+Dec 1974

Summary of research – Jan. 1975 – concerning Schubert cells and buildings.

Nature of Schubert cells: $G = GL_n k$, $B \overset{\text{"}T \ltimes B^u\text{"}}{=}$ Borel subgroup, $Y \cong G/P$ a flag manifold. The B-orbits on Y are cells ($\cong k^m$) with unique T-fixpts. The theory classifies and parametrizes these B-orbits (essentially row-echelon forms). Schubert cycles are the closures of these orbits. Desingularization of Schubert cycles + their cohomology classes (paper of Gelfand + co.).

$B \backslash T(V) \simeq \partial \Delta(n-1)$. Bruhat decomposition

Version for a field K with d.v. $\mathcal{B}$ = Iwahori subgroup. One classifies and parametrizes the $\mathcal{B}$-orbits on the simplices of the building. This interests me because $\Omega U_n \sim \mathcal{L}$ = space of lattices in $\mathbb{C}[z, z^{-1}]^n$, and the cell decomposition gives the minimal cell complex structure for ΩU_n.

Poset of Schubert cells. I wanted to determine the homotopy type of this poset, but too hard

December 28, 1974 Schubert cycles.

Let's examine carefully Schubert cells and cycles in the Grassmannian of d-planes in V. Here $\alpha : \{1, .., n\} \longrightarrow \{1,2\}$ is determined by $\alpha^{-1}(1) = \{a_1 < \cdots < a_d\}$. The cell C_α in $Y_d(V)$ consists of all d-dimensional subspaces F such that

$$\dim (F \cap V_p) = \operatorname{card} \{a \leq p \mid \alpha(a) = 1\}$$

i.e. such that the filtration $F \cap V_p$ of F has jumps at $p = a_1, \ldots, a_d$.

My candidate for a resolution of $\overline{C}_\alpha$ is the manifold $\tilde{C}_\alpha$ consisting of all flags

$$F^1 < \cdots < F^d \quad \text{in } V$$

such that $\dim F^i = i$
and $F^i \subset V_{a_i}$

Better approach — define $\overline{C}_\alpha$ = subset of Y_d consisting of F such that $\dim (F \cap V_p) \geq \operatorname{card} \{a \leq p \mid \alpha(a) = 1\}$, i.e. $\dim (F \cap V_{a_i}) \geq i$. Then $\overline{C}_\alpha$ is closed and it contains C_α. Also $\tilde{C}_\alpha$ maps to $\overline{C}_\alpha$ and fibres over C_α are single points. It's also clear that C_α is open in $\overline{C}_\alpha$

It's clear that $\overline{C}_\alpha = \bigcup C_\beta$ where β is any map corresp. to a sequence $b_1 < \cdots < b_d$ such that $b_i \le a_i$. But now I can see easily that C_β is in the closure of C_α. For given $F \in C_\beta$ write $F = L_1 \oplus \cdots \oplus L_d$ where $L_i \in \mathbb{P}V_{b_i} - \mathbb{P}V_{b_i - 1}$. Now since $b_i \le a_i$, I can find $L_i' \in \mathbb{P}V_{a_i} - \mathbb{P}V_{a_i - 1}$ as close to L_i as I like. Then $F' = L_1' \oplus \cdots \oplus L_d'$ is in C_α and it can be made as close to F as I want.

~~$Y_{1,\ldots,d}(V)$ = set of $F^1 < \cdots < F^d$ with $\dim(F^i) = i$.~~
~~To each such flag I can associate a sequence $b_1 < \cdots < b_d$ with b_i = least p such that $F^i \subset V_p$~~

Suppose $F^1 < \cdots < F^d$ is in $\tilde{C}_\alpha$, i.e. $F^i \subset V_{a_i}$ and choose L_i such that $F^i = L_1 \oplus \cdots \oplus L_i$. Then L_i can be approximated by $L_i' \in \mathbb{P}V_{a_i} - \mathbb{P}V_{a_i - 1}$. Put

$$F^{i'} = L_1' \oplus \cdots \oplus L_i'$$

Then $F^{d'} \cap V_p$ has jumps at $p = a_1, \ldots, a_d$ and $F^{d'} \cap V_{a_i} = F^{i'}$, so $F' \in C_\alpha$. Thus C_α is dense in $\tilde{C}_\alpha$. Note that C_α in $Y_{1,\ldots,d}$ is the Schubert cell in $Y_{1,\ldots,d}$ corresponding to the map $\{1,\ldots,n\} \to \{1,\ldots,d+1\}$ sending $a_1, \ldots, a_d$ to $1, \ldots, d$ and the rest to $d+1$.

Here is how I have computed the cohomology class associated to $\overline{C}_\alpha$. Since $\tilde{C}_\alpha \longrightarrow C_\alpha$ is birational I ~~compute~~ the class is $f_* 1$ where $f : \tilde{C}_\alpha \longrightarrow Y_d$. I factor f into

$$\tilde{C}_\alpha \overset{\iota}{\subset} Y_{1,\ldots,d} \xrightarrow{\ p\ } Y_d$$

where p is the full flag bundle of $\mathcal{F}_d$ on Y_d. Now I propose to compute $\iota_* 1$ and then use my res. formula for p_*.

Recall the computation of $\iota_* 1$. First choice in the construction of $\underline{F} = (F^1, \ldots, F^d)$ in $\tilde{C}_\alpha$ is the choice of F^1 which can be any element of $\mathbb{P}V_{a_1}$. What this means is that I am looking at:

$$\begin{tikzcd}
\tilde{C}_\alpha \arrow[d, dashed] & \subset & Y_{1,\ldots,d} \arrow[d, dashed] \\
\{F^1 < F^2 \mid F^i \subset V_{a_i}\} \arrow[d] & \subset & Y_{12} \arrow[d] \\
\mathbb{P}V_{a_1} & \subset & Y_1 = \mathbb{P}V \arrow[d] \\
 & & pt
\end{tikzcd}$$

or better I should write a triangular array

$$\begin{tikzcd}[column sep=small]
\tilde{C}_\alpha = \left\{\begin{smallmatrix} F^1 & < \cdots < & F^d \\ \cap & & \cap \\ V_{a_1} & & V_{a_d}\end{smallmatrix}\right\} \arrow[r, phantom, "\subset"] & \cdots\cdots \arrow[r, phantom, "\subset"] & \left\{\begin{smallmatrix} F^1 & < \cdots < F^d \\ \cap & \\ V_{a_1} & \end{smallmatrix}\right\} \arrow[r, phantom, "\subset"] \arrow[d] & \{F^1 < \cdots < F^d\} = Y_{1\cdots d} \arrow[d] \\
 & \left\{\begin{smallmatrix} F^1 & < & F^2 \\ \cap & & \cap \\ V_{a_1} & & V_{a_2}\end{smallmatrix}\right\} \arrow[r, phantom, "\subset"] \arrow[u, leftarrow] & \left\{\begin{smallmatrix} F^1 & < F^2 \\ \cap & \\ V_{a_1} & \end{smallmatrix}\right\} \arrow[r, phantom, "\subset"] \arrow[d] & \{F^1 < F^2\} \arrow[d] \\
 & & \left\{\begin{smallmatrix} F^1 \\ \cap \\ V_{a_1}\end{smallmatrix}\right\} \arrow[r, phantom, "\subset"] & \{F^1\}
\end{tikzcd}$$

vanishing of a trans. section of $\mathrm{Hom}(\mathcal{F}^1, V/V_{a_1})$

<u>Lemma</u>: Suppose one has $Z \overset{j}{\subset} Y \overset{i}{\subset} X$ where Y is the zero-set of a trans. section of a vector bundle $\mathcal{F}$ on X, and where Z is the zero-set of a trans. section of $i^*\mathcal{E}$, where $\mathcal{E}$ is a v.b. on X. Then

$$i_* j_* 1 = e(\mathcal{E})\, e(\mathcal{F}).$$

<u>Proof</u>: $\quad j_* 1 = e(i^*\mathcal{E}) = i^* e(\mathcal{E})$

$$i_* j_* 1 = i_*(i^* e(\mathcal{E})) = i_* 1 \cdot e(\mathcal{E}) = e(\mathcal{F})\, e(\mathcal{E}).$$

QED.

As a consequence the class of $\tilde{C}_\alpha$ in $H^*(Y_{1\dots d})$ is

$$\prod_{i=1}^{d} e\left(\mathrm{Hom}\left(\mathcal{F}^i/\mathcal{F}^{i-1}, V/V_{a_i}\right)\right) = \prod_{i=1}^{d} \xi_i^{\,n-a_i}$$

Residue formula: $f : \mathbb{P}E \longrightarrow X$ $\quad d = \dim(E)$

$$\xi = e(\mathcal{O}(1)).$$

$$f_* a(\xi) = \mathrm{res} \frac{a(T)\,dT}{T^d + \dots + c_d E}$$

~~Generalizes to~~ $\cancel{f_k : Y_{1,\dots,k}(E) \to X}$

$\cancel{\xi_j = e(\mathcal{F}_j/\mathcal{F}_{j-1}^{\vee})}$ $\quad \cancel{j = 1,\dots,d.}$

$\cancel{f_{d*}(a(\xi_1,\dots,\xi_k)) = (f_{d-1})_* \,\mathrm{res}\, a(\xi_1,\dots,\xi_{d-1},T_d)\,dT_d}$

Call this f_1 and generalize to $f_d : Y_{1,\dots,d}(E) \to X$.

Put $\xi_j = e\left(\mathcal{F}_j/\mathcal{F}_{j-1}^{\vee}\right)$

$$(f_d)_* a(\xi_1,\dots,\xi_d) = (f_{d-1})_* \,\mathrm{res}\, \frac{a(\xi_1 \dots \xi_{d-1}, T_d)\,dT_d}{T_d^{\,n-d+1} + \dots + c_{n-d+1}(E/\mathcal{F}_{d-1})}$$

$$= (f_{d-1})_* \,\mathrm{res}\, \frac{a(\xi_1,\dots,\xi_{d-1},T_d)\,\prod_{j<d}(T_d - \xi_j)\cdot dT_d}{T_d^{\,n} + \dots + c_n(E)}$$

So iterating:

$$(f_d)_* \, a(\xi_1, \dots, \xi_d) = \operatorname{res}\left[\begin{matrix} a(T_1, \dots, T_d) \prod_{i<j\le d} (T_j - T_i)\, dT_1 \cdots dT_d \\ T_1^n + \cdots + c_n E, \quad \dots, \quad T_d^n + \cdots + c_n E \end{matrix}\right]$$

So taking $E = \mathcal{F}_d$ on $Y_d(V)$ I get for the class of C_α:

$$\text{coeff of } (T_1 \cdots T_d)^{n-1} \text{ in } \prod_{i=1}^{d} T_i^{n-a_i} \prod_{i<j\le d} (T_j - T_i) \prod_{i=1}^{d} \left(T_i^{n-d} + \cdots + c_{n-d}(V/\mathcal{F}_d)\right)$$

Now $$\prod T_j - T_i = \begin{vmatrix} 1 & T_1 & T_1^{d-1} \\ 1 & T_d \cdots & T_d^{d-1} \end{vmatrix}$$

and so calculation shows this coeff. to be

$$\begin{vmatrix} c_{n-d-a_1+1}(V/\mathcal{F}_d) & c_{n-d-a_1+2}(V/\mathcal{F}_d) & \cdots \\ c_{n-d-a_2+1}(V/\mathcal{F}_d) & c_{n-d-a_2+2}(V/\mathcal{F}_d) & \cdots \\ & & \\ & & \end{vmatrix}$$

Can we generalize this calculation to other flag manifolds.

We will now work with $Y = Y_{1,\dots,n}(V)$, and now $\alpha : \{1,\dots,n\} \xrightarrow{\sim} \{1,\dots,n\}$. C_α consist of flags $F_1 < \dots < F_n$, $\dim F_i = i$ such that

$$\dim F_j \cap V_p = \operatorname{card}\{a \le p \mid \alpha(a) \le j\}$$

$\tilde{C}_\alpha$ will consist of families F_{jp} monotone in j and p such that $F_{np} = V_p$ and

$$\dim F_{jp} = \operatorname{card}\{a \le p \mid \alpha(a) \le j\}.$$

I want to see $\tilde{C}_\alpha$ as a non-singular subvariety of a fibre bundle Z_α over Y. Z_α has the same elements as $\tilde{C}_\alpha$ except that I drop the condition $F_{np} = V_p$. Z_α will be smooth over Y in two ways

$$Z_\alpha \longrightarrow Y \times Y$$
$$(F_{jp}) \longmapsto (F_{n*}), (F_{*n}).$$

First I will have to get an integration formula for $Z_\alpha \longrightarrow Y$ $\quad (F_{**}) \longmapsto F_{*n}$. I have to build Z_α up as successive projective fibre bundles. So Y gives us F_{jn}, $1 \le j \le n$.

I pick $F_{1,n-1} \subset F_{1,n}$; $\quad \dim F_{jp}/F_{j,p-1} = \begin{cases} 0 & \alpha(p) > j \\ 1 & \alpha(p) \le j. \end{cases}$

Thus since $F_{1,n}$ is a line, there is only one possibility for $F_{1,n-1}$. ~~Put $j = \alpha(n)$, so that~~ As $F_{j,n-1} = F_{j,n}$ for $j < \alpha(n)$, the first choice we have comes with $j = \alpha(n)$, except that then $F_{\alpha(n)-1,n-1} = F_{\alpha(n),n-1}$. But the next time $F_{\alpha(n)+1,n-1}$ is a hyperplane in $F_{\alpha(n)+1,n}$ containing $F_{\alpha(n),n-1}$. Thus the possibilities form a projective line.

Thus Z_α will be the end of a succession of fibre bundles with $\mathbb{P}^1$-fibres.

~~If $\beta \le j \le \alpha(p)$ denote by $Z_{j,p}$~~

~~the manifold consisting of choices~~

For every j, p let Z_{jp} denote the manifold consisting of choices for $F_{j'p'}$ for $p' > p$ or $p' = p$ and $j' \leq j$. Thus the fibres of $Z_{jp} \to Z_{j-1,p}$ (or $Z_{1p} \to Z_{n,p+1}$) are the choices for F_{jp}. The conditions on F_{jp} are what?

$$\dim\left(F_{j,p+1} / F_{j,p}\right) = \begin{cases} 0 & \alpha(p+1) > j \\ 1 & \alpha(p+1) \leq j. \end{cases}$$

$$\dim\left(F_{j,p} / F_{j-1,p}\right) = \begin{cases} 0 & \nexists\, a \leq p \quad \alpha(a) = j \quad \text{or} \quad \alpha^{-1}(j) > p \\ 1 & \exists\, a \leq p \quad \alpha(a) = j \quad \text{or} \quad \alpha^{-1}(j) \leq p \end{cases}$$

We get two 1's when $\alpha(p+1) \leq j$, and $\alpha^{-1}(j) \leq p$. Thus the pair $\alpha^{-1}(j) < p+1$ will have $j = \alpha(\alpha^{-1} j) > \alpha(p+1)$. So we see the choice of F_{jp} matters exactly when

$\alpha^{-1}(j) < p+1$ and $j > \alpha(p+1)$. Thus in the construction of Z_α we count pairs $a<p$ such that $\alpha(a) > \alpha(p)$, first by decreasing p and then increasing $\alpha(a)$.

Basic line bundles on Z_α will be the quotients $\mathcal{F}_{j,p+1}/\mathcal{F}_{j,p}$ for $\alpha^{-1}(j) < p+1$, $j > \alpha(p+1)$

If I put $\xi_{j,p} = c(\ \text{"}\ )$, then for the map

$$f : Z_{j,p} \longrightarrow Z_{j,p}$$

one has

$$f_*(1) = 0$$
$$f_*(\xi_{jp}) = 1.$$

Thus it would seem that if we use the notation $\gamma \in S$ for any of the pairs (j,p) as as, then $H^*(Z_\alpha)$ has as basis over $H^*(Y)$, the monomials $\prod_{\gamma\in S} \xi_\gamma^{\varepsilon_\gamma}$ where $\varepsilon_\gamma = 0, 1$ and all of these but the top one are killed by the integration map to $H^*(Y)$.

So to finish I have to express the cohomology class of $\tilde{C}_\alpha \subset Z_\alpha$ in terms of the basis $\prod \xi_\gamma^{\varepsilon_\gamma}$ and integrate.

Now $\tilde{C}_\alpha$ is where $\mathcal{F}_{n,p} \subset V_p$ for all p, so its class is

$$\prod_p e\left(\mathrm{Hom}\left(\mathcal{F}_{n,p}/\mathcal{F}_{n,p-1}, V/V_p\right)\right)$$

Rest looks like fun!

Summary: Given the permutation α, I let Z_α be the manifold consisting of systems F_{jp}, $1 \le j, p \le n$, of subspaces of V, which are monotone in j and p, and such that $\dim(F_{jp}) = \mathrm{card}\{a \le p \mid \alpha(a) \le j\}$. On Z_α I have tautological vector bundles $\mathcal{F}_{jp}$.

Let $p_1 : Z_\alpha \to Y$ send (F_{jp}) to the flag $(F_{jn}, 1 \le j \le n)$. and $p_2 : Z_\alpha \to Y$ send (F_{jp}) to the flag $(F_{np}, 1 \le p \le n)$. $\tilde{C}_\alpha$ is the fibre of p_2 over the base-flag (V_p). I know the restriction of p_1 to $\tilde{C}_\alpha$ is a birational map $\tilde{C}_\alpha \to C_\alpha$. Thus if I want to integrate a coh. class over C_α I can pull it back to Z_α, multiply by the class of $\tilde{C}_\alpha$ and integrate over Z_α.

The class of $\tilde{C}_\alpha$ in Z_α is the inverse image of the top class of $H^*(Y)$ under p_2.

$$\prod_{p=1}^{n} e\left(\mathrm{Hom}\left(\mathcal{F}_{n,p}/\mathcal{F}_{n,p-1}, V/V_p\right)\right) = \prod_{p=1}^{n} \xi_{n,p}^{n-p}$$

where $\xi_{j,p} = e\left(\mathcal{F}_{j,p}/\mathcal{F}_{j,p-1}\right)^\vee$; as $\dim(F_{jp}/F_{j,p-1}) = \begin{cases} 1 & \alpha(p) \le j \\ 0 & \alpha(p) > j \end{cases}$

we consider ξ_{jp} only when $j \geq \alpha(p)$.

A typical class in $H^*(Y)$ is of the form $\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Better: one gets a basis for $H^*(Y)$ of the form $\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ with $0 \leq \alpha_i < n-i$. When lifted up to Z_α via p_1 it becomes

$$\prod_{p=1}^{n} \xi_{p,n}^{\alpha_p}$$

So the problem is to compute

$$\int_{Z_\alpha} \prod_p \xi_{n,p}^{n-p} \cdot \prod_p \xi_{p,n}^{\alpha_p}$$

Alternate approach to integrating over Y: Consider Y as U/T, i.e. perpendicular lines, i.e. I embed Y inside of $(\mathbb{P}V)^n$. I want to compute the class of Y.

$Y =$ subset of $(L_1, \dots, L_n)$ in $(\mathbb{P}V)^n$ where $L_i \perp L_j$ for each $i<j$. But if H_{ij} is the set where $L_i \perp L_j$, then H_{ij} is where

$$\mathcal{L}_i \subset \mathcal{O} \otimes V \xrightarrow[\text{proj.}]{\text{orth}} \mathcal{L}_j$$

vanishes, so $[H_{ij}] = c_1(\mathcal{L}_j \otimes \check{\mathcal{L}}_i) = \xi_i - \xi_j$. These hyperplanes intersect transversally at Y, so the cohomology class of Y is $\prod_{i<j} (\xi_i - \xi_j)$ up to sign.

~~But suppose given α I might try to describe the image of the Schubert cell indexed by α inside of $(\mathbb{P}V)^n$~~

Review what I've understood about $Y_d(V)$. Here a Schubert cell C_α is described by $\alpha : \{1, .., n\} \to \{1, 2\}$ card $\alpha^{-1}(1) = d$, or equiv. $\alpha^{-1}(1) = 1 \le a_1 < \cdots < a_d \le n$. $\overline{C}_\alpha = \bigcup C_\beta$, where β runs over sequences $b_1 < \cdots < b_d$ such that $b_j \le a_j$, $\forall\, 1 \le j \le d$. $\widetilde{C}_\alpha$ consists of flags $F^1 < \cdots < F^d$ in Y_d such that $F^i \subset V_{a_i}$. A nice way to describe an element of $\widetilde{C}_\alpha$ is to give independent lines $L_1, .., L_d$ with $L_i \in V_{a_i}$, and put $F^i = L_1 \oplus \cdots \oplus L_i$; this description is 1-1 if one requires the L_i to be perpendicular.

Next I want to understand $Y_{12}(V)$, where C_α is described by $\alpha : \{1, .., n\} \to \{1, 2, 3\}$, which may be identified with $\alpha^{-1}(1)$, $\alpha^{-1}(2)$. C_α consists of $F^1 < F^2$ such that

$$\dim F^j \cap V_p = \operatorname{card}\{a \le p \mid \alpha(a) \le j\}.$$

Thus $F^1 \cap V_p$ jumps at $\alpha(p) = 1$, or $p = \alpha^{-1}(1)$

and $F^2 \cap V_p$ jumps at $p = \alpha^{-1}(1), \alpha^{-1}(2)$. There are two cases:

1) $\alpha^{-1}(1) < \alpha^{-1}(2)$. Here I can describe C_α as generated by lines L_1, L_2 such that $L_1 \in \mathbb{P}V_{\alpha^{-1}(1)} - \mathbb{P}V_{\alpha^{-1}(1)-1}$, $L_2 \in \mathbb{P}V_{\alpha^{-1}(2)} - \mathbb{P}V_{\alpha^{-1}(2)-1}$; and the description is 1-1 if I require L_2 to be perpendicular to L_1. This sort of cell I encountered before as the lift of the cell in $Y_2(V)$ corresponding to the sequence $\alpha^{-1}(1) < \alpha^{-1}(2)$.

2) $\alpha^{-1}(1) > \alpha^{-1}(2)$. Here elements of C_α are in 1-1 correspondences with pairs of lines L_1, L_2 such that $L_i \in \mathbb{P}V_{\alpha^{-1}(i)} - \mathbb{P}V_{\alpha^{-1}(i)-1}$.

In the first case, the closure of C_α is the set of $F^1 < F^2$ such that $F^1 \subset V_{\alpha^{-1}(1)}$, $F^2 \subset V_{\alpha^{-1}(2)}$, and it is a non-singular subvariety whose cohomology class is $\xi_1^{\,n-\alpha^{-1}(1)}\ \xi_2^{\,n-\alpha^{-1}(2)}$.
Here $\tilde{C}_\alpha = \bar{C}_\alpha$.

In the second case, $\tilde{C}_\alpha$ consists of triples $F^1 < F^2 > L_2$, where $F^2 \subset V_{\alpha^{-1}(1)}$, $L_2 \subset V_{\alpha^{-1}(2)}$. In effect, an element of $\tilde{C}_\alpha$ is given by (F_{jp}), where F_{1p} has a jump at $\alpha^{-1}(1)$, hence (F_{1p}) is given by $F^1 = F_{1,\alpha^{-1}(1)}$; $F_{2,p}$ has jumps at $\alpha^{-1}(2)$, and $\alpha^{-1}(1)$, hence is determined by $L_2 = F_{2,\alpha^{-1}(2)}$, $F^2 = F_{2,\alpha^{-1}(1)}$.

Note that the image of C_α in $Y_2(V)$ consists of all F^2 such that $F^2 \cap V_p$ jumps at $p = a_1$ and $p = a_2$, where I put $a_1 = \alpha^{-1}(2) < a_2 = \alpha^{-1}(1)$. Thus the inverse image of the cell in $Y_2(V)$ described by $a_1 < a_2$ becomes 2-cells in $Y_{1,2}(V)$, namely given by $\alpha^{-1}(1,2) = (a_1, a_2)$ and (a_2, a_1) respectively.

Change notation: Suppose we fix $1 \le a_1 < a_2 \le n$ and define α, β, γ by

$$\alpha : \{1, \dots, n\} \longrightarrow \{1, 2, 3\} \qquad \alpha^{-1}(1) = a_2 \qquad \alpha^{-1}(2) = a_1$$

$$\beta : \quad '' \qquad\qquad '' \qquad \beta^{-1}(1) = a_1 \qquad \beta^{-1}(2) = a_2$$

$$\gamma : \{1, \dots, n\} \longrightarrow \{1, 2\} \qquad \gamma^{-1}(1) = \{a_1, a_2\}.$$

so that $C_\gamma \subset Y_2$; $C_\alpha, C_\beta \subset Y_{1,2}$. I know

$$\tilde{C}_\gamma = \tilde{C}_\beta = \overline{C}_\beta = \{ F^1 < F^2 \mid F^1 \subset V_{a_1}, F^2 \subset V_{a_2} \}$$

has the cohomology class $\xi_1^{n-a_1} \xi_2^{n-a_2} \in H^*(Y_{1,2})$. On the other hand $\tilde{C}_\alpha$ consists of $F^1 < F^2 > L_2$ as above where $F^2 > L_2$ is a point of $\overline{C}_\beta$. Thus

$$\tilde{C}_\alpha = \text{projective bundle } \mathbb{P}(\mathcal{F}_2) \text{ over } \overline{C}_\beta,$$

which means that $\overline{C}_\alpha$ is the inverse image of $\overline{C}_\gamma$.

To calculate the coh. class of C_α, we want to use

$$\begin{tikzcd}[column sep=small]
\tilde{C}_\alpha \arrow[rr] \arrow[dr, hook] \arrow[dd] & & \bar{C}_\beta \arrow[dr, hook] \arrow[dd] & \\
& Y_{1,2} \times_{Y_2} Y_{12} \ (F^1 < F^2 > L_1^E) \arrow[rr] \arrow[dl] & & Y_{12} \ (F^2 > L_1^E) \arrow[dl] \\
Y_{1,2} \arrow[rr] & & Y_2 \ (F^2) &
\end{tikzcd}$$

Better – we want to use

$(F^1 < F^2 > L_1)$
$L_1 \subset V_{a_1}$
$F_2 \subset V_{a_2}$

$$\begin{tikzcd}
\tilde{C}_\alpha \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & Y_{12} \times_{Y_2} Y_{12} = Z \ (F^1 < F^2 > L_1^*) \arrow[dl, "p"] \\
& Y_{1,2} \ (F^1 < F^2) &
\end{tikzcd}$$

$[C_\alpha] = f_* 1 = p_* i_* 1$. Now

$$i_* 1 = e(\mathcal{L}_1^\vee)^{n-a_1} \, e(\mathcal{F}_2/\mathcal{L}_1^\vee)^{n-a_2}$$

and p is the projective fibre bundle $\mathbb{P}\mathcal{F}_2$ over Y_{12} where $\mathcal{O}(-1) = \mathcal{L}_1$.

Put $z_1 = e(\mathcal{L}_1^\vee)$, $z_2 = e(\mathcal{F}_2/\mathcal{L}_1^\vee)$ whence

$$z_1 + z_2 = c_1(\mathcal{F}_2^\vee) = c_1(\mathcal{F}_1^\vee) + c_1(\mathcal{F}_2/\mathcal{F}_1^\vee) = \xi_1 + \xi_2$$

$$z_1 z_2 = c_2(\mathcal{F}_2^\vee) = \xi_1 \xi_2 .$$

Now $\iota_* 1 = z_1^{n-a_1} z_2^{n-a_2}$ we should rewrite this in the form $M(\xi_1, \xi_2) + N(\xi_1, \xi_2) z_1$, so we can compute p_*.

$$\iota_* 1 = (z_1 z_2)^{n-a_2} z_1^{a_2 - a_1}$$

Working with universal polynomials, we can write

$$f(z_1, z_2) = \frac{g}{2} + \frac{h}{2}(z_1 - z_2)$$

where g, h are symmetric:

$$g = f(z_1, z_2) + f(z_2, z_1)$$

$$h = \frac{f(z_1, z_2) - f(z_2, z_1)}{z_1 - z_2}$$

Then integrating and using that $p_*(z_1) = 1$, $p_*(z_2) = -1$ one gets

$$p_* f = h$$

So for example if $f = z_1^k$, then

$$h = \frac{z_1^k - z_2^k}{z_1 - z_2} = +\left(z_2^{k-1} + \cdots + z_1^{k-1}\right)$$

$$= +\left(\xi_2^{k-1} + \cdots + \xi_1^{k-1}\right)$$

Thus $p_* \iota_* 1 = (\xi_1 \xi_2)^{n-a_2} \left(\xi_1^{a_2 - a_1 - 1} + \cdots + \xi_2^{a_2 - a_1 - 1}\right)$

So we find

$$[C_\alpha] = $$

$$= \xi_1^{n-a_1-1}\xi_2^{n-a_2} + \xi_1^{n-a_1-2}\xi_2^{n-a_2+1} + \cdots + \xi_1^{n-a_2}\xi_2^{n-a_1-1}$$

Next try to generalize to $Y_{1,2,\ldots d}$. A Schubert cell C_α in $Y_{1,\ldots,d}$ is described by $\alpha : \{1,\ldots,n\} \to \{1,\ldots,d+1\}$ such that $\alpha^{-1}(1),\ldots,\alpha^{-1}(d)$ have one elements. Let $a_1 < \cdots < a_d$ be the set $\alpha^{-1}(1),\ldots,\alpha^{-1}(d)$ arranged in order. Then if $(F^1 < \cdots < F^d) \in C_\alpha$ one has

$$\dim(F^d \cap V_p) = \operatorname{card}\{a \leq p \mid \alpha(a) \leq d\}$$

hence $F^d \cap V_p$ has jumps at $p = a_1,\ldots,a_d$. Note that if $F^d \cap V_p = F^d \cap V_{p+1} \Longrightarrow F^j \cap V_p = F^j \cap V_{p+1}$ for $j \leq d$. Thus once the jumps in $F^d \cap V_p$ are given, one has the flag $F^d \cap V_{a_i}$ in F^d, and then the orbit type of $(F^1 < \cdots < F^d)$ is determined by the relation of these two flags in F^d.

December 31, 1974

Problem: Find formulas for the coh. classes belonging to the Schubert cells in $Y_{1\ldots d}$.

Given $\alpha : \{1, \ldots, n\} \longrightarrow \{1, \ldots, d+1\}$ such that card $\alpha^{-1}(j) = 1$ for $1 \le j \le d$, one has the Schubert cell

$$C_\alpha = \left\{ (F_1 < \cdots < F_d) \in Y_{1,\ldots,d} \;\middle|\; \begin{array}{l} \dim (F_j \cap V_p) \\ = \mathrm{card}\{a \le p \mid \alpha(a) \le j\} \end{array} \right\}.$$

I propose to resolve $\overline{C}_\alpha$ using the manifold $\tilde{C}_\alpha$ consisting of systems of subspaces F_{jp} $1 \le j \le d$, $1 \le p \le n$, monotone in j and p, of dim $= \mathrm{card}\{a \le p \mid \alpha(a) \le j\}$, such that $F_{dp} < V_p$. That $\tilde{C}_\alpha$ is a manifold I have shown by constructing the system (F_{jp}) by decreasing j, and for j fixed by increasing p; Once $F_{j+1,p}$ $1 \le p \le n$ is given the construction of F_{jp} proceeds by induction on p.

I want to embed $\tilde{C}_\alpha$ in the fibre bundle Z over $Y_{1,\ldots,d}$ whose fibre over $F_1 < \cdots < F_d$ is $Y_1(F_1) \times Y_{1,2}(F_2) \times \cdots \times Y_{1,\ldots,d}(F_d)$. The point is that for each j, (F_{jp}) is essentially a full flag in $F_{jn} = F_j$, except that it has been indexed strangely.

A point of Z will be a system F^{ij}

$1 \le i \le j \le d$ of subspaces, such that F^{ij}, $1 \le i \le j$ is a full flag in $F^{jj} = F_j$. Given (F_{jp}) in $\tilde{C}_\alpha$ I send it to the element (F^{ij}) of Z defined as follows

$$F^{ij} = F_{jp} \qquad b_{ij} \le p < b_{i+1,j}$$

where $b_{1,j}, \ldots\ldots, b_{j,j}$ is the set $\alpha^{-1}(1), \ldots, \alpha^{-1}(j)$ arranged in order.

I want next to find the image of $\tilde{C}_\alpha$ inside of Z. The condition $F_{dp} \subset V_p$ becomes simply

$$F^{id} \subset V_{a_i} \qquad a_i = b_{id}$$

where $a_1, \ldots, a_d$ is the sequence $\alpha^{-1}(1), \ldots, \alpha^{-1}(d)$ arranged in order.

Condition next the condition $F_{j-1,p} \subset F_{j,p}$. Let $b_1, \ldots, b_j$ be the sequence $\alpha^{-1}(1), \ldots, \alpha^{-1}(j)$ arranged in order, so that

$$F_{jp} = F^{ij} \qquad b_i \le p < b_{i+1}$$

Put $b_k = \alpha^{-1}(j)$, so that $b_1, \ldots, \hat{b}_k, \ldots, b_j$ is the sequence $\alpha^{-1}(1), \ldots, \alpha^{-1}(j-1)$ arranged in orders. Then

$$\begin{array}{l}
0 = \cdots < F_{j b_1} = \cdots < F_{j b_2} = \cdots < F_{j b_{k-1}} = \cdots < F_{j b_k} = \cdots < F_{j b_{k+1}} = \cdots \\
0 = \cdots < F_{j-1, b_1} = \cdots < F
\end{array}$$

$$\begin{array}{ccccccccc}
F^{1j} & & F^{k-1,j} & & F^{k,j} & & F^{k+1,j} & & F^{jj} = F_j \\
\| & & \| & & \| & & \| & & \| \\
0 = \cdots < F_{j,b_1} & = \cdots\cdots & < F_{j,b_{k-1}} & = \cdots < & F_{j b_k} & = \cdots\cdots < & F_{j b_{k+1}} & = \cdots\cdots & F_{jn} = F_j \\
\| & & \| & & & & & & \vee \\
0 = \cdots < F_{j-1,b_1} & = \cdots\cdots & F_{j-1,b_{k-1}} & = \cdots = & F_{j-1,b_k} & = \cdots\cdots < & F_{j-1,b_{k+1}} & = \cdots\cdots & F_{j-1,n} = F_{j-1} \\
\| & & \| & & & & \| & & \| \\
F^{1,j-1} & & F^{k-1,j-1} & & & & F^{k,j-1} & & F^{j-1,j-1}
\end{array}$$

Hence the conditions are:

$$F^{i,j-1} = F^{i,j} \qquad i < k$$

$$F^{i,j-1} \subset F^{i+1,j} \qquad i \geq k$$

where $\quad k = \operatorname{card}\{\, a \leq \alpha^{-1}(j) \mid \alpha(a) \leq j \,\}$

To compute $[\bar{C}_\alpha]$ in $H^*(Y_{1,\ldots,d})$ I have two steps: 1) integration formula for $Z \longrightarrow Y_{1,\ldots,d}$
2) class of $\tilde{C}_\alpha \subset Z$.

It seems desirable to change Z at this point. The point is that the choice of $F^{i,j-1}$ involves at most choosing a point in a projective line. Because we have already chosen $F^{i-1,j-1}$, $F^{i+1,j}$

I think we will get a simpler integration formula using the following. [redacted]

[redacted] Let W be the set of systems of subspaces F^{ij} $\quad 1 \leq i \leq j \leq d$ in V such that

$$\dim F^{ij} = i$$

$$F^{ij} \subset F^{i+1,j}$$

$$F^{ij} \subset F^{i+1,j+1}$$

Picture:

$$\begin{array}{ccccc} < & F^{d-1,d} & < & F^{dd} \\ & \vee & & \vee \\ < & F^{d-2,d-1} & < & F^{d-1,d-1} \\ & \vee & & \vee \end{array}$$

Map this to $Y_{1,\dots,d}$ by $(F^{ij}) \longmapsto (F^{jj})$.

Example: $d=2$, it is the gadget $(F^1 < F^2 > L_2)$ encountered before

$$\dim (W / Y_{1,\dots,d}) = \frac{d(d-1)}{2}$$

In effect the choice of F^{ij} for $1 \leq i < j \leq d$ is a projective line.

Write $W \longrightarrow Y_{1\ldots d}$ as an iteration of $\mathbb{P}^1$-bundles. Denote by W_b the set of (F^{ij}) of the preceding type with $j \leq b$, whence $W = W_d$. To lift an element of W_{b-1} into W_b we choose F^{ab}, $1 \leq a < b$, starting with F^{bb} which is given. Let W_{ab} be the set of systems (F^{ij}) given for $j \leq b$ and $a \leq i \leq b$, so that the map

$$\pi_{ab} : W_{ab} \longrightarrow W_{a+1,b} \qquad 1 \leq a < b$$

forgets F^{ab}. The fibre is the set of F^{ab} of dim a such that

$$F^{a-1,b-1} \subset F^{a,b} \subset F^{a+1,b}$$

hence the fibre is isomorphic to $\mathbb{P}^1$. Denote by

$$\mathcal{L}^{a,b} = \mathcal{F}^{a,b} / \mathcal{F}^{a-1,b-1}$$

$$\xi_{ab} = e\left(\mathcal{L}^{ab\,\vee}\right).$$

Then $H^*(W_{ab})$ is a free module over $H^*(W_{a+1,b})$ with basis $1, \xi_{ab}$; moreover

$$(\pi_{ab})_* : H^*(W_{ab}) \longrightarrow H^*(W_{a+1,b})$$

is given by

$$(\pi_{ab})_* \begin{Bmatrix} 1 \\ \xi_{ab} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

$$W_{1b} = W_b$$

$$W_{bb} = W_{b-1} \times_{Y_{1,\dots,b-1}} Y_{1,\dots,b}$$

What I am interested in are the bundles

$$W'_{ab} = W_{ab} \times_{Y_{1,\dots,b}} Y_{1,\dots,d} \qquad \text{over } Y_{1,\dots,d}$$

Then $\quad W'_{bb} = W'_{1,b-1}$

consists of systems F^{ij}, $1 \le i \le j < b$, F_j, $1 \le j \le d$, such that $F^{jj} = F_j$ for $1 \le j \le b$.

Anyway it is now clear that $H^*(W'_{ab})$ admits as a $H^*(Y_{1,\dots,d})$ module the simple system of generators ξ_{ij} with $j < b$, or $j = b, i \ge a$

<u>Review</u>: Over $Y_{1\dots d} = \{(F_1 < \dots < F_d)\}$ I introduce the fibre bundle W whose fibre over $F_1 < \dots < F_d$ consists of systems (F_{ij}) $1 \le i \le j \le d$ of subspaces such that

$$\dim F_{ij} = i$$

$$F_{i,j} \subset F_{i+1,j} \qquad F_{jj} = F_j.$$

$$F_{i,j} \subset F_{i+1,j+1}$$

If $1 \leq a \leq b \leq d$, I let W_{ab} be the quotient of W which forgets F_{ij} for $j > b$ and for $j = b$, $i < a$. Thus the fibre of

$$W_{ab} \longrightarrow W_{a+1,b} \qquad 1 \leq a < b$$

consists of all F_{ab} of dimension a such that

$$F_{a-1,b-1} < F_{ab} < F_{a+1,b} \quad ;$$

the fibre is therefore $\cong \mathbb{P}^1$. The corresp. $\mathcal{O}(-1)$ is $\mathcal{L}_{ab} = \mathcal{F}_{ab}/\mathcal{F}_{a-1,b-1}$, so if we put $\xi_{ab} = c_1(\mathcal{L}_{ab}^\vee)$, then $H^*(W_{ab})$ is a free $H^*(W_{a+1,b})$ module with basis $1, \xi_{ab}$. If $a = b$, one has

$$W_{b,b} = W_{1,b-1} .$$

Now suppose $\alpha : \{1, .., n\} \twoheadrightarrow \{1, -, d\}$ is given with $\operatorname{card} \alpha^{-1}(j) = 1$ for $1 \leq j \leq d$. Let

$$C_\alpha = \{ F_1 < \cdots < F_d \in Y_{1,..,d} \mid \dim(F_j \cap V_p) = \operatorname{card}\{a \leq p \mid \alpha(a) \leq j\} \}$$

This is one of the Schubert cells in $Y_{1,..,d}$ whose coh. class I want to compute. I want to find inside of W a manifold $\tilde{C}_\alpha$ which maps birationally to C_α.