## Notes on + construction:

Let $f: X \to Y$ be a map of cell complexes. Suppose $Y$ connected to simplify, let $y_0$ be a basepoint of $Y$. Let $F$ = hom. fibre of $f$ over $y_0$ = space of pairs $(x, \lambda)$ $\lambda$ a path joining $f(x)$ to $y_0$. Let $\tilde{Y}$ be the universal covering of $Y$.

Proposition: TFAE

(i) $F$ acyclic (i.e. $\tilde{H}_*(F, \mathbb{Z}) = 0$, understood: $H_0(F, \mathbb{Z}) = \mathbb{Z}$)

(ii) $\forall$ local system $L$ on $Y$ we have $H_*(X, f^*L) \simeq H_*(Y, L)$

(iii) $X \times_Y \tilde{Y} \to \tilde{Y}$ induces isos. on integral homology.

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Pf: (i) $\Rightarrow$ (ii). Consider Leray spectral sequence

$$E^2_{pq} = H_p(Y, H_q(F, E)) \Longrightarrow H_{p+q}(X, E)$$

for $E$ any local system on $X$. Take $E = f^*L$, whence $f^*L$ is trivial on $F$, so $H_*(F, f^*L) = L$ (use univ. coeffs.) Thus spec. seq. degenerates yielding (ii).

(ii) $\Rightarrow$ (iii). Have

$$\begin{tikzcd} X \times_Y \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where vertical maps are principal coverings group $\pi_1 Y$. Thus have

$$\begin{array}{ccc} H_*(X \times_Y \tilde{Y}, \mathbb{Z}) & \longrightarrow & H_*(\tilde{Y}, \mathbb{Z}) \\ \| & & \| \\ H_*(X, \mathbb{Z}[\pi_1 Y]) & \longrightarrow & H_*(Y, \mathbb{Z}[\pi_1 Y]) \end{array}$$

so clear.

(iii) $\Rightarrow$ (i). Since homot. fibre doesn't change under pulling back via $\tilde{Y} \to Y$, we can suppose $Y$ simply-connected. Now stare at the spectral seq. $E^2_{pq} = H_p(Y, H_q(F, \mathbb{Z})) \to H_{p+q}(X, \mathbb{Z})$.

Def: Such a map will be called acyclic.

Cor 1: Acyclic maps stable under composition (use ii) homotopy base change, (use (i)) + homotopy cobase change.

Proof: For last suppose have cocart:

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d] & Y \arrow[d, "g"] \\ X' \arrow[r, hook, "f'"] & Y' \end{tikzcd}$$

with $f$ acyclic + a cofibration. Then (ii) $\Rightarrow$ $H_*(Y, X; L) = 0$ for all local systems on $Y$. Since $H_*(Y', X'; L) \xleftarrow{\sim} H_*(Y, X; g^*L)$ (consider cellular chains), one gets $f'$ is acyclic.

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Cor. 2: If $f$ is acyclic, then $\pi_1(f) : \pi_1 X \to \pi_1 Y$ is onto and its kernel is perfect. $f$ is a heq $\Leftrightarrow$ $\pi_1(f)$ is an isomorphism.

Proof: $\pi_0(F) = 0 \Rightarrow \pi_1(f)$ onto. $F$ acyclic $\Rightarrow H_1(F, \mathbb{Z}) = \pi_1 F^{ab} = 0$ $\Rightarrow \pi_1(F)$ perfect $\Rightarrow \operatorname{Ker} \pi_1(f)$ perfect. If $\pi_1(f)$ is an isom, then $X \times_Y \tilde{Y} = \tilde{X}$, so by Whitehead $\Rightarrow \tilde{X} \to \tilde{Y}$ is a heq $\Rightarrow$ $\pi_q(X) \xrightarrow{\sim} \pi_q(Y)$ all $q \geq 2$. Thus $f$ is a heq.

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~~Proposition: Let $X \to Y$ be acyclic~~

~~From now on to simplify, suppose~~

From now on, work with connected ptd CW cxs. and put $[X, Y]$ for ptd. homot. classes.

Proposition: Let $f : X \to Y$ be acyclic. Then

$$f^* : [Y, Z] \longrightarrow \{ \underset{\substack{\uparrow \\ \text{class}}}{cl(g)} \in [X, Z] \mid \operatorname{Ker} \pi_1(f) \subset \operatorname{Ker} \pi_1(g) \}$$

Proof: Surjectivity: Can suppose $f$ is a cofibration. Given $g: X \to Z$ $\ni$ $\text{Ker}\, \pi_1(f) \subset \text{Ker}\, \pi_1(g)$, form pushout

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "f'"] \\ Y \arrow[r, "g'"'] & Y \cup_X Z \end{tikzcd}$$

By van Kampen $\pi_1(Y \cup_X Z) = \pi_1 Y *_{\pi_1 X} \pi_1 Z \xleftarrow{\sim} \pi_1 Z$. But $f$ acyclic $\Rightarrow$ $f'$ acyclic. $\therefore$ $f'$ heq and so $g$ factors thru $f$.

Injectivity. Assume we have $g_1, g_2 : Y \to Z$ $\ni g_1 f \sim g_2 f$. By HET can homotop $g_2$ until $g_1 f = g_2 f$; call this map $g$, and form $Y \cup_X Z$ as before. Then $g_1, g_2$ induce maps $h_1, h_2 : Y \cup_X Z$ $\ni$ $g_i = h_i g'$ and $h_i f' = id_Z$. But $f'$ is a heq and $h_i f' = id \Rightarrow h_1 \sim h_2 \Rightarrow g_1 \sim g_2$

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~~Prop: Given N perfect $\triangleleft$ $\pi_1(X)$, $\exists$ acyclic $f: X \to Y$ $\ni$ Ker $\pi_1(f) = N$. Moreover $f$ is unique up to homotopy in the sense that given another $g: X \to Z$, then $\exists$ heq $Y \to$~~

Cor: Let $f: X \to Y$ and $f': X \to Y'$ are acyclic with $\text{Ker}\, \pi_1(f) = \text{Ker}\, \pi_1(f')$, then $\exists$ heq $h: Y \to Y'$ $\ni$ $hf = f'$.

This is clear (Y and Y' both represent the same functor).

Prop. Given $N$ perfect $\triangleleft$ $\pi_1(X)$, $\exists$ $f: X \to Y$ acyclic with $\text{Ker}\, \pi_1(f) = N$.

Proof: First suppose $\pi_1(X)$ is perfect. Then choose element $\alpha_i \in \pi_1(X)$ which normally generate $\pi_1(X)$ and let $X'$ be the result of attaching 2 cells to kill the $\alpha_i$. Then $X'$ is simply-connected by van Kampen and

$$\bigvee_{i\in I} S^1 \xrightarrow{\alpha_i} X \longrightarrow X' \longrightarrow \bigvee_{i\in I} S^2$$

$$\underset{i\geq 3}{H_i X \simeq H_i X'} \qquad 0 \longrightarrow H_2(X) \longrightarrow H_2(X') \longrightarrow \bigoplus_{i\in I} \mathbb{Z} \longrightarrow \underset{\substack{\parallel \\ 0}}{H_1(X)}$$

Thus $H_2(X', X)$ is free with base $e_i$, $i \in I$. Since $\pi_2 X' \underset{\substack{\uparrow \\ \text{Hurewicz}}}{=} H_2(X') \twoheadrightarrow H_2(X', X)$ is onto, we can find $\beta_i \in \pi_2 X'$ such that $\beta_i$ goes to $e_i$. Then define $Y$ by attaching 3-cells

$$\bigvee S^2 \xrightarrow{\beta_i} X' \longrightarrow Y$$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & H_2(X) \arrow[d] & & \\
{} \arrow[r] & H_3(Y) \arrow[r] & \bigoplus_{i\in I} \mathbb{Z} \arrow[r] \arrow[dr, "\sim"'] & H_2(X') \arrow[r] \arrow[d] & H_2(Y) \arrow[r] & 0 \\
 & & & H_2(X,X') \arrow[d] & & \\
 & & & 0 & &
\end{tikzcd}$$

so $H_2(X) \simeq H_2(Y)$. It follows then that $H_n(X) \xrightarrow{\sim} H_n(Y)$ for all $n$. This proves the prop. when $N = \pi_1(X)$.

Now in the general case, let $X'$ be the covering space of $X$ with $\pi_1 X' = N$, let $X' \xrightarrow{f'} Y'$ be acyclic with $\pi_1 Y' = 0$, and form the pushout

$$\begin{tikzcd}
X' \arrow[r, hook, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, hook, "f"] & Y
\end{tikzcd}$$

Then $f'$ acyclic $\Rightarrow$ $f$ acyclic. Also Van Kampen $\Rightarrow$ $\pi_1(Y) = \pi_1(X) *_N e = \pi_1(X)/N$. done.

<u>Remark:</u> Clear from the proof that if $N$ is normally generated by a finite number of elements, it is necessary to attach only a finite number of 2 & 3 cells to get $Y$. Actually if $N$ has a finite no. of gen. as a normal subgroup of $G$, this remains true. One has to go back to the proof, but use cellular chains on the universal covering. (Thus set $\tilde{X}$ = covering corresp. to $N$. Then $0 \to H_2(\tilde{X}) \to H_2(\tilde{X}') \to \bigoplus_I \mathbb{Z}[\pi_1 X/N] \to 0$ so again can find $\beta_i \in \pi_2(\tilde{X}')$ mapping onto a basis for $H_2(\tilde{X}, \tilde{X}')$. etc.)

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Let $N$ be the largest perfect subgroup of $\pi_1 X$. (A group gen. by perfect subgps. is perfect - consider a homo. to any abelian group.) The acyclic map in this case will be denoted $X \to X^+$. It is universal for maps of $X$ to spaces having no perfect subgps. $\neq e$.

Formula:

$$(X \times Y)^+ = X^+ \times Y^+$$

because the product of two acyclic maps is acyclic

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Deligne's question: Given a fibration, can the plus construction be performed fibrewise.

~~Prop. Let $F \to E \to B$ be a fibration of connected pted. spaces~~

Let $F \to E \to B$ be a fibration (of conn. ptd. spaces as always) and suppose we have a map of fibrations

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, equal] \\ F' \arrow[r] & E' \arrow[r] & B \end{tikzcd}$$

with $F \to F'$ acyclic. Then from homology spectral sequences, one can see $E \to E'$ is acyclic, hence it is determined by a perf. normal subgp $N$ of $\pi_1(E)$ which goes to $0$ in $B$. Now we know $\pi_2 B$ maps into the center of $\pi_1(F)$, hence $\pi_1(F)$ is a central extension of its image in $\pi_1(E)$, and one knows there is a unique perf. subgp. $M$ of $\pi_1(F)$ mapping onto $N$, namely the commutator subgp. of the inverse image of $N$.

$$\begin{tikzcd} & \pi_1 F \arrow[r] \arrow[d] & \pi_1 E \arrow[d] \arrow[dr] & \\ \pi_2 B \arrow[ur] \arrow[dr] & & & \pi_1 B \\ & \pi_1 F' \arrow[r] & \pi_1 E/N \arrow[ur] & \end{tikzcd}$$

Diagram chasing shows that $\mathrm{Ker}(\pi_1 F \to \pi_1 F')$ maps onto $N$. Hence $\mathrm{Ker}(\pi_1 F \to \pi_1 F') = M$.

Thus we see that we can kill any perfect normal subgroup $M$ of $\pi_1 F$ whose image in $\pi_1(E)$ is normal, or equiv. which is stable under the action of $\pi_1(B)$ on $\pi_1(F)$ mod. inner autos.

<u>Prop</u>: <u>Given</u> a fibration $F \to E \to B$ <u>and a perf. normal</u> $M \subset \pi_1 F$ <u>stable under the $\pi_1 B$-action</u>, <u>there exists a map of fibrations over $B$</u>

$$\begin{tikzcd} F^* \arrow[r] \arrow[d, "f"'] & E \arrow[r] \arrow[d, "g"'] & B \arrow[d, equal] \\ F' \arrow[r] & E' \arrow[r] & B \end{tikzcd}$$

where $f : F^* \to F'$ is acyclic with $\operatorname{Ker} \pi_1(f) = N$, and $g$ is acyclic with $\operatorname{Ker} \pi_1(g) =$ Image of $N$ in $\pi_1 E$.

(Clearer: Given $E \to B$ with fibre $F$, those acyclic maps $E \to E'$ over $B$ are classified by perf. normal subgroups $N$ of $\operatorname{Ker} \pi_1(E) \to \pi_1(B)$. If $F' =$ Fibre of $E'$ over $B$, then $F = F' \times_{E'} E$ and as we can suppose $E \to E'$ is a fibn, we have

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"'] & E \arrow[d, "g"] \\ F' \arrow[r] & E' \end{tikzcd}$$

is h-cart. $\therefore$ $f$ is acyclic and $\pi_1(F) \twoheadrightarrow \pi_1(F') \times_{\pi_1(E')} \pi_1(E)$, so $\operatorname{Ker} \pi_1(f) \twoheadrightarrow \operatorname{Ker} \pi_1(g)$. But as $\pi_1(F)$ is a central extension of $\operatorname{Ker}(\pi_1 E \to \pi_1 B)$, $\operatorname{Ker} \pi_1(f)$ is the unique perf. subgroup of $\pi_1(F)$ with image $N$. Conversely given $M$ perf. $\triangleleft$ in $\pi_1(F)$ stable under $\pi_1 B$ action, taking $N = \operatorname{Im} M$ in $\pi_1 E$, this process kills $N$ in the fibre.)

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# Dror approach:

(All spaces conn. fntd. CW cxs., [ ] = basept. classes)

We begin by recalling a standard construction in homotopy theory introduced in Serre's thesis.

Let $K(A,n)$ be an Eilenberg-Maclane space with $n \geq 2$. From the Hurewicz theorem we have the formulas.

$$H_i(K(A,n)) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & 0<i<n \\ A & i=n \\ 0 & i=n+1 \end{cases}$$

(last follows because $\pi_{n+1}K(A,n) \longrightarrow H_{n+1}K(A,n)$ is onto.)

Lemma 1: $\exists$ map $X \to K(H_nX, n)$ which induces the canonical isom. $\theta: H_nX \simeq H_n(K(H_nX,n))$. If $H_{n-1}X = 0$, this map is unique.

Proof. Start with U.C. formula

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}X, A) \longrightarrow H^n(X,A) \xrightarrow{\varphi} \mathrm{Hom}(H_nX, A) \longrightarrow 0$$

~~Applying this to $X = K(A,n)$ we get [illegible]~~ and

$$[X, K(A,n)] \xrightarrow{\sim} H^n(X,A) \qquad f \longmapsto f^*(u_n)$$

where $u_n \in H^n(K(A,n), n)$ is the unique class such that $\varphi(u_n)$ is the canon. isom. $\theta: H_n(K(A,n)) = A$. It follows that $\varphi$ is isomorphic to the map

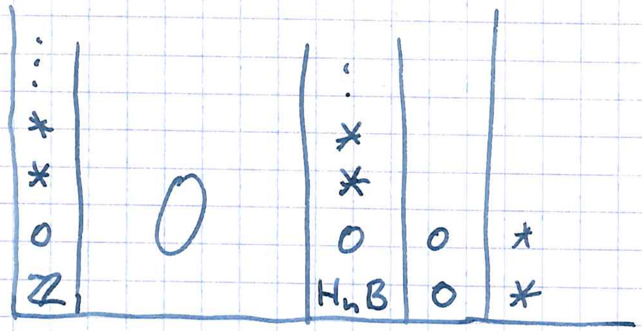
$$[X, K(A,n)] \longrightarrow \mathrm{Hom}(H_nX, A) \qquad f \longmapsto \theta H_n f$$

and so the latter is always surjective and bijective when $H_{n-1}X = 0$. Taking $A = H_nX$, the lemma follows.

Lemma 2: Assume $H_1(X) = H_{n-1}(X) = 0$, $n \geq 2$, and let $F$ be the homotopy-fibre of the map $v: X \longrightarrow K(H_nX, n)$ of Lemma 1. Then $H_i(F) \simeq H_i(X)$ for $i \leq n-1$, and $H_nF = 0$.

<u>Proof</u>: Put $B = K(H_n X, n)$ and consider the spec. seq

$$E^2_{pq} = H_p(B, H_q F) \Rightarrow H_{p+q} X$$



which gives $H_i F \xrightarrow{\sim} H_i X \quad i \leq n-2$ and an exact sequence

$$0 \leftarrow H_{n-1} X \leftarrow H_{n-1} F \leftarrow H_n B \leftarrow H_n X \leftarrow H_n F \leftarrow 0$$

(the last 0 results from the fact that all $E^2$ terms of total degree $n$ are zero except for $E^2_{0n} = H_n F$). Now using the fact that $H_{n-1} X = 0$, and that $H_n X \xrightarrow{\sim} H_n B$ the lemma follows.

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Now use this lemma as follows. Given a space $X$ such that $H_1 X = 0$ and an integer $n \geq 2$ such that $H_{n-1} X$ we construct recursively a tower of spaces

$$\longrightarrow X_{n+2} \longrightarrow X_{n+1} \longrightarrow X_n = X$$

such that

$$H_i(X_p) \xrightarrow{\sim} H_i(X) \quad \text{for } i < n$$
$$= 0 \quad \text{for } n-1 \leq i < p$$

by letting $X_{p+1}$ be the fibre of the canonical arrow $X_p \to K(H_p X_p, p)$ of Lemma 1. ~~Put $X' = \text{holim}(X_p)$. Since fibre of $X_\infty \to X_p$ becomes increasingly connected with $p$, we have~~

$$\cancel{H_i(X_\infty) \longrightarrow \begin{cases} H_i(X) & i < n \\ 0 & i \geq n-1 \end{cases}}$$

$$X_{p+1} \longrightarrow X_p \longrightarrow K(H_pX_p, p)$$

$$\Rightarrow \quad \pi_i(X_{p+1}) \longrightarrow \pi_i(X_p) \qquad \begin{array}{ll} \text{iso} & i<p-1 \\ \text{onto} & i=p-1 \end{array}$$

$$\Rightarrow \quad \pi_i(X_{p+k}) \longrightarrow \pi_i(X_p) \qquad \begin{array}{ll} \text{iso} & i<p-1 \\ \text{onto} & i=p-1 \end{array}$$

(meaning: fibre begins in dim $p-1$).

Put $X' = \operatorname{holim}_{\overleftarrow{p}} X_p$. Then because the system $\{\pi_i(X_p)\}$ is essentially constant, one has for each $p$

$$\pi_i(X') \longrightarrow \pi_i(X_p) \qquad \begin{array}{ll} \text{iso} & i<p-1 \\ \text{onto} & i=p-1 \end{array}$$

$$\Rightarrow \quad H_i(X') \longrightarrow H_i(X_p) \qquad \text{"}$$

So we have proved:

Proposition: Let $X$ be a space with $H_1X=0$ and $n$ an integer $\geq 2$ such that $H_{n-1}X=0$. Then there exists a map $X' \longrightarrow X$ such that

i) $\pi_i X' \longrightarrow \pi_i X$ $\begin{cases} \text{iso} & i<n-1 \\ \text{onto} & i=n-1 \end{cases}$

ii) $H_iX' = 0$ $\quad i \geq n-1$
$\quad\xrightarrow{\sim} H_iX$ $\quad i \leq n-1$

Examples: 1) If $\pi_1X=0$ and $n=2$, then the tower $X_p$ is just the Postnikov tower of $X$, so $X' \sim pt$.

2) $X = K(A,m)$, $m \geq 2$, $n = m+2$. ✓ Then $X'$ is a Moore space $M(A,m)$. $(H_{m+1}(K(A,m))=0)$

Now consider the general case $n=2$:

<u>Prop</u>: <u>Let $X$ be a space with $H_1 X = 0$</u> (i.e. $\pi_1 X$ <u>perfect</u>). <u>Then</u> <u>the map $X' \to X$ constructed above starting with</u> $n=2$ <u>is a universal map from an acyclic space to $X$</u>.

<u>Proof</u>: Clearly the space $X'$ is acyclic. If now $Y$ is an acyclic space, one has $[Y, X_{n+1}] \xrightarrow{\sim} [Y, X_n]$ as $[Y, K(A,p)] = 0$ for all $p \geq 1$, and $A$ abelian. Thus passing to the limit $[Y, X'] \xrightarrow{\sim} [Y, X]$ proving the assertion.

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<u>Cor</u>. $X' \to X$ <u>is the fibre over $X \to X^+$</u>.

<u>Proof</u>: If $F$ is this fibre, then we know it is acyclic (first prop.). Also for $Y$ acyclic we have $[Y, X^+] = 0$ $[Y, \Omega X^+] = 0$ (universal property for $Y \to pt$). Thus from

$$[Y, \Omega X^+] \longrightarrow [Y, F] \longrightarrow [Y, X] \longrightarrow [Y, X^+]$$

one concludes that $[Y, F] \xrightarrow{\sim} [Y, X]$. Thus $F \to X$ has the universal property of $X' \to X$.

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So now here is Dror method for proving the existence of an acyclic map $f: X \to Y$ killing a perf. $N \triangleleft \pi_1 X$. He constructs $\tilde{X}$ = covering corresponding to $N$ and then the universal acyclic space $A(\tilde{X}) \to \tilde{X}$ which has the property that its $\pi_1$ maps on $N$. Then he forms pushouts

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] & \tilde{X}/A(\tilde{X}) = \tilde{X}^+ \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$\boxed{\text{Summary}}$ Dror's version which gives some useful information

**Lemma 1:** $H_1 X = 0$, $H_{n-1} X = 0$.

i) $\exists !$ map $X \xrightarrow{u} K(H_n X, n)$ inducing canon. isom. $H_n X \xrightarrow{\sim} H_n K(H_n X, n)$.

ii) If $F$ is fibre of $u$, then

$\pi_i F \longrightarrow \pi_i X$ isom. $i \leqq n-1$, onto $i = n-1$

$$H_{n-1} F = H_n F = 0$$

Dror tower of a space $X \ni H_1 X = 0$. Let

$$\begin{array}{ccccccc} X_{p+1} \longrightarrow & X_p & \longrightarrow \cdots\cdots \longrightarrow & X_4 & \longrightarrow X_3 \longrightarrow X_2 = X \\ & \downarrow & & & \\ & K(H_p X_p, p) & & & \end{array}$$

Has to be generalized!!

Properties:

(i) $\tilde{H}_i(X_p) = 0 \quad i < p$

(ii) $[Y, X_p] \xrightarrow{\sim} [Y, X]$ if $\tilde{H}_i(Y) = 0 \quad i < p$.

(iii) $\pi_i(X_{p+r}) \longrightarrow \pi_i(X_p)$ isom $i < p-1$, onto $i = p-1$.

~~$p-1 \geq 2$ $H^{p-2}(Y, H_{p-1}X_{p-1}) \to [Y, X_p] \to [Y, X_{p-1}] \to H^{p-1}(Y, H_{p-1}X_{p-1})$~~

Put: $X_\infty = \varprojlim X_p$

(i) $\tilde{H}_i(X_\infty) = 0$

(ii) $[Y, X_\infty] \xrightarrow{\sim} [Y, X]$ if $\tilde{H}_*(Y) = 0$

Thus $X_\infty \longrightarrow X$ universal map from an acyclic space to $X$

Cor: $X_\infty =$ fibre of $X \longrightarrow X^+$.

<u>Example</u>: $X = BG$ $G$ perfect.

Claim $X_3 = B\tilde{G}$, $\tilde{G}$ - covering group of $G$.

<u>Proof</u>: From homot. sequence finds $\pi_q X_3 = \pi_q BG = 0$ $q \geq 2$ hence $X_3 = BG'$, $G' = \pi_1 X_3$. Also get central ext.

$$1 \to H_2 G \to \pi_1 X_3 \to G \to 1$$

Finally $H_1(G') = H_2(G') = 0 \Rightarrow G' = \tilde{G}$ by theory of Shur mult..

Next one sees inductively: since $\pi_q X_2 = 0$ $q \geq 2$ that for $p \geq 3$

$$\pi_i X_p = \begin{cases} G' & i = 1 \\ H_{i+1} X_{i+1} & 2 \leq i \leq p-2 \\ 0 & p-1 \leq i \end{cases}$$

$$\begin{array}{c} K(H_p X_p, p-1) \\ \downarrow \\ X_{p+1} \\ \downarrow \\ X_p \end{array} \quad \begin{array}{l} \text{h.gps} \\ \text{dim} \\ \leq p-2 \end{array}$$

so

$$\pi_i X_\infty = \begin{cases} G' & i = 1 \\ H_{i+1} X_{i+1} & i \geq 2 \end{cases}$$

Now look at fibration

$$X_\infty \to BG \to BG^+$$

and one gets

$$\boxed{\pi_q(BG^+) = H_q X_q}$$

In particular

$$\pi_2(BG^+) = H_2 G$$
$$\pi_3(BG^+) = H_3 \tilde{G}$$

<u>Proposition</u>: Let $G$ be a perfect group, and let $\{X_p\}$ be the Dror tower over $BG$.

(i) $X_3 = B\tilde{G}$ where $\tilde{G}$ is the universal covering group of $G$ in the sense of the Shur mult. theory.

(ii) $\pi_q(BG^+) = H_q(X_q)$.

(iii) $\pi_2(BG^+) = H_2(G)$, $\pi_3(BG^+) = H_3(\tilde{G})$.

Complements: Given $X$ ptd. + connected but $H_1X$ not necessarily zero, let $N$ be the max. perfect subgroup of $\pi_1 X$, and put $X_1 = X$, $X_2$ = covering of $X$ with $\pi_1 X_2 = N$. ~~This extends the~~ So get still

$X_\infty \longrightarrow X$ universal map from an acyclic space to $X$

$X_\infty = \text{Fibre}\{X \longrightarrow X^+\}$

$X^+ = \text{Cone}\{X_\infty \longrightarrow X\}$

Last formula gives another construction of $X^+$.

## Lecture:

I will begin by proving a basic result on infinite matrix groups which has many applications in algebraic K-theory.

Recall the following result: Let $H, G$ be the subgps

$$\begin{pmatrix} I_r & 0 \\ 0 & GL_n(R) \end{pmatrix}, \qquad \begin{pmatrix} I_r & M_{r,n}(R) \\ & GL_n(R) \end{pmatrix}$$

of $GL_{n+r}(R)$. Then $BH \longrightarrow BG$ is a homotopy equivalence. Indeed $BH$ is hom. equiv. to the assoc. fibre space over $BG$ with fibre $G/H$, and $G/H$ is contractible.

The analogue of this result in alg. K-theory goes as follows. Let $A$ be a ring (always supposed assoc. with 1) and let $GL_\infty(A) = \bigcup GL_n(A)$, $M_{r,\infty}(A) = \bigcup M_{r,n}(A)$ under the standard inclusions.

**Thm:** The inclusion $\begin{pmatrix} I_r & \\ & GL_\infty(A) \end{pmatrix} \subset \begin{pmatrix} I_r & M_{r,\infty}(A) \\ & GL_{r,\infty}(A) \end{pmatrix}$ induces isomorphisms on homology with coefficients in any abelian group $\Lambda$ equipped with trivial action.

(Improvement: If $\Lambda$ is an abelian gp, let $H_*(G,\Lambda)$ denotes the homol. of $G$ with coefficients in $\Lambda$ equipped with the trivial $G$-action.) Say that $H \to G$ induces isos. on homology with constant coefficients if $H_*(H,\Lambda) \xrightarrow{\sim} H_*(G,\Lambda)$ for every abel. gp. $\Lambda$. It is enough to check this for $\Lambda = \mathbb{Z}$, or also for each of the prime fields $\mathbb{Q}, \mathbb{F}_p$ $p$ prime).

Before begining the proof, recall that $H_*(GL_\infty(A),\Lambda)$ has a canonical ring structure when $\Lambda$ is a ring defined as follows. One starts with the homos.

$$GL_p(A)\times GL_q(A) \xrightarrow{\oplus} GL_{p+q}(A)$$

$$\alpha\oplus\beta = \begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}$$

which induce pairings

$$\mu_{pq} : H_*(GL_p(A),\Lambda)\otimes H_*(GL_q(A),\Lambda) \longrightarrow H_*(GL_{p+q}(A),\Lambda)$$

Example to show $n=\infty$ is necessary

$$H_*\left(\begin{pmatrix}1 & \\ & \mathbb{F}_p^*\end{pmatrix},\mathbb{F}_p\right) \neq H_*\left(\begin{pmatrix}1 & \mathbb{F}_p\\ & \mathbb{F}_p^*\end{pmatrix},\mathbb{F}_p\right)$$

trivial as finite order prime to $p$      non-zero in $\infty$ many degrees.

and since inner autos of a group act trivially in homology one gets that $\mu_{pq}$ is commutative (assuming $\Lambda$ commutative.) Precisely one has a comm. diag.

$$\begin{tikzcd}
H_*(G_p)\otimes H_*(G_q) \arrow[r] \arrow[d, "\tau"] & H_*(G_p\times G_q) \arrow[r, "\oplus_*"] \arrow[d, "T_*"] & H_*(G_{p+q}) \arrow[d, "id"] \\
H_*(G_q)\otimes H_*(G_p) \arrow[r] & H_*(G_q\times G_p) \arrow[r, "\oplus_*"] & H_*(G_{p+q})
\end{tikzcd}$$

where $\tau(x\otimes y) = (-1)^{d^\circ x\, d^\circ y}\, y\otimes x$, so $\mu_{pq}\tau = \mu_{qp}$.

The point is that because of this commutativity $\mu_{pq}$ is compatible with passing from $p$ to $p+1$, $q$ to $q+1$

$$\begin{tikzcd}
G_p\times G_q \arrow[r] \arrow[d] & G_{p+q} \arrow[d] \\
G_{p+1}\times G_q \arrow[r] & G_{p+q+1}
\end{tikzcd}
\qquad (\alpha\oplus\varepsilon)\oplus\beta \simeq (\alpha\oplus\beta)\oplus\varepsilon$$

Have assoc.

$$(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$$

and unity

$$\alpha \oplus id_0 = id_0 \oplus \alpha = \alpha$$

and commutativity up to conjugacy:

$$\alpha \oplus \beta \sim \beta \oplus \alpha$$

Thus if $\varepsilon = (1)$, we have

$$\begin{array}{ccc} G_p \times G_q & \longrightarrow & G_{p+q} \\ \downarrow & & \downarrow \\ G_{p+1} \times G_q & \longrightarrow & G_{p+q+1} \end{array} \qquad \begin{array}{ccc} (\alpha, \beta) & \longmapsto & \alpha \oplus \beta \\ \downarrow & & \downarrow \\ (\alpha \oplus \varepsilon, \beta) & & (\alpha \oplus \beta) \oplus \varepsilon \\ & \searrow & \wr \\ & & (\alpha \oplus \varepsilon) \oplus \beta \end{array}$$

so we get that $\{\mu_{p,q}\}$ are compatible with stabilization and define in the limit a map

$$\mu : H_*(G_\infty) \otimes H_*(G_\infty) \longrightarrow H_*(G_\infty)$$

$\mu$ has following properties:

i) assoc. $\quad \mu(\mu \otimes id) = \mu(id \otimes \mu)$

ii) unity $\quad \mu(1 \otimes x) = x$, where $1$ denotes the image of the basepoint in $H_0(G_\infty)$, or canonical generator

iii) commutativity $\quad \mu T = \mu. \qquad T(x \otimes y) = (-1)^{d^\circ x \, d^\circ y} y \otimes x$

---

Demonstration of thm. Put $G_n = \begin{pmatrix} I_r & M_{rn} \\ & GL_n \end{pmatrix}$ and define

$$G_p \times G_q \xrightarrow{\perp} G_{p+q}$$

$$\begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \perp \begin{pmatrix} 1 & v \\ & \beta \end{pmatrix} = \begin{pmatrix} 1 & u & v \\ & \alpha & \\ & & \beta \end{pmatrix}$$

This is assoc. + comm. up to conjugacy, so it induces a product on $H_*(G_\infty, \Lambda)$ as before.

$$GL_n \xrightarrow{i_n} G_n \xrightarrow{k_n} GL_n$$

$$\alpha \longmapsto \begin{pmatrix} 1 & \\ & \alpha \end{pmatrix}$$

$$\begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \longmapsto \alpha$$

compatible with $\oplus, \perp$ hence induce alg. homos.

$$H_*(GL_\infty) \xrightarrow{i_*} H_*(G_\infty) \xrightarrow{k_*} H_*(GL_\infty)$$

$$\ni \quad k_* i_* = id.$$

Thus have to show that $i_* k_* = id$. Let $\varphi_n = i_n k_n$

$$\varphi_n \begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} = \begin{pmatrix} 1 & \\ & \alpha \end{pmatrix}.$$

so that $\varphi_* = i_* k_*$.

Identity:

$$\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & u & u \\ & \alpha & \\ & & \alpha \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & u & \\ & \alpha & \\ & & \alpha \end{pmatrix}$$

$$\begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \perp \begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \sim \begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix} \perp \varphi_p \begin{pmatrix} 1 & u \\ & \alpha \end{pmatrix}$$

$$H_*(G_p) \xrightarrow{\Delta} H_*(G_p) \otimes H_*(G_p) \underset{id \otimes (\varphi_p)_*}{\overset{id}{\rightrightarrows}} H_*(G_p) \times H_*(G_p) \xrightarrow{\perp} H_*(G_{2p})$$

Reduce to case $\Lambda =$ field so that $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$ and such that

$$\Delta(x) = 1 \otimes x + \sum_i x_i' \otimes x_i'' \qquad \deg(x_i'') < \deg(x)$$

1 denoting image of basepoint in $H_0(X)$.

Can pass to limit + get

$$\mu \Delta = \mu (id \otimes \varphi_*) \Delta \quad \text{on } H_*(G_\infty)$$

So now can show $x = \varphi_*(x)$ for $x \in H_n(G_\infty)$ by ind. on n.

$$x + \sum_i x_i' \, \mu(x_i'') = \varphi(x) + \sum x_i' \, \varphi(x_i'')$$

$+$ ind. $\Longrightarrow$ $x = \varphi(x)$.

---

<u>Questions:</u> Would this work for $E(A) = \bigcup E_n(A)$?
Or better $SL_n(A)$? Permutative category is a bit tricky.
$V$, $\Lambda^{\dim(V)} V \underset{\omega}{\cong} A$. Define exact sequence to consist of a normal exact sequence

$$0 \to V' \to V \to V'' \to 0$$

such that

$$\begin{tikzcd}
\Lambda^{p+q} V \arrow[rr, equal] \arrow[dr, "\omega"', "\wr"] & & \Lambda^p V' \otimes \Lambda^q V'' \arrow[dl, "\wr", "\omega' \cdot \omega''"'] \\
& A &
\end{tikzcd}$$

commutes.

This is not a perm. cat. because

$$V \oplus W \simeq W \oplus V$$

is not compatible with orient.

$$\begin{tikzcd}
v_1 \cdots v_p \, w_1 \cdots w_q \in \Lambda^{p+q}(V \oplus W) \arrow[rr, equal] & & \Lambda^{p+q}(W \oplus V) \\
v_1 \cdots v_p \otimes w_1 \cdots w_q \in \Lambda^p V \otimes \Lambda^q W \arrow[u] \arrow[dr, "\omega_V \cdot \omega_W"', "\wr"] & & \Lambda^q W \otimes \Lambda^p V \arrow[u, "(-1)^{qp} w_1 \cdots w_q \, v_1 \cdots v_p"'] \arrow[dl, "\wr", "\omega_W \cdot \omega_V"'] \\
& A &
\end{tikzcd}$$

$$(\omega_V \cdot \omega_W)(v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q) = (-1)^{qp} \omega_V(v_1 \cdots v_p) \, \omega_W(w_1 \cdots w_q)$$

$$(-1)^{qp} \omega_W \omega_V (w_1 \cdots w_q \, v_1 \wedge \cdots \wedge v_p) = \omega_W(w_1 \cdots w_q) \, \omega_V(v_1 \cdots v_p)$$

Maybe there is a moral here. It seems that the cat. of vector spaces with volume is not a perm. cat.

$$SL_p \qquad SL_q \qquad\qquad SL_{p+q}$$

What this example shows is that one cannot kill the $K_1$ without first killing $K_0$.

$$\begin{array}{ccc} K_2F & K_1F & K_0F \\ & F^* & \mathbb{Z} \end{array}$$

Thus the problem is to construct a model for the theory beginning in dim 2.

---

<u>Applications</u>:

<u>Cor. 1</u>: $$H_*\begin{pmatrix} GL_r(A) & \\ & GL_\infty(A) \end{pmatrix} \xrightarrow{\sim} H_*\begin{pmatrix} GL_r(A) & M_{r\infty}(A) \\ & GL_\infty(A) \end{pmatrix} \qquad 0 \le r \le \infty$$

<u>Pf</u>: Let $G'$ $\qquad$ $G$.

$$\begin{array}{ccccccccc} 1 & \to & H' & \to & G' & \to & GL_r & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & H & \to & G & \to & GL_r & \to & 1 \end{array}$$

$$\begin{array}{ccc} E^2_{pq} = H_p(GL_r, H_q(H')) & \Longrightarrow & H_{p+q}(G') \\ \downarrow & & \downarrow \\ E^2_{pq} = H_p(GL_r, H_q(H)) & \Longrightarrow & H_{p+q}(G) \end{array}$$

Preceding $\Rightarrow$ $H_*(H') \cong H_*(H)$.

<u>Cor. 2</u>: $$H_*\begin{pmatrix} GL_\infty & & \\ & \ddots & \\ & & GL_\infty \end{pmatrix} \cong H_*\begin{pmatrix} GL_\infty & & * \\ & \ddots & \\ 0 & & GL_\infty \end{pmatrix}$$

Remark:
$$GL_n\begin{pmatrix}A & A\\ 0 & A\end{pmatrix} \cong \begin{pmatrix}GL_n(A) & M_{n,n}(A)\\ 0 & GL_n(A)\end{pmatrix}$$

$$\begin{pmatrix}A & A\\ 0 & A\end{pmatrix} = \mathrm{End}(A \subset A^2) \qquad \parallel$$

$$GL_n(\;\text{"}\;) = \mathrm{Aut}\left((A\subset A^2)^{\oplus n}\right) \cong \mathrm{Aut}(A^n \subset A^{2n})$$

first n coords

$$\begin{pmatrix} a_1 & b_1 & a_2 & b_2\\ & c_1 & & c_2\\ a_3 & b_3 & a_4 & b_4\\ & c_3 & & c_4\end{pmatrix} \sim \begin{pmatrix} a_1 & a_2 & b_1 & b_2\\ a_3 & a_4 & b_3 & b_4\\ & & c_1 & c_2\\ & & c_3 & c_4\end{pmatrix}.$$

Cor 3: $H_*\left(GL_\infty\begin{pmatrix}A & A\\ & A\end{pmatrix}\right) \xleftarrow{\sim} H_*\left(GL_\infty\begin{pmatrix}A & \\ & A\end{pmatrix}\right)$.

---

Claim that to $G \xrightarrow{\rho} \mathrm{Aut}(P)$ $P\in\mathcal{P}(A)$, one has
$$\rho_*: H_*(G) \longrightarrow H_*(GL(A)).$$

such that

$$(\rho_1)_* = (\rho_2)_* \qquad \rho_1\oplus\varepsilon_1 \simeq \rho_2\oplus\varepsilon_2$$

$\varepsilon_1,\varepsilon_2$ trivial reps. In effect

$$\mathrm{rep}(G,A) = \coprod_P \mathrm{Hom}(G,\mathrm{Aut}(P))_{\mathrm{inn.}}$$ P runs over iso. classes

$$\mathrm{strep}(G,A) = \varinjlim_P \mathrm{Hom}(G,\mathrm{Aut}(P))_{\mathrm{inn.}}$$ limit is taken over trans. cat

$$= \varinjlim_n \mathrm{Hom}(G, GL_nA)_{\mathrm{inn.}}$$ cofinality

$$\longrightarrow \mathrm{Hom}(G, GL_\infty A)_{\mathrm{inn.}}$$

$$\longrightarrow \mathrm{Hom}(H_*(G), H_*(GL_\infty A))$$

In down-to-earth terms, one chooses $Q$ + $P\oplus Q \xrightarrow{\sim} A^n$, and lets $G$ act trivially on $Q$, thus getting $G \to GL_nA \subset GL(A)$.

Cor. 4: Let $\rho$ be a repn. of $G$ on $P$ and suppose $G$ leaves stable a flag

$$0 = P_0 \subset P_1 \subset \cdots\cdots \subset P_n = P$$

s.t. $P_i / P_{i-1} \in \mathcal{P}(A)$. Let $\rho_i$ be the ind. rep. of $G$ on $P_i/P_{i-1}$. Then

$$\rho_* = (\rho_1 \oplus \cdots \oplus \rho_n)_* .$$

Proof: Induction reduces me to $n=2$: If $\rho' =$ ind. rep. on $P_{n-1}$, then $\rho_* = (\rho' \oplus \rho_n)_* \underset{\text{ind.}}{=} (\rho_1 \oplus \cdots \oplus (\rho_{n-1} \oplus \rho_n))_*$.

$$0 \subset P_1 \subset \cdots\cdots \subset P_{n-2} \subset P_{n-1} \oplus P_n/P_{n-1} .$$

Can suppose $n=2$, $P_1 = A^r$, $P_2 = A^{r+n}$.

$$\begin{tikzcd}
H_* G \arrow[dr] \arrow[rr, bend left, "\rho_*"] \arrow[rr, bend left=15, "(\rho_1+\rho_2)_*"'] \arrow[drr, bend left=40] & & \\
& H_*\begin{pmatrix} GL_r & M_{rn} \\ & GL_n \end{pmatrix} \arrow[r, shift left, "i_*"] \arrow[r, shift right, "i_*\varphi_*"'] \arrow[d] & H_* GL_{r+n} \arrow[d] \\
H_*\begin{pmatrix} GL_r & \\ & GL_\infty \end{pmatrix} \arrow[r, "\sim"] & H_*\begin{pmatrix} GL_r & M_{r\infty} \\ & GL_\infty \end{pmatrix} \arrow[r, shift left, "i_*"] \arrow[r, shift right, "i_*\varphi_*"'] & H_* GL_\infty
\end{tikzcd}$$

---

Cor. 5: $H_i(GL_\infty(\mathbb{F}_q), \mathbb{F}_p) = 0$, $q = p^d$, $i > 0$.

Proof: Sylow subgroup of $GL_n(\mathbb{F}_q)$ — enough to show this is zero

$$\begin{tikzcd}
H_i\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \arrow[r, "\text{Sylow}", "\text{theory}"'] \arrow[rr, bend right, "\rho_*"'] & H_i(\underbrace{GL_n \mathbb{F}_q}_{\text{finite}}, \mathbb{F}_p) \arrow[r] & H_i(GL_\infty(\mathbb{F}_q), \mathbb{F}_p)
\end{tikzcd}$$

$\rho$ canon. rep on $\mathbb{F}_q^n$

$\rho_* = (\rho_1 \oplus \cdots \oplus \rho_n)_*$

$= (\text{trivial rep on } \mathbb{F}_q^n)_*$

$= 0$ in degrees $> 0$.