate $\xi_m | V_n$ and ξ_n by giving an embedding

$$\xi_m | V_n \hookrightarrow \xi_n$$

compatible with

$$s_1(x), \ldots, s_m(x) \subset s_1(x) \cdots s_n(x)$$

for $\forall x \in V_n$. This should be transitive.

Possible method: Try to relate the $(s_k(x))$ to the bundles given.

Over $U_k = \{x \mid s_k(x) < s_{k+1}(x)\}$ I give a rank k bundle ξ_k parameterized by $T-t_0$ such that $s_k(x)$ is the maximum eigenvalue of ξ_k.

Perhaps from this point of view the basic thing to give is the sequence

$$0 \leq s_1(x) \leq s_2(x) \leq s_3(x) \leq \cdots\cdots \quad 1$$

Thus I suppose given the map $X \longrightarrow SP^\infty([0,1])$ which I think of as represented by ~~?~~ continuous functions $0 \leq s_1(x) \leq s_2(x) \leq \cdots\cdots$ almost all equal to one. ~~[illegible]~~ And now I think I get the following classifying space. ~~at great~~

Suppose I und

So let me suppose that I have managed to give over X the map $X \longrightarrow SP^{\infty}([0,1])$ assoc. to my bundle ξ. Thus I have a partition

$$\sum \rho_k(x) = 1.$$

and open sets U_k where the eigenvalues change. Now what more do I have to give in order to have a bundle. Example: suppose over X I give globally a rank n bundle decomposed according to T, so that I have now $X \longrightarrow SP^{n}([0,1])$. ~~and~~ Then over all of X I have this rank n-bundle E

(Recall over U_k I give ξ_k of rank k, and in some sense these get glued together so that if x is a point with support $k_0 < \cdots < k_p$ then the bundle ξ_{k_p} determines the others.)

~~[illegible]~~

Basic thing to keep in mind is the stratification

$$\bar{X}_0 \subset \bar{X}_1 \subset \bar{X}_2 \subset \cdots\cdots$$

$\rho_0 = 1$ ~~[illegible]~~

$\rho_1 + \rho_2 = 1$

because over X_k we have at most a bundle of rank k.
(i) $x \in X_k$
(ii) $s_k(x) < s_{k+1}(x) = 1$
(iii) k is the largest $\ni x \in U_k$

I want to express the fact that as I specialize $x \in X_k$ to a lower point, then my bundle shifts

Problem: Describe a class. space for T mod t_0 bundles.

~~This classifying space should sit over the semi-direct product~~ If I give $\rho : T \longrightarrow [0,1]$ barycentric coordinate of t_0 say, then a rank n-bundle det. a map $X \longrightarrow SP^n(T) \longrightarrow SP^n([0,1])$ (namely you apply ρ to the eigenvalues), and trans. by a bundle with support at basepoint gives map

$$SP^n([0,1]) \longrightarrow SP^{n+k}([0,1])$$

$$0 \le s_1 \le \dots \le s_n \le 1 \longmapsto 0 \le s_1 \le s_2 \le \dots \le s_n \le \underbrace{1 \le \dots \le 1}_{k} \le 1$$

this is a better description.

$$SP^\infty([0,1]) : \quad 0 \le s_1 \le s_2 \le \cdots \quad 1$$

think of these as simplices with baryc. coords

$$s_{k+1} - s_k$$

So let $s_1 - s_0$, $s_2 - s_1$, $s_3 - s_2$, $s_4 - s_3$, ... be the baryc. coords and these define the open sets $U_0, U_1, U_2, \dots$ ~~On the other hand we have the strata~~ Thus on

U_0 $\quad s_1 > 0$ so all eigen. can be pushed to t_0

U_1 $\quad s_2 > s_1$ so all but one eigenvalue ..

So over U_k we will have $s_{k+1} > s_k$ a break in the eigenvalues sequences. and now I can define

---

~~Maybe one wants to define~~

Over U_k I get a bundle of rank k param. by $T - t_0$ - but more for I get a map

K simplicial complex

$\sigma \longmapsto F_\sigma$ contra. functor from simplices to spaces

U_σ = open star of σ = $\bigcap_{i \in \sigma} U_i$ $\quad i \in I$ = vertices

$\doteq \bigcup_{\tau \geq \sigma} \tau$

Then I can form a space over K by taking

$$E = \coprod \sigma \times F_\sigma \xrightarrow{p} \coprod \sigma = K$$

and ~~stratifying~~ topologizing so that

$$p^{-1}(U_\sigma) = \coprod_{\tau \geq \sigma} \tau \times F_\tau \quad \text{is open}$$

and such that the canon. map

$$p^{-1}(U_\sigma) \longrightarrow F_\sigma$$

is continuous. Put least such topology

I ~~want to~~ understand this construction very carefully. So what does it mean to give an open covering U_σ of K and a functor $\sigma \mapsto F_\sigma$ and to ask for a space E over K ~~with compat~~ maps $E|U_\sigma \longrightarrow F_\sigma$ universal with this property.

$\mathrm{Hom}(T, \varprojlim U_\sigma \times F_\sigma)$ means I have to give

in spaces over K,

$T \longrightarrow U_\sigma \times F_\sigma$ NO

~~Also I have to give~~

$U_\sigma \times F_\sigma$

F sheaf on V

$\Gamma(U, j_* F) = \Gamma(U \cap V, F)$

so define $(j_{U_\sigma})_*(U_\sigma \times F_\sigma)$ to be the space over X such that

$$\mathrm{Hom}_{/X}(T, (j_{U_\sigma})_*(U_\sigma \times F_\sigma))$$

$$= \mathrm{Hom}_{/U_\sigma}(U_\sigma \times_X T, U_\sigma \times F_\sigma)$$

Then if $\sigma \subset \tau$ $U_\tau \supset U_\sigma$ and so we have a map

$$\mathrm{Hom}_{/U_\tau}(U_\tau \times_X T, U_\tau \times F_\tau)$$

$$\longrightarrow \mathrm{Hom}_{/U_\sigma}(U_\sigma \times_X T, U_\sigma \times F_\sigma)$$

for any T/X, hence a map

$$(j_{U_\tau})_*(U_\tau \times F_\tau) \longrightarrow (j_{U_\sigma})_*(U_\sigma \times F_\sigma)$$

and so now my guess is that

$$[illegible] \quad E(T) = [illegible] \varinjlim_\sigma (j_{U_\sigma})_*(U_\sigma \times F_\sigma)$$

$X \longrightarrow \Delta(n) = SP^n[0,1]$.

X_k = part ~~[illegible]~~ where k eigenvalues are at basepoint, i.e. $\xi_k | X_k$ is of rank $n-k$. Now

X_0 is open i.e. $s_1 > 0$

$$X_0 \underset{f_0}{\cup} X_{01} \times [0,1] \underset{f_1}{\cup} X_1$$

But I have

V_k = part where $s_k < s_{k+1}$
i.e. kth barycentric coordinate $s_k > 0$.

Then $\bigcup_{k=0}^{n} V_k = X$. ~~the result goes as follows~~

$$\begin{array}{ccc} \eta^{-1}[0,1) & \longrightarrow & X_0 \\ \uparrow & & \uparrow f_0 \\ \eta^{-1}(0,1) & \longrightarrow & X_{01} \\ \downarrow & & \downarrow f_1 \\ \eta^{-1}(0,1] & \longrightarrow & X_1 \end{array}$$

$X_0 = V_0 \qquad 0 < s_1$

$X_1 = V_1 - V_0 \qquad 0 = s_1 < s_2$

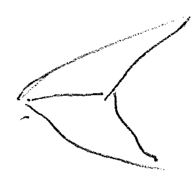
$X_2 = V_2 - (V_1 \cup V_0) \qquad 0 = s_1 = s_2 < s_3$

So it might be more natural to consider the open sets

$V_0 = X_0 \qquad 0 < s_1$

$V_0 \cup V_1 = X_0 \cup X_1 \qquad 0 \quad < s_2$

$V_0 \cup V_1 \cup V_2 = X_0 \cup X_1 \cup X_2 \qquad 0 \quad < s_3$



which increase and whose complements give the strata

~~Suppose then I find the difference~~

On V_k where $s_k < s_{k+1}$ I can take the ~~[illegible]~~ associated rank $(n-k)$-bundle obtained by pushing the eigenvalues $s_1 \cdots s_k$ to the basepoint.

space of self-adjoint Fredholm operators

Better: let $\mathcal{F}$ = self-adjoint Fredholm operators with essential spectrum ± 1, stratified in the usual way. [redacted] I want to understand why $\mathcal{F}$ is the classifying space of the [redacted] Q-cat of unitary vector spaces.

[redacted] What I have at the moment: [redacted] Given $A \in \mathcal{F}$ suppose [redacted] that [redacted] I arrange the eigenvalues of A in order of increasing abs. value

$$|\lambda_1| \leq |\lambda_2| \leq \cdots\cdots$$

~~suppose that A is different from~~

$A \in \mathcal{F}$	

T compact with basepoint t_0

$\mathrm{Vect}_n(X;T)$ homotopy classes of rank n bundles on X [redacted] decomposed with respect to T. Represented by $PU_n \times^{U_n} D(\mathbb{C}^n;T)$ where $D(\mathbb{C}^n;T)$ = space of decompositions of $\mathbb{C}^n$ wrt T.

Now I want to stabilize, hence I want to describe what I might mean by a [redacted] T mod t_0 bundle. Intuitively this means that X is stratified and on X_n one gives a ~~stratification wrt T~~ rank n bdle E_n decomposed wrt $T - t_0$.

Guess: suppose [redacted] I give myself ρ [redacted]: $T \longrightarrow [0,1]$ such that $\rho^{-1}(0) = t_0$. Then taking ranks I will get a map [redacted] $SP^\infty[0,1]$ = infinite symplex.

given E_n over X_n will take x into ρ of eigenvalues

So what comes next. You have an idea now of how to consider the space one can form over a simplicial complex starting from a contravariant functor $\sigma \mapsto F_\sigma$. This variance is the sort of thing you see from a simplicial map $E \longrightarrow K$. Here F_σ = fibre of E over the barycenter of σ.

So the next stage would be to understand the space of s.a. Fredholm operators and K-homology theory

---

define open set V_k to consist of those operators such that $|\lambda_k| < |\lambda_{k+1}|$

$V_0 =$

In what sense is Q a classifying space for of fin. dim'l Hilbert spaces?

~~E quadratic~~

~~be a coherent sheaf~~

$$V_k = \{A \mid |\lambda_k(A)| < |\lambda_{k+1}(A)|\}.$$

$\exists \varepsilon \ni$ mult of A in $(-\varepsilon, \varepsilon)$ is k

Then V_k is of the homotopy type BU_k and one considers the nerve of the open covering V_k

$i_0 < \cdots < i_q$ $\quad V_{i_0 \cdots i_q} =$

etc.

Topological category:

The objects are unitary vector spaces decomposed wrt T i.e. $\mathbb{C}^n$ + an element α of $D(\mathbb{C}^n; T)$.

The maps are unitary injections $\ni$ complement is supported at the basepoint.

~~The objects~~ Topologize in evident way.

What this top. cat. should classify over X. First we have a functor to $\mathbb{N}$ given by rang. So we might expect to have an open covering

$$\mathcal{U}_k \text{ of } X$$

and then for each $\mathcal{U}_k$ a bundle of rank k, ξ_k decomposed wrt T. ~~And for each ...~~

On $\mathcal{U}_k \cap \mathcal{U}_\ell$ we then have to relate ξ_k and ξ_ℓ. So if $k < \ell$ we can ask for an embedding

$$\xi_k|_{\mathcal{U}_k \cap \mathcal{U}_\ell} \hookrightarrow \xi_\ell|_{\mathcal{U}_k \cap \mathcal{U}_\ell}$$

compatible with T-decompositions, such that the complement is in the open star of the basepoint.

To be a bit more precise, one might want ~~...~~ the covering $\mathcal{U}_k$ to come from a partition $\sum \rho_k = 1$ and then

Example: F = self-adjoint Fredholm operator with ess. spec. $\{\pm 1\}$.

Q category of unitary vector spaces.

Describe for me what I should mean by a Q-bundle over a space X. Should be an open covering $U_0, U_1, U_2, \ldots$ together with a unitary bundle of ξ_k rank k over U_k together with over $U_k \cap U_\ell$ ~~existing the~~ a Q-morphism from $\xi_k | U_k \cap U_\ell$ to $\xi_\ell | U_k \cap U_\ell$ i.e. a splitting $\xi_\ell | U_{k\ell} = \alpha \oplus \xi_k | U_{k\ell} \oplus \beta$ together with compatibility data. Yes.

Example: F. Put U_k = those A such that $|\lambda_k| < |\lambda_{k+1}|$. Then over U_k one gets a bundle of rank k namely the eigenspace for first k eigenvalues.

~~Can I describe this reasonably. Thus it is necessary perhaps possible to prove~~

Thus the categories I consider tends to map nicely to $\mathbb{N}$ and the fibres are groupoids

Suppose then I give an open covering U_i of K $i \in I$ and for each σ in the nerve of this covering I give a space F_σ depending contrav. in σ. Pose the problem

$$\mathrm{Hom}_{/K}(T, R) = \mathrm{Ker}\left\{ \prod_\sigma \mathrm{Hom}(U_\sigma \times_K T, F_\sigma) \rightrightarrows \right.$$

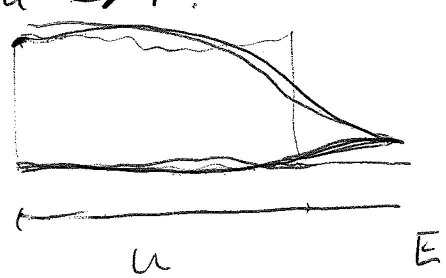
Suppose then I give a single open set U in K and then I ask for

$$\mathrm{Hom}_{/K}(T,$$

~~In what sense is U_i of K~~

~~If it can invers~~

U open in K, F single space. Consider trying to construct a space E over K with a map $E/U \longrightarrow F$.



$$\begin{array}{ccc} U_\sigma \times F_\sigma & \longleftarrow & E/U_\sigma \\ \searrow & & \swarrow \\ & U_\sigma \longrightarrow K & \end{array}$$

$E(T) =$ nat. transf.

$$\begin{array}{ccc} U_\sigma \times_K T & \longrightarrow & U_\sigma \times F_\sigma \\ & \searrow \quad \swarrow & \\ & U_\sigma & \end{array}$$

$\sigma \mapsto U_\sigma \times F_\sigma$ contravariant system of spaces over K. Take their inverse limit as a space over K.

At any point of V_k I have an interval of $n-k$ eigenvalues and I can kill the comp. to get a rank $n-k$ bundle ξ_k over V_k param. by $T-t_0$; note ξ_k extends the given ξ_k on X_k.

Now given $x \in V_k \cap V_\ell$ $k < \ell$ meaning

$$s_0(x) \qquad s_k(x) < s_{k+1}(x) \cdots s_\ell(x) < s_{\ell+1}(x)$$

Then over $V_k \cap V_\ell$ I have the restrictions of ξ_k and ξ_ℓ and I have an embeding $\xi_\ell | V_k \cap V_\ell \hookrightarrow \xi_k | V_k \cap V_\ell$ with complement supported at the basepoint. Now I can continue to higher $V_{k_0 k_1 \cdots k_p}$. So I get an open covering V_k (which is simply the intersection of the one from $\Delta(u)$), and a cocycle on V_k with values in my top. cat ?

Example

~~and now I would guess that I will obtain~~

So over X_k I have ξ_k of rank $n-k$ par. by $T-t_0$. Better over $V_k = \{x \mid s_k(x) < s_{k+1}(x)\}$ I have ξ_k. V_k is tubular around X_k. Given a point x it gives an eigenvalue sequence

such that $s_0(x) \leq \cdots\cdots$

So because of this nice function $\rho : T \longrightarrow [0,1]$
$\rho^{-1}(0) = 0$

I get a map $X \longrightarrow SP^\infty([0,1])$ which associates to each point x the image under ρ of the eigenvalue sequence of T acting on the fibre of ξ over x. Then I can define ?

I have how many eigenvalues

I have X stratified $X = X_0 \amalg X_1 \amalg \cdots$ where on X_k I have a bundle of rank $n-k$ param. by $T-t_0$. Thus X_0 is open by this definition. Then I define $X \longrightarrow SP^n([0,1])$ by defining the sequence to start with k eigenvalues at 0 over X_k. Thus at each point x we have the sequence

$$0 = s_0(x) = \cdots = s_k(x) < s_{k+1}(x) \leq \cdots \leq s_n(x) \leq 1$$

if $x \in X_k$. I define $V_k \ni s_k(x) < s_{k+1}(x)$ so that $X_k \subset V_k$ is a tubular nbd. Now over V_k I give ξ_k by killing the part with $s_0(x) \leq \cdots \leq s_k(x)$.

higher alg. K-theory.

9-06 Alg. K-theory starts with:

A ring (ass. with 1)

$\mathcal{P}_A$ = cat. of fin. gen. proj. A-modules

$K_0 A$ = Groth. grp of $\mathcal{P}_A$ = monoid of iso. classes made into an abelian group.

$GL_n A$ = group of $n \times n$ inv. matrices

$GL_n A \subset GL_{n+1}(A)$ embed via $\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$

Put $GL(A) = \bigcup_n GL_n A$

$E(A)$ = subgroup gen. by $I + a e_{ij}$ $\quad i \neq j$

$$\begin{pmatrix} 1 & & a \\ & 1 & \\ & & 1 \end{pmatrix}$$

Whitehead lemma: $E(A) = (GL(A), GL(A)) = (E(A), E(A))$

$$K_1 A \simeq GL(A)/E(A) = H_1(GL(A), \mathbb{Z})$$

The ~~[illegible]~~ use of this notation to connect the work of Groth. & Whitehead is due to Bass, who justified it by producing exact sequences, etc. Almost imm. problem of extending to a seq. of K_n's arose

~~Milnor $K_2 A = H_2(E(A), \mathbb{Z})$~~

Negative direction:

Def: $K_{n-1} A = \operatorname{Coker}\{K_n(A[t]) \oplus K_n(A[t^{-1}]) \longrightarrow K_n(A[t,t^{-1}])\}$.

V fin. dim. H. s.

form simplicial space: a p-simplex will be a sequence

$E_1, \ldots, E_p$

of orthogonal projections $E_i^2 = E_i$, $E_i E_j = E_j E_i$, $E_i^* = E_i$

with face operators

$$d_0(E_1, \ldots, E_p) = E_2, \ldots, E_p \qquad d_i(E_0, \ldots, E_p) = E_0, \ldots E_i + E_{i+1}, \ldots, E_p$$

$$d_1(E_1, E_2, \ldots, E_p) = E_1 + E_2, \ldots, E_p \qquad d_p(\quad) = E_p + E_0, E_1 \ldots E_{p-1}$$

$$d_p(E_1, \ldots, E_p) = E_1, \ldots, E_{p-1}$$

Claim the realization of this simplicial space is the unitary group of V. A point in the realization is a pair consisting of a non-degenerate p-simplex $E_1, \ldots, E_p$ (i.e. $E_i \neq 0 \ \forall i$) and a sequence $0 < t_1 < \ldots < t_p < 1$. Assoc. to this pair the unitary operator which has eigenvalue $\exp(2\pi i t_j)$ on $\operatorname{Im} E_j$ and 1 on the orth. comp. Obvious 1-1 correspondence. Now to check OKAY have to show the map

$$(E_0, \ldots, E_p) \times \{0 \le t_1 \le \ldots \le t_p\} \longmapsto \sum_{j=0}^{p} \exp(2\pi i t_j) E_j$$

is compatible with faces. This is clear.

V vector space of dim n over k alg. closed

Can you describe $\operatorname{Aut}(V) = GL(V)$ as a contraction in some sense?

~~[illegible]~~

<u>Question</u>: How to make 1-par. subgps. [redacted] of GL_n into a simplicial complex. Thus if it corresponds to a grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$, then we consider the flag in some order, and we have to worry about the same flag param by diff. exponents.

X space

$SP^n(X) = X^n / \Sigma_n$ $\qquad n_1 x_1 + \cdots + n_p x_p$

$SP^\infty(X) = \bigcup SP^n(X)$ $\qquad \boxed{\sum_i n_i x_i}$ $\qquad$ basept ~ 0

DT thm: $A \subset X \quad X/A$ $\qquad \mathcal{P}_A$

$SP(A) \longrightarrow SP(X) \longrightarrow SP(X/A)$ $\qquad$ quasi-fibration

$X \longmapsto \pi_*[SP(X)]$ gen. homology theory

M top. abel. monoid.

S set with basepoint $\qquad S \mapsto M_S$ $\quad$ covariant

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad$ $\|$

chains on S with coeff. in

$\sum_{s \in S - \{*\}} m_s s$

$Ch(X; M) = \{S \mapsto X^S\} \otimes_{\Gamma} \{S \mapsto M_S\}$

$\mathcal{P} = \mathcal{P}_A \qquad S \qquad \mathcal{P}_S$ cat $\quad \sigma \quad P_\sigma$ Ob

$\mathcal{BP}_S$ family of projections

1

6.04

higher alg. K-theory

A ring (assoc. with 1)

P_A = cat. of f.g. proj. A-modules

→ $K_0 A$ = Groth. grp. of P_A = ab. group assoc. to monoid of isom. classes of P_A

$GL_n A$ = group of $(n \times n)$-invertible matrices

$\bigcup GL_n A = GL(A)$ $\qquad GL_n \subset GL_{n+1} \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$

$E(A)$ = subgp gen. by elem. matrices $\begin{pmatrix} 1 & & x \\ & \ddots & \\ & & 1 \end{pmatrix}$

Wh. Lemma: $\quad E(A) = (GL(A), GL(A)) = (E(A), E(A))$

→ $K_1 A = GL(A)/E(A) = H_1(GL(A), \mathbb{Z})$

These notations were introduced by Bass, who justified them by showing functors K_0, K_1 are related by exact sequences. Raises problem: Produce a satisfactory theory of K_n for all $n \in \mathbb{Z}$.

Negative direction can be done recursively.

Bass def: $\quad K_{n-1} A = \text{Coker}\, \{ K_n(A[t]) \oplus K_n(A[t^{-1}]) \longrightarrow K_n(A[t, t^{-1}])$

(this is a theorem for $n = 1$) Karoubi, proceeding differently, introduced for any ring A, a new ring called the suspension of A, SA and put

$$K_{-n}(A) = K_0(S^n A)$$

leading to the same groups as Bass.

Positive direction: Milnor

$K_2 A =$ Shur multiplier of $E(A)$.

However to proceed further one (at least at the moment) has to [redacted] use alg. topology, i.e. the groups $K_n A$ are homotopy groups of a suitable space constructed from the ring A.

<u>Lemma</u>: Let X be a CW complex, basept, conn. Let $N \subset \pi_1 X$, N normal + perfect ($N = (N,N)$). Then $\exists$ CW complex Y and a map $f: X \to Y$ s.t.

(i) $f_*: \pi_1(X)/N \xrightarrow{\sim} \pi_1 Y$

(ii) for any local coeff. system L on Y,

$$f_*: H_q(X, f^*L) \xrightarrow{\sim} H_q(Y, L).$$

Further, the pair (Y, f) is uniquely determined up to homotopy equivalence by (i), (ii).

[redacted] To apply this, take

$$X = BGL(A) \qquad \pi_q BGL(A) = \begin{cases} GL(A) & q = 1 \\ 0 & q \neq 0. \end{cases}$$

Take $N = E(A)$, [redacted] and [redacted] the resulting space Y, unique up to $\simeq$ will be denoted $BGL(A)^+$.

<u>Def</u>: $K_i A = \pi_i BGL(A)^+ \qquad i \geq 1.$

~~$\mathcal{C} \ni p \mapsto \Delta(p)$ = stand. p-simplex cov.~~

$[p] \longmapsto$ ~~...~~ = set of diagrams
$X_0 \to \cdots \to X_p$ in $\mathcal{C}$

contrav. functor from Δ to sets. Contracting this with the covariant functor

$[p] \longmapsto \Delta(p)$ = simplex with vertices $\{0, \dots, p\}$

gives ~~geometric realization~~ the classifying space $B\mathcal{C}$

So given S, $\mathcal{P}_A[S]$ = cat. consisting of an object P of $\mathcal{P}_A$ decomp. $P = \bigoplus_{\sigma \in S} P_\sigma$, + $P_* = 0$. and isom. of these.
$S \longmapsto B\mathcal{P}_A[S]$ covariant

$B\mathcal{P}_A[X]$ = Contraction

Key point: $Y \subset X$ pt. conn. CW cxs.

$\Rightarrow$ $B\mathcal{P}_A[Y] \longrightarrow B\mathcal{P}_A[X] \longrightarrow B\mathcal{P}_A[Y/X]$ quasi-fibration

---

<u>Th. 1</u>: $X \longmapsto \pi_q(B\mathcal{P}_A[X])$ hom. theory. X connected.

<u>Th. 2</u>: $\Omega^1 B\mathcal{P}_A[S^1] \sim K_0 A \times BGL(A)^+$

<u>negative K-groups.</u>

Bass: ~~...~~

$$\begin{array}{ccc} A & \subset & A[t] \\ \cap & & \cap \\ A[t^{-1}] & \subset & A[t, t^{-1}] = \{\sum_n a_n t^n\} \end{array}$$

Th. $\operatorname{Coker}\{K_1(A[t]) \oplus K_1(A[t^{-1}]) \longrightarrow K_1(A[t, t^{-1}])\} \cong K_0(A)$

K_q

D $\quad q+1$

What kind of paths in U_n can we realize by maps $\mathbb{G}_m \to \mathrm{Aut}(V)$.

why are Fredholm ops. loops on $\hat{\mathcal{F}}$? In A why is the space of units A^* closely related to the loops on the projectors?

e.g. why is $GL_n(\mathbb{C})$ related to $\Omega\{\text{projectors}\}$.

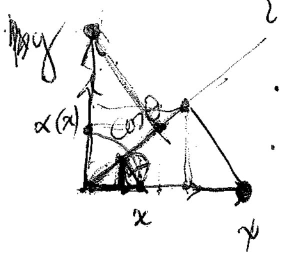
Observe that the space of projectors E of rank n in $\mathrm{End}(V)$ V large is BU_n whose loop space is U_n.

$$SM \longrightarrow BM$$
$$M \longrightarrow \Omega BM.$$

If BM = k planes in H, one should see U_k inside

plane graph of A. $\qquad (\cos\theta)x$

$\{(x, Ax)\,|$ $\qquad (x, Ax) \quad (-A^*y, y)$



$$(x, A^*y) = (Ax, y)$$

$$x \longmapsto (\cos\theta)x,\ (\sin\theta)Ax$$
$$y \longmapsto (-\sin\theta)A^*y,\ (\cos\theta)y$$

$$\begin{pmatrix} \cos\theta & -A^*\sin\theta \\ A\sin\theta & \cos\theta \end{pmatrix}$$

$$(\cos\theta\, x, (\sin\theta)A\cdot x) = \Gamma$$
$$(-\sin\theta\, A^*y, \cos\theta\, y) = \Gamma^\perp$$

$$\cos^2\theta \qquad \sin\theta\cos\theta\, A$$
$$\sin\theta\cos\theta\, A^* \qquad \sin^2\theta$$

Outline

1.) K_n $\quad n \geq 0$

$BGL(A)^+$

[redacted] Segal – Anderson

2.) K_n $\quad n \leq 0.$

Bass Def: $\quad K_{n-1}(A) = \text{Coker} \; \{K_n A[t] \oplus K_n A[t^{-1}] \longrightarrow K_n A[t, t^{-1}]\}$

Karoubi.

Segal - Anderson approach.

X finite cx. with basepoint, M top. ab. monoid

Let $M[X]$ be the set of reduced 0-chains on X coeff. in M

$$\sum_{x\in X} m_x\, x \qquad m_* = 0.$$

Can make $M[X]$ into a space as follows.

~~S finite set with basepoint~~

Γ = cat of finite sets with basepoint

$\underline{n} = \{0, 1, \dots, n\}$ basept 0. $n \geq 0$.

S object of Γ

Segal - Anderson approach - I like to think of this as a gen. of D-T theory of symm. products. Ex.

Γ = cat. of fin. sets with basept

$\underline{n} = \{0, 1, \dots, n\}$

X space with basept. , $S \in \mathrm{Ob}\,\Gamma$

$$X^S = \mathrm{Hom}_{pt}(S, X) = \prod_{S-*} X$$

$S \longmapsto X^S$ contrav. from Γ to Spaces

M top. ab. monoid

$$M[S] = \prod_{S-*} M$$

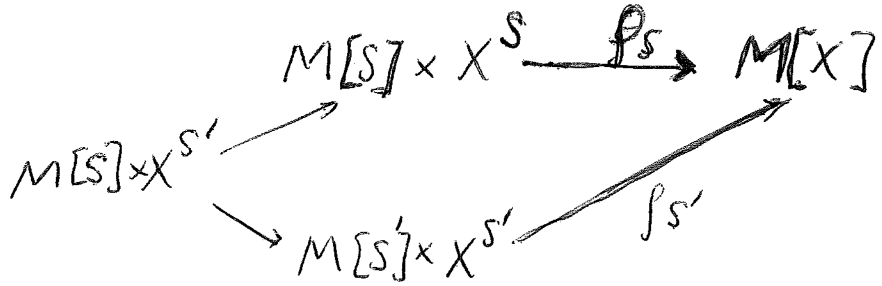
covariant in S: $\quad S \xrightarrow{u} S'$

$$\{m_s\}_{s\in S} \in M[S]$$

$$(u_*\{m_s\})_{s'} = \sum_{s \in u^{-1}\{s'\}} m_s$$

covariant functor $S \mapsto M[S]$ from Γ to spaces.

Put $M[X] = \coprod_{\underline{n}} M[\underline{n}] \times X^{\underline{n}} \Big/ (\theta_* \alpha, \beta) \sim (\alpha, \theta^* \beta)$
all θ in Γ.



Easy to see that a point of $M[X]$ is a 0-chain $\sum' m_x x$ on X with coeff in M $\ni$ $m_* = 0$.

Ex: $M = \mathbb{N}$ $\quad M[X] = SP(X)$

~~D-Thom thm: $A \subset X$ finite complex basept.~~ [illegible]

D-Thom: $X \mapsto \pi_q(M[X])$ gen. hom. theory

Idea in K-A theory is to replace chain $\sum'_{x \in X} m_x \cdot x$ by $\sum' P_x \cdot x$, ~~proj modules over A~~ [illegible] $\mathcal{P}_A$.

For S given, let $\mathcal{P}_A[S]$ be the cat. cons. of an object P of $\mathcal{P}_A$ together with submod. P_s $s \in S$ such that $P = \bigoplus_{s \in S} P_s$ $\quad P_* = 0$.

$\mathcal{C}$ cat. $\quad \Delta$ = cat. of p.o. sets and (weakly) monot. maps of the forms
$[n] = \{0, 1, \dots, n\}$ usual ordering.

SU_n conjugacy classes are points of a simplex $(n-1)$
Point is that if I have a divisor on S^1 of degree 0
then ? Given $\lambda_1, \dots, \lambda_n$ in $\mathbb{R}/\mathbb{Z}$ $\ni$ $\sum \lambda_i = 0$
$\exists!$ lifting to $x_1, \dots, x_n$ $\ni$ $x_1 \le x_2 \le \dots \le x_n \le x_1 + 1$, $\sum x_i = 0$.
and this gives the corresp. with a simplex..

$$x_1 \le \dots \le x_1 + 1 \longmapsto (x_2 - x_1) + (x_3 - x_2) + \dots + (1 + x_1 - x_n)$$

$$t_0 + t_1 + \dots + t_n = 1$$

Thus it should be so that SU_n is the realization of a simplicial complex. Vertices occur when

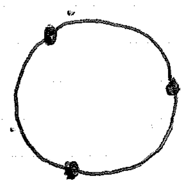
$$\underbrace{x_1 = \dots = x_k}_{a} < \underbrace{x_{k+1} = \dots = x_n}_{a+1} =$$

$$ka + (n-k)(a+1) = 0$$

$$na + n - k = 0$$

$$\text{or } a = -\frac{n-k}{n}$$

$$= \frac{k}{n} - 1$$



and the fibre over the vertex is a single point, namely the point $\exp\left\{2\pi i \frac{k}{n}\right\}$ $k = 0, \dots, n-1$

<u>Idea</u>: periodicity map classically.

$$A[t, t^{-1}]^* \longrightarrow SA$$

Start with a path in U ΩU_n i.e. a map $S^1 \longrightarrow U_n$ possibly depending on parameters.
Then you ~~...~~ approximate by a Laurent ~~series~~ poly map

$$\sum \alpha_n z^n \qquad \alpha_n \in End(V)$$

and <u>linearize</u> to get $\alpha + \beta z$; non-sing. on S^1

Higher algebraic K-theory.

$A\,,\quad \mathcal{P}_A$

K_0A = Groth group of $\mathcal{P}_A$

$GL_n(A) \subset GL_{n+1}(A)$ & $GL(A) = \bigcup GL_n(A)$

$\alpha \longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad E(A) = (GL(A), GL(A))$

$K_1A = GL(A)/E(A) = H_1(GL(A), \mathbb{Z})$

Problem: Construct a "good" theory of K_nA, $n \in \mathbb{Z}$

Milnor K_2A = Shur multiplier of $E(A)$ = $H_2(E(A), \mathbb{Z})$.

Lemma: X ptd. conn. CW cx.
$N \subset \pi_1 X$ $\quad$ N normal, perfect $\quad N = (N, N)$

Then $\exists$ CW cx. Y and a map $f: X \longrightarrow Y$ such that

i) f_* induces isom. $\pi_1 X/N \xrightarrow{\sim} \pi_1 Y$

ii) for all local coeff. systems L on Y one has

$$f_* : H_q(X, f^*L) \xrightarrow{\sim} H_q(Y, L).$$

Further the homot. type of the pair (Y, f) is uniquely det. by these properties.

$X = BGL(A) \qquad \pi_1(X) = GL(A) \supset E(A) = N$

$$f: BGL(A) \longrightarrow BGL(A)^+$$

Def. $K_i A = \pi_i\left(BGL(A)^+\right) \qquad i \geq 1$