December 15, 1974 Bruhat decomp. for GL_n.

$k = \mathbb{C}$, $\mathcal{F}$ = manifold of (complete) flags in k^n, $B = \begin{pmatrix} \nabla \\ 0 \end{pmatrix}$ = the stabilizer in GL_n of the flag

$$\underline{F}_0 = \underbrace{ke_1}_{V_1} < \underbrace{ke_1 + ke_2}_{V_2} < \cdots\cdots$$

Given $\underline{F} = (0 < F_1 < \cdots < F_n = k^n) \in \mathcal{F}$ define a sequence of integers $p(s)$, $1 \leq s \leq n$, by

$p(1)$ = least p such that $ke_1 \subset F_p$

$p(2)$ = least p ——— $ke_2 \subset ke_1 + F_p$ etc.

Choose $x_1 \in F_{p(1)}$ such that $e_1 = x_1$

$x_2 \in F_{p(2)}$ ——— $e_2 \in ke_1 + x_2 = kx_1 + x_2$

$x_3 \in F_{p(3)}$ ——— $e_3 \in ke_1 + ke_2 + x_3 = kx_1 + kx_2 + x_3$

Then
$$x_s \in F_{p(s)} - (kx_1 + \cdots + kx_{s-1} + F_{p(s)-1})$$
which shows that the elements of $\{x_s \mid p(s) = p\}$ have independent images in F_p / F_{p-1}.

$$n = \sum_p \operatorname{card}\{s \mid p(s) = p\} \leq \sum_p \dim F_p/F_{p-1} = n$$

It follows that $s \mapsto p(s)$ is a permutation σ of $\{1, \ldots, n\}$. As it depends only on $\underline{F}$ and $\underline{F}_0 = ke_1 < \cdots$ it

is an invariant of the orbit BF in $B\backslash\mathcal{F}$.
Also

$$F_p = \sum_{\sigma(s)\le p} k x_s .$$

Since there is a unique element in B^u $\left(=\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right)$ carrying $\{e_i\}$ into $\{x_i\}$, F is in the B^u orbit of the flag

$$\left\{p \longmapsto \sum_{\sigma(s)\le p} k e_s\right\} = \left\{p \longmapsto \sum_{i\le p} k e_{\sigma^{-1}(i)}\right\}$$

$$= \sigma^{-1}\left\{p \longmapsto \sum_{i\le p} k e_i\right\}$$

where σ is the permutation matrix

$$\sigma\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix} = \begin{pmatrix}a_{\sigma^{-1}(1)}\\ \vdots \\ a_{\sigma^{-1}(n)}\end{pmatrix} \qquad \sigma = \begin{array}{c} i \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} \sigma(i)$$

It follows that

$$B^u\backslash\mathcal{F} \xrightarrow{\sim} B\backslash\mathcal{F} \xrightarrow{\sim} \Sigma_n$$

To compute the <u>dimension</u> of the orbit indexed by $\sigma \in \Sigma_n$. This orbit is $B^u / B^u \cap \sigma^{-1} B \sigma$.

Recall $\sigma^{-1} e_s = e_{\sigma^{-1}(s)}$. If $a = (a_{ij})$ is a matrix, then $\quad a\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \Sigma a_{1i} x_i \\ \vdots \\ \Sigma a_{ni} x_i \end{pmatrix}$ and $a e_i = \sum_j a_{ji} e_j$.

Thus
$$(\sigma^{-1} a \sigma) e_s = \sigma a e_{\sigma s}$$
$$= \sigma\Big(\sum_i a_{i,\sigma s} e_i\Big)$$
$$= \sum_i a_{i,\sigma s} e_{\sigma^{-1} i} = \sum_j a_{\sigma j, \sigma s} e_j$$

$\therefore$
$$(\sigma^{-1} a \sigma)_{ij} = a_{\sigma i, \sigma j}$$

So if $a \in B$ ($\Leftrightarrow a_{ij} = 0$ for $i > j$), then
$$(\sigma^{-1} a \sigma)_{ij} = a_{\sigma i, \sigma j} = 0 \quad \text{if } \sigma i > \sigma j$$

Hence $\quad b \in \sigma^{-1} B \sigma \iff b_{ij} = 0$ for $\sigma i > \sigma j$.

So
$$B^u \cap \sigma^{-1} B \sigma = \left\{ (b_{ij}) \;\middle|\; \begin{array}{l} b_{ii} = 1 \\ b_{ij} = 0 \quad \text{if } i > j \text{ or } \sigma i > \sigma j \end{array} \right\}$$

$$\therefore \boxed{\dim\left(B^u / B^u \cap \sigma^{-1} B \sigma\right) = \operatorname{card}\{(i<j) \mid \sigma i > \sigma j\}}$$

Note: You want $\sigma e_s = e_{\sigma(s)}$ so that
$(\sigma\tau) e_s = e_{\sigma\tau(s)} = \sigma e_{\tau(s)} = \sigma(\tau e_s)$.

Normal structure to $B\sigma B$.

It is clear that if H = diagonal matrices in GL_n, then the flags τF_0 as τ runs over Σ_n are the H-invariant flags exactly. Hence each B orbit has a unique H-fixpoint.

Put B_- = the opposite Borel to B wrt H

$$B_- = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$$

so that $B_- \cap B = H$. Because $\mathfrak{gl}_n = \mathfrak{b}_- + \mathfrak{b}$ the B_- and B orbits intersect transversally. Let's see what happens at τF_0.

$$\tau B \tau^{-1} = \{ a \mid a_{ij} = 0 \Leftarrow \tau^{-1}(i) > \tau^{-1}(j) \}.$$

$$B^u \cap \tau B \tau^{-1} = \left\{ a \;\middle|\; \begin{array}{l} a_{ii} = 1 \\ a_{ij} = 0 \Leftarrow \begin{Bmatrix} \tau^{-1}(i) > \tau^{-1}(j) \\ \text{or } i > j \end{Bmatrix} \end{array} \right\}$$

$$B^u_- \cap \tau B \tau^{-1} = \left\{ a \;\middle|\; \begin{array}{l} a_{ii} = 1 \\ a_{ij} = 0 \Leftarrow \begin{Bmatrix} \tau^{-1}(i) > \tau^{-1}(j) \\ \text{or } i < j \end{Bmatrix} \end{array} \right\}$$

$$\dim B^u / B^u \cap \tau B \tau^{-1} = \operatorname{card} \{ (i \leq j) \mid \tau^{-1}(i) > \tau^{-1}(j) \}$$

$$\dim B^u_- / B^u_- \cap \tau B \tau^{-1} = \operatorname{card} \{ (i > j) \mid \tau^{-1}(i) > \tau^{-1}(j) \}.$$

Thus these two orbits have complementary dimensions, and so they intersect in a point.

Alternatively we can work in the Lie algebra. The tangent space to G/B at τB may be identified with

$$\mathfrak{g}/\mathrm{Ad}\,\tau(\mathfrak{b}) = \mathfrak{g}/\tau\mathfrak{b}\tau^{-1}$$

The tangent space to $B^u\tau B$ $= B^u/B^u \cap \tau B \tau^{-1}$ is

$$\mathfrak{b}^u/\mathfrak{b}^u \cap \tau\mathfrak{b}\tau^{-1} \simeq \mathfrak{b}^u + \tau\mathfrak{b}\tau^{-1}/\tau\mathfrak{b}\tau^{-1}$$

$$\mathfrak{b}^u_-/\mathfrak{b}^u_- \cap \tau\mathfrak{b}\tau^{-1} \simeq \mathfrak{b}^u_- + \tau\mathfrak{b}\tau^{-1}/\tau\mathfrak{b}\tau^{-1}$$

So the thing to see is $(\mathfrak{b}^u + \tau\mathfrak{b}\tau^{-1}) \cap (\mathfrak{b}^u_- + \tau\mathfrak{b}\tau^{-1}) = \tau\mathfrak{b}\tau^{-1}$ which is <u>clear from the usual basis</u> for $\mathfrak{g}$.

Homotopy type of poset of Schubert cells in $\mathcal{F}$.

Let B be a Borel subgroup and let $x \in \mathcal{F}$. Can I recover B from the orbit Bx? Put

$$P = \{ g \in G \mid gBx = Bx \}$$

This is a subgroup containing B. If $x = \sigma B$, then

$P\sigma B = B\sigma B$. Now $P = \bigcup B\tau B$, and if $B\tau B\sigma B = B\sigma B$, we have $\tau\sigma \subset B\sigma B$, hence $\tau\sigma = \sigma$ because any double coset contains a unique element of the Weyl group. Thus $P = B$.

So from the orbit $\mathcal{O} = Bx$ we can recover B as $\{g \mid gBx = Bx\}$. If H is a maximal torus of B, there is a unique point of the orbit Bx fixed under H, which I may as well assume is x. In fact identifying $\mathcal{F}$ with G/B, then $x = \sigma B$ for a unique $\sigma \in W$.

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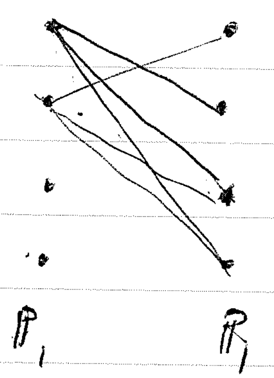
$\mathcal{F}$ = manifold of flags in k^n. A Borel subgroup B of GL_n may be identified with a point in $\mathcal{F}$, same the unique point of $\mathcal{F}^B$. The orbits of B on $\mathcal{F}$ are cells. Subsets of $\mathcal{F}$ arising this way I call Schubert cells, and I would like to determine the homotopy type of the poset $\mathcal{S}$ of Schubert cells.

Ex. $n=2$. Here $\mathcal{F} = \mathbb{P}_1(k)$ and the Schubert cells are either points or complements of points. So the poset $\mathcal{S}$ has elements $(L,0)$, $(L,1)$ $L \in \mathbb{P}_1(k)$ with $(L,0) < (L',1) \iff L \neq L'$. The associated simp. cx. is my friend



put in all segments projecting non-degenerately.

Suppose $\mathcal{O}, \mathcal{O}'$ are in $\mathcal{S}$. I have seen that one can recover a Borel from any of its orbits: $B = \{g \mid g\mathcal{O} = \mathcal{O}\}$. Let B' belong to $\mathcal{O}'$. I want to understand when $\mathcal{O}' \subset \overline{\mathcal{O}}$.

Special case where B, B' are opposite so that $B \cap B' =$ max. torus T. Let $\sigma \in$ Weyl group be such that $\sigma \cdot B = (\mathcal{O}')^T = (\mathcal{O})^T$. Thus

$$\mathcal{O}' = B'\sigma B$$
$$\mathcal{O} = B\sigma B$$

Now because B, B' are opposite, I know these orbits $\mathcal{O}, \mathcal{O}'$ meet transversally at σB, hence $\mathcal{O}' \subset \overline{\mathcal{O}}$ imply that $\mathcal{O}$ is open. Hence $\mathcal{O}$ must be the open B-orbit which means σ is the Coxeter element and $B' = \sigma B \sigma^{-1}$. Thus the case where B, B' are opposite corresponds to the case of a point sitting in an open cell.

In general start with $\mathcal{O}' \subset \overline{\mathcal{O}}$ where $\mathcal{O}, \mathcal{O}'$ determine Borels B, B'. Choose $T \subset B \cap B'$; whence $(\mathcal{O}')^T = \sigma B$, $\sigma \in W$, and we have

$$B'\sigma B \subset \overline{B\sigma B}.$$

Now $\exists !\ \tau \in W$ such that $B' = \tau B \tau^{-1}$, in fact

$$(G/B)^{B'} = \tau B \qquad (B'\tau B = \tau B \iff B' = \tau B \tau^{-1}).$$

Thus we have $\quad \tau B \tau^{-1} \sigma B \subset B \sigma B \quad$ or

$$B \tau B \tau^{-1} \sigma B = B \sigma B.$$

Now in general I believe that one has
$B \alpha B \beta B = B \alpha \beta B \iff \ell(\alpha\beta) = \ell(\alpha) + \ell(\beta)$,
where ℓ is the length. Assuming this, one gets

<u>Assertion</u>: Let $\mathcal{O}' \subset \mathcal{O}$ be Schubert cells belonging to Borel's B' and B. If $x \in \mathcal{O}'$ then

$$d(x, B) = d(x, B') + d(B', B)$$

Conversely if this holds one has $B'x \subset Bx$.

~~Given two points $B, B' \in \mathcal{F}$, one~~ ~~$d(B, B')$~~ ~~$\dim B$~~

Given two points $x, y \in \mathcal{F}$ one defines $d(x,y)$ by either

i) $\dim (B_x y) = \dim (B_y x)$

ii) choose $T \subset B_x \cap B_y$, whence B_x, B_y determines Weyl chambres in $\mathrm{Hom}(\mathbb{G}_m, T) \otimes \mathbb{R}$. $d(x,y)$ is the number of root hyperplanes crossed in going from one chambre to another.

Why these are the same: Can assume $B_x = B = \left(\begin{smallmatrix} & * \\ 0 & \end{smallmatrix}\right)$ and $y = \sigma B \subset G/B \cong \mathcal{F}$. So I want the dimension of $B\sigma B/B$, which I computed to be

$$\operatorname{card}\{i<j \mid \sigma i > \sigma j\}$$

which is exactly the number of hyperplanes crossed in going from the positive Weyl chambre to the one described by σ.

Suppose $x, y, z \in \mathcal{F}$, put $B = B_x$ and $x = B/B$ $y = fB/B$, $z = gB/B \in G/B \simeq \mathcal{F}$. Then $B_y = fBf^{-1}$, $B_z = gBg^{-1}$.
and

$$B_x y = BfB/B$$

$$B_x z = BgB/B$$

$$B_y z = fBf^{-1}gB/B$$

$$BgB \subset BfB \times^B Bf^{-1}gB$$

$$\therefore \dim BgB/B \leq \dim BfB/B + \dim Bf^{-1}gB/B$$

$$\|$$

$$\dim(B_x z) \leq \dim(B_x y) + \dim(B_y z)$$

Thus if equality holds for these dimensions we have

$$BgB = BfB\,Bf^{-1}gB$$

equality of double cosets.

So I want to be sure that if $l(\alpha\beta) < l(\alpha)+l(\beta)$, then $B\alpha B\beta B$ consists of more than one double coset. ~~But write $\alpha = s_1 \cdots s_l$ and consider the first i such that $l(s_i \cdots s_l\beta) <$ $i + l(\beta)$. It is enough to show $Bs_i \cdots s_l B\beta B$ contains more than the one element of $s_i \cdots s_l\beta$ of W.~~

Use induction on $l(\alpha)$; write $\alpha = s\alpha'$, $l(\alpha') = l(\alpha)-1$. Then $B\alpha B = BsB \cdot B\alpha' B$. If $l(\alpha')+l(\beta) > l(\alpha'\beta)$, then already $B\alpha' B\beta B$ contains more elements of W than $\alpha'\beta$, so done. If $l(\alpha')+l(\beta) = l(\alpha'\beta)$, then $B\alpha' B\beta B = B\alpha'\beta B$, and so we are reduced to showing

$l(s\beta) < l(\beta)+1 \implies BsB\beta B$ contains more elements of W than $s\beta$. But one knows then that $\beta = s\beta'$ so $BsB\beta B = BsBsB\beta' B$, and $BsBsB = B \sqcup BsB$.

So the simplicial complex associated to the poset S of Schubert cells in $\mathcal{F}$ has for its p-simplices $(\mathcal{O}, B_0, \dots, B_p)$ where B_i are Borels, $\mathcal{O}$ is a B_0-orbit, and

$$d(B', B_0) + \cdots + d(B_{p-1}, B_p) = d(B', B_p)$$

for some (hence any) $B' \in \mathcal{O}$.

<u>Question</u>: Given a simplex $(\mathcal{O}, B_0, \dots, B_p)$ does there exist a $T \subset \bigcap B_i$?

Question: Given Borels $B_0, \ldots, B_p$ such that
$$d(B_0, B_1) + \cdots + d(B_{p-1}, B_p) = d(B_0, B_p),$$
does there exist $T \subset \bigcap B_i$?

Suppose B_0, B_p are opposite, i.e. $B_0 \cap B_p = T$. Consider the set Z of Borels B_1 such that $d(B_0, B_p) = d(B_0, B_1) + d(B_1, B_p)$ with $d(B_0, B_1)$ a given integer t. For each $z \in Z$,
$$\dim(B_0 z) = t$$
$$\dim(B_p z) = d(B_0, B_p) - t$$
and I know the orbits $B_0 z$, $B_p z$ are transversals. Thus the orbits $B_0 z$ and $B_p z$ have a zero dimensional intersection, which is T-stable; hence T being connected, z is a T-fixed point. This means that any B_1 with $d(B_0, B_p) = d(B_0, B_1) + d(B_1, B_p)$ contains T.

Now given $B_0, \ldots, B_p$ as above, choose a maximal torus $T \subset B_0 \cap B_p$, and let B_{p+1} be the opposite Borel to B_0 containing T. Then
$$d(B_0, B_p) + d(B_p, B_{p+1}) = d(B_0, B_{p+1}).$$
Thus $d(B_0, B_1) + \cdots + d(B_p, B_{p+1}) = d(B_0, B_{p+1})$, and so by what I have already done I see that

$B_1, \ldots, B_p$ contain T.

Suppose I fix T and consider only cells fixed under T. Each cell has a unique T-fixpoint, so a p-simplex is a sequence of Borels $(B_{-1}, B_0, \ldots, B_p)$ containing T such that the distance condition holds. So this simplicial complex is clearly a union of cones, one for each T-fixpt, i.e. for each element of W.

Generalization — Consider orbits of the various Borels on all the simplices of the building as Schubert cells.

Example $n=2$. Here the building is the set of lines in k^2 with the discrete topology. But if you "topologize" the set of lines using Shubert cells you get something which is connected. You get a connected graph whose H_1 has rank q^2-q-1 for $k=\mathbb{F}_q$.

$$h_0 - h_1 = 2(q+1) - (q+1)q$$

$$1 - h_1 = 2q+2-q^2-q = 2-q^2+q$$

$$h_1 = q^2-q-1$$

December 19, 1974

Lets first classify the B-orbits on the building X. Given a simplex $0 < F_1 < \cdots < F_q = V$, better to call this a flag, we can define a map $\alpha : \{1, \dots, n\} \longrightarrow \{1, \dots, q\}$ as follows:

$$\alpha(1) = \text{least } p \text{ such that } e_1 \in F_p$$

$$\alpha(2) = \text{least } p \text{ ——— } e_2 \in ke_1 + F_p$$

Choose $x_1 = e_1 \in F_{\alpha(1)}$, $e_2 \in ke_1 + x_2$ $x_2 \in F_{\alpha(2)}$, etc.

Then by old arguments I know F_p/F_{p-1} has as basis the images of the x_s, $\alpha(s) = p$. Hence

$$F_p = \sum_{\alpha(s) \leq p} k x_s$$

One sees that $\alpha : \{1, \dots, n\} \to \{1, \dots, q\}$ is surjective with $\operatorname{card} \alpha^{-1}(p) = \dim(F_p/F_{p-1})$. α is an invariant of the B-orbit of the flag, and determines the B-orbit. So if I denote by $D_{r_1 \dots r_q}(V)$ the flags with jumps $r_1, \dots, r_q$, I see that

$$B \backslash D_{r_1 \dots r_q}(V) \simeq \{\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, \dots, q\} \mid \operatorname{card} \alpha^{-1}(p) = r_p\}$$

B-orbits are same as B^u-orbits. Also the B-orbit indexed by $\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, \dots, q\}$ contains the flag with

$$F_p = \sum_{\alpha(s) \le p}' k e_s$$

which is T-invariant. This is the unique T-invariant flag in the orbit, so again each B-orbit has a center, the unique T-fixpt.

Thus each B-orbit of simplices in X contains a unique T-fixpt, which means that $B\backslash X$ is the subcomplex of T-fixpoints of X. (In particular $B\backslash X$ is a simplicial complex.)

Recall that you have identified the simplicial complex of all subspaces of V with self-adjoint operators A on V such that $0 \le A \le 1$. If $0 < \lambda_1 < \dots < \lambda_q < 1$ are the eigenvalues of A except for $0, 1$, then these are the simplicial coordinates, and the simplex has as vertices

$$E_0^A V < E_{\lambda_1}^A V < \dots < E_{\lambda_q}^A V$$

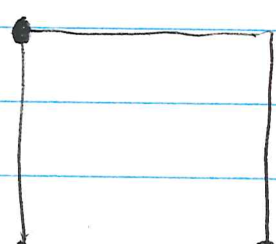
where $A = \int \lambda \, dE_\lambda^A$. (Thus $E_0^A V = \operatorname{Ker} A$, $E_{\lambda_1}^A V$ is where $A \le \lambda_1$, etc.). (Thought: The vertices are the projectors; every point is an average of commuting projectors. Maybe projectors are the extreme points of the convex set of A, $0 \le A \le 1$.)

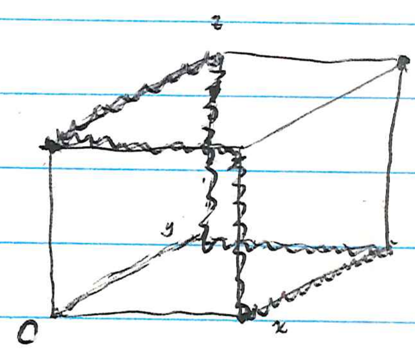
So now let me compute the B-orbits on the big building of all subspaces of V. Same as the subcomplex of T-invariant simplices, i.e. operators A commuting with T. Such A are of the form $A = \sum_i \lambda_i e_i$ $\quad 0 \le \lambda_i \le 1$, so we get I^n for the orbit space, triangulated in the standard way.

Next consider the proper building X of proper subspaces. By the preceding formulas the simplex

$$E_0^A V < E_{\lambda_1}^A V < \cdots < E_{\lambda_q} V$$

will be a chain of proper subspaces, provided A has the eigenvalues 0 and 1. So after taking T-orbits we get the closed subspace of I^n consisting of points (λ_i) such that $\lambda_i = 0$ $\quad \lambda_j = 1$ for some i, j.





So its should be a triangulation of a sphere.

Given $\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, \dots, q\}$ what is the dimension of the B-orbit corresponding to α? Compute the stabilizer of the flag

$$\{1, \dots, q\} \ni p \longmapsto F_p = \sum_{\alpha(s) \le p} k e_s = \{x \in k^n \mid x_s = 0 \text{ if } \alpha(s) > p\}$$

$$\text{Stabilizer of } F_p = \{a \in Gl_n \mid \alpha(i) > p,\ \alpha(t) \le p \Rightarrow a_{it} = 0\}.$$

$$\text{Stabilizer of } \{F_p\}_{1 \le p \le q} = \{a \in Gl_n \mid \alpha(i) > \alpha(t) \Longrightarrow a_{it} = 0\}$$

$$B \cap \text{———————} = \left\{ a \in M_n \;\middle|\; \begin{array}{l} a_{ii} \in k^* \\ \left.\begin{array}{l} i > t \\ \alpha(i) > \alpha(t) \end{array}\right\} \Longrightarrow a_{it} = 0 \end{array} \right\}$$

$$\text{codim of } B \cap St.\{F_p\} \text{ in } B = \text{card}\,\{(i < t) \mid \alpha(i) > \alpha(t)\}$$

<u>Formula:</u> The dimension of the B-orbit indexed by $\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, \dots, q\}$ is $\text{card}\,\{(i < j) \mid \alpha(i) > \alpha(j)\}$.

Addition to page 10. The

$$\begin{array}{ccccc}
\left|(\text{all subspaces of } V)\right| & = & \{0 \le A \le 1\} & \xrightarrow{B\text{-orbit}} & I^n \\
\cup & & \cup & & \cup \\
\left|\begin{array}{l}\text{Subcomplex of} \\ \text{chains } V_0 < \dots < V_p \\ \text{s.t. } \dim(V_p/V_0) < n\end{array}\right| & = & \left\{\begin{array}{l} 0 \le A \le 1 \\ 0 \text{ or } 1 \in sp(A) \end{array}\right\} = \partial\{0 \le A \le 1\} & \longrightarrow & \partial I^n \\
\cup & & & & \cup \\
|T(V)| & = & \left\{\begin{array}{l} 0 \le A \le 1 \\ 0 \text{ and } 1 \in sp A \end{array}\right\} & \longrightarrow & \left\{(\lambda_i) \in I^n \;\middle|\; \begin{array}{l}\lambda_i = 0 \\ \lambda_j = \end{array}\right\}
\end{array}$$

So I can now identify a B-orbit on the big building with a point $(\lambda_i) \in I^n$. The associated α is the surjection one gets by arranging the λ_i in order. Thus if $0 \leq \mu_1 < \cdots < \mu_q \leq 1$ is the sequence arranged in order with repetitions, $\lambda_i = \mu_{\alpha(i)}$.

Take (λ_i) and any point $0 < z_1 < z_2 < \cdots < z_n < 1$ of the interior of the positive Weyl chambre. Consider the straight line segment

$$(*) \qquad (1-t)\lambda + tz \quad \text{in } I^n$$

Take the root $x_j - x_i$, $i<j$ and consider the image of this straight line under this root function. It is the segment going from $\lambda_j - \lambda_i$ to $z_j - z_i > 0$. Thus the line segment $(*)$ crosses the hyperplane $X_i = X_j$ iff $\lambda_j - \lambda_i < 0$ i.e. $\mu_{\alpha(j)} < \mu_{\alpha(i)}$ or $\alpha(i) > \alpha(j)$.

<u>Formula:</u> The dimension of the B-orbit indexed by α is the number of root hyperplanes crossed in going along a straight line joining any point of the α-stratum of $\mathbb{R}^n$ to the positive Weyl chambre

Now the program will be to show the (proper) building X has the homotopy type of a sphere by Serre's method. This method involves filtering X according to the length of elements in the Weyl group.

So one lists the elements of W, $w_0, w_1, w_2, \ldots$ such that $0 = l(w_0) \leq l(w_1) \leq l(w_2) \leq \ldots$ etc.

Let $S_p \subset I^n$ be the subcomplex containing all the closed n-simplices associated to $w_0, \ldots, w_p$; and let X_p be the inverse image of S_p inside of X.

Suppose we have a point $(\lambda_i) \in I^n$ such that $\lambda_i = \mu_{\alpha(i)}$, where $\alpha : \{1, \ldots, n\} \longrightarrow \{1, \ldots, g\}$ and $0 \leq \mu_1 < \cdots < \mu_g \leq 1$. Then we can take all of the i such that $\alpha(i) = p$ and perturb the λ_i so as to get a new point $(\lambda_i') \in I^n$ with all λ_i' distinct and such that $\alpha(i) = \alpha(j), i < j \Rightarrow \lambda_i' < \lambda_j'$.

I can also arrange none of the λ_i' to be 0 or 1. Let w be the permutation assoc. to λ_i', whence we see λ is on the boundary of the n-simplex of I^n

described by ω. ω is obviously the unique permutation refining α of the same length. ("refine" means $\omega(i) \leq \omega(j) \Rightarrow \alpha(i) \leq \alpha(j)$ or equival. $\alpha(i) > \alpha(j) \Rightarrow \omega(i) > \omega(j)$). Thus I see that each stratum of I^n is in the boundary of a unique chambre of the same length. It's also clear that the α-stratum is in the boundary of the β-stratum iff β refines α.

So let us now consider how X_p is obtained from X_{p-1}. First consider the passage from S_{p-1} to S_p, where we add on the closure of the stratum belonging to the permutation $\sigma = \omega_p$. This stratum is the chambre of $(\lambda_i) \in I^n$ where the λ_i are distinct and are arranged in the order given by σ $\lambda_i \leq \lambda_j$ iff $\sigma(i) \leq \sigma(j)$). This stratum is open except that $\lambda_{\sigma^{-1}(1)}$ can be 0 and $\lambda_{\sigma^{-1}(n)}$ can be 1. The boundary of this stratum will consist of strata refined by σ. Such things are of the form

$$\begin{tikzcd} \{1,\dots,n\} \arrow[r, "\sigma"] \arrow[dr, "\alpha"'] & \{1,\dots,n\} \arrow[d, "\eta\ \text{surj monotone}"] \\ & \{1,\dots,q\}. \end{tikzcd}$$

and we know $l(\alpha) \leq l(\sigma)$, with equality iff σ^{-1}

preserves order on the fibres of η. I now have to figure out which α have already occurred in S_{p-1}. Certainly all α with $l(\alpha) < l(\sigma)$ have occurred; if $l(\alpha) = l(\sigma)$, then α has not occurred before because then α would be refined by some w_j with $j < p$, hence $l(w_j) \leq l(\sigma) = l(\alpha)$, so $w_j = \sigma$ since we know there is a unique refining permutation of the same length. Thus I must see what sort of geometric object is the union of the α-strata with α refined by σ and $l(\alpha) < l(\sigma)$. It is a closed subcomplex.

Suppose we have η such that σ^{-1} does not preserve the order on the fibres. Pick one such bad fibre, say the interval $[a, b] = \{a, a+1, .., b\}$, and choose i least such that $\sigma^{-1}(a+i) > \sigma^{-1}(a+i+1)$. Then I find η is refined by the degeneracy η_{a+i} coallescing $a+i$ and $a+i+1$. This shows that the α's involved form a union of the codim 1 faces η_i such that σ^{-1} reverses $i, i+1$. This is the union of not all the codim 1 faces, except when σ is the Coxeter element. So to make this all airtight, I have to take the closed σ-stratum, and to retract it onto the subcomplex with smaller length.

So next I try the same game on X.

Changing notation let me denote by X_p the union of all strata of length $\leq p$. [redacted] Then in going from X_{p-1} to X_p I attach all strata with indices α of length p. Each such α is attached to a unique permutation σ of length [redacted] p. So what I want to see is that if I [redacted] put Z_σ = closure of σ-stratum, then Z_σ deforms strongly down to $Z_\sigma \cap X_{p-1}$. Have to be careful that Z_σ contains new stuff besides α-strata with $\alpha < \sigma$ (σ refines α).

Let $U_\sigma = \sigma$-stratum. This will consist of s.a. operators A with distinct eigenvalues $0 \leq \lambda_1 < \dots < \lambda_n \leq 1$, satisfying a condition relative to the flag $\underset{V_1}{\mathbb{C}e_1} < \underset{V_2}{\mathbb{C}e_1 + \mathbb{C}e_2} < \dots\dots$
A typical A is of the form

$$A = \sum_{i=1}^{n} \lambda_i \, pr_{L_i}$$

where $L_1 \in \mathbb{P}V_{\sigma(1)} - \mathbb{P}V_{\sigma(1)-1}$, $L_2 \in \mathbb{P}(V_{\sigma(2)} + L_1) - \mathbb{P}(V_{\sigma(2)-1} + L_1)$ and $L_2 \perp L_1$, etc. Something is wrong.

December 22, 1974. Cohomology classes associated to Schubert cells.

Let Y be the manifold of full flags in $V \simeq k^n$; $Y = GL_n/B$ where $B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ is the stabilizer of the basic flag $\{V_p = ke_1 + \dots + ke_p\}$. Y is an iterated projective bundle

$$(*) \qquad Y = \{(0<F_1<\dots<F_{n-1}<V)\} \longrightarrow \{(0<F_1<\dots<F_{n-2}<V)\} \longrightarrow \dots \longrightarrow \underset{\mathbb{P}V}{\underset{\shortparallel}{\{0<F_1<V\}}} \longrightarrow pt$$

Recall for $f: \mathbb{P}E \longrightarrow X$ the formula

$$f_* a(\xi) = \operatorname{res}\left(\frac{a(T)\,dT}{T^q + (c_1E)T^{q-1} + \dots + (c_qE)}\right) \qquad q = \dim E$$

where $\xi = c_1(\mathcal{O}(1))$.

$$= \operatorname{res}\left(\frac{a(T)\,(T^p + (c_1Q)T^{p-1} + \dots + c_pQ)}{T^n}\,dT\right)$$

if $E \oplus Q \simeq n$ $\qquad p+q=n$.

~~Applying this to each of the maps (*), I get~~

$$\cancel{\int_Y a(\xi_1,\dots,\xi_n) = \operatorname{res}\left(\frac{dT_1}{T_1^n + \dots + c_nV}\right)\left(\operatorname{res}\frac{dT_2}{T_2^{n-1}}\right.}$$

Put $Y_i = \{(F_1<\dots<F_i)\}$. Over Y_i we have

the bundle $\mathcal{F}_i$ with fibre F_i at $(F_1 < \cdots < F_i)$, and $\mathcal{F}_i \subset \mathcal{O}_{Y_i} \times V$. Denote all by $\mathcal{F}_i$ the pull-back of $\mathcal{F}_i$ to Y_j for $j \geq i$. Put $\mathcal{L}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$, $\xi_i = e(\check{\mathcal{L}}_i)$. For the map $f_i : Y_i \longrightarrow Y_{i-1}$, we have $Y_i = \mathbb{P}(V/\mathcal{F}_{i-1})$, $\mathcal{O}(-1) = \mathcal{L}_i$, so

$$(f_i)_* \, a(\xi_i) = \operatorname{res} \frac{a(T_i)\, dT_i}{T_i^{n-i+1} + \cdots + c_{n-i+1}(V/\mathcal{F}_{i-1})}$$

$$= \operatorname{res} \frac{a(T_i)\left(T_i^{i-1} + \cdots + c_{i-1}(\mathcal{F}_{i-1})\right)}{T_i^n} \, dT_i$$

$$= \operatorname{res} \; a(T_i) \prod_{j<i} (T_i - \xi_j) \, \frac{dT_i}{T_i^n}$$

Thus we get the formula

$$\star \quad \int_Y a(\xi_1, \ldots, \xi_n) = \text{coefficient of } (T_1 \cdots T_n)^{n-1} \text{ in } a(T_1, \ldots, T_n) \prod_{i<j} (T_j - T_i).$$

Consider now the map

$$Y = U/T \xrightarrow{\gamma} BT$$

which classifies the line bundles $\check{\mathcal{L}}_i$ on Y.

since $H^*(BT) = \mathbb{Z}[[T_1, \dots, T_n]]$ $\quad T_i = c_1(pr_i : T \to S^1)$.
and $\gamma^*(T_i) = \xi_i$, the formula ☆ tells us that

$$\langle \gamma_*[Y], a(T_1, \dots, T_n) \rangle = \text{coeff. of } (T_1 \cdots T_n)^{n-1} \text{ in } a(T_1, \dots, T_n) \prod_{i<j} (T_j - T_i)$$

Now

$$\prod_{i<j} (T_j - T_i) = \begin{vmatrix} 1 & T_1 & T_1^2 & \cdots & T_1^{n-1} \\ & & & & \\ 1 & T_n & T_n^2 & \cdots & T_n^{n-1} \end{vmatrix} = \sum_{\sigma \in \Sigma_n} (-1)^\sigma T_{\sigma(2)} T_{\sigma(3)}^2 \cdots T_{\sigma(n)}^{n-1}$$

Therefore $$\langle \gamma_*[Y], T_1^{\alpha_1} \cdots T_n^{\alpha_n} \rangle = \begin{cases} (-1)^\sigma & \text{if } n - \alpha_i = \sigma^{-1}(i) \text{ for } i = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

<u>Question:</u> Can you construct resolutions of the Schubert cells?

<u>Example:</u> Take $G_p(k^n)$. A B-orbit on this is described by $\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, 2\}$ such that $\alpha^{-1}(1)$ has p elements. It is therefore described by a sequence $1 \leq s_1 < \cdots < s_p \leq n$, and specifically is the subset consisting of A in V, $\dim A = p$, such that

the induced filt. $V_i \cap A$ has jumps at $i = s_1, \dots, s_p$. The closure of this cell consists of A such that $\dim(V_{s_j} \cap A) \geq j$. I resolve the closure by considering the manifold of flags $F_1 < \cdots < F_p$ in V such that $F_j \subset V_{s_j}$.

But we can also get another resolution as follows. Let $\alpha^{-1}(2) = \{t_1 \cdots t_q\}$ with $1 \le t_1 < \cdots < t_q \leq n$. Then the cell under consideration can be described as consisting of A such that $V_i + A / A$ has its jumps at $t_1, \dots, t_q$, that is $V_{t_j - 1} + A < V_{t_j} + A$ has dim $p+j$.

So the closure of the cell would seem to consist of A such that $\dim(V_{t_j} + A) \leq p+j$. I should be able to resolve this by the manifold of flags $F_p < F_{p+1} < \cdots < F_n = V$ such that $F_{p+j} \supset V_{t_j}$ for $j = 1, \dots, q$. Questions: Are these two resolutions the same or different?

December 23, 1974

V is a vector space of dim. n with a given full flag $0 < V_1 < \cdots < V_n$, B = corresp. Borel subgroup $\subset G = \mathrm{Aut}(V)$.

Given $1 \leq s_1 \leq \cdots \leq s_\mu = n$, I let $D_{\underline{s}}(V)$ be the manifold of flags $0 < F_1 < \cdots < F_\mu = V$ such that $\dim(F_j) = s_j$. Here $\underline{s} = (s_1, \ldots, s_\mu)$. I know that the B-orbits on $D_{\underline{s}}(V)$ are classified by functions $\alpha : \{1, \ldots, n\} \twoheadrightarrow \{1, \ldots, \mu\}$ such that $\mathrm{card}[\alpha^{-1}\{1, \ldots, j\}] = s_j$. More precisely, suppose a flag $(0 < F_1 < \cdots < F_\mu = V)$ in $D_{\underline{s}}(V)$ is given. Then for each p, $1 \leq p \leq n$, the quotient V_p/V_{p-1} "appears" in one of the quotients F_j/F_{j-1}, and then $\alpha(p) = j$. α being fixed, the corresp. cell in $D_{\underline{s}}(V)$ consists of all $\underline{F}$ such that

$$\dim(F_j \cap V_p) = \mathrm{card}\{i \leq p \mid \alpha(i) \leq j\}.$$

Call this cell C_α.

I have the following candidate for a resolution of C_α. Consider families (F_{jp}) $j = 1, \ldots, \mu$; $p = 1, \ldots, n$ of subspaces monotone in both j and p such that

$$F_{jp} \subset V_p$$

$$\dim F_{jp} = \mathrm{card}\{a \leq p \mid \alpha(a) \leq j\}.$$

These families form a closed subvariety $\tilde{C}_\alpha$ of a product of Grassmannians, hence $\tilde{C}_\alpha$ is a complete variety. By sending F_{jp} into F_{jn} for $1 \le j \le \mu$ we get a map $\tilde{C}_\alpha \longrightarrow D_\Delta(V)$.

If $(F_{jn})_{1 \le j \le \mu}$ is in C_α, then

$$\begin{array}{ccc} \dim F_{jn} \cap V_p & = & \operatorname{card}\{a \le p \mid \alpha(a) \le j\} \\ \vee\!\mid & & \\ \dim (F_{jp}) & & \end{array}$$

hence $F_{jp} = F_{jn} \cap V_p$ showing that over C_α, $\tilde{C}_\alpha$ has fibres reduced to a point.

Examples: 1: Take $\alpha : \{1, \dots, n\} \twoheadrightarrow \{1, 2\}$. Then $\tilde{C}_\alpha$ consists of flags in $A = F_{1,n}$

$$\begin{array}{ccccc} V_1 & < \cdots\cdots < & V_n & = & V \\ \cup & & \cup & & \\ F_{1,1} & \subset \cdots\cdots \subset & F_{1,n} & = & A \end{array}$$

such that $F_{1,p} = \{a \le p \mid \alpha(a) = 1\}$ has the jumps at the points of $\alpha^{-1}(1)$. So this is one of the resolutions used before for Schubert cells in the Grassmannians.

2: Take $\alpha : \{1, \dots, n\} \to \{1, \dots, n\}$ to be the Coxeter permutation $\alpha(i) = n-i+1$. Take $n=2$

$$
\begin{array}{ccccc}
0 & < & V_1 & < & V \\
 & & & & \vee \\
 & & 0 & & F_1 \\
 & & & & \vee \\
 & & & & 0
\end{array}
$$

Try $n=3$.

$$
\begin{array}{ccccc}
V_1 & < & V_2 & < & V_3 \\
 & & & & \vee \\
0 & < & F_{2,2} & \leq & F_{2,3} \\
 & & & & \vee \\
0 & = & 0 & < & F_{1,3}
\end{array}
$$

$F_{2,2}$ can be any line in $V_2 \cap F_{2,3}$, hence it is not uniquely determined. Thus $\tilde{C}_\alpha$ is not Y in this case, but bigger.

Is $\tilde{C}_\alpha$ non-singular? Consider for example the case $\mu = 2$, where I have a filtration F_p with $\dim F_p = \{ i \leq p \mid \alpha(i) = 1 \}$ and $F_p \subset V_p$. Start by mapping

$$(F_i) \longmapsto F_1 .$$

Let $\{t_1 < \cdots < t_\nu\} = \alpha^{-1}(1)$. Then $0 = F_1 = \cdots = F_{t_1 - 1}$ and one chooses F_{t_1} to be any line in V_{t_1}. Then $F_{t_1} = \cdots = F_{t_2 - 1}$, and the next jump occurs at F_{s_2} which can be any 2-plane in V_{s_2} containing F_{s_1}. Then F_{s_3} can be any 3-plane is V_{s_3} containing F_{s_2}. So we see that C_α is non-singular (more or less). It's dimension is

$$(s_1-1)+(s_2-2)+\cdots+(s_\nu-\nu).$$

$$\operatorname{card}\left\{(i<j) \,\middle|\, \begin{matrix}\alpha(i)>\alpha(j)\\ j\le s_1\end{matrix}\right\} = s_1-1$$

$$\left\{(i<j) \,\middle|\, \begin{matrix}\alpha(i)>\alpha(j)\\ s_1<j\le s_2\end{matrix}\right\} = s_2-2$$

So its dimension is $l(\alpha)$. (Pairs out of order are of the form (a, s_j), where $a \neq s_1, \dots, s_{j-1}$. The number of these is $s_j-1-(j-1)=s_j-j$.)

[redacted] See if we can do the same computation without introducing $s_1, \dots, s_\nu$. Consider the [redacted] choice of F_p once $F_1 \cdots F_{p-1}$ have been chosen. The conditions are $F_{p-1}\subset F_p \subset V_p$ and that F_p have dimension $\operatorname{card}\{i\le p \mid \alpha(i)=1\}$. So it's clearly non-singular, this fibre is, and its dimension is

$$\begin{cases} 0 & \alpha(p) > 1 \\ p-1-\operatorname{card}\{i<p \mid \alpha(i)=1\} & \text{if } \alpha(p)=1 \end{cases}$$

$$= \operatorname{card}\{i<p \mid \alpha(i) > \alpha(p)\}$$

~~In the general case we work vertically, i.e doing $F_{11} \cdots F_{\mu 1} = V_1$ then $F_{12}, \ldots, F_{\mu 2} = V_2$. In a given column F_{jp} there is exactly one j where $F_{jp}/F_{j,p-1}$ jumps, namely $j = \alpha(p)$ because~~

~~$$\dim F_{jp} = \operatorname{card}\{a \le p \mid \alpha(a) \le j\}$$~~

~~$$\dim F_{jp}/F_{j,p-1} = \begin{cases} 1 & \alpha(p) \le j \\ 0 & \alpha(p) < j \end{cases}$$~~

~~Only problem is that I don't see why for $j \ge \alpha(p)$~~

~~$$F_{jp}/F_{j,p-1} \xrightarrow{\sim} V_p/V_{p-1}$$~~

General case:

$$\dim F_{jp} = \operatorname{card}\{a \le p \mid \alpha(a) \le j\}$$

$$\dim (F_{jp}/F_{j,p-1}) = \begin{cases} 1 & \alpha(p) \le j \\ 0 & \alpha(p) > j \end{cases}$$

So we choose $F_{11}, \dots, F_{1n}$, then $F_{21}, \dots, F_{2n}$, etc. Consider the possible choices for F_{jp}, once $F_{j'p'}$ has been chosen for all j', p' such that $j' < j$ or $j' = j$ and $p' < p$. F_{jp} is a subspace of V_p of $\dim = \text{card}\{a \le p \mid \alpha(a) \le j\}$, subject to the conditions $F_{jp} \supset F_{j-1,p}, F_{j,p-1}$. So the dimensions of the possible F_{jp} will depend on $F_{j-1,p} + F_{j,p-1}$, and this method won't work.

Try instead choosing the columns $F_{1p}, \dots, F_{\mu p}$ inductively starting with $F_{\mu p} = V_p$. Assume the columns $F_{jp'}$ chosen for $p' < p$, and that I have also chosen $F_{j+1,p}, \dots, F_{\mu p} = V_p$. Consider the possible choices for F_{jp}. This is a subspace of $\dim = \text{card}\{a \le p \mid \alpha(a) \le j\}$, such that $F_{jp} \supset F_{j,p-1}$, $F_{jp} \subset F_{j+1,p}$. ~~This is the set of lines in $F_{j+1,p}/F_{j,p-1}$ whose dimension doesn't vary.~~

$$\dim\left(F_{j+1,p}/F_{j,p-1}\right) =$$

$$= \dim\left(F_{j+1,p}/F_{j,p}\right) + \dim\left(F_{j,p}/F_{j,p-1}\right)$$

$$= \text{card}\{a \le p \mid \alpha(a) = j+1\} +$$

Two cases: $\alpha(p) > j \Rightarrow F_{jp} = F_{j,p-1}$ $\quad$ 0 dim'l choice for F_{jp}

$$\alpha(p) \leq j \implies F_{jp}/F_{j,p-1} \text{ is any line in } F_{j+1,p}/F_{j,p-1}$$

$$-1 + \dim\left(F_{j+1,p}/F_{j,p-1}\right) = \dim\left(F_{j+1,p}/F_{j,p}\right)$$

$$= \operatorname{card}\{a \leq p \mid \alpha(a) = j+1\},$$

so we see now that $\tilde{C}_\alpha$ is non-singular of dimension

$$\sum_{\substack{1 \leq j < \mu \\ 1 \leq p \leq n}} \begin{Bmatrix} 0 & \alpha(p) > j \\ \operatorname{card}\{a \leq p \mid \alpha(a) = j+1\} & \alpha(p) \leq j \end{Bmatrix}$$

$$= \sum_{\substack{1 \leq a \leq p \\ 1 \leq p \leq n}} \sum_{1 \leq j < \mu} \begin{Bmatrix} 1 & \alpha(p) \leq j,\ \alpha(a) = j+1 \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$= \operatorname{card}\{a < p \mid \alpha(a) > \alpha(p)\}.$$

Next thing to understand is the image of $\tilde{C}_\alpha \longrightarrow D_{\underline{d}}(V)$, which should be the closure of C_α. In particular I want to understand which β are such that $C_\beta \subset \overline{C_\alpha}$.

Change notation and replace $s_1, \ldots, s_\mu$ by $\underline{d} = (d_1, \ldots, d_\mu)$, $\quad 0 < d_1 < \cdots < d_\mu = n$.

Then $D_{\underline{d}}(V) \cong G/P_{\underline{d}}$ where $P_{\underline{d}}$ is the stabilizer of the flag

$$F_j = \sum_{p \le d_j} k e_p .$$

$$P_{\underline{d}} = \begin{pmatrix} \boxed{d_1} & & & * \\ & \boxed{d_2 - d_1} & & \\ & & \boxed{} & \\ 0 & & & \boxed{d_\mu - d_{\mu-1}} \end{pmatrix}$$

Can also describe $P_{\underline{d}}$ as the subgroup of Gl_n generated by B and those simple roots s_i (s_i transposes i and $i+1$, $1 \le i < n$) such that $\alpha_{\underline{d}}(i) = \alpha(i+1)$, where $\alpha_{\underline{d}}$ is the monotone map $\{1, \ldots, n\} \twoheadrightarrow \{1, \ldots, \mu\}$ such that $d_j = \operatorname{card}\{p \mid \alpha(p) \le j\}$.

$$W = \Sigma_n , \quad W_{\underline{d}} = \Sigma_{d_1} \times \Sigma_{d_2 - d_1} \times \cdots \times \Sigma_{d_\mu - d_{\mu-1}}$$

$=$ subgroup gen. by s_i such that $\alpha_{\underline{d}} s_i = \alpha_{\underline{d}}$, $\alpha_{\underline{d}}$ as above. $W/W_{\underline{d}}$ can be identified with the set of $\alpha : \{1, \ldots, n\} \twoheadrightarrow \{1, \ldots, \mu\}$ such that $\operatorname{card}\{p \mid \alpha(p) \le j\} = d_j$.
So I understand the Bruhat decomposition for

$G/P_{\underline{d}}$:

$$G = \coprod_{\alpha \in W/W_{\underline{d}}} B\alpha P_{\underline{d}}$$

(Recall that $\sigma \in W$ is interpreted as the matrix such that $\sigma(e_p) = e_{\sigma(p)}$, whence

$$\sigma\Big(\sum_{p \le d_j} k e_p\Big)_{1\le j\le\mu} = \sigma\Big(\sum_{\alpha_{\underline{d}}(p)\le j} k e_p\Big)_{1\le j\le\mu}$$

$$= \Big(\sum_{\alpha_{\underline{d}}(p)\le j} k e_{\sigma(p)}\Big)_{1\le j\le\mu}$$

$$= \Big(\sum_{(\alpha_{\underline{d}}\sigma^{-1})(p)\le j} k e_p\Big)_{1\le j\le\mu}.$$

In other words applying σ to the basic T-inv. flag $F_{\underline{d}}$ gives the T-inv. flag corresponding to $\alpha_{\underline{d}}\sigma^{-1}$. Therefore the identification of W/W_d with the set of α proceeds by making σ act on α by $\sigma\cdot\alpha = \alpha\cdot\sigma^{-1}$.)

Observe that if s is a simple root, then BsB contains B in its closure. Hence if $\ell(s\sigma) = 1 + \ell(\sigma)$, which is equivalent to $Bs\sigma B = BsB\cdot B\sigma B$, then $\overline{Bs\sigma B}$ contains $B\sigma B$.

More generally B is contained in the closure of every cell $B\sigma B$, hence one sees that

$$\ell(\sigma)+\ell(\tau)=\ell(\sigma\tau) \Rightarrow B\sigma B \subset \overline{B\sigma\tau B}.$$

If $\sigma \in W$, then $\sigma P_{\underline{d}}$ is the flag

$$(\sigma P_{\underline{d}})_j = \sigma (P_{\underline{d}})_j = \sigma \sum_{\alpha_{\underline{d}}(p)\leq j} k e_p$$

$$= \sum_{\alpha_{\underline{d}}(p)\leq j} k e_{\sigma(p)}$$

$$= \sum_{(\alpha_{\underline{d}}\sigma^{-1})(p)\leq j} k e_p$$

Now recall we saw that any $\alpha : \{1,\dots,n\} \twoheadrightarrow \{1,\dots,\mu\}$ factored uniquely

$$\begin{tikzcd}
\{1,\dots,n\} \arrow[r, "\sigma"] \arrow[dr, "\alpha"'] & \{1,\dots,n\} \arrow[d, "\alpha_{\underline{d}}"] \\
& \{1,\dots,\mu\}
\end{tikzcd} \quad \text{monotone}$$

where $\ell(\sigma)=\ell(\alpha)$ (meaning σ preserves order on fibres over each j, $1\leq j\leq \mu$). Thus if I convert α to a permutation σ, the B orbit indexed by α is $B\sigma^{-1}P_{\underline{d}}$. In particular a flag in $B\sigma B \subset G/B$ is indexed by $\sigma^{-1} : \{1,\dots,n\} \to \{1,\dots,n\}$.

Formula: $$G = \coprod_{\sigma W_{\underline{d}} \in W/W_{\underline{d}}} B\sigma P_{\underline{d}}$$

where $B\sigma P_{\underline{d}}$ is indexed by $\alpha_{\underline{d}}\sigma^{-1}$.

Now suppose we consider a product $B\tau B \cdot B\sigma P_{\underline{d}}$. This is a quotient of

$$B\tau B \times^{B} B\sigma P_{\underline{d}}$$

which has $\dim = l(\tau) + l(\alpha_{\underline{d}}\sigma^{-1}) + \dim P_{\underline{d}}$

Thus

$$l(\alpha_{\underline{d}}\sigma^{-1}\tau^{-1}) = \dim(B\tau\sigma P_{\underline{d}}) - \dim P_{\underline{d}}$$

$$\leq l(\tau) + l(\alpha_{\underline{d}}\sigma^{-1})$$

In particular

$$l(\tau) + l(\alpha_{\underline{d}}\sigma^{-1}) = l(\alpha_{\underline{d}}\sigma^{-1}\tau^{-1})$$

$$\Rightarrow \quad B\tau B \cdot B\sigma P_{\underline{d}} = B\tau\sigma P_{\underline{d}}$$

~~To establish the converse it suffices to do so when τ is one of the s_i. Use induction on $l(\tau)$, Assuming true if $l(\tau) = 1$. If $l(\tau) + l(\alpha_{\underline{d}}\sigma^{-1}) > l(\alpha_{\underline{d}}\sigma^{-1}\tau^{-1})$ write $\tau = s\tau'$ with $l(\tau) = 1 + l(\tau')$, whence~~

Simpler proof: Can suppose σ such that $l(\sigma) = l(\alpha_d \sigma^{-1})$.
Then $l(\tau) + l(\sigma) = l(\alpha_d \sigma^{-1}\tau^{-1}) \leq l(\tau\sigma) \implies B\tau B \cdot B\sigma B = B\tau\sigma B$
hence $B\tau B \cdot B\sigma P_d = B\tau\sigma P_d$.

~~$B\tau B = BsB \cdot B\tau B$. Assuming $B\tau B \cdot B\sigma P_d = B\tau\sigma P_d$, it follows the same must hold~~

Converse isn't true.

Given $\alpha : \{1, .., n\} \to \{1, .., \mu\}$, let $s = s_i =$ the transposition interchanging i and $i+1$.

$$l(\alpha s) = \text{card}\,\{ a < p \mid \alpha(sa) > \alpha(sp)\}$$

Now $\{(a,p) \mid a < p\}$ and $\{(a,p) \mid sa < sp\}$ are the same except that $(i, i+1)$ has been replaced by $(i+1, i)$. Thus

$$l(\alpha s) - l(\alpha) = \text{card}\,\{(a,p) \mid sa < sp,\ \alpha a > \alpha p\}$$
$$- \text{card}\,\{(a,p) \mid a < p,\ \alpha a > \alpha p\}$$

$$= \begin{cases} 1 & \alpha(i+1) > \alpha(i) \\ 0 & \alpha(i+1) \leq \alpha(i) \end{cases} - \begin{cases} 1 & \alpha(i) > \alpha(i+1) \\ 0 & \alpha(i) \leq \alpha(i+1) \end{cases}$$

$$= \begin{cases} 1 & \alpha(i) < \alpha(i+1) \\ 0 & \alpha(i) = \alpha(i+1) \quad \text{whence} \implies \alpha s = \alpha. \\ -1 & \alpha(i) > \alpha(i+1) \end{cases}$$

So one can have $BsB \cdot C_\alpha = C_{\alpha s}$ where $\alpha s = \alpha$, even though $\ell(s) + \ell(\alpha) = 1 + \ell(\alpha) \neq \ell(\alpha s) = \ell(\alpha)$.

I would like to know when $C_\beta \subset \overline{C}_\alpha$. Recall that B is in the closure of any cell $B\tau B$, hence C_β is in the closure of $B\tau B \cdot C_\beta = C_{\beta\tau^{-1}}$ if $\ell(\tau) + \ell(\beta) = \ell(\beta\tau^{-1})$.

Hence I see that $C_\beta \subset \overline{C}_\alpha$ when $\alpha = \beta\tau^{-1}$ where $\ell(\alpha) = \ell(\beta) + \ell(\tau)$. I conjecture this condition is also necessary.

Compute $H_\alpha = \{g \in G \mid g C_\alpha = C_\alpha\}$. This subgroup contains B, hence it should be generated by B and those reflections s which it contains. Now $sC_\alpha = C_{\alpha s}$ and $\dim(C_{\alpha s}) = \ell(\alpha s)$. If $C_{\alpha s} = C_\alpha$, then we have $\ell(\alpha s) = \ell(\alpha)$, which by a preceding calculation shows that $\alpha s = \alpha$. Thus H_α is generated by B and those transp. s_i such that $\alpha(i) = \alpha(i+1)$.

Example: Take $\underline{d} = (1, n)$, whence $D_{\underline{d}}(V) = \mathbb{P}V$. $\alpha : \{1, \cdots, n\} \longrightarrow \{1, 2\}$ is given by $\alpha^{-1}(1) = \{k\}$ $1 \leq k \leq n$. The corresponding Schubert cell is $\mathbb{P}V_k - \mathbb{P}V_{k-1}$

and its stabilizer is the parabolic subgroup fixing the flag $0 \leq V_{k-1} < V_k \leq V$, which is indeed generated by ~~...~~ B and the transpositions $s_1, \ldots, s_{k-2}, s_{k+1}, \ldots, s_{n-1}$.

<u>Assertion:</u> Let $Y_{\underline{d}}$ = flag manifold of type $\underline{d}$, let C be a Schubert cell in $Y_{\underline{d}}$, let $P = \{g \mid gC = C\}$, ~~...~~ and let B' be any Borel subgroup of P. Then B' acts transitively on C.

<u>Proof:</u> We know C is an orbit of $Y_{\underline{d}}$ for some Borel subgroup B of P. Let T be a maximal torus contained in $B \cap B'$. We know there is a unique point of C fixed under T, because C is a B-orbit. C ~~...~~ is a union of B'-orbits, each having a unique T-fixpt. Hence C must be a single B'-orbit.

Problem: To understand the poset of Schubert cells fixed by a given maximal torus T.

The maximal torus T is equivalent to a set of axes, indep. lines spanning V; An element (F_j) of $Y_{\underline{d}}$ fixed under T may be identified with a map $\alpha : \Phi \to \{1,\dots,\mu\}$ such that card $\{p \in \Phi \mid \alpha(p) \leq j\} = d_j$. To give a Borel subgroup of G containing T^d is the same as giving a linear or total ordering of Φ, i.e. an isomorphism $\{1,\dots,n\} \xrightarrow{\sim} \Phi$.

$\alpha : \Phi \longrightarrow \{1,\dots,\mu\}$ being fixed, we can now try to classify the different orbits BF_α as B runs through the different Borels containing T.

$$\dim(BF_\alpha) = \mathrm{card}\left\{ (a,p) \;\middle|\; \begin{array}{l} a<p \quad \text{wrt } B \\ \alpha(a) > \alpha(p) \end{array} \right\}.$$

This will be zero provided the B-ordering on Φ refines the α-ordering.

Suppose we choose B so that F_α is a fixpt for B. Then we can identify Φ with $\{1,\dots,n\}$ and $\alpha : \{1,\dots,n\} \longrightarrow \{1,\dots,\mu\}$ is monotone, i.e. $\alpha = \alpha_{\underline{d}}$. Also $Y_{\underline{d}} = G/P_{\underline{d}}$ where $P_{\underline{d}} = \{g \in GL_n \mid \alpha(i) > \alpha(j) \Rightarrow g_{ij} = 0\}$.

Consider the Borel subgroup $B' = \sigma B \sigma^{-1}$ preserving the flag $\sigma(V_j) = \left(\sum_{p \leq j} e_{\sigma(p)}\right) = \left(\sum_{\sigma^{-1}(p) \leq j} e_p\right)$. Then B' consists of matrices b such that

$$\sigma^{-1}(a) > \sigma^{-1}(p) \implies b_{ap} = 0$$

And [redacted] $B' \cap P_{\underline{d}} = \left\{ b \;\middle|\; \left\{ \begin{matrix} \sigma^{-1}(a) > \sigma^{-1}(p) \text{ or} \\ \alpha(a) > \alpha(p) \end{matrix} \right\} \Rightarrow b_{ap} = 0 \right\}$.

So $B'/B' \cap P_{\underline{d}}$ can be [redacted] represented by matrices (b_{ap}) [redacted] with non-zero entries at positions (a,p) $\sigma^{-1}(a) < \sigma^{-1}(p)$, $\alpha(a) > \alpha(p)$, and 1's on the diagonal.

Note that these representatives for the Schubert cell $B'F_\alpha$ form a subgroup of the unipotent N group with non-zero entries in the positions (a,p), $\alpha(a) > \alpha(p)$, and 1's on the diagonal.

N :