January 19, 1974.

Lemma: Let X be an infinite set, let J_X be the partially ordered set consisting of non-empty totally ordered finite subsets of X. (J_X is thus fibred over the poset of non-empty finite subsets of X, the fibre over σ being the set of total orderings of σ). Then J_X is contractible.

Proof: One has $J_X = \bigcup_S J_S$ where S runs over the finite subsets of X. It suffices to show each J_S contracts to a point in J_X. But as X is infinite ~~there~~ we can choose $x \in X - S$. Given $\sigma \in J_S$ let $\sigma \cup x$ denote the union of σ and x ordered so that x is the maximum object. Then clearly we have natural transformations

$$\begin{array}{ccc} \sigma & \longmapsto & \sigma \\ & & \rotatebox[origin=c]{90}{$\geq$} \\ & \longmapsto & \sigma \cup x \\ & & \rotatebox[origin=c]{90}{$\leq$} \\ & \longmapsto & x \end{array}$$

from J_S to J_X.

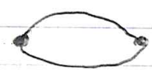
Conjecture: If X is finite, then J_X is spherical.

Examples:

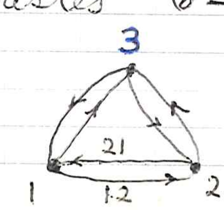
$X = \emptyset \qquad J_X = \emptyset$

$X = pt \qquad J_X = pt$

$\mathrm{card}(X) = 2 \qquad J_X \sim S^1$



$\mathrm{card}(X) = 3$. One pastes 6 - 2 simplices onto



The result is simply-connected. In effect take the maximal tree (1,2) (1,3). We then have to contract the loops (1,2)-(2,1), (1,3)-(3,1), $\underbrace{(1,2)-(2,3)-(1,3)}_{\partial(1,2,3)}$, and $\underbrace{(1,2)-(3,2)-(1,3)}_{\partial(1,3,2)}$.

$$\begin{array}{lll} \text{As for } (1,3) \text{ it deforms via } (1,3,2) & \text{to} & (1,2)-(3,2) \\ \phantom{\text{As for }} (3,1) \longrightarrow (3,1,2) & \text{to} & (1,2)-(3,2) \end{array}$$

so the loop (1,3)-(3,1) is homotopic to zero. Similarly for (1,2)-(2,1).

Now consider the complex of chains on J_X:

$$\cdots \longrightarrow \coprod_{\sigma_0<\sigma_1} \mathbb{Z} \longrightarrow \coprod_{\sigma_0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Better. I know that given any element σ of J_X the set of $\sigma'<\sigma$ is the barycentric subdivision of the boundary of the simplex with vertices σ. Thus by Lusztig's observation I should have a resolution

$$\coprod_{\substack{x_0,x_1,x_2\\ \text{distinct}}} \mathbb{Z} \cdots \longrightarrow \coprod_{(x_0\neq x_1)} \mathbb{Z} \longrightarrow \coprod_{x\in X} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

In any case I should be able to check this directly. X is an infinite set and I define the above complex by setting

$$d(x_0,\dots,x_s) = \sum_{i=0}^{s} (-1)^i (x_0,\dots,\hat{x}_i,\dots,x_s)$$

It is therefore a subcomplex of the complex of singular chains for X. Now the homotopy operator should work because given any finite chain we can always find a vertex outside the support of my chain. Clear.

Now take $X = \mathbb{P}_1(k)$, k an infinite field, and let $G = GL_2(k)/\text{center}$ act on X. Then from the above resolution we get a spectral sequence. Now the point is that G acts simply-transitively on triples of distinct points in $\mathbb{P}_1(k)$, and that on quadruples there is a single invariant – the cross-ratio which is any element of $k - \{0, 1\}$. So in the spectral sequence

$$E^1_{0*} = H_*(G)$$

$$E^1_{1*} = H_*(B)$$

$$E^1_{2*} = H_*(T)$$

$$E^1_{3*} = H_*(pt) = \mathbb{Z}$$

$$E^1_{4*} = \bigoplus_{k^\times - \{1\}} \mathbb{Z}$$

Thus it would seem that the first two terms are:

$$\bigoplus_{k^\times - \{1\}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow H_*(T) \xrightarrow{1-\sigma} H_*(B) \longrightarrow H_*(G)$$

Suppose one tries to use this to get information on the low dimensional homology of G. For example, when k is infinite, or better, when one has a splitting theorem, one knows $H_*(T) \xrightarrow{\sim} H_*(B)$, hence

$W =$ Weyl group $\mathbb{Z}/2$.

$$E^2_{2*} = H_*(T)^W$$

$$E^2_{1*} = \operatorname{Ker}\{H_*(T)_W \longrightarrow H_*(G)\}$$

~~So if k is algebraically closed and we only look at torsion, then $H_*(T)_W$ is $(\mathbb{Q}/\mathbb{Z})^r$ in odd degrees.~~

~~So the result seems to be that for an algebraically~~

One should examine this spectral sequence carefully ignoring 2 torsion. (In effect one has a homomorphism

$$GL_2(k) \longrightarrow k^* \times PGL_2(k)$$

first component is the determinant. The kernel is $\{\pm 1\}$ and the cokernel is $k^*/(k^*)^2 = H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) =$ quadratic extensions of k. This sequence can be used to show $GL_2(k)$ and $k^* \times PGL_2(k)$ have the same homology except for 2-torsion.)

$$H_1(T) = T = k^* = (k^* \times k^* / \Delta k^*)$$

and W acts as -1, so that ignoring 2-torsion

$$H_1(T)_W = H_1(T)^W = 0.$$

Picture of the spectral sequence

$$
\begin{array}{ccccccccc}
 & & & & & & H_3B & & H_3G \\
 & & & & H_2T & & H_2B & & H_2G \\
 & & & & \cancel{\ } & & \cancel{H_1B} & & \cancel{H_1G} \quad \leftarrow 0 \text{ in } E^2 \\
\bigoplus_{k^\circ-\{1\}} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z}
\end{array}
$$

So one sees that ignoring 2-torsion one has

$$H_2B \twoheadrightarrow H_2G$$

~~Modulo torsion one has that $H_2T = \Lambda^2 T$ and W acts trivially on this. In general $T \cong C \times F$ where C is a union of cyclic groups and F is torsion-free. Then~~

$$\cancel{H_2(C \times F) = H_2C + H_1C \otimes H_1F + H_2F}$$
$$\cancel{= 0 + C \otimes F + \Lambda^2 F}$$

But for any abelian group T one has

$$H_2T = \Lambda^2 T$$

~~Hence in the case k infinite, where I know that~~ ~~[illegible]~~ and the Weyl group acts trivially on this. Thus what we find is an exact sequence

$$\bigoplus_{k^\circ-\{1\}} \mathbb{Z} \xrightarrow{d_3} \underset{\substack{\parallel \\ H_2B}}{\Lambda^2(k^\circ)} \longrightarrow H_2G \longrightarrow 0$$

ignoring 2-torsion, k infinite.

~~[illegible]~~ If we do not ignore 2-torsion, ~~[illegible]~~ but still assume k infinite so that

$H_*(T) = H_*(B)$, then our E^1_{*1} row is

$$\cdots\ 0 \longrightarrow k^* \xrightarrow{2} k^* \longrightarrow H_1(PGL_2(k))$$

whereas the E^1_{*2} row is

$$\longrightarrow 0 \longrightarrow \Lambda^2 k^* \xrightarrow{0} \Lambda^2 k^* \longrightarrow H_2(PGL_2(k))$$

But one should observe that if we look at the spectral sequence for the central extension

$$1 \longrightarrow k^* \longrightarrow GL_2(k) \longrightarrow PGL_2(k) \longrightarrow 1$$

its 5 term exact sequence gives

$$\begin{array}{ccccccccc} H_2(GL_2(k)) & \longrightarrow & H_2(PGL_2(k)) & \longrightarrow & H_1(k^*) & \longrightarrow & H_1(GL_2 k) & \longrightarrow & H_1(PGL_2 k) \\ & & & & \| & & \| & & \| \\ & & & & k^* & \xrightarrow{2} & k^* & & 0 \end{array}$$

hence we see $H_2(PGL_2(k))$ maps onto μ_2. This suggests

<u>Conjecture</u>: $d_3(\lambda) = \lambda \wedge (1-\lambda)$ $\qquad \lambda \in k^* - \{1\}$,

hence one has an exact sequence

$$0 \longrightarrow K_2(k) \longrightarrow H_2(PGL_2(k)) \longrightarrow \mu_2 \longrightarrow 0$$

for any infinite field.

This should also be true for a finite field with a "large" no. of elements. In fact ignoring 2-torsion one has $H_2(SL_2(k)) \xrightarrow{\sim} H_2(PGL_2(k))$ is zero because $SL_2(k)$ is "simply-connected."

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] & SL_2 \arrow[r] \arrow[d, phantom, "\cap"] & PSL_2 \arrow[r] \arrow[d, phantom, "\cap"] & 1 \\
1 \arrow[r] & k^\cdot \arrow[r] & GL_2 \arrow[r] & PGL_2 \arrow[r] & 1
\end{tikzcd}$$

$$\begin{tikzcd}
\cancel{H_2(SL_2)} \arrow[r] & H_2(PSL_2) \arrow[r, "\sim"] \arrow[d] & H_1(\mu_2) \arrow[r] & \cancel{H_1(SL_2)} \\
H_2(GL_2) \arrow[r] & H_2(PGL_2) \arrow[r] & H_1(k^\cdot) \arrow[r] & {}
\end{tikzcd}$$

shows that when SL_2 is simply-connected that

$$\begin{tikzcd}
H_2(PSL_2) \arrow[d] \arrow[dr, "\sim"] & \\
H_2(PGL_2) \arrow[r] & \mu_2
\end{tikzcd}$$

On the other hand from the extension

$$0 \longrightarrow PSL_2 \longrightarrow PGL_2 \longrightarrow k^\cdot/k^{\cdot 2} \longrightarrow 0$$

$$k^\cdot/k^{\cdot 2} = \mathbb{Z}/2 \quad \text{if } k \text{ finite}$$

one get

$$\begin{array}{|llll}
\mu_2 & & & \\
0 & 0 & 0 & \\
\mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\
\hline
\end{array}$$

so one sees that $H_2(PSL_2) \xrightarrow{\sim} H_2(PGL_2)$. Thus the conjecture is true for a finite field when $SL_2(k)$ is simply-connected.

But wait: If $k = \mathbb{R}$, then I remember vaguely that the symbol relations were different. The point was that in $H_2(SL_2\mathbb{R})$ there is the Euler class, which is of infinite order and which becomes a order 2 class in $H_2(SL_3\mathbb{R})$. But this class should already be a mod 2 class in $H_2(GL_2\mathbb{R})$.

Further stability results. Let V be a vector space of dimension n over an infinite field k. Then it should be possible now to prove that one has an acyclic complex

$$\longrightarrow \coprod_{x_1,x_2} \mathbb{Z} \longrightarrow \coprod_{x_1\neq 0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where in dimension q I have subsets of V of card q which are in general position, i.e. every subset has rank = min of its cardinality and n. The proof works as before - given a finite subset of card q subsets in general position, one can always find an element of V which can be added as last vertex, etc.

Now I use the resolution above to get [redacted] a spectral [redacted] sequence converging to zero with

$$E^1_{st} = H_*\left(\left(\begin{array}{c|c} I_s & * \\ \hline 0 & GL_{n-s}\end{array}\right)\right) \qquad s\leq n$$

$$\bigoplus_{k^n - \text{coordinate hyperplanes}} \mathbb{Z} \qquad s=n+1, \ t=0$$

So it will look like this (recall k infinite)

	0	$H_1(GL_1)$	$H_1(GL_n)$
$\boxed{\text{mess}}$	$\mathbb{Z}$	$\cancel{\mathbb{Z}}$	$\mathbb{Z}$

$n=2$:

$$\begin{array}{cccc}
0 & 0 & H_2(GL_1) \longrightarrow & H_2(GL_2) \\
0 & 0 & H_1(GL_1) \longrightarrow & H_1(GL_2) \\
\boxed{\text{mess}} \twoheadrightarrow \mathbb{Z} & \xrightarrow{0} \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}
\end{array}$$

gives $H_1(GL_1) \twoheadrightarrow H_1(GL_2)$, but not trivially

$n=3$:

$$\begin{array}{ccccc}
 & H_2(GL_1) & \xrightarrow{0} & H_2(GL_2) \longrightarrow & H_2(GL_3) \\
0 & H_1(GL_1) & \xrightarrow{0} & H_1(GL_2) \longrightarrow & H_1(GL_3) \\
\mathbb{Z} \xrightarrow{\sim} & \mathbb{Z} & & \mathbb{Z} \xrightarrow{\sim} & \mathbb{Z}
\end{array}$$

So it is clear from this that we get

$$H_1(GL_2) \xrightarrow{\sim} H_1(GL_3) \xrightarrow{\sim} \cdots\cdots$$

However we cannot get $H_2(GL_2)$ onto $H_2(GL_3)$.

$n=4$

$$\begin{array}{ccc}
 & & H_2(GL_3) \longrightarrow H_2(GL_4) \\
 & H_1(GL_1) \twoheadrightarrow H_1(GL_2) & H_1(GL_3) \xrightarrow{\sim} H_1(GL_4) \\
\twoheadrightarrow \mathbb{Z} & \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} & \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}
\end{array}$$

It seems that without further information, I get the stability range this way that I obtained before using the unimodular complexes mod 2. i.e.

$$H_n(GL_{2n}) \xrightarrow{\sim} H_n(GL_{2n+1}) \xrightarrow{\sim} \cdots$$

January 23, 1974

Let $G = PGL_3(k) = GL_3(k)/\text{center}$ act on $\mathbb{P}^2 =$ lines $L \subset k^3 = V$. Claim G acts simply transitively on 4-tuples of points in general position. In effect if L_1, L_2, L_3 are ~~points~~ lines in general position, then $V = L_1 \oplus L_2 \oplus L_3$, so if L is a fourth line ~~independent of these~~ in general position with resp. to the L_i, one has $L = k(x,y,z)$ with x,y,z all $\neq 0$. Thus there is a unique diagonal matrix making x,y,z equal to 1. In other words we get a unique element of G carrying L_1, L_2, L_3, L to the 4-tuple $(ke_1, ke_2, ke_3, k(e_1+e_2+e_3))$.

So now consider the complex

$$\longrightarrow \bigoplus_{(L_1,L_2,L_3)} \mathbb{Z} \longrightarrow \bigoplus_{(L_1,L_2)} \mathbb{Z} \longrightarrow \bigoplus_{L_1} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where in degree s the sum is taken over s-tuples of lines in $\mathbb{P}^2$ with the standard boundary operators. This gives a spectral sequence

$$\cancel{0 \to H_2(T) \to H_2(G_2) \to H_2(G_1) \to H_2(G)}$$

$$E^1_{st} = \begin{cases} H_t\begin{pmatrix} \begin{matrix} k^* & & \\ & \ddots & \\ & & k^* \end{matrix} & * \\ 0 & GL_{3-s} \end{pmatrix} & s \leq 3 \\ \mathbb{Z} & s = 4 \\ \text{meas} & s \geq 5 \end{cases} \implies 0$$

$$\begin{array}{cccccccc}
 & & & & & H_2\left(\begin{vmatrix} k^* & \\ & GL_2 \end{vmatrix} /\text{cent}\right) & \longrightarrow & H_2(PGL_3) \\
 & (k^*)^2 & & (k^*)^2 & & & k^* & 0 \\
 & \| & & \| & & & \| & \| \\
0 & H_1\left(\begin{matrix} k^* & & \\ & k^* & \\ & & k^* \end{matrix} /\text{cent}\right) & \longrightarrow & H_1\left(\begin{vmatrix} k^* & & \\ & k^* & \\ & & k^* \end{vmatrix} /\text{cent}\right) & \longrightarrow & H_1(\qquad\qquad) & \longrightarrow & H_1(PGL_3) \\
\mathbb{Z} & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}
\end{array}$$

Perhaps this is a bit too hard. Instead let us consider the spectral sequence with Gl_3 and we get

$$\begin{array}{ccccccccc}
\to H_1(C) & \xrightarrow{0} & H_1(T) & \longrightarrow & H_1(T) & \longrightarrow & H_1(k^* \times G_2) & \longrightarrow & H_1(G_3) \\
\| & & \| & & \| & & \| & & \\
k^* & & (k^*)^3 & & (k^*)^3 & & (k^*)^2 & & k^*
\end{array}$$

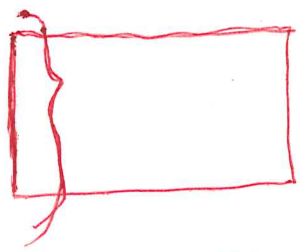
One hopes this is exact, but this doesn't seem to be the cases.

Thus what seems to happen is that there is a kernel to the map $H_2(GL_2) \twoheadrightarrow H_2(GL_3)$ which is generated by k^*. So it seems that one will not ~~[illegible]~~ have $H_2(GL_2, GL_1) = K_2$ as I would have hoped, and that moreover you don't get a simplified proof of the Matsumoto result this way.

June 24, 1974. Classifying spaces.

Recall the two possible ways of topologizing the suspension of a space F:

1) fine: quotient space of $I \times F$. Here an open nbd of 0 is given by $\{(t,f) \mid f < \alpha(t)\}$ where α is a (- - -) semi-continuous function.



2) coarse: Take a nbd of 0 to be one containing a product nbd. $\{(t,f) \mid t < \varepsilon\}$.

Version 1) is suitable for maps into other spaces, while 2) is good for maps into the suspension. In fact for 2) one has

$$\mathrm{Hom}_{\text{spaces}/I}(T, \mathrm{Susp}(F)) = \mathrm{Hom}\left((0,1) \times_I T, F\right)$$

Generalization: Let K be a simplicial complex

January 27, 1974. $\boxed{\text{Stability}}$.

Preliminaries:

Let $u: P \longrightarrow P'$ be a map of complexes in an abelian category. The cone of u, denoted $C(u)$, is defined to be the complex with $C(u)_k = P'_k \times P_{k-1}$, $d(x', x) = (dx' + u(x), -dx)$, $x \in P'$, $x \in P$. If T_* is an exact ∂-functor on complexes, one has an exact seq.

$$\longrightarrow T_q(P) \longrightarrow T_q(P') \longrightarrow T_q(C(u)) \longrightarrow \cdots\cdots$$

Define an increasing filtration of $C(u)$ by putting $\mathrm{Filt}_s(C(u))$ equal to

$$\longrightarrow \begin{matrix} P'_{s+1} \\ \oplus \\ P_s \end{matrix} \longrightarrow \begin{matrix} P'_s \\ \oplus \\ P_{s-1} \end{matrix} \longrightarrow \begin{matrix} P'_{s-1} \\ \oplus \\ P_{s-2} \end{matrix} \longrightarrow \cdots$$

Then
$$\mathrm{Filt}_s / \mathrm{Filt}_{s-1} = C(u_s : P_s \longrightarrow P'_s)[s]$$
where as usual we identify an object with a complex concentrated in degree zero, and $[s]$ denotes s-fold suspension. This filtration gives rise to a spec. sequence

$$E^1_{st} = T_{s+t}(\mathrm{Filt}_s / \mathrm{Filt}_{s-1}) \Longrightarrow T_*(C(u))$$

or

$$\boxed{E^1_{st} = T_t(C(u_s : P_s \to P'_s)) \Longrightarrow T_*(C(u))}$$

Example: Let G be a group, H a subgroup, and M a G-module. By Shapiro's lemma one has a canon. isom.

$$H_*(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M) = H_*(H, M)$$

Hence if we define relative homology gps

$$H_*(G, H; M) = H_*(G, C(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \longrightarrow M))$$

one gets a long exact sequence

$$\cdots H_t(H,M) \xrightarrow{\text{res}} H_t(G,M) \longrightarrow H_t(G,H;M) \longrightarrow H_{t-1}(H,M) \cdots$$

Let V be a vector space over a field k, and $I(V) = \tilde{H}_{n-2}(T(V))$, where $T(V)$ = Tits building and $n = \dim V$. According to Lusztig, one has an exact sequence

$$L(V):\ 0 \to I(V) \to \bigoplus_{\dim(W)=n-1} I(W) \longrightarrow \cdots\cdots \longrightarrow \bigoplus_{\dim(W)=1} \mathbb{Z} \longrightarrow \mathbb{Z} \to 0$$

on which $GL(V)$ operates. Recall this is ~~obtained~~ just the E^1-term of the homology spectral sequence obtained by filtering the ordered set of subspace of V by dim + 1. Hence, if $V \subset V'$, then one has an embedding $L(V) \subset L(V')$ compatible with the action of $GL(V', V) = \{\alpha \in GL(V') \mid \alpha V = V\}$.

Consider now the standard embedding of k^n in k^{n+1}. This induces a map of complexes $L(k^n) \hookrightarrow L(k^{n+1})$ which is equivariant with respect to the standard embedding of GL_n into GL_{n+1}, hence it extends to a map of GL_{n+1} module complexes

$$(*) \qquad \mathbb{Z}[GL_{n+1}] \otimes_{\mathbb{Z}[GL_n]} L(k^n) \longrightarrow L(k^{n+1})$$

which we denote $u : P \longrightarrow P'$.

<u>Assertion:</u> The spectral sequence on page 1 for (*) ~~...~~ has

$$E^1_{st} \cong \begin{cases} H_t\left(\left(\begin{array}{c|c} GL_s & * \\ \hline & GL_{n+1-s} \end{array}\right), \left(\begin{array}{c|c|c} GL_s & * & \\ \hline & GL_{n-s} & 0 \\ \hline & & 1 \end{array}\right); I(k^s)\right) & 0 \le s \le n \\ H_t\left(GL_{n+1}, I(k^{n+1})\right) & s = n+1 \\ 0 & s > n+1 \end{cases}$$

and it abuts to zero

<u>Proof:</u> The complex $L(V)$ is acyclic, and $\mathbb{Z}[GL_{n+1}]$ is flat over $\mathbb{Z}[GL_n]$ $\Longrightarrow$ P, P' are acyclic so the abutment is zero.

Statements about E^1_{st} $s \ge n+1$ are obvious, so suppose $0 \le s \le n$. Then

$$L_s(k^n) = \bigoplus_{d(W)=s} I(W)$$

$$= \mathbb{Z}[GL_n] \otimes_{\mathbb{Z}\left[\begin{array}{c|c} GL_s & * \\ \hline & GL_{n-s} \end{array}\right]} I(k^s)$$

where the subgroup $\begin{pmatrix} GL_s & * \\ 0 & GL_{n-s} \end{pmatrix}$ acts on $I(k^s)$ via the obvious surjection onto GL_s. So

$$P_s = \mathbb{Z}[GL_{n+1}] \otimes_{\mathbb{Z}\left[\begin{array}{c|c|c} GL_s & * & \\ \hline & GL_{n-s} & 0 \\ \hline & & 1 \end{array}\right]} I(k^s)$$

$$P'_s = \mathbb{Z}[GL_{n+1}] \otimes_{\mathbb{Z}\left[\begin{array}{c|c} GL_s & 0 \\ \hline & GL_{n+1-s}\end{array}\right]} I(k^s)$$

and it is clear that the map $u_s : P_s \to P'_s$ is induced by the inclusion.

$$\left(\begin{array}{c|c|c} GL_s & * & \\ \hline & GL_{n-s} & \\ \hline & & 1\end{array}\right) \subset \left(\begin{array}{c|c} GL_s & * \\ \hline & GL_{n+1-s}\end{array}\right).$$

Therefore

$$(P_s \to P_s') = \mathbb{Z}[GL_{n+1}] \otimes_{\mathbb{Z}\begin{bmatrix} GL_s & * \\ & GL_{n+1-s}\end{bmatrix}} \left(\begin{array}{c}\mathbb{Z}\begin{bmatrix} GL_s & * \\ & GL_{n+1-s}\end{bmatrix} \otimes_{\mathbb{Z}\begin{bmatrix} GL_s & * & \\ & GL_{n-s} & \\ & & 1\end{bmatrix}} I(k^s) \\ \downarrow \\ I(k^s)\end{array}\right)$$

and so via Shapiro's lemma

$$E^1_{st} = H_t(GL_{n+1}, C(u_s)) = H_t\left(\begin{pmatrix} GL_s & * \\ & GL_{n+1-s}\end{pmatrix}, C(\uparrow)\right)$$

which by defn $= H_t\left(\begin{pmatrix} GL_s & * \\ & GL_{n+1-s}\end{pmatrix}, \begin{pmatrix} GL_s & * & \\ & GL_{n-s} & \\ & & 1\end{pmatrix}; I(k^s)\right)$

Q.E.D.

Assume now

Prop. 1: $H_0(GL_s, I(k^s)) = 0$ for $s \geq 2$.

Prop. 2: needs to be strengthened
k infinite $\Longrightarrow$ $H_*\left(\begin{array}{c|c} 1_s & * \\ \hline 0 & GL_r\end{array}\right) \xleftarrow{\sim} H_*\left(\begin{array}{c|c} 1_s & 0 \\ \hline 0 & GL_r\end{array}\right)$.
see page 13

Then I claim I can prove

$$H_i(GL_{n+1}, GL_n) = 0 \quad \text{if } i \leq n$$

by induction on i. Assume this is true for $i < 8$

Using Lemma 2 and H-S spec. seq. of the extension

$$1 \longrightarrow \begin{pmatrix} 1_s & * \\ & GL_{n+1-s} \end{pmatrix} \to \begin{pmatrix} GL_s & * \\ & GL_{n+1} \end{pmatrix} \longrightarrow GL_s \longrightarrow 1$$

one finds that for $0 \leq s \leq n$

$$E^1_{st} = H_t\left(\left(\begin{pmatrix} GL_s & * \\ & GL_{n+1-s} \end{pmatrix}, \begin{pmatrix} GL_s & * \\ & GL_{n-s} & \\ & & 1 \end{pmatrix}\right), I(k^s)\right)$$

$$\cong H_t\left(\left(\begin{pmatrix} GL_s & 0 \\ 0 & GL_{n+1-s} \end{pmatrix}, \begin{pmatrix} GL_s & 0 \\ & GL_{n-s} & \\ & & 1 \end{pmatrix}\right); I(k^s)\right)$$

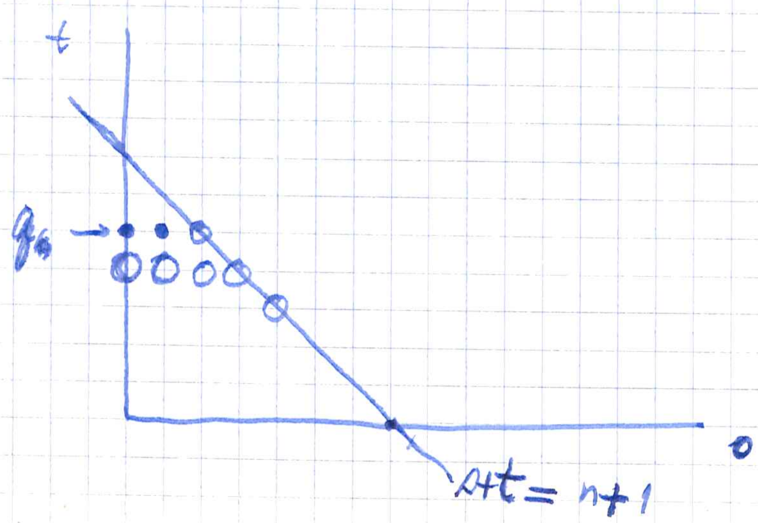
which by Kunneth can be written

$$E^1_{st} \cong \bigoplus_{i+j=t} H_i(GL_s, I_s) \otimes H_j(GL_{n+1-s}, GL_{n-s})$$

$$\oplus \bigoplus_{i+j=t-1} \mathrm{Tor}_1\big(H_i(GL_s, I_s), H_j(GL_{n+1-s}, GL_{n-s})\big).$$

Suppose $s \geq 2$, so that $H_0(GL_s, I_s) = 0$; then as $H_j(GL_{n+1-s}, GL_{n-s}) = 0$ for $j \leq \min(q_0 - 1, n-s)$, one has $E^1_{st} = 0$ for $t \leq \min(q_0, n-s+1)$. Here $0 \leq n$.

If $s = 1, 0$ then we get $E^1_{st} = 0$ for $t \leq \min(q_0 - 1, n-s)$. Thus the 0 range is



Note, for $s = n+1$, $t = 0$

$$E^1_{n+1,0} = H_0(GL_{n+1}, I(k^{n+1})) = 0$$

so we are still OKAY.

So we can conclude that for $n \geq q$, one has

$$E^1_{st} = 0 \qquad s+t = q+1 \;, \quad s \geq 2.$$

As the spectral sequence abuts to zero this implies $E^2_{0q} = 0$ i.e. that the map

$$H_q(GL_1 \times GL_n, GL_1 \times GL_{n-1}) \twoheadrightarrow H_q(GL_{n+1}, GL_n)$$

induced by the inclusion of $GL_1 \times GL_n \subset GL_{n+1}$ is surjective for $n \geq q$. But by Künneth one has

$$H_q(GL_n, GL_{n-1}) \xrightarrow{\sim} H_q(GL_1 \times GL_n, GL_1 \times GL_{n-1})$$

as $H_i(GL_n, GL_{n-1}) = 0$ for $i < q \leq n$. Thus we find that the map

$$H_q(GL_n, GL_{n-1}) \longrightarrow H_q(GL_{n+1}, GL_n)$$

induced by the embedding $\alpha \mapsto 1 \oplus \alpha : GL_n \to GL_{n+1}$ is surjective. However it is also zero by commutativity hopefully. BE CAREFUL - see page 8.

~~Lemma: Let $H' \subset H$, $G' \subset G$ and let u, v be homorphisms from H to G carrying H' into G'. Suppose q~~

Simpler argument would be to suppose that $n > q$ whence $E^1_{st} = 0$ for $s+t = q+2$, $s \geq 2$. Since the spectral sequence abuts to zero this implies (the groups represent all positions mapping by diff to E^r_{1q}) that $E^1_{1q} \hookrightarrow E^1_{0q}$ i.e. that

$$n > q \Rightarrow \underset{\substack{\| \\ H_q(GL_n, GL_{n-1})}}{H_q(GL_1 \times GL_n, GL_1 \times GL_{n-1})} \hookrightarrow H_q(GL_{n+1}, GL_n)$$

Thus for $n > q$ we have

$$H_q(GL_n, GL_{n-1}) \xrightarrow{\sim} H_q(GL_{n+1}, GL_n) \xrightarrow{\sim} \cdots$$

where the maps are induced by $\alpha \mapsto 1 \oplus \alpha$, + where $GL_{n-1} \subset GL_n$ is embedded via $\alpha \mapsto \alpha \oplus 1$. Now write down the exact sequences

$$\begin{tikzcd}
H_q(GL_{n-1}) \arrow[r, "j_*"] \arrow[d] & H_q(GL_n) \arrow[r] \arrow[d, "i_*"] & H_q(GL_n, GL_{n-1}) \arrow[r] \arrow[d, "\wr"] & H_{q-1}(GL_{n-1}) \arrow[r, "j_*"] \arrow[d, "i_*"] & H_{q-1}(GL_n) \arrow[d] \\
H_q(GL_n) \arrow[r, "j_*"] & H_q(GL_{n+1}) \arrow[r] \arrow[d] & H_q(GL_{n+1}, GL_n) \arrow[r] \arrow[d, "\wr"] & H_{q-1}(GL_n) \arrow[r] \arrow[d] & {} \\
 & {} \arrow[r] & H_q(GL_{n+2}, GL_{n+1}) \arrow[r] & {} &
\end{tikzcd}$$

Using the fact that $i_* = j_*$ (Here i is induced by $\alpha \mapsto 1 \oplus \alpha$, and j by $\alpha \mapsto \alpha \oplus 1$), one diagram chases and finds

$$H_q(GL_n, GL_{n-1}) = 0 \quad \text{for} \quad q < n$$

and so we have proved (modulo Prop's 1 and 2).

Theorem: k an ~~infinite~~ field. see page 13 Then

$$H_i(GL_i) \twoheadrightarrow H_i(GL_{i+1}) \xrightarrow{\sim} H_i(GL_{i+2}) \xrightarrow{\sim} \cdots$$

On $H_*(G, H; M)$:

First of all one has ~~from the exact sequence~~

$$H_q(G,H;M) = \mathrm{Tor}_{q-1}^{\mathbb{Z}[G]}\left(\bar{\mathbb{Z}}[H\backslash G], M\right)$$

where $0 \longrightarrow \bar{\mathbb{Z}}[H\backslash G] \longrightarrow \mathbb{Z}[H\backslash G] \longrightarrow \mathbb{Z} \longrightarrow 0$

and $\mathbb{Z}$ acts on the right. Similarly

$$H^q(G,H;M) = \mathrm{Ext}^{q-1}_{\mathbb{Z}[G]}\left(\bar{\mathbb{Z}}[G/H], M\right).$$

<u>Question:</u> Let $x \in G$ centralize H; does x act trivially on $H_*(G,H;A)$, A ~~abelian~~ trivial action?

This is false. Here is an example in cohomology. An element $\alpha \in H^2(G,H;A)$ is an iso. class of central extensions E of G by A trivialized over H. If we we lift x to E and conjugate we don't change the isom. class of E but we might change the lifting of H. Ex:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \begin{pmatrix}\text{dihedral}\\ \text{group of}\\ \text{order } 8\end{pmatrix} \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] & 1 \\
& & & \mathbb{Z}/2 = H = \text{subgp gen by } x \arrow[u, "x"] \arrow[ul, "\text{lift via } \tilde{x}"]
\end{tikzcd}$$

$(\tilde{x}\tilde{y})^2$ $\quad$ $\tilde{x}, \tilde{y}$ $\quad$ $\tilde{x}^2 = \tilde{y}^2 = 1$ $\quad$ $(\tilde{x}\tilde{y})^4 = 1$ $\quad$ $\mathbb{Z}/2 \times \mathbb{Z}/2 = G$

Then conjugating by y ~~which~~ which centralizes x in G, changes $\tilde{x}$ to $\tilde{y}\tilde{x}\tilde{y} \neq \tilde{x}$.

<u>Proposition 1</u>: k any field: $H_0(GL_n(k), I(k^n)) = 0$ for $n \geq 2$.

<u>Proof</u>: First take $n=2$. Then one has

$$0 \longrightarrow I(k^2) \longrightarrow \bigoplus_{L \subset k^2} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and by Shapiro

$$H_*\Big(GL_2, \bigoplus_{L\subset k^2} \mathbb{Z}\Big) = H_*(B)$$

where $B = \begin{pmatrix} GL_1 & * \\ 0 & GL_1 \end{pmatrix}$ = stabilizer of $ke_1 \subset k^2$. Thus

$$H_1(B) \longrightarrow H_1(GL_2) \longrightarrow H_0(GL_2, I(k^2)) \longrightarrow \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$$

so we need that $H_1(B) \twoheadrightarrow H_1(GL_2)$. But this is clear as $GL_2 = B \cup BwB$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ itself is a product of elementary matrices which are in the normal subgp. generated by B.

~~Now for $n \geq 2$ we need~~

<u>Lemma</u>: If $A \subset V$, then there is a canonical isom of $GL(V,A)$-modules

$$I(V) \cong \bigoplus_{\substack{\text{splittings} \\ \text{of } 0\to A \to V \to V/A \to 0}} I(A) \otimes I(V/A)$$

Assuming this for the moment if $\dim(V) \geq 2$, take A to be a 2-diml. subspace. Then $H_0(GL(V), I(V))$ is a quotient of

$$H_0(GL(V,A), I(V)) \overset{\text{Shapiro}}{=} H_0(GL(A) \times GL(V/A), I(A) \otimes I(V/A))$$

$$= H_0(GL(A), I(A)) \otimes H_0(GL(V/A), I(V/A))$$

$$= 0$$

Now for $n > 2$ we use induction on n. Recall that if L is a line in V we have a canonical homotopy equivalence

$$T(V) \sim \bigvee_{H} \operatorname{Susp} I(V/L)$$

where H runs over the hyperplanes complementary to L. Thus one has a $GL(V,L)$-module isomorphism

$$I(V) \simeq \bigoplus_{\{H\}} \mathbb{Z} \otimes I(V/L).$$

Hence if we look at $I(k^n)$ as a $\begin{pmatrix} 1 & * \\ & GL_{n-1} \end{pmatrix}$-module, it is the module induced from the subgp $\begin{pmatrix} 1 & 0 \\ 0 & GL_{n-1} \end{pmatrix}$ and the module $I(k^{n-1})$. Thus

$$\begin{aligned} H_0\left(\begin{pmatrix} 1 & * \\ & GL_{n-1} \end{pmatrix}, I(k^n)\right) &= H_0\left(GL_{n-1}, I(k^{n-1})\right) \\ &= 0 \quad \text{by induction.} \end{aligned}$$

But $H_0(G, M)$ is a quotient of $H_0(H, M)$ for $H \subset G$, so we are done.

<u>Remark</u>: Still need a proof of the canon. isom

$$I(V) = \mathbb{Z}[GL(V,A)] \otimes_{\mathbb{Z}[GL(A) \times GL(B)]} \left(I(A) \otimes I(B)\right)$$

where $V = A \oplus B$.

Proposition 2: k infinite field $\Rightarrow$ (NO)

$$H_*\left(\begin{array}{c|c} 1_k & 0 \\ \hline & GL_n \end{array}\right) \xrightarrow{\sim} H_*\left(\begin{array}{c|c} 1_k & * \\ \hline & GL_n \end{array}\right)$$

More generally consider the ~~fo~~ following ~~s~~ situation. Let

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$$

be a group extension with N abelian. We would like conditions ~~ge~~ implying $H_*(E) \xrightarrow{\sim} H_*(G)$. By universal coefficient it is enough to show $H_*(E,\Delta) \xrightarrow{\sim} H_*(G,\Delta)$ for some field Δ of each characteristic; (here Δ is given the trivial group action). In view of the spec. seq.

$$E^2_{pq} = H_p(G, H_q(N,\Delta)) \Longrightarrow H_{p+q}(E,\Delta)$$

it suffices to show $E^2_{pq} = 0$ for $q > 0$.

Now in general if M is a $\Delta[G]$-module and c is an element of the center of $\Delta[G]$, one knows that the ~~endo~~ endo of $H_q(G,M)$ induced by mult. by c on M is the same as mult. by $\varepsilon(c)$ where $\varepsilon : \Delta[G] \to \Delta$ is the augmentation. Thus we have

Lemma: If c is in the center of $\Delta[G]$ and multiplication by $c - \varepsilon(c)$ is ~~an auto~~ an autom. of M, then $H_*(G,M) = 0$.

So we therefore get the following

~~Theorem~~ Principle: If there exists an element c in the center of G such that $c-1$ is an automorphism of $H_g(N,\Delta)$ for $0<g\leq d$, then $H_g(E,\Delta) \xrightarrow{\sim} H_g(G,\Delta)$ for $0\leq g\leq d$.

Next we need to know how to compute the homology of an abelian group with coefficients in the field Δ. Formulas:

$\mathrm{char}(\Delta)=0$: $\quad H_i(N;\Delta) = \Lambda^i(N\otimes_{\mathbb{Z}}\Delta)$

$\mathrm{char}(\Delta)=p>0$. Here there is a filtration such that

$$gr\{H_*(N,\Delta)\} = \Lambda^*(N\otimes_{\mathbb{Z}}\Delta[1]) \otimes \Gamma^*({}_pN\otimes_{\mathbb{F}_p}\Delta[2])$$

where ${}_pN = \mathrm{Ker}\{p: N \to N\}$. (For p odd, the gr can be dropped.)

$$gr\{H_n(N,\Delta)\} \simeq \bigoplus_{i+2j=n} \Lambda^i(N_p\otimes\Delta)\otimes\Gamma_j({}_pN\otimes\Delta)$$

So I am interested now in proving this: MUST BE MODIFIED.

Lemma: Let k be an ~~infinite field~~ NO, let N be a k-module, and let Δ be a field. ~~Then given an integer d, there exists a $c\in k^*$ such that~~ Let k^* act on N by multiplication, and consider the induced action on $H_g(N,\Delta)$. Then given any integer d there exists a $c\in k^*$ such that $c-1$ is an autom. of $H_g(N,\Delta)$ for $0<g\leq d$.

<u>Proof</u>: <u>Case 1</u>: $\mathrm{char}(k) \neq \mathrm{char}(\Delta)$. In this case $H_*(N,\Delta) = \Delta$.

<u>Case 2</u>: $\mathrm{char}(k) = \mathrm{char}(\Delta) = 0$. Take c to be a prime number p in $\mathbb{Q}^* \subset k^*$. Then the action of c on $N\otimes\Lambda$ is mult. by p, hence on $H_i(N,\Delta) = \Lambda^i(N\otimes\Delta)$ it is mult. by p^i. Since $p^i - 1 \neq 0$ done.

<u>Case 3</u>: $\mathrm{char}(k) = \mathrm{char}(\Delta) = p > 0$, ~~and k is a trans. extension of $\mathbb{F}_p$. Can suppose $k =$~~ and k is a transcendental extension of $\mathbb{F}_p$. Say $k = \mathbb{F}_p(T)$.

~~If $N = k$, then taking $\Delta = k$ one has $N\otimes\Delta = \mathbb{F}_p(t_1)\otimes\mathbb{F}_p(t_2) = \mathbb{F}_p[t_1,t_2]$ localized wrt $f(t_1)g(t_2)$, $f,g\neq 0$.~~

If $N = k$, then $\Lambda^2 N \oplus \Gamma_2 N = N \otimes N$ $(p\neq 2)$ and

$$N\otimes N = \mathbb{F}_p(T)\otimes_{\mathbb{F}_p}\mathbb{F}_p(T)$$

$$= \mathbb{F}_p[T_1,T_2] \text{ localized wrt } \{f(T_1)g(T_2) \mid f,g\neq 0\}$$

I wanted to take $c = T$, but then c on $N\otimes N$ is mult. by T_1T_2 and $T_1T_2 - 1$ is not invertible in $\mathbb{F}_p(T_1)\otimes\mathbb{F}_p(T_2)$. Thus method doesn't work.

$\Longrightarrow$ HYPOTHESIS k infinite should be replaced by either k of char. 0 or k of charac. p and $\mu(k)$ infinite.

Case 3: $\mathrm{char}(k) = \mathrm{char}(\Delta) = p$ and $k \supset \mathbb{F}_{p^d}$ for arbitrarily large d. Here we take c to be a generator of $(\mathbb{F}_{p^d})^*$ which is cyclic of order $p^d - 1$. Suppose N is a vector space over $F = \mathbb{F}_{p^d}$ for the moment and that $\Delta = \overline{\mathbb{F}}_p$. Then as a rep. of F^* over Δ

$$N \underset{\mathbb{F}_p}{\otimes} \Delta = \text{direct sum of } F \underset{\mathbb{F}_p}{\otimes} \Delta$$

and
$$F \otimes_{\mathbb{F}_p} \Delta \xrightarrow{\sim} \Delta^d$$
$$x \otimes y \longmapsto (x^{p^a} y)_{a=0,\dots,d-1}$$

i.e. $N \otimes \Delta$ is a direct sum of the characters $\chi, \chi^p, \dots, \chi^{p^{d-1}}$. ~~It follows that~~ where $\chi : F^* \longrightarrow \Delta^*$ sends x to x.
Thus
$$\Lambda^i(N\otimes\Delta) \otimes \Gamma_j(N\otimes\Delta)$$
is a sum of ~~the~~ characters occurring in $(N\otimes\Delta)^{\otimes(i+j)}$ which are

$$\chi^{a_0 + a_1 p + \cdots + a_{d-1} p^{d-1}} \qquad \sum_{i=0}^{d-1} a_i = i+j$$

~~This will give these to contain the t~~ Thus $\Lambda^i(N\otimes\Delta)\otimes\Gamma^j(N\otimes\Delta)$ will contain the trivial character of F^* only if $\exists$ $a_0, \dots, a_{d-1} \geq 0$ $\ni$

$$\sum a_i = i+j$$

$$\sum a_i p^i \equiv 0 \mod p^d - 1$$

And I've seen before that this happens with $i+j > 0$ only if $\sum a_i \geq d(p-1)$. Thus as $i+2j \geq i+j$, one has that $H_q(N,\Delta)$ doesn't contain the trivial repn. for $0 < q < d(p-1)$.

It seems therefore that the good statement is:

<u>Theorem</u>: Let k be a field. Then one has

$$(*) \qquad H_i(GL_i) \twoheadrightarrow H_i(GL_{i+1}) \xrightarrow{\sim} H_i(GL_{i+2}) \xrightarrow{\sim} \cdots\cdots$$

if either k is of char. 0, or if $\mathrm{char}(k) = p > 0$ and k contains arbitrarily large finite subfields. In any case, one has $(*)$ modulo p-torsion when $\mathrm{char}(k) = p > 0$.

January 30, 1974. Crude stability.

Let k be a field. If k is of char $p>0$, either ignore p-torsion in what follows, or else suppose $\mu(k)$ is infinite. With these assumptions I can ignore unipotent subgroups of parabolic gps.

Suppose k is infinite ~~and consider the complex~~ let V be a vector space of dimension n over k, and let $L(V)$ be the complex given by

$$L(V)_s = \text{free abelian group generated by independent sequences } (v_1, v_2, \dots, v_s) \text{ in } V.$$

Then because k is infinite I know that the complex $L(V)$ is acyclic in degrees $<n$. Since $GL(V)$ acts transitively on the indep. sequences of a given length s, one has

$$H_*(GL(V), L(V)_s) = H_*\begin{pmatrix} 1_s & * \\ & GL_{n-s} \end{pmatrix}$$

and so I get a spectral sequence

$$E^1_{st}(n) = H_t\begin{pmatrix} 1_s & * \\ & GL_{n-s} \end{pmatrix} \Longrightarrow 0 \quad \text{in degrees} < n.$$

$$\Vert \ \text{hyp. on } k$$

$$H_t(GL_{n-s}).$$

The differential $d_1 : E^1_{st} \longrightarrow E^1_{s-1,t}$ is an alternating sum of face operators which are all the same, hence

~~[illegible]~~ it is zero where s is even. Thus one finds that the E^1_{**} term looks as follows

$$H_t(GL_0) \to \cdots \xrightarrow{res} H_t(GL_{n-2}) \xrightarrow{0} H_t(GL_{n-1}) \xrightarrow{res} H_t(GL_n) .$$

n $\qquad\qquad\qquad\qquad\qquad\qquad$ 0

In particular if one assumes inductively that

$$H_t(GL_{n-1}) \xrightarrow{\sim} H_t(GL_n) \qquad \text{for } t<q, \ n \text{ large}$$

One has that $E^2_{st} = 0 \qquad t<q, \ s \le q-t+2$

hence $E^2_{1q} = E^2_{0q}$ i.e.

$$H_q(GL_{n-1}) \xrightarrow{\sim} H_t(GL_n)$$

for n large enough.

But even if we don't care to compute d, we can argue as follows. ~~[illegible]~~ We have an obvious map of $L(k^n)$ to $L(k^{n+1})$ compatible with the actions of GL_n and GL_{n+1}, hence a map of spectral sequences. Assuming $H_t(GL_{n-1}) \xrightarrow{\sim} H_t(GL_n)$ for $t<q$ and n large, one has that the map $E^1_{st}(n) \longrightarrow E^1_{st}(n+1)$ is an isomorphism for $s \le q+1$, $t<q$, so as abutments are trivial, the comparison thm tells us that $E^2_{0q}(n) \xrightarrow{\sim} E^2_{0q}(n+1)$ and

$$E^2_{1q}(n) \longrightarrow E^2_{1q}(n+1)$$

for n large. Thus

$$\begin{tikzcd} H_q(GL_{n-1}) \arrow[r] \arrow[d] & H_q(GL_n) \arrow[r] \arrow[d] & E^2_{0q}(n) \arrow[r] \arrow[d, "\wr"] & 0 \\ H_q(GL_n) \arrow[r] & H_q(GL_{n+1}) \arrow[r] & E^2_{0q}(n+1) \arrow[r] & 0 \end{tikzcd}$$

and since the two maps from $H_*(GL_n)$ to $H_*(GL_{n+1})$ coincide, this gives $H_q(GL_n) \twoheadrightarrow H_q(GL_{n+1})$ for n large. So this implies

$$\begin{array}{ccc} E^1_{2q}(n) & \longrightarrow & E^1_{2q}(n+1) \\ \parallel & & \parallel \\ H_q(GL_{n-2}) & \longrightarrow & H_q(GL_{n-1}) \end{array}$$

is onto, so $B^1_{2q}(n) \twoheadrightarrow B^1_{1q}(n+1)$ for large n.
~~E^2_{1n}~~ From

$$\begin{tikzcd} B^1_{1q}(n) \arrow[r] \arrow[d] & Z^1_{2q}(n) \arrow[r] \arrow[d] & E^1_{2q}(n) \arrow[r] \arrow[d] & 0 \\ B^1_{1q}(n+1) \arrow[r] & Z^1_{2q}(n+1) \arrow[r] & E^1_{2q}(n+1) \arrow[r] & 0 \end{tikzcd}$$

we get $Z^1_{2q}(n) \twoheadrightarrow Z^1_{2q}(n+1)$. From

$$\begin{tikzcd} 0 \arrow[r] & Z^1_{2q}(n) \arrow[r] \arrow[d] & H_q(GL_{n-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{same map}"] & H_q(GL_n) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z^1_{1q}(n+1) \arrow[r] & H_q(GL_n) \arrow[r] & H_q(GL_{n+1}) \arrow[r] & 0 \end{tikzcd}$$

One deduces $H_q(GL_n) \xrightarrow{\sim} H_q(GL_{n+1})$ for large n as desired.

<u>Question:</u> Is $H_*(GL_n, GL_{n-1}) \longrightarrow H_*(GL_{n+1}, GL_n)$ zero where the map is the embedding $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ from GL_n to GL_{n+1}?

<u>Partial answer:</u>

$$H_q(GL_n, GL_{n-1}) \longrightarrow H_q(GL_{n+2}, GL_{n+1}) \quad \text{is zero.}$$

<u>Proof:</u> Diagram chase in

$$\begin{tikzcd}
H_q(GL_{n-1}) \arrow[r] \arrow[d] & H_q(GL_n) \arrow[r] \arrow[d] & H_q(GL_n, GL_{n-1}) \arrow[r] \arrow[d] & H_{q-1}(GL_{n-1}) \arrow[r] \arrow[d] & H_{q-1}(GL_n) \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & H_q(GL_{n+1}, GL_n) \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
H_q(GL_{n+1}) \arrow[r] & {} \arrow[r] & H_q(GL_{n+2}, GL_n) \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

and use always that the two embeddings $GL_n \to GL_{n+1}$ are conjugate, hence induce the same map on homology.

January 31, 1974. Weak stability for a Dedekind domain.

Let A be a Dedekind domain with quotient field K, let E be a proj. f.t. A-module of rank n and let $GL(E) = Aut(E)$ act on the exact sequence

$$0 \longrightarrow I(V) \longrightarrow \bigoplus_{\substack{\dim(W)=n-1\\ W\subset V}} I(W) \longrightarrow \cdots \longrightarrow \bigoplus_{L\subset V} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $V = K\otimes_A E$. Because A is a Dedekind domain to give a subspace W of $K\otimes_A E$ is the same as given a direct summand $E' \subset E$. If we agree to put $I(E) = I(k\otimes_A E)$ for any $E \in P(A)$, then we get the exact sequence

$$0 \longrightarrow I(E) \longrightarrow \bigoplus_{F\in G_{n-1}(E)} I(F) \longrightarrow \cdots \longrightarrow \bigoplus_{F\in G_1(E)} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $G_p(E)$ = direct summands of rank p. This exact sequence gives rise to a spectral sequence

$$E^1_{st} = H_t\Big(GL(E), \bigoplus_{F\in G_s(E)} I(F)\Big) \Longrightarrow 0$$

But the orbits of $GL(E)$ on $G_s(E)$ are easy to classify. In effect given $F_1, F_2 \in G_s(E)$, then F_1, F_2 are isom. $\iff [\Lambda^s F_1] = [\Lambda^s F_2]$ in $Pic(A)$. This implies $E/F_1 \simeq E/F_2$ and since $E = F_1 \oplus E/F_1$ it follows that ~~we get a~~ $F_1, F_2 \in G_s(E)$ are $GL(E)$-conjugate $\iff \Lambda^s F_1 = \Lambda^s F_2$ in $Pic(A)$. On the other hand provided $0<s<n$ ~~n~~ ≥ 2 given $\alpha \in Pic(A)$ one can find an $F \in G_s(E)$ with $\Lambda^s F \in \alpha$. In effect, one can choose L invertible $\ni$

$F \oplus L \oplus A^{n-s-1}$ has same determinant in $\mathrm{Pic}(A) \Longrightarrow$
$F \oplus L \oplus A^{n-s-1} \simeq E$.

Let me denote by E [redacted] $_{n,\alpha}$ a representative for the iso. class of proj. f.t. A-module of rank n and with $[\Lambda^n E$ [redacted] $_{n,\alpha}]$ [redacted] $= \alpha$ in $\mathrm{Pic}(A)$ and $\alpha \in \mathrm{Pic}(A)$. Then for $0 < s < n = \mathrm{rank}(E)$, one [redacted] can choose an isom

$$E_{s,\alpha} \oplus E_{n-s,\, c_1(E)-\alpha} \cong E_{n,\, c_1(E)} = E$$

and for each α one gets a representation for $GL(E)\backslash G_s(E)$. The stabilizer of this is the "matrix" group

$$\left(\begin{array}{c|c} GL(E_{s,\alpha}) & * \\ \hline 0 & GL(E_{n-s,\, c_1(E)-\alpha}) \end{array}\right) \qquad * = \mathrm{Hom}(E_{s,\alpha}, E_{n-s,\cdot})$$

Thus for each (n,α) we get a spectral sequence

$$E^1_{st}(n,\alpha) \Longrightarrow 0$$

with

$$E^1_{[redacted]\,0t}(n,\alpha) = H_t(GL(E_{n,\alpha}))$$

$$E^1_{st}(n,\alpha) = \bigoplus_{\substack{[redacted] \\ \beta \in \mathrm{Pic}(A)}} H_t\left(\begin{pmatrix} GL(E_{s,\beta}) & * \\ 0 & GL(E_{n-s,\, \alpha-\beta}) \end{pmatrix}, I(E_{s,\beta})\right)$$

for $0 < s < n$

$$E^1_{nt}(n,\alpha) = H_t(GL(E_{n,\alpha}), I(E_{n,\alpha}))$$

$$E^1_{st}(n,\alpha) = 0 \qquad s > n$$