Problem: The space $\mathcal{L} = GL_n(\mathbb{C}[z,z^{-1}])' / GL_n(\mathbb{C}[z])'$ we have seen is a union of algebraic varieties, hence it has an etale homotopy type. The problem is to give an algebraic proof that $\mathcal{L} \sim \Omega GL_n$.

Put $\Omega = GL_n(\mathbb{C}[z,z^{-1}])'$, so that we have a homotopy equivalence $\Omega \to \mathcal{L}$. Then we have algebraic maps

$$\Omega \times \mathbb{G}_m \longrightarrow GL_n \qquad (\Omega \times 1) \longmapsto 1 .$$

So if we give $S^1 \to (\mathbb{G}_m)_{et}$, then we ought to get a map

$$\Omega_{et} \times S^1 \longrightarrow (GL_n)_{et}$$

which should give the desired homotopy equivalence $\Omega_{et} \longrightarrow \Omega (GL_n)_{et}$.

Slight problem: Consider $n = 1$, whence $\Omega = \mathbb{Z}$. Then we would be trying to prove that $\Omega \mathbb{G}_m = \mathbb{Z}$ which isn't true in the algebraic context. What in fact one has is

$$\mu_m \longrightarrow \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m$$

together with $\Omega = \mathbb{Z} \twoheadrightarrow \mathbb{Z}/m \simeq \mu_m$ given by a choice of primitive m-th root of unity.

The principal bundle^{P} of $A(\omega) = e^{2\pi i \omega P} F(z)$ over GL_n I have constructed is not algebraic over GL_n. The problem is to somehow approximate

P by an inverse system of algebraic gadgets.

Let us determine where

$$(*) \qquad GL_n \xrightarrow{m} GL_n \qquad A \longmapsto A^m$$

is etale. A tangent vector to GL_n at A is of the form $(I+\varepsilon B)A$ $\quad B \in M_n$.

$$[(I+\varepsilon B)A]^m = (A+\varepsilon BA)^m$$
$$= A^m + \varepsilon\left[(BA)A^{m-1} + A(BA)A^{m-2} + \cdots + A^{m-1}(BA)\right]$$

Right translate by A^{-m} to get a tangent vector at O

$$[(I+\varepsilon B)A]^m A^{-m} = I + \varepsilon\left[B + ABA^{-1} + \cdots + A^{m-1}BA^{-(m-1)}\right]$$
$$= I + \varepsilon\left[\frac{Ad(A)^m - 1}{Ad(A)}(B)\right]$$

Thus we see by reasoning analogous to before that $(*)$ is etale at A iff ~~the eigen~~ no eigenvalue of $Ad(A)$ is an m-th root of $1 \neq 1$, hence iff no two eigenvalues of A have as ratio an m-th root of $1 \neq 1$.

Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ so that A^m has the eigenvalues $\lambda_1^m, \ldots, \lambda_n^m$. If $\lambda_i^m = \lambda_j^m$ for some $i \neq j$, then $(\lambda_i \lambda_j^{-1})^m = 1$, and so if $(*)$ is etale at A, we must have $\lambda_i = \lambda_j$. Thus when we have an etale solution of $A^m = B$, the multiplicity of the eigenvalues λ of A equal the multiplicity of the eigenvalues λ^m of B. Moreover one has the following possibilities for A. If one breaks up $\mathbb{C}^n$ according to the eigenspaces of B, so that $B_s = \mu I$, then $A_s = \lambda I$ where λ is one of the m-th roots of μ. Of course $A_u = B_u^{1/m}$. So we see any two etale solutions of $A^m = B$ commute. Summarizing:

<u>Proposition:</u> (i) The map $A \mapsto A^m$ from $GL_n \to GL_n$ is etale at A iff for any two eigenvalues λ, λ' of A one has $\lambda^m = \lambda'^m \Rightarrow \lambda = \lambda'$.

(ii) Given $B \in GL_n$, let B have k distinct eigenvalues $\mu_1, \ldots, \mu_k$, say

$$B_s = \sum_{i=1}^{k} \mu_i E_i$$

Then any solution of $A^m = B$ which is "etale" (i.e. a "simple" root) is of the form $A = A_s A_u$ where

$$A_u = (B_u)^{1/m}$$

$$A_s = \sum_{i=1}^{k} \lambda_i E_i \quad \text{where } \lambda_i^m = \mu_i$$

In particular there are mk etale solutions and all these solutions commute with each other. ~~Note that if $A_1^m = B = A_2^m$ are two etale solutions, then $(A_1^{-1}A_2)^m = I$~~

November 21, 1974.

Notation: $\Omega = GL_n(\mathbb{C}[z,z^{-1}])'$, $\Omega^{(m)} = GL_n(\mathbb{C}[z^m, z^{-m}])'$.

$\mathcal{P} =$ holom maps $w \mapsto A(w)$, $\mathbb{C} \to GL_n$ of the form $e^{2\pi i w X} F(e^{2\pi i w})$ where $X \in \mathfrak{gl}_n$, $F \in \Omega$.

$\mathcal{P}$ is a principal Ω-bundle over GL_n the map $\mathcal{P} \to GL_n$ being $A \mapsto A(1)$.

Introduce an equivalence relation in $\mathcal{P}$. Call $A_1(w)$ and $A_2(w)$ equivalent if $A_1(w)A_2(w)^{-1} \in \Omega^{(m)}$. Because $\Omega^{(m)}$ is a group, this is an equivalence relation. Let $\mathcal{P}_m$ denote the set of equivalence classes. Observe that

$$A_1 \sim A_2 \implies A_1 F \sim A_2 F \quad \text{for any } F \in \Omega$$

hence Ω acts to the right on $\mathcal{P}_m$. Observe also that $A_1 \sim A_2$ means

$$A_1(w) = F(z^m) A_2(w) \qquad z^m = e^{2\pi i m w}$$

$$\implies A_1(1) = F(1) A_2(1) = A_2(1)$$

hence $A_1 \sim A_2 \implies A_1, A_2$ lie in the same fibre over GL_n. Thus we have an induced map

$$\mathcal{P}_m \longrightarrow GL_n.$$

~~Assume~~

Let $C \in GL_n$, and choose $X \in \mathfrak{gl}_n$ so that

$$e^{2\pi i \frac{1}{m} X} = C.$$

If $e^{2\pi i \frac{1}{m} Y} = C$ also, then we have seen that

$$e^{2\pi i \omega X} e^{-2\pi i \omega Y} = e^{2\pi i (m\omega)\frac{1}{m} X} e^{-2\pi i (m\omega)\frac{1}{m} Y}$$

$$= F(e^{2\pi i m\omega}) = F(z^m) \in \Omega^{(m)}.$$

Thus $e^{2\pi i \omega X} \sim e^{2\pi i \omega Y}$. Thus we have a section

$$\begin{tikzcd} & GL_n \arrow[dl, dashed] \arrow[d, "m"] \\ \mathcal{P}_m \arrow[r] & GL_n \end{tikzcd} \qquad \begin{tikzcd} & e^{2\pi i \frac{1}{m} X} \arrow[dl, dashed, mapsto] \arrow[d, mapsto, "m"] \\ e^{2\pi i \omega X} \arrow[r, mapsto] & e^{2\pi i X} \end{tikzcd}$$

I next want to compute the fibres of $\mathcal{P}_m$ over GL_n. Fix $C \in GL_n$, and choose X so that $e^{2\pi i X} = C$. Then since Ω acts transitively on the fibre of $\mathcal{P}_m$ over C, this fibre is $\sigma \backslash \Omega$ where σ is the stabilizer of a point in the fibre, e.g. $e^{2\pi i \omega X}$. But $F \in \sigma \iff$

$$e^{2\pi i \omega X} \sim e^{2\pi i \omega X} F$$

$$\iff F(z^m) e^{2\pi i \omega X} = e^{2\pi i \omega X} F$$

$$\iff F \in e^{-2\pi i \omega X} \, \Omega^{(m)} e^{2\pi i \omega X}$$

Thus $\sigma = \Omega \cap e^{-2\pi i \omega X} \Omega^{(m)} e^{2\pi i \omega X}$ which shows that $\mathcal{P}_m$ isn't a fibre bundle over GL_n, probably i.e. if $f(\omega) = e^{-2\pi i \omega X} F(z^m) e^{2\pi i \omega X}$ then

$$f(\omega + 1) = C^{-1} f(\omega) C$$

won't usually be periodic. So this makes me lose confidence in P_m.

Let V be the open subset of GL_n consisting of matrices such that no two eigenvalues have as ratio an m-th root $\zeta \neq 1$; equiv. V is the open set where the mth power map $GL_n \to GL_n$ is etale.

The analogous subset $U \subset gl_n$ is defined to be where $e^{2\pi i}$ is etale; thus on U no two eigenvalues differ by an integer $\neq 0$. Given two points on U with the same image C in GL_n, there difference is a semi-simple matrix with integer coeffs. which is the same as a 1-parameter subgroup $G_m \to GL_n$.

Given $A, B \in V$ with $A^m = B^m = C$, we know that A, B commute and that $A^{-1}B$ is a matrix of order m, i.e. a homomorphism $\mathbb{Z}/m\mathbb{Z} \to GL_n$.

November 22, 1974.

Problem: Find the universal top. group G equipped with a continuous map $\varphi : \{A \in GL_n \mid A^m = 1\} \longrightarrow G$ such that $\varphi(AB) = \varphi(A)\varphi(B)$ if A, B commute. As an example of such G, we have

$$G = \prod_{i=1}^{m-1} GL_n \qquad \varphi(A) = (A^i)$$

but the hope is that the universal G is somehow richer.

Analogous problem: Find the universal map

$$\varphi : \{X \in \mathfrak{gl}_n \mid \exp(2\pi i X) = 1\} \longrightarrow G$$

such that $\varphi(X+Y) = \varphi(X)\varphi(Y)$ if $(X,Y) = 1$

Example: $G = \Omega = GL_n(\mathbb{C}[z, z^{-1}])'$, $\varphi(X) = z^X$. I hope this is universal.

$$\begin{tikzcd}
\{X \mid e^{2\pi i X} = 1\} \arrow[r] \arrow[d, "e^{2\pi i \frac{1}{m}(?)}"] & \Omega \arrow[d, dashed, "\text{exists if } \Omega \text{ is universal}"] \\
\{A \mid A^m = 1\} \arrow[r] & G
\end{tikzcd}$$

whence we find that $\Omega /$ normal subgroup generated by z^X, $e^{2\pi i \frac{1}{m} X} = 1$ is the universal group for the mod m problem. I am fairly certain that the subgroup of Ω generated by z^X with $e^{2\pi i \frac{1}{m} X} = 1$, is $\Omega^{(m)} = GL_n(\mathbb{C}[z^m, z^{-m}])'$. ~~and that the normal subgroup by Bass's thm would have to~~

~~contain all of Ker $SL_n \mathbb{C}[z,z^{-1}] \to$? WAIT ? think Bass classifies normal subgroups of ... in terms of ideals of Λ and certain K_1 ...~~

Let N be the normal subgp of Ω generated by $\Omega^{(m)}$. Then since $\Omega^{(m)}$ is stable under conjugation by constant matrices, so must N be, hence N is normal in $\Omega \rtimes GL_n = GL_n \mathbb{C}[z,z^{-1}]$. Now Bass has proved, I think, that normal subgroups of $GL_n \mathbb{C}[z,z^{-1}]$ are all congruence subgps essentially, hence it should follow that $N = \mathrm{Ker}\,\{GL_n(\mathbb{C}[z,z^{-1}]) \to GL_n(\mathbb{C}[z]/(z^m-1))\}$. Therefore the mod m universal group should be

$$\text{image of } GL_n(\mathbb{C}[z,z^{-1}])' \text{ in } \prod_{i=1}^{m-1} GL_n = GL_n(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])'$$

$$\text{functions } \{\mu_n \to \mathbb{C}\}$$

Conclusion: <u>This isn't going to work.</u>

So far we have tried to form over GL_n a fibre bundle with fibre Ω mod m which would be algebraic. One might also examine the restriction of P to the points of order m in GL_n. This restriction is the subset of P consisting of $A(\omega)$ such that $A(1)^m = A(m) = 1$, i.e. $A(\omega+m) = A(\omega)$, which means $A(\omega) = B(e^{2\pi i \frac{\omega}{m}}) = B(z^{1/m})$. Call this restriction $P^{(m)}$. Thus

$$P^{(m)} \subseteq \Omega^{(\frac{1}{m})}$$

~~one sees immediately~~ and for $B(y) \in \Omega^{(\frac{1}{m})}$, $y^m = z$

to be in $\mathcal{P}^{(m)}$ means that $B(\zeta y) = B(\zeta)B(y)$,
(ζ generator of μ_m.)

Assertion: $\Omega/\Omega^{(m)}$ = h-fibre of $m: GL_n \longrightarrow GL_n$.

Proof: We have a map of fibrations

$$\begin{tikzcd}
\Omega \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d, "\alpha_m"] & GL_n \arrow[d, "m"] \\
\Omega \arrow[r] & \mathcal{P} \arrow[r] & GL_n
\end{tikzcd}$$

$$\alpha_m: A \longmapsto A(m\omega)$$

and $\alpha_m \Omega = \Omega^{(m)}$. Thus one has a fibration

$$\begin{tikzcd}
\Omega/\Omega^{(m)} \arrow[r] & \mathcal{P}/\Omega^{(m)} \arrow[r] & GL_n \\
& \alpha_m\mathcal{P}/\Omega^{(m)} \arrow[u, "\text{heq as } \mathcal{P},\ \alpha_m\mathcal{P} \text{ are contractible}"'] & \\
& \mathcal{P}/\Omega = GL_n \arrow[u, equal] &
\end{tikzcd}$$

and the induced map lifts $C \in GL_n$ to A in $\mathcal{P}$, $A(1)=C$, which goes into $\alpha_m A = A(m\omega)$, which projects to $A(m) = C^m$ in GL_n. QED.

This raises the question of whether I can algebraically map $\Omega/\Omega^{(m)}$ into GL_n, and homotop the composition with m to the basepoint.

$$\begin{tikzcd}
\Omega \arrow[r] \arrow[drr, dashed, "?"] & \mathcal{P} & \\
& \alpha_m \mathcal{P} \arrow[u, hook] \arrow[r] & GL_n
\end{tikzcd}$$

Example: Suppose we have an element of Ω given ~~by~~ a 1-parameter subgroup $\chi : \mathbb{G}_m \longrightarrow GL_n$. Then maybe the element of GL_n I want is $\chi(\zeta)$, $\zeta = \exp(2\pi i \frac{1}{m})$. Yes, because the image of χ $\mathcal{P}$ actually sits in $\alpha_m \mathcal{P}$. $\alpha_m \mathcal{P}$ is the set of $A \in \mathcal{P}$ such that $A(\omega + \frac{1}{m}) = A(\frac{1}{m}) A(\omega)$ and this includes 1-parameter subgroups.

Problem: To define "algebraically" a map $\Omega/\Omega^{(m)} \longrightarrow GL_n$ whose comp. with m is homotopically trivial. In the case $n=1$ this amounts to giving a ζ generating μ_m.

Example: If $F \in \Omega$ satisfies $F(\zeta z) = C F(z)$ (so $C = F(\zeta)$), then we want to map F to $F(\zeta)$.

One thing you might try is to take $F \in \Omega$ send it to $F(\zeta)$, then use the universal homotopy

$$F(z^m) \sim F(z)^m.$$

Another thing to try is the following. As we've done before let us identify a ptd. path $I \to GL_n$ with a continuous map $\omega \mapsto f(\omega)$ from $\mathbb{R}$ to GL_n such that $f(\omega+1) = f(1)f(\omega)$. Then for any $\lambda \in \mathbb{R}$, we have

$$\omega \longmapsto f(\omega+\lambda)f(\lambda)^{-1} \quad \text{satisfies}$$

$$\begin{aligned} f(\omega+1+\lambda)f(\lambda)^{-1} &= f(1)\, f(\omega+\lambda)f(\lambda)^{-1} \\ &= \left[f(1+\lambda)f(\lambda)^{-1}\right] f(\omega+\lambda)f(\lambda)^{-1} \end{aligned}$$

hence $f \longmapsto f(\ +\lambda)f(\lambda)^{-1}$ is a map of the path space of GL_n to itself, which covers the identity map of GL_n. Thus the induced map on the fibre ΩGL_n must be homotopic to the identity. Thus we have proved:

<u>Assertion</u>: The map $F(z) \longmapsto F(\alpha z)F(\alpha)^{-1}$ from Ω to itself is homotopic to the identity map. Here $\alpha \in \mathbb{C}^*$.

Direct proof: Pick a path α_t joining α to 1. Then clearly $F(\alpha_t z)F(\alpha_t)^{-1}$ is the path we need.

Another version:

$$A(\omega) \longmapsto A(\omega+\lambda)A(\lambda)^{-1} \qquad \lambda \in \mathbb{C}$$

maps P into itself: $\left(e^{2\pi i(\omega+\lambda)X} F(e^{2\pi i(\omega+\lambda)}) F(e^{2\pi i\lambda})^{-1} e^{2\pi i\lambda X}\right)$

$$= e^{2\pi i\omega X}\; e^{2\pi i\lambda X} F(e^{2\pi i\lambda} z) \cdots$$

and covers the identity ~~map~~ of GL_n.

Thus we know that $F \mapsto F(\zeta z)F(\zeta)^{-1}$ is homotopic to the identity. Hence

$$F \longmapsto F(z),\ F(\zeta z)F(\zeta)^{-1},\ F(\zeta^2 z)F(\zeta)^{-2}, \ldots, F(z)F(\zeta)^{m}$$

are homotopic, giving a canonical homotopy of $F(\zeta)^m$ to 1.

Nov. 24, 1974 – Review.

Over $\mathbb{C}$ I can show that $\Omega = GL_n(\mathbb{C}[z,z^{-1}])'$ is homotopy equivalent to ΩGL_n by exhibiting [redacted] a [redacted] principal bundle.

$$\Omega \longrightarrow P \longrightarrow GL_n$$

with P contractible. P is a holomorphic gadget which [redacted] is trivialized by $\exp 2\pi i : gl_n \longrightarrow GL_n$. ($P$ can probably be described algebraically, but not the map $P \longrightarrow GL_n$.)

(Alg. description of P: Take pairs (X, F) with $X \in gl_n$, $F \in \Omega$. Introduce an equivalence relation $(X,F) \sim (X_1, F_1) \iff e^{2\pi i w X} F(z) = e^{2\pi i w X_1} F_1(z)$ as functions, i.e. if their [redacted] power series at $w = 0$ coincide. For example, $e^{2\pi i w X} F(z) = 1$ [redacted] $\iff$

$$\frac{1}{2\pi i}\frac{d}{dw}\left(e^{2\pi i w X} F(z)\right) = e^{2\pi i w X}\left(XF(z) + zF'(z)\right) = 0$$

$$\iff \quad XF(z) + zF'(z) = 0.\ )$$

[redacted]

Basic problem is to [redacted] give an algebraic proof that $(\Omega)_{et}$ and $\Omega(GL_n)_{et}$ have the same profinite completions, or maybe even that $\Omega(GL_n)_{et}$ is the profinite completion of $(\Omega)_{et}$.

I know that the map $\Omega \to \Omega$, $F(z) \mapsto F(z^m)$ corresponds [redacted] to looping the map $GL_n \to GL_n$, $A \mapsto A^m$.

If I write $\Omega^{(m)}$ for the image, then I know that $\Omega/\Omega^{(m)}$ is homotopy equivalent to the fibre of $Gl_n \xrightarrow{m} Gl_n$. So the point therefore is to show that one has $(\Omega/\Omega^{(m)})_{et}$ = fibre of $m : (Gl_n)_{et} \circlearrowleft$.

Action of $\mathbb{G}_m$ on Ω:

$$(\varphi_\lambda F)(z) = F(\lambda z)F(\lambda)^{-1}$$

$$(\varphi_\lambda \varphi_\mu F)(z) = (\varphi_\mu F)(\lambda z)\,(\varphi_\mu F)(\lambda)^{-1}$$

$$= F(\mu\lambda z)\,F(\mu)^{-1}\left[F(\mu\lambda)\,F(\mu)^{-1}\right]^{-1}$$

$$= (\varphi_{\mu\lambda} F)(z).$$

So I am now interested in the action of μ_m on Ω.

~~Change notation:~~

$$\cancel{(\varphi_\lambda F)(z) = F(\lambda)^{-1}F(\lambda z).}$$

~~Now what are the fixpts of~~ $\cancel{\mu_m}$.

$$\cancel{F(z) = (\varphi_\zeta F)(z) = F(\zeta)^{-1}F(\zeta z)}$$

~~means~~ $\cancel{F(\zeta z) = F(\zeta)\,F(z)}$

Anyway, I know (?) that the map

$$\Omega^{(m)} \longrightarrow \Omega \longrightarrow \Omega/\Omega^{(m)} \longrightarrow Gl_n \xrightarrow{m} Gl_n$$

is given by sending F to $F(\zeta)$. Of couse you want to understand why composing with $F(\zeta) \mapsto F(\zeta)^m$ is null-homotopic.

Problem: Show: $\Omega/\Omega^{(m)} \xrightarrow{ev_\zeta} GL_n \xrightarrow{m} GL_n$ is null-homotopic.

Method:

$$\begin{tikzcd} \Omega/\Omega^{(m)} \arrow[r] & P/\Omega^{(m)} \arrow[r] & GL_n \\ & P^{(m)}/\Omega^{(m)} \arrow[u, phantom, "\cup\ \text{heq}"] \arrow[r, equal] & GL_n \arrow[u, "m"] \end{tikzcd}$$

$*$

So we have $F \in \Omega$ and want to homotop it to something in $P^{(m)} = \{A \mid A(\omega + \frac{1}{m}) = A(\frac{1}{m})A(\omega)\}$ and then take $A(\frac{1}{m})$.

Problem: I define a map $\Omega/\Omega^{(m)} \longrightarrow GL_n$ in the homotopy category using

$$\begin{tikzcd} \Omega/\Omega^{(m)} \arrow[r] & P/\Omega^{(m)} \\ & P^{(m)}/\Omega^{(m)} \arrow[u, "\text{heq}"] \arrow[r, equal] & GL_n \end{tikzcd}$$

Show this map is given by $F(z)\Omega^{(m)} \longmapsto F(\zeta)$.

Prop: i) $\mathbb{C}$ acts on P by

if $\alpha \in \mathbb{C}$ $\quad (\varphi_\alpha A)(\omega) = A(\alpha)^{-1} A(\alpha + \omega)$

ii) $P^{(m)} = \{A \mid \varphi_{\frac{1}{m}} A = A\}$.