K-theory of a ring:

A ring with unit $\quad GL(A) = \bigcup GL_n(A)$ ~~[illegible]~~

GL_n embedded in GL_{n+1} via $\alpha \mapsto \alpha \oplus \varepsilon$. $\quad E(A) =$ subgp. gen. ~~[illegible]~~ by elem. matrices. Whitehead lemma ~~[illegible]~~ $E(A)$ is perfect, and it is the comm. subgp. of $GL(A)$. Following Bass put

$$(\ ) \qquad K_1 A = GL(A)/E(A) = H_1(GL(A))$$

Let $St(A)$ = Steinberg group of A; according to (Milnor ---) it is the universal covering group of $E(A)$. Following Milnor put

$$(\ ) \qquad K_2 A = \operatorname{Ker}\{St(A) \to E(A)\} = H_2(E(A)).$$

To obtain a gen. consider the Dror tower ~~[illegible]~~ of $BGL(A)$:

$$\cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow X_1 = BGL(A)$$

Because $E(A)$ is the largest perfect subgp of $GL(A)$, and $St(A)$ is its universal cover, one has from the discussion on the Dror tower

$$X_2 = BE(A)$$
$$X_3 = BSt(A)$$

$$\pi_1 BGL(A)^+ = H_1 X_1 = H_1(GL(A))$$
$$\pi_2 BGL(A)^+ = H_2 X_2 = H_2(E(A)).$$

Therefore we can extend the Bass-Milnor definitions by putting

Definition: $\quad K_i A = \pi_i BGL(A)^+ \qquad i \geq 1.$

Note that

$$(\ ) \qquad K_3 A = H_3 X_3 = H_3(St(A)).$$

$$\pi_1\, BGL(A)^+ = H_1 X_1 = H_1(GL(A))$$

$$\pi_2\, BGL(A)^+ = H_2 X_2 = H_2(E(A))$$

$$\pi_3\, BGL(A)^+ = H_3 X_3 = H_3(St(A))$$

In view of ( ), ( ) this shows that

<u>Def:</u> $K_i A = \pi_i\left(BGL(A)^+\right) \qquad i \geq 1$

is a reasonable definition.

<u>Proposition:</u> $BGL(A \times B)^+ \cong BGL(A)^+ \times BGL(B)^+$, hence

$$K_i(A \times B) = K_i A \oplus K_i B$$

<u>Proof:</u> $BGL(A \times B) = B\left(GL(A) \times GL(B)\right)$

$$= BGL(A) \times BGL(B)$$

Now use that $(X \times Y)^+ \Rightarrow X^+ \times Y^+$.

<u>**Prop:**</u> $\cancel{K_i(A \times B) = K_i A \oplus}$

$BGL(A \times B)^+$ is heq to $BGL(A)^+ \times BGL(B)^+$, hence

$K_i(A \times B) \cong K_i A \oplus K_i B$

<u>Proof:</u> One has heq:

$$BGL(A \times B) = B\left(GL(A) \times GL(B)\right) \longrightarrow BGL(A) \times BGL(B)$$

and also $(X \times Y)^+ \simeq X^+ \times Y^+$.

K-theory of a ring.

Whitehead lemma: $(GL(A), GL(A))$ is perfect.

Proof: Have $\Sigma_n \subset GL_n(A)$ as permutation matrices (provided $A \neq 0$). The alternating group $\mathfrak{A}_n$ is perfect for $n \geq 5$. Let N be the normal subgroup of $GL(A)$ generated by $\mathfrak{A}_5$. Then N contains the subgroups $\mathfrak{A}_n \subset GL_n(A) \subset GL(A)$ for every n, and N is perfect, so that it only remains to show that N contains the commutator subgroup of $GL(A)$. So suppose $\alpha, \beta \in GL(A)$ and let n be $\ni$ $\alpha, \beta \in GL_n(A)$, ~~and such that n is even. Let σ, τ ∈~~ $n \geq 2$.

Let $\sigma, \tau \in \mathfrak{A}_{3n}$ be such that

$$\sigma^{-1}(i+n) = i \qquad i = 1, \ldots, n$$

$$\tau^{-1}(i+2n) = i \qquad ''$$

Then $\alpha\sigma\alpha^{-1}\sigma^{-1} \in N$, $\beta\tau\beta^{-1}\tau^{-1} \in N$ and

$$(\alpha, \beta) = (\alpha\sigma\alpha^{-1}\sigma^{-1}, \beta\tau\beta^{-1}\tau^{-1})$$

Proof 2: $\alpha, \beta \in GL_n(A)$, $\sigma \in \mathfrak{A}_{2n}$ $\ni$, $\sigma^{-1}(i+n) = i$, $1 \leq i \leq n$. Then working modulo normal subgroup of $GL_{2n}(A)$ generated by $\mathfrak{A}_{2n}$ we have $\alpha \cdot \beta \equiv \alpha \cdot \sigma\beta\sigma^{-1} = \sigma\beta\sigma^{-1} \cdot \alpha \equiv \beta \cdot \alpha$ showing that $(\alpha, \beta) \equiv e$.

$\boxed{(GL_n, GL_n) \subset \text{normal subgrp in } GL_{2n} \text{ containing } \sigma}$

Universal Property:

~~Have seen that $E(A)$ is the normal subgp of $GL(A)$~~

~~Need more rigid picture of $BGL(A)^+$. Will take $BG = |\text{Nerv}(G)|$. If then G is the inductive limit of a filtered inductive system of gr~~

Need more rigid construction of $BGL(A)^+$. Will take $BG = |\text{Nerv}\, G|$. Then if G_i is a filtered ind. sys. of groups

$$B(\varinjlim G_i) = \varinjlim BG_i .$$

~~From proof of the Whitehead lemma we know $E(A)$ is normally gen. by $\mathfrak{A}_5 \subset GL_5(A) \cap E(A) \subset GL(A)$. Thus if we let F_0 = 2-skeleton of $B\mathfrak{A}_5$, F_0 is a finite complex and~~

Let F_0 = 2 skeleton of $B\mathfrak{A}_5$. Since $\mathfrak{A}_5 = \pi_1(F_0)$ is gen. as a normal subgp by 1 element we can attach one 2-cell and one 3-cell to F_0 to obtain an acyclic map $F_0 \subset F_0'$ of finite complexes with $\pi_1(F_0') = 0$. View $\mathfrak{A}_5$ as a subgp of $GL_5(A) \subset GL(A)$ in the obvious way, so that $B\mathfrak{A}_5$ is a subcomplex of $BGL(A)$. Then the inclusion

$$BGL(A) \subset BGL(A) \cup_{F_0} F_0'$$

is acyclic (prev. prop.), and π_1 of latter space is $GL(A)/E(A)$ because $E(A)$ = normal subgp of $GL(A)$ gen. by $\mathfrak{A}_5$. Thus we can take specific CW complex of the h-type of $BGL(A)^+$

$$BGL(A)^+ = BGL(A) \cup_{F_0} F_0'$$

whence it is clear that $A \mapsto BGL(A)^+$ is functorial in the ring A and

$$\boxed{BGL(\varinjlim A_i)^+ = \varinjlim BGL(A_i)^+}$$

X finite complex.

Prop: $\varinjlim [X, BGL(A_i)^+] \Rightarrow [X, BGL(\varinjlim A_i)^+]$.

Proof. Any finite subcomplex of $BGL(\varinjlim A_i)^+$ lifts to $BGL(A_i)^+$ for some i, and the lifting is unique up to enlarging the index i.

From now on let X range over fin. complexes.

Proposition: Let Z be a space $\ni$ $\pi_1 Z$ has no perf. subgps $\neq e$. Then any nat. transf. $\theta : [X, BGL(A)] \to [X, Z]$ factors uniquely ~~Nat transf~~ $\cancel{([\ , BGL(A)^+], [\ , Z])}$ thru the canonical map $[X, BGL(A)] \longrightarrow [X, BGL(A)^+]$.

Proof. If F_α runs over the finite subcomplexes of $BGL(A)$ containing F_0, then we have

$$BGL(A)^+ = \bigcup (F_\alpha \cup_{F_0} F_0')$$

so for X finite we have

$$[X, BGL(A)^+] = \varinjlim [X, F_\alpha \cup_{F_0} F_0'].$$

So

$$\text{Nat transf}([\ , BGL(A)^+], [\ , Z])$$

$$= \varprojlim_\alpha [F_\alpha \cup_{F_0} F_0', Z]$$

by Yoneda's lemma. and similarly $\text{Nat transf}([\ , BGL(A)], [\ , Z]) = \varprojlim [F_\alpha, Z]$.

Because $\pi_1 Z$ has no perf. subgps $\neq 1$ and $F_\alpha \subset F_\alpha \cup_{F_0} F_0'$ is acyclic, we have

$$[F_\alpha \cup_{F_0} F', Z] \simeq [F_\alpha, Z],$$

so the prop. is clear.

Recall: X pointed and connected. Claim $\pi_1(X)$ acts on X considered as an object of the homotopy category of ptd. conn. spaces. To see this cons. fibration

$$Map(Z, z_0; X, x_0) \longrightarrow Map(Z, X) \longrightarrow Map(z_0, X)$$

Then for this fibration $\pi_1(\text{base}) = \pi_1 X$ acts on $\pi_0(\text{fibre}) = [Z, X]$, and

$$[Z,X]/\pi_1 X = [Z \cup pt, X] = \text{hom. classes of maps } Z \to X \text{ not nec. pres. basept.}$$

Thus $\pi_1(X)$ acts on the functor $Z \mapsto [Z, X]$, hence $\pi_1 X$ acts on X.

Alternative versions:

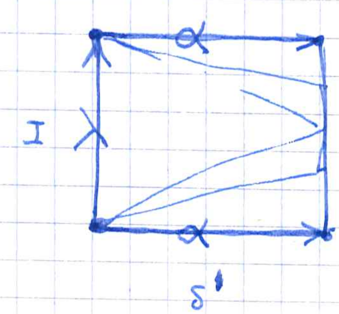
apply CHT: Given a ~~path~~ loop $\lambda : I \longrightarrow X$ starting at x_0

$$\begin{tikzcd} X \times 0 \cup x_0 \times I \arrow[r] \arrow[d] & X \\ X \times I \arrow[ur, dotted] & \end{tikzcd}$$

and you get a homotopy $H : X \times I \longrightarrow X$ such that $H_t(x_0) = \lambda(t)$. Then $H_1 : X \longrightarrow X$ is a basept-preserving self-map of X to itself, and

Assertion: With this action $\pi_1 X$ acts on $\pi_1 X$ by conjugation: Let $\alpha \in \pi_1(X)$ $\lambda \in \pi_1 X$

$$\begin{tikzcd} S^1 \times 0 \cup * \times I \arrow[r, "\alpha + \lambda"] \arrow[d, phantom, "\cap"] & X \\ S^1 \times I \arrow[ur, dotted] & \end{tikzcd}$$



so H_1 here is clearly $\alpha \lambda \alpha^{-1}$. (convention is paths comp. like morphisms)

So now if N is a perf. normal subgp. of $\pi_1 X$ corresp to $f: X \longrightarrow Y$, then by functoriality it is clear that

$$\begin{tikzcd} X \arrow[d, "\lambda"'] \arrow[dr, "f"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes for each $\lambda \in N$. Conversely given $X \xrightarrow{g} Z$ $\ni$ $g\lambda = g$ for all $\lambda \in N$, then $\pi_1(g)(\lambda\alpha\lambda^{-1}) = \pi_1(g)(\alpha)$ all $\alpha \in \pi_1 X$, $\lambda \in N$ $\Longrightarrow$ $\operatorname{Ker} \pi_1(g) \supset (\pi_1 X, N) \supset (N, N) = N$ $\Longrightarrow$ g factors thru f. Thus if $f: X \longrightarrow Y$ is acyclic, Y is the quotient of X by the action of N.

<u>**Assertion to be used:**</u> $\exists$ canon. action of $\pi_1(X)$ on X up to basepoint-preserving homotopy. Notation: $\lambda \in \pi_1 X$ put $\theta_\lambda : X \longrightarrow X$ for the induced element. If $f: X \longrightarrow Y$ is any map

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta_\lambda"'] & Y \arrow[d, "\theta_{f_*(\lambda)}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} {[Z,X]} \arrow[r] \arrow[d, "\theta_\lambda"] & {[Z,Y]} \arrow[d, "\theta_{f_*(\lambda)}"] \\ {[Z,X]} \arrow[r] & {[Z,Y]} \end{tikzcd}$$

commutes up to homotopy. Also if $\alpha \in \pi_1 X$, then $\theta_{\lambda *}(\alpha) = \lambda(\alpha)\lambda^{-1}$.

Granted this assertion one sees that if $\lambda \in \pi_1(BGL(A)) = GL(A)$, then

$$\begin{array}{ccc} [X, BGL(A)] & \longrightarrow & [X, BGL(A)^+] \\ \downarrow{\scriptstyle \theta_\lambda} & & \downarrow{\scriptstyle \theta_{\bar\lambda}} \\ [X, BGL(A)] & \longrightarrow & [X, BGL(A)^+] \end{array}$$

commutes, where $\bar\lambda$ = image of λ in $\pi_1 BGL(A)^+ = K_1 A$. In particular θ_λ is compatible with id if $\lambda \in E(A)$.

~~Now can prove:~~

~~Proposition: Let $u: \mathbb{N}-\{0\} \hookrightarrow \mathbb{N}-\{0\}$ and $u: GL(A) \to GL(A)$ be the induced map sending (α_{ij}) to $(\alpha_{u^{-1}(i), u^{-1}(j)})$. Then the map induced by u on~~

In the following, let M be the monoid of injective endos. of $\{1, 2, \dots\}$. If $u \in M$ it induces an endom. $\bar u: GL(A) \longrightarrow GL(A)$ $\bar u(\alpha)$ = matrix having entry α_{ij} in the $(u(i), u(j))$-th position, and zeros elsewhere. We are going to show that the self-map of $BGL(A)^+$ induced by $\bar u$ is the identity.

<u>Lemma 1</u>: <u>If</u> X <u>is a finite complex, then</u> $\bar u$ <u>induces the identity on</u> $[X, BGL(A)^+]$.

~~Proof. For any n $\exists$ $\sigma \in E(A)$ such that $\sigma \alpha \sigma^{-1} = \bar u(\alpha)$ for all $\alpha \in GL_n A$. This shows that if f_n is the restriction of the~~

~~canonical map $BGL(A) \to BGL(A)^+$~~

<u>Proof:</u> We know that if $\sigma \in E(A)$, then

$$\begin{tikzcd} BGL(A) \arrow[d, "\theta_\sigma"] \arrow[dr, "f"] & \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

is homot. commutative where θ_σ is the map induced by conjugation $\alpha \longmapsto \sigma\alpha\sigma^{-1}$ on $GL(A)$.

$$\begin{tikzcd} BGL(A) \arrow[r, "f"] \arrow[d, "B\bar{u}"] & BGL(A)^+ \arrow[d, "(B\bar{u})^+"] \\ BGL(A) \arrow[r, "f"] & BGL(A)^+ \end{tikzcd}$$

Now for each n, $\exists\ \sigma \in E(A)$ such that $\sigma\alpha\sigma^{-1} = \bar{u}(\alpha)$ for all ~~$\alpha \in GL_n A$.~~ $\alpha \in i_n GL_n(A) \subset GL(A)$. Thus

$$\begin{aligned} (B\bar{u})^+ f B i_n &= f\, Bu\, Bi_n \\ &= f\, \theta_\sigma\, Bi_n \\ &= f\, Bi_n \end{aligned}$$

showing that ~~$(B\bar{u})$~~

$$\begin{tikzcd} & {[X, BGL(A)^+]} \arrow[d, "(B\bar{u})_*^+"] \\ {[X, BGL_n A]} \arrow[ur, "fBi_n"] \arrow[r, "fBi_n"'] & {[X, BGL(A)^+]} \end{tikzcd}$$

commutes for every n. Passing to limit over $n \Longrightarrow$ $(B\bar{u})^+ f = f$ and this + universal property $\Longrightarrow$ $(B\bar{u})^+ = id$

<u>Lemma 2:</u> Any monoid homom. $\varphi : M \longrightarrow G$ where G is a group is trivial.

<u>Proof:</u> Let $v, w \in M$ have disjoint images. If

$u \in M$, let $v_*(u)$ denote the unique element of M such that $v_*(u)\,v = vu$ and $v_*(u) = id$ on the complement of the image of v. Then $v_*(u)\,w = w$

$\Rightarrow \varphi(v_* u) = e \Rightarrow \varphi(v)\varphi(u) = \varphi(v_*(u)v) = \varphi(v)$

$\Rightarrow \varphi(u) = e$ for all u.

~~Proposition~~

Lemma 3: If $u \in M$, then the map induced by u on $BGL(A)^+$ is the identity.

Proof: By lemma 1, if u^+ denotes the induced endom. of $BGL(A)^+$, then u^+ induces isos on the homotopy groups of $BGL(A)^+$, hence by Whitehead it is a heq. But it is clear that $u \mapsto u^+$ is a homo. from M to the group of heq of $BGL(A)^+$, so by lemma 2 $u_*^+ = id$.

Proposition: $BGL(A)^+$ is a homotopy associative + commutative H-space.

Proof: Choose an embedding $N' \sqcup N' \hookrightarrow N'$, i.e. two elements v, w of M with disjoint images. Then one gets homo

$$\begin{array}{ccc} GL \times GL & \longrightarrow & GL \\ \alpha \quad \beta & & \bar{u}_*(\alpha)\,\bar{v}_*(\alpha) \end{array}$$

hence

$$\begin{tikzcd} (BGL \times BGL)^+ \arrow[r] & BGL^+ \\ BGL^+ \times BGL^+ \arrow[u, "\wr\ heq"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

Rest clear. Restriction of μ to each factor is identity by lemma 3.

so we do get an H-space. On the other hand, it is clear also by the ~~lem~~ lemma 3 that the choice of the elements v, w is irrelevant, from which one easily gets ~~that~~ the h-assoc. + comm.

~~[illegible]~~. Retake:

Theorem: $BGL(A)^+$ is a homotopy assoc + comm. H-space.

Proof: