Karoubi's thm: $K_1(SA) = K_0 A$

1) $A \xrightarrow{\pi} \!\!\!\rightarrow B$ surjective

M = set of iso classes of triples (E, F, α) $\quad \alpha : \pi E \xrightarrow{\sim} \pi F$

Lemma 1: (A, A, id) is a basic element in M

Def: $K_0(\pi) = M \Big/ (E, F, \alpha) \simeq (E', F', \alpha')$
iff
$\exists\ P, P' \in \mathcal{P}(A)$
$(E \oplus P, F \oplus P, \alpha \oplus id) \simeq (E' \oplus P', F' \oplus P', \alpha' \oplus id)$

Put $[E, F, \alpha]$ = image of (E, F, α) in $K_0(\pi)$

Lemma 2: $[E, F, \alpha] + [F, G, \beta] = [E, G, \beta\alpha]$
$K_0(\pi)$ is an abelian group.

Prop 1: $K_1 A \longrightarrow K_1 B \xrightarrow{\partial} K_0(\pi) \xrightarrow{i} K_0 A \longrightarrow K_0 B$ exact

$\partial\theta = [A^m, A^m, \theta]$ if $\theta \in GL_m A$

$i[E, F, \alpha] = [E] - [F]$.

2) Assume $\pi : A \twoheadrightarrow B$ satisfies

(*) given any finite subset a_i of $\operatorname{Ker} \pi$, $\exists\ e$ in $\operatorname{Ker} \pi$, $e^2 = e$, $ea_i = a_i$

Put $\mathcal{P}$ = full subcat of P in $\mathcal{P}(A)$ such that $\pi P = 0$.

Lemma 3: $u : E \to F$ a map in $\mathcal{P}(A)$. Then
$\pi(u) = 0 \iff u$ factors $E \to P \to F$ with $P \in \mathcal{P}$.

Prop 2: Assuming (*), then $K_0 \mathcal{P} \xrightarrow{\sim} K_0(\pi)$, $[P] \mapsto [P, 0, 0]$.

Pf: One defines an inverse map

$\operatorname{Ind}[E, F, \alpha] = [\operatorname{Ker}\ u pr_1 + v pr_2] - [P]$ $\quad$ where $\pi(u) = \alpha$ and $E \oplus P \xrightarrow{u pr_1 + v pr_2} \!\!\!\rightarrow F$

3) Def: a ring R is *flask* if $\exists$ $\tau : \mathcal{P}(R) \to \mathcal{P}(R)$
(additive) such that $id \oplus \tau \simeq \tau$.

Prop 3: $K_i R = 0$ if R is flask.

4) $N = \{1, 2, \ldots\}$ $\Gamma : \text{Mod } A \longrightarrow \text{Mod } A$, $\Gamma M = \bigoplus_N M$
shift isom $\alpha : M \oplus \Gamma M \simeq \Gamma M$
choose $N \cong N \times N$, whence get
$$\theta : \Gamma(\Gamma M) \simeq \Gamma M$$

Prop. 4: $R = \text{End}(\Gamma M)$ is flask
Pf: $\varphi : R \to R$ $\qquad \varphi(x) = \theta \Gamma(x) \theta^{-1}$

$$\Gamma M \oplus \Gamma(\Gamma M) \xrightarrow[\sim]{\alpha} \Gamma(\Gamma M) \xrightarrow[\sim]{\theta} \Gamma M$$

$$i_j = \theta \alpha \, in_j \qquad p_j = pr_j \alpha^{-1} \theta^{-1}$$

Then define $\qquad R \oplus {}_\varphi R \xrightarrow{\sim} {}_\varphi R$

$$r_1 \oplus r_2 \longmapsto i_1 r_1 + i_2 r_2$$

and $\tau : \mathcal{P}(R) \to \mathcal{P}(R)$ $\qquad \tau(P) = P \otimes_R {}_\varphi R$
Here $\mathcal{P}(R)$ = right R-modules.

Any subring of R closed under φ containing i_1, i_2, p_1, p_2 is also flask. Take $M = A$, examples

bA = matrices with finite rows + columns.

cone $CA = \sum a_i m_i$ $\qquad m_i$ = matrix with at most a single 1 in each row + column (partial permutation)

$SA = CA/$ ideal of finite matrices.

Karoubi thm: $K_1(SA) \xrightarrow{\sim} K_0(A)$

$$\alpha \in GL_m(SA) \longmapsto \mathrm{Ind}(\alpha)$$

October 25, 1974. Karoubi's periodicity theorem

A ring with an involution $a \mapsto \bar{a}$

If $P \in \mathcal{P}(A)$, then ${}^tP = \mathrm{Hom}_A(P, A)$ is a $\mathcal{P}(A^\circ)$-module with $(\lambda a)(x) = \lambda(x) a$. Via $a \mapsto \bar{a}$ it becomes a $\mathcal{P}_A$-mod which we denote P^*. Put $\langle x, \lambda \rangle = \lambda(x)$ for $x \in P$, $\lambda \in P^*$ so that $\langle ax, \lambda \rangle = a \langle x, \lambda \rangle$, $\langle x, a\lambda \rangle = \langle x, \lambda\bar{a} \rangle = \langle x, \lambda \rangle \bar{a}$.

A sesquilinear form on P is a biadditive map $f : P \times P \longrightarrow A$ such that $f(ax, y) = af(x,y)$, $f(x, ay) = f(x,y)\bar{a}$. This is the same as a linear map $P \longrightarrow P^*$, $y \mapsto f(?, y)$. f is called non-deg. if $P \xrightarrow{\sim} P^*$, and ε-symmetric if

$$f(y, x) = \varepsilon \overline{f(x,y)}$$

where ε is in the center of A such that $\varepsilon\bar{\varepsilon} = 1$. ${}_\varepsilon\mathcal{Q}(A)$ is the category of non-degenerate ε-symm. quadratic A-modules, and

$$ {}_\varepsilon L(A) = K({}_\varepsilon\mathcal{Q}(A)). $$

Hyperbolic functor $H : P(A) \longrightarrow {}_\varepsilon\mathcal{Q}(A)$ sends P to $P \oplus P^*$ with form

$$ (x + \lambda, x' + \lambda') = (\langle x, \lambda' \rangle + \varepsilon \overline{\langle x', \lambda \rangle}) $$

Assume $\frac{1}{2} \in A$. Forgetful functor

$$ J : {}_\varepsilon\mathcal{Q}(A) \longrightarrow \mathcal{P}(A) $$

Clearly $\boxed{JH(P) = P \oplus P^*.}$

If $Q \in {}_\varepsilon\mathcal{Q}(A)$, let the isom $Q \xrightarrow{\sim} Q^*$ given by the form on Q be denoted $x \longmapsto x^*$ so that

$$(x, y) = \langle x, y^* \rangle .$$

Then for $HJ(Q) = Q \oplus Q^*$ one has

$$\begin{aligned}
(x + \tfrac{1}{2}x^*, \; y + \tfrac{1}{2}y^*) &= \tfrac{1}{2}\left(\langle x, y^* \rangle + \varepsilon \overline{\langle y, x^* \rangle} \right) \\
&= \tfrac{1}{2}\left((x, y) + \varepsilon \overline{(y, x)} \right) \\
&= (x, y)
\end{aligned}$$

and similarly $(x - \frac{1}{2}x^*, y - \frac{1}{2}y^*) = -(x, y)$. Thus (as $\frac{1}{2} \in A$) we have

$$\boxed{HJ(Q) = Q \oplus (-Q)}$$

where $(-Q)$ is Q with form $-(x, y)$.

Note that $(-Q)$ is canonically isomorphic to Q if $\exists$ α in the center of A such that $\alpha\bar{\alpha} = -1$, the isom being $x \longmapsto \alpha x$

Ex: $A = \mathbb{C}$ with trivial involution.

If $\exists$ $\beta \in (\text{center } A)^*$ such that $\beta = -\bar{\beta}$, then ${}_\varepsilon\mathcal{Q}(A) \simeq {}_{-\varepsilon}\mathcal{Q}(A)$ by sending $(Q, f) \longmapsto (Q, \beta f)$

Example: $A = \mathbb{C}$ with involution given by conjugation.

Karoubi's periodicity thm. Defines

$${}_\varepsilon U(A) = \text{grothendieck group of } H: P(A) \to {}_\varepsilon Q(A)$$
$${}_\varepsilon V(A) = \text{——————} \quad J: {}_\varepsilon Q(A) \to P(A).$$

One can describe ${}_\varepsilon U(A)$ in terms of triples $(P_1, P_2, H(P_1) \simeq H(P_2))$, or what amounts (essentially) to the same thing, formations (Q, L_1, L_2); here L_1, L_2 are Lagrangians in the ε-quadratic module Q. (I will ignore momentarily the equivalence relation one puts on these formations)

Next we consider the map $[P] \mapsto [P] - [P^*]$ from $K(A)$ to $K(A)$. Since $H(P)$ is canonically isomorphic to $H(P^*)$, this map factors thru ${}_\varepsilon U(A)$ giving us a map

$$(*) \qquad \begin{array}{ccc} K(A) & \longrightarrow & {}_\varepsilon U(A) \\ P & \longmapsto & \text{formation } (H_\varepsilon(P), P, P^*) \end{array}$$

Claim this map composed with $J: {}_{-\varepsilon}L(A) \to K(A)$ gives zero. In effect if Q is a $-\varepsilon$ quadratic module, with canon. map $x \mapsto x^*$, $Q \xrightarrow{\sim} Q^*$

Then in $H_\varepsilon(Q)$ we have

$$(x+x^*, y+y^*) = \left(\langle x, y^*\rangle + \varepsilon\,\overline{\langle y, x^*\rangle}\right)$$

$$= \left[(x,y) + \varepsilon\,\overline{(y,x)}\right]$$

$$= 0$$

Hence the graph of the duality map $Q \longrightarrow Q^*$ is a Lagrangian in $H_\varepsilon(Q)$. (More generally the graph of a map $u: P \longrightarrow P^*$ in $H_\varepsilon(P)$ is a Lagrangian iff u is $(-\varepsilon)$-symmetric, meaning

$$\langle x, u(y)\rangle + \varepsilon\,\overline{\langle y, u(x)\rangle} = 0.\,)$$

Thus in the formation $(H(Q), Q, Q^*)$ one has the Lagrangian Γ_f, $f: Q \xrightarrow{\sim} Q^*$ is the $(-\varepsilon)$-quad. form on Q. Γ_f is complementary to both Q, Q^*, hence one knows this formation gives zero in ${}_\varepsilon U(A)$.

So because of the sequence

$$\longrightarrow {}_{-\varepsilon}L(A) \xrightarrow{J} K(A) \longrightarrow {}_{-\varepsilon}V(SA) \longrightarrow {}_{-\varepsilon}L(SA) \longrightarrow \cdots$$

one morally has to have an induced map

$${}_{-\varepsilon}V(SA) \longrightarrow {}_\varepsilon U(A)$$

compatible with (*) on pg 3.

Thm of Karoubi: $\quad {}_{-\varepsilon}V(SA) \xrightarrow{\sim} {}_{\varepsilon}U(A)$

Suppose A is (~~regular noetherian, so we know~~) such that $K(S^nA) = 0$ $\quad n \geq 1$. Then we have.

$$K(A) \xrightarrow{H} {}_{\varepsilon}L(A) \longrightarrow {}_{\varepsilon}U(SA) \longrightarrow 0$$

so that $\quad {}_{\varepsilon}U(SA) = {}_{\varepsilon}W(A) \quad$ Witt group.

and $\quad {}_{\varepsilon}U(S^{n+1}A) = {}_{\varepsilon}L(S^{n}A) \quad n \geq 1$.

We also have

$$L_{\varepsilon}(A) \xrightarrow{J} K(A) \longrightarrow {}_{\varepsilon}V(SA) \longrightarrow {}_{\varepsilon}L(SA) \longrightarrow 0$$

$$V_{\varepsilon}(S^nA) = {}_{\varepsilon}L(S^nA) \qquad n \geq 2.$$

So we get

$$\begin{aligned} {}_{\varepsilon}W(A) = {}_{\varepsilon}U(SA) &= {}_{-\varepsilon}V(S^2A) = {}_{-\varepsilon}L(S^2A) \\ &= {}_{-\varepsilon}U(S^3A) = {}_{\varepsilon}V(S^4A) = {}_{\varepsilon}L(S^4A) \\ &= {}_{\varepsilon}U(S^5A) = {}_{-\varepsilon}V(S^6A) = {}_{-\varepsilon}L(S^6A) = \cdots \end{aligned}$$

and in particular $\quad {}_{-\varepsilon}W(S^2A) = {}_{\varepsilon}L(S^4A) = {}_{\varepsilon}W(A)$

This shows the sequence ${}_\varepsilon W(S^nA)$ $n \geq 0$ ($= {}_\varepsilon L(S^nA)$ for $n \geq 1$) has period 4 (assuming $K(S^nA) = 0$ $n \geq 1$). Modulo 2 torsion, the result holds in general, because one finds elements in ${}_{-1}W(S^2\mathbb{Z}[\tfrac{1}{2}])$ ${}_{-1}W(S^2\mathbb{Z}[\tfrac{1}{2}])$ with cup product 4 in ${}_1W(\mathbb{Z}[\tfrac{1}{2}])$

Goal – I would like to understand clearly enough about the suspension of a ring to see that this theorem is true. ~~[illegible]~~ What I basically don't understand is the whole business about negative K-groups. ~~[illegible]~~

~~[illegible]~~

Suppose X is a ~~compact metric space~~ finite complex. Then I can consider the Banach alg. $\mathbb{C}(X)$ of cont. complex-valued fns on X. The category $P(\mathbb{C}(X))$ is the category of vector bundles over X.

Can I speak of Fredholm operators over $\mathbb{C}(X)$, and Kuiper's theorem?

So any vector bundle E over X.

Grothendieck group of a cofinal functor $F: \mathcal{P} \to \mathcal{Q}$

Suppose $\mathcal{P}, \mathcal{Q}$ additive, F additive, & F cofinal (i.e. $\forall Q$, $\exists Q', P$ with $Q \oplus Q' \simeq FP$). Define $K_0(F)$ to be the abel. gp. with generators $(P, P', \alpha : FP \simeq FP')$ and the relations

i) direct sum

ii) $[P, P', \alpha] + [P', P'', \beta] = [P, P'', \beta\alpha]$

Note ii) forces $[P, P', \alpha] = [P_1, P_1', \alpha_1]$ if $(P, P', \alpha) \sim (P_1, P_1', \alpha_1)$. Also it forces $[P, P', \alpha] = 0$ if α lifts to an isom $P \simeq P'$, because it forces $[P, P, id]$ to be zero.

Claim

$$K_0 F \xrightarrow{\varepsilon} K_0 \mathcal{P} \xrightarrow{F_*} K_0 \mathcal{Q}$$
$$[P, P', \alpha] \longmapsto [P] - [P']$$

is exact. Clearly $F_* \varepsilon = 0$. Given $x \in \mathrm{Ker}(F_*)$ we can represent x as $[P] - [P']$. Then $[FP] = [FP']$, hence $FP \oplus Q \simeq FP' \oplus Q$. As F is cofinal $Q \oplus Q' \simeq FP_0$, so we get $\alpha : F(P \oplus P_0) \simeq F(P' \oplus P_0)$, and $x = \varepsilon[P \oplus P_0, P' \oplus P_0, \alpha]$.

Recall $K_1 \mathcal{Q}$ has generators (Q, θ) $\theta \in \mathrm{Aut}(Q)$ with relations i) direct sum ii) composition of autos. Given (Q, θ) choose $\gamma : Q \oplus Q_0 \simeq F(P_0)$, and put

$$\partial(Q, \theta) = [P_0, P_0, \gamma(\theta \oplus id)\gamma^{-1}] \quad \in K_0 F$$

Indep. of choice of γ, and

also to replacing γ, Q_0, P_0 by $\gamma \oplus id_{F(P_1)}$; $Q_0 \oplus F(P_1)$, $P_0 \oplus P_1$.
If $Q \oplus Q_0 \simeq F(P_0)$, $Q \oplus Q_1 \simeq F(P_1)$, then

$$\underbrace{(Q \oplus Q_0)}_{Q \oplus Q_0 \,\simeq\, F(P_0)} \oplus (Q_1 \oplus Q) \simeq F(P_0 \oplus P_1)$$

is

$$(Q \oplus Q_1) \oplus (Q_0 + Q) \simeq F(P_1 \oplus P_0)$$

$$Q \oplus Q_1 \simeq F(P_1).$$

Thus one sees $\partial(Q, \theta)$ depends only on (Q, θ). Obviously satisfies i) and ii) so we get $\partial : K_1 \mathcal{Q} \longrightarrow K_0(F)$.

Exactness of $K_1\mathcal{Q} \xrightarrow{\partial} K_0 F \xrightarrow{\varepsilon} K_0 \mathcal{P}$.
Any $x \in K_0 F$ can be represented $x = [P, P', \alpha]$.
If $\varepsilon(x) = 0$, then clearly $[P] = [P']$ so $P \oplus P_0 \simeq P' \oplus P_0$, and $x = [P \oplus P_0, P' \oplus P_0, \alpha \oplus id_{FP_0}]$; but then x is isomorphic to something in the image of ∂.

<u>Remark</u>: The proof would be simplified by first replacing $\mathcal{Q}$ by the image $\mathcal{J}$ of $\mathcal{Q}$ in $\mathcal{P}$. One then instantly constructs

$$K_1 \mathcal{J} \longrightarrow K_1 F \longrightarrow K_0 \mathcal{P} \longrightarrow K_0 \mathcal{J} \longrightarrow 0$$

and then one separately shows that $K_1 \mathcal{J} = K_1 \mathcal{Q}$ and $K_0 \mathcal{J} \hookrightarrow K_0 \mathcal{Q}$ using cofinality.

Finally, one wants exactness of $K_1 \mathcal{P} \xrightarrow{F} K_1 \mathcal{Q} \xrightarrow{\partial} K_0 F$.

Can suppose $\mathcal{Q} = \mathcal{J}$.
So if $\partial[FP, \theta] = [P, P, \theta]$ is zero. This seems to be involved, since one must construct another description of $K_0 F$.

Milnor approach is to form a cart. square

$$\begin{tikzcd}
\mathcal{P} \times_{\mathcal{Q}} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\
\mathcal{P} \arrow[r] & \mathcal{Q}
\end{tikzcd}$$

and to produce a Mayer-Vietoris sequence, then define the relative group

$$K_0(F) = \text{Cokernel}\left\{ K_0\mathcal{P} \xrightarrow{\Delta} K_0(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}) \right\}.$$

Seems ~~[illegible]~~ to be much easier.

Work more generally

$$\begin{tikzcd}
\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2 \arrow[r] \arrow[d] & \mathcal{P}_2 \arrow[d, "G"] \\
\mathcal{P}_1 \arrow[r, "F"] & \mathcal{Q}
\end{tikzcd}$$

$[P_i] \in K_0\mathcal{P}_i$, $[FP_1] = [GP_2]$ i.e. $FP_1 \oplus Q \simeq GP_2 \oplus Q$.

~~[illegible] can suppose $FP_1 \oplus Q \simeq GP_2 \oplus Q$~~

In the case where F, G come from ring homomorphisms we know $\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2 \longrightarrow \mathcal{Q}$ is cofinal, hence we can suppose, modifying $([P_1], [P_2])$ by something coming from $K_0(\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2)$, that $FP_1 \simeq FP_2$, whence we have exactness of

$$K_0(\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2) \longrightarrow K_0(\mathcal{P}_1) \oplus K_0(\mathcal{P}_2) \longrightarrow K_0(\mathcal{Q}).$$

Next I want to show $A_1 \times_B A_2$ is cofinal in $\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2$. Start with $(P_1, P_2, \alpha : FP_1 \simeq GP_2)$, choose inverses P_1', P_2', whence FP_1', FP_2' are stably isomorphic, so modifying, they become isom., so (P_1, P_2, α) is a summand of (A_1^n, A_2^n, α). Can suppose α in the elementary group, whence if $A_1 \twoheadrightarrow B$, α lifts to A_1, and one wins.

Remaining parts of exactness.

$$K_1 \mathcal{Q} \longrightarrow K_0(\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2) \longrightarrow K_0\mathcal{P}_1 \times K_0 \mathcal{P}_2$$

$$(P_1, P_2, \alpha) - (A_1^m, A_2^m, id) \longmapsto 0$$

Can suppose $P_1 \simeq A_1^m$, $P_2 \simeq A_2^m$.

And

$$K_1 \mathcal{P}_1 \times K_1 \mathcal{P}_2 \longrightarrow K_1 \mathcal{Q} \longrightarrow K_0(\mathcal{P}_1 \times_{\mathcal{Q}} \mathcal{P}_2)$$

$$(B^m, \theta)$$

$$(A_1^m, A_2^m, \theta) \simeq (A_1^m, A_2^m, id)$$

OKAY.

Rest is clear

Something good with this approach is that $K_0(F)$ is described with generators $[P_1, P_2, \alpha]$ and the relations o) isom. i) $\oplus$ ii) $[P, P, id] = 0$. Any element of $K_0(F)$ is represented by $[P_1, P_2, \alpha]$ and $[P_1, P_2, \alpha] = 0$ iff $\alpha \oplus id_{B^m}$ lifts to an isomorphism $P_1 \oplus A^m \cong P_2 \oplus A^m$. Assuming here that $F : A \longrightarrow B$ maps $E(A)$ onto $E(B)$, i.e. F onto.

More precisely, if $A \overset{F}{\twoheadrightarrow} B$, then $K_0(F)$ is the triples $[P_1, P_2, \alpha]$ modulo the relation

$$[P_1, P_2, \alpha] \sim [P_1', P_2', \alpha'] \iff (P_1 \oplus P, P_2 \oplus P, \alpha + id) \cong (P_1' \oplus Q, P_2' \oplus Q, \alpha' + id).$$

Thus it is exactly the equivalence generated by isomorphism & diagonal action. Now if this is the case, the exact sequence

$$K_1 A \longrightarrow K_1 B \longrightarrow K_0 F \longrightarrow K_0 A \longrightarrow K_0 B$$

is trivial.

Procedure: Define $K_0(F)$ to be the quotient of the monoid of iso. classes of triples $[P_1, P_2, \alpha]$ by the monoid of triples $[P, P, id]$.

Define $K_1 B \xrightarrow{\partial} K_0 F$ by sending (B^m, θ) to (A^m, A^m, θ); since $E(A) \twoheadrightarrow E(B)$, this gives zero if $\theta \in E(B)$, so ∂ is well-defined. Next verify exactness of

$$K_1 A \to K_1 B \longrightarrow K_0 F \longrightarrow K_0 A \longrightarrow K_0 B$$

by hand. So now you can see that $K_0 F$ is a group; or equivalently that the triples $[P, P, id]$ are cofinal.

<u>Remark</u>: $E(A) \twoheadrightarrow E(B) \iff A \twoheadrightarrow B$.

<u>Question</u>: I know how to make triples $[P_1, P_2, \alpha]$ modulo action of diagonal triples into a space, which realizes the relative theory for $\Delta : \mathcal{P}_A \longrightarrow \mathcal{P}_{A \times_B A}$ or for maybe $pr_1 : \mathcal{P}_{A \times_B A} \to \mathcal{P}_A$. The extent to which this coincides with the relative theory for $\mathcal{P}_A \to \mathcal{P}_B$ is exactly whether one has a M-V sequence for

$$\begin{tikzcd} \mathcal{P}_{A \times_B A} \arrow[r] \arrow[d] & \mathcal{P}_A \arrow[d] \\ \mathcal{P}_A \arrow[r] & \mathcal{P}_B \end{tikzcd}$$

It will probably be essential to know some better conditions for this to work beside $E(A) \twoheadrightarrow E(B)$.

What I seem to need is that for $\theta \in E_m B$,

$[A^m, A^m, \theta]$ is zero. [redacted] This means that $[A^{m+n}, A^{m+n}, \theta \oplus id_{B^n}] \simeq [A^{m+n}, A^{m+n}, id]$ for some n, or in other words that $\theta \oplus id_{B^n}$ lifts to A. This more or less means I can't expect things to work unless $E(A) \twoheadrightarrow E(B)$ hence [redacted] $A \twoheadrightarrow B$.

[redacted] ~~(Raising idea: Let F be a field and let A be a central simple algebra over F of dim [illegible] n an integer ≥ 1 invertible in F such that $\mu_n \subset F^*$. Then~~

Now I am above all interested in the [redacted] functor $\mathcal{P}_A \longrightarrow \mathcal{P}_A$ sending P to $P \oplus \cdots \oplus P$ n-times. This functor is cofinal. ~~[illegible]~~ Since $End(A^n) = M_n(A)$, this functor is that induced by the diagonal

$$A \longrightarrow M_n(A)$$

$$a \longmapsto \begin{pmatrix} a & & \\ & \ddots_{n} & \\ & & a \end{pmatrix}$$

And as $A \rightarrow M_n(A)$ is not surjective, this functor is not E-surjective. Too bad.

$H : \mathcal{P}(A) \longrightarrow {}_\varepsilon\mathcal{Q}(A)$ cofinal because $Q \oplus (-Q) = H(Q)$

$GL_n A \longrightarrow O_{n,n}(A)$ surjective on commutator subgroups in the limit?

Seems unlikely because $E_n A$ leaves invariant a Lagrangian, and $EO_{n,n}(A)$ shouldn't.

Is $J : {}_\varepsilon\mathcal{Q}(A) \longrightarrow \mathcal{P}(A)$ E-surjective

$$O_{n,n}(A) \longrightarrow GL_{2n}(A)$$

Again this is very unlikely. (e.g. take A to be a field).

Conclusion: It becomes important to understand $K_0 F$ in the general case when F is not E-surjective.

Suppose A is a Banach alg with involution over $\mathbb{C}$, compatible with conjugation on $\mathbb{C}$. What does Karoubi's theorem say?

1) $A = \mathbb{C}$ $\qquad {}_{\varepsilon}O(A) = \left\{ a \in GL(\mathbb{C}) \;\middle|\; a\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} a^* = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \right\}$

If θ is unitary with $\theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\theta^{-1} = i\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, then $a \mapsto \theta a \theta^{-1}$ is an isomorphism of ${}_{-1}O(A)$ with $O(A) = \left\{ a \in GL(\mathbb{C}) \;\middle|\; a\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} a^* = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$.

Now $\{ a \mid aJa^* = J \}$ is a subgroup of $GL(\mathbb{C})$ which ought to retract to the unitary subgroup $\{ a \in U(\mathbb{C}) \mid aJa^{-1} = J \} \sim U \times U$. Thus

$$O(\mathbb{C}) \sim U \times U$$

Pretty sure that: $\quad \Phi : O(\mathbb{C}) \longrightarrow GL(\mathbb{C}) \quad$ is $\quad U \times U \xrightarrow{\oplus} U$

$\qquad\qquad\qquad\quad H : GL(\mathbb{C}) \longrightarrow O(\mathbb{C}) \quad$ is $\quad U \longrightarrow U \times U$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \alpha \longmapsto (\alpha, \alpha^{-1})$

Hence

$$V = \text{fibre of } B\Phi = BU$$

$${}_{\varepsilon}U = \text{fibre of } BH = U$$

So his thm. gives $V = \Omega\, {}_{\varepsilon}U$ or $BU = \Omega U$ which is OKAY up to connected components.

2) A = Calkin algebra. Fix $J \in A$ $\quad J^2 = 1$ with two big eigenspaces. Then again $O(A) \sim A^* \times A^*$ so the theorem would say in this case that .

$V = A^*/A^* \times A^* = BA^*$ is the conn. comp. of loop space on ${}_{\varepsilon}\mathcal{U} = A^* \times A^*/A^* = A^*$.
This seems to be quite general for a big hermitian complex Banach algebra.

Steps in proof of periodicity now —

1) Fibration

$$\begin{array}{ccccc} \left\{\begin{array}{l}\text{unitaries}\\ \equiv 1 \ \text{mod } K\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{big}\\ \text{unitary}\\ \text{gp of } H\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{unitaries}\\ \text{in } A^*\\ \text{of index } 0\end{array}\right\} \\ \wr & & \wr \ \text{Kuiper thm} & & \| \\ \mathcal{U} & & 0 & & A^{un}_{(0)} \end{array}$$

shows that $\mathcal{U} \sim \Omega(A^{un})$

2) Fibration

$$A^{un} \longrightarrow \left\{ a \in A \;\middle|\; \begin{array}{l} a^*a = 1 \\ aa^* \neq 1 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l}\text{self-adjoint}\\ \text{projectors } e\\ \neq 0, 1\end{array} \right\}$$

$$a \longmapsto aa^*$$

+ contractibility of middle
shows that $A^{un} \sim \Omega\{\text{s.a. proj. } e \neq 0, 1 \text{ on } A\}$

3) $\{\text{s.a. proj. } e\} \sim \{\text{s.a. } A,\ 0 \leq A \leq 1,\ \text{ess. spec } \{0,1\}\} \xrightarrow{\exp 2\pi i} \left\{\begin{array}{l}\text{unit}\\ \equiv 1\ (K)\end{array}\right\}$

In this proof there are two "general" steps (hopefully "general"), namely the fact that $\mathcal{U} = \widetilde{GL}(F)$ is the loop space of $(SF)^* = \widetilde{GL}(SF)$, and the fact that $(SF)^*$ is the loop space of projectors in SF. But what is missing is the exponential map relating projectors in SF to the group $\mathcal{U}$.

Possible proof of periodicity theorem might go as follows. Let V be a finite diml. Hilbert space, and consider the map

$$f : \left\{ \begin{matrix} A = A^* \in \mathrm{End}(V) \\ 0 \le A \le 1 \end{matrix} \right\} \xrightarrow{\exp 2\pi i} \mathcal{U}(V).$$

Then we get a spectral sequence in cohomology

$$H^p(\mathcal{U}_n, R^q f_*(\mathbb{Z})) \Longrightarrow H^{p+q}(pt, \mathbb{Z})$$

where $R^q f_*(\mathbb{Z})_\theta = H^q(f^{-1}(\theta), \mathbb{Z})$. This sheaf certainly isn't locally constant, but ?

Let H be Hilbert space. Fix a splitting $H = V_0 \oplus V_0^\perp$ where $V_0, V_0^\perp$ are inf. dimensional, and let E_0 = orth proj on V_0.

Infinite Grassmannian $\mathbb{Z} \times BU$ can be roughly identified with the space of orthogonal proj. E such that $E \equiv E_0$ mod compact operators.

In Atiyah-Singer paper one uses the exp. map

$$\begin{pmatrix} \text{s.a. } A,\ 0 \le A \le 1 \\ \text{ess. sp. } (A) = \{0,1\} \end{pmatrix} \xrightarrow{\exp 2\pi i} \begin{pmatrix} \text{unitaries} \\ \equiv 1 \mod K \end{pmatrix}$$

whereas I want to use the exp. map

$$\begin{pmatrix} 0 \le A \le 1 \\ A \equiv E_0 \ (\text{mod } \mathcal{K}) \end{pmatrix} \xrightarrow{\exp 2\pi i} \begin{pmatrix} \text{unitaries} \\ \equiv 1 \mod K \end{pmatrix}$$

which embeds inside. Notation

$$\mathcal{U} = \{\text{unitaries} \equiv 1 \mod \mathcal{K}\}$$

$$\mathcal{T} = \{0 \le A \le 1,\ \text{ess. sp}(A) = \{0,1\}\}$$

$$\overline{\mathcal{P}} = \{e \in \text{Calkin} = \mathcal{L}/\mathcal{K} \mid e = e^*,\ e^2 = e,\ e \ne 0, 1\}$$

maps

$$\exp : \mathcal{T} \longrightarrow \mathcal{U} \qquad A \longmapsto \exp 2\pi i A$$

$$\pi : \mathcal{T} \longrightarrow \overline{\mathcal{P}} \qquad A \longmapsto A \mod \mathcal{K}.$$

These are homotopy equivalences. Put

$\mathcal{P} = \{E \in \mathcal{L} \mid E = E^*, E^2 = E, E \not\equiv 0,1 \mod \mathcal{K}\}$. Thus $\mathcal{P} = \{A \in \mathcal{T} \mid \mathrm{sp}\, A = \{0,1\}\}$. Fix $E_0 \in \mathcal{P}$ and let $e_0 = \pi(E_0)$. Then we can play around with various fibrations:

$$\mathcal{P} \longrightarrow \mathcal{T} \xrightarrow[\sim]{\exp} \mathcal{U} \qquad \mathcal{P} \text{ contractible.}$$

$$\mathcal{P}_{e_0} = \pi^{-1}(e_0) \longrightarrow \mathcal{P} \longrightarrow \overline{\mathcal{P}}$$

$\mathcal{P}_{e_0} = \{E \in \mathcal{P} \mid E \equiv E_0 \mod \mathcal{K}\}$ this is my space $\mathbb{Z} \times BU$.

$$\underset{\text{contractible}}{\mathcal{T}_{e_0}} \longrightarrow \mathcal{T} \xrightarrow{\pi} \overline{\mathcal{P}}$$

$$\begin{tikzcd}
\mathcal{P}_{e_0} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r, "\sim *"] \arrow[d, hook] & \overline{\mathcal{P}} \arrow[d, equal] \\
{* \sim}\ \mathcal{T}_{e_0} \arrow[r, hook] \arrow[d] & \mathcal{T} \arrow[r, "\pi", "\sim"'] \arrow[d, "\exp", "\wr"'] & \overline{\mathcal{P}} \\
\mathcal{U} \arrow[r, equal] & \mathcal{U} &
\end{tikzcd}$$

$\exists$ Technical problems with map $\mathcal{T}_{e_0} \xrightarrow{\exp} \mathcal{U}$ which make it somewhat unpleasant (e.g. if $\Theta \in \mathcal{U}$ has all eigenvalues $\neq 1$, then $\exp^{-1}(\Theta)$ is reduced to a point). Thus the really nice procedure is to use the fibration

$$\begin{array}{ccccc} \mathcal{P}_{e_0} & \hookrightarrow & \mathcal{P} & \xrightarrow{\pi} & \overline{\mathcal{P}} \\ \| & & \| & & \| \\ UN_{e_0}/UN_{E_0} & & UN/UN_{E_0} & & UN/UN_{e_0} \end{array}$$

together with the homotopy equivalences

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\exp} & \mathcal{U} \\ {\scriptstyle \pi}\downarrow & & \\ \overline{\mathcal{P}} & & \end{array}$$

~~[illegible] it is better to think [illegible]~~

Further analysis:

$$(\mathcal{P}_{e_0})_{(0)} = \mathcal{U}/\mathcal{U}_{E_0} \qquad \mathcal{U}_{E_0} = \mathcal{U}(\operatorname{Im} E_0) \times \mathcal{U}(\operatorname{Im} I - E_0)$$

so one gets fibration

$$\mathcal{U} \longrightarrow \mathcal{U}/\mathcal{U}(\operatorname{Im} I - E_0) \longrightarrow (\mathcal{P}_{e_0})_{(0)}$$

showing that $\mathcal{U} = \Omega \mathcal{P}_{e_0}$

This proof makes no use of Fredholm ops.

Simpler proof of periodicity:

Put $\mathcal{U} = \{\text{unitaries} \equiv 1 \mod \mathcal{K}\}$

$\mathcal{P} = \{\text{projectors}_{\text{s.a.}}\ E \text{ in } \mathrm{End}(\mathcal{H}) \not\equiv 0,1 \mod \mathcal{K}\}$.

$\overline{\mathcal{P}} = \{\text{s.a. proj. } e \text{ in } \mathcal{S} = \mathrm{End}\,\mathcal{H}/\mathcal{K},\ e \neq 0,1\}$

$\mathcal{T} = \{\text{s.a. } A,\ 0 \leq A \leq 1,\ \text{ess. spec. } 0,1\}$

① $\mathcal{T} \longrightarrow \overline{\mathcal{P}}$ homotopy equivalence for fibres are convex

② $\exp 2\pi i : \mathcal{T} \to \mathcal{U}$ heq for one stratifies by the dimension of the -1 eigenspace and the fibres are contractible by Kuiper. (fibres ~~[illegible]~~ like $\mathcal{P}$).

③ fibration

$$\mathcal{P}_{e_0} \hookrightarrow \underset{\underset{pt}{\wr}}{\mathcal{P}} \longrightarrow \overline{\mathcal{P}}$$

E_0 basept of $\mathcal{P}$

$\pi E_0 = e_0$ basept of $\overline{\mathcal{P}}$.

shows $\mathcal{P}_{e_0} \sim \Omega \overline{\mathcal{P}}$

④ fibration

$$\underset{\underset{\mathcal{U}(E_0)}{\parallel}}{\mathcal{U}^{E_0}/\mathcal{U}(I-E_0)} \longrightarrow \underset{\underset{pt}{\wr}}{\mathcal{U}/\mathcal{U}(\mathrm{Im}\ I-E_0)} \longrightarrow \mathcal{U}/\mathcal{U}^{E_0} = (\mathcal{P}_{e_0})_{\text{component of } E_0}$$

?

shows that $\mathcal{U} \sim \Omega \mathcal{P}_{e_0}$

The main point seems to be showing $\overline{\mathcal{P}} \sim \mathcal{U}$ via the exponential map.

Why CA is flask:

Let $N = \{1, 2, \cdots\}$ and let $\Gamma : \mathrm{Mod}\, A \longrightarrow \mathrm{Mod}\, A$ be the functor $\Gamma(M) = \bigoplus_N M$. Choosing a bijection $N \simeq N \times N$ we get an isom of functors

$$\theta : \Gamma(\Gamma M) \xrightarrow{\sim} \Gamma M$$

We also have the shift autom

$$\alpha : M \oplus \Gamma M \xrightarrow{\sim} \Gamma M$$

Let $\varphi : \mathrm{End}(\Gamma M) \longrightarrow \mathrm{End}(\Gamma M)$ be the homom. of rings given by $\varphi(x) = \theta\, \Gamma(x)\, \theta^{-1}$.

$$\begin{tikzcd}
\Gamma M \oplus \Gamma M \arrow[d, "x \oplus \varphi(x)"] & \Gamma M \oplus \Gamma\Gamma M \arrow[l, "id \oplus \theta"', "\sim"] \arrow[r, "\alpha", "\sim"'] \arrow[d, "x \oplus \Gamma(x)"] & \Gamma\Gamma M \arrow[r, "\theta", "\sim"'] \arrow[d, "\Gamma(x)"] & \Gamma M \arrow[d, "\varphi(x)"] \\
\Gamma M \oplus \Gamma M & \Gamma M \oplus \Gamma\Gamma M \arrow[l, "id \oplus \theta"', "\sim"] \arrow[r, "\alpha", "\sim"'] & \Gamma\Gamma M \arrow[r, "\theta", "\sim"'] & \Gamma M
\end{tikzcd}$$

Define elements in $\mathrm{End}(\Gamma M)$ as follows:

$$\iota_1 = \theta \alpha\, in_1 \qquad \iota_2 = \theta\alpha\, in_2\, \theta^{-1}$$

$$p_1 = pr_1\, \alpha^{-1} \theta^{-1} \qquad p_2 = \theta\, pr_2\, \alpha^{-1} \theta^{-1}$$

so that we have the identities

$$p_1 \iota_1 = p_2 \iota_2 = \iota_1 p_1 + \iota_2 p_2 = 1$$

$$p_2 \iota_1 = p_1 \iota_2 = 0$$

and

$$\varphi(x)\iota_1 = \iota_1 x \qquad\qquad \varphi(x)\iota_2 = \iota_2\varphi(x)$$

$$x p_1 = p_1\varphi(x) \qquad\qquad \varphi(x)p_2 = p_2\varphi(x).$$

for all $x \in \text{End}(\Gamma M)$.

Now I can show $\overset{R}{\overset{\shortparallel}{\text{End}(\Gamma M)}}$ is flask as follows. Associated to the ring hom. $\varphi : \text{End}(\Gamma M) \circlearrowleft$ is its base change

$$\tau : \mathcal{P}(\text{End}\,\Gamma M) \circlearrowleft$$

which is "the" additive functor such that $\tau(R) = R$ and $\tau(\text{mult by } x) = \text{mult. by } \varphi(x)$. Here we use right R-modules, so mult. is always left multiplication. To define an isom $\text{id} \oplus \tau \simeq \tau$, it suffices(*) to give an isomorphism of right R-mods

$$\eta : R \oplus R \simeq R$$

such that $\eta(x \oplus \varphi(x)) = \varphi(x)\eta$. But

$$\eta(r_1 \oplus r_2) = \iota_1 r_1 + \iota_2 r_2$$

works.

(*). $\tau(P) = P \otimes_{R\varphi} R$ hence

$$P \oplus \tau(P) = P \otimes_R (R \oplus_\varphi R) \overset{id \otimes \gamma}{\simeq} P \otimes_{R\varphi} R = \tau(P).$$

As a consequence of this argument it follows that any subring of $R = \mathrm{End}(\Gamma M)$ containing i_1, i_2, p_1, p_2 and closed under φ is also flask. So we should figure out the minimal gadget.

Karoubi's cone CA: ~~He takes inside of~~ Interpret $\mathrm{End}(\Gamma A)$ as the ring of matrices (a_{ij}) with finite columns. Inside this one has the set of matrices with at most a single 1 in each row + column – partial permutation matrices, i.e. isom of one subset of N with another. This set of matrices is a monoid and includes i_1, i_2, p_1, p_2 and is closed under φ. CA is the set of A-linear combinations of such matrices. It is thus the smallest subring of $\mathrm{End}(\Gamma A)$ containing the partial permutation operators and $\Gamma(a)$ for any $a \in A$. And it is flask, by what we've proved.

Next point: Define $SA = CA/$ideal of finite matrices. One has

$$\mathcal{P}(A) \longrightarrow \mathcal{P}(CA) \xrightarrow{\pi} \mathcal{P}(SA)$$

where ~~[illegible]~~ π is induced by the surjection $CA \twoheadrightarrow SA$. One ~~[illegible]~~ has an exact sequence

$$K_1 CA \longrightarrow K_1 SA \longrightarrow K_0(\pi) \longrightarrow K_0 CA$$

where $K_0(\pi)$ is generated by triples (E, F, α) $E, F \in \mathcal{P}(CA)$, $\alpha : \pi E \xrightarrow{\sim} \pi F$, modulo the relations of isom., direct sum, and vanishing of (E, E, id).

One has

$$K_0 A \longrightarrow K_0(\pi)$$
$$P \longmapsto (P, P, 0)$$

and the point is to prove this is an ~~[illegible]~~ isomorphism. Thus I have to define

$$\text{Index} : K_0(\pi) \longrightarrow K_0 A$$

So I start with a "Fredholm operator" i.e. a homom. $u : E \to F$ in $\mathcal{P}(CA)$ which is invertible in $\mathcal{P}(SA)$, hence $\exists$ $v : F \to E$ such that $vu - id_E$, $uv - id_F$ factor through an object of $\mathcal{P}(A)$.

~~[illegible]~~ Check this last statement. A map $E \to F$

in ~~S~~ $\mathcal{P}(CA)$ is zero in SA iff it factors $E \to P \to F$ with P in $\mathcal{P}(A)$. Only have to show $\Rightarrow$. Can reduce to the case of $E = (CA)^m$, $F = (CA)^n$, hence to the case of an element x of CA. Then x is in the ideal of finite matrices, so it is clear.

So we have $u : E \to F$ Fredholm. If u is surjective, then $\mathrm{Ker}(u)$ exists in $\mathcal{P}(CA)$ and the identity map from $\mathrm{Ker}(u)$ to itself factors through a P in $\mathcal{P}(A)$, hence $\mathrm{Ker}(u)$ is in $\mathcal{P}(A)$. Thus I can define $\mathrm{Ind}(u) = [\mathrm{Ker}(u)] \in K_0 A$.

Given a general Fredholm $u : E \to F$, I want to find $P \to F$ such that $E \oplus P \to F$ is onto. I have $v : F \to E$ such that $uv - id_F : F \to F$ is "compact, hence it factors through some P.

$$uv - id_F = \text{comp}\quad F \xrightarrow{q} P \xrightarrow{r} F$$

But then $E \oplus P \xrightarrow{u,r} F$ is onto, because $\forall f \in F$, $f = u(v(f)) + f - uvf \in \mathrm{Im}\, u + \mathrm{Im}\, r$. Having chosen such a ~~map~~ couple P, r I define

$$\mathrm{Ind}(u) = [\mathrm{Ker}(u+r)] - [P].$$

To see well-defined suppose P', r' is another couple.

where $(P \oplus P', r+r')$ is also.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}(u+r) \arrow[r] \arrow[d] & E \oplus P \arrow[r, "u+r"] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(u+r+r') \arrow[r] \arrow[d] & E \oplus P \oplus P' \arrow[r] \arrow[d] & F \arrow[d] & \\
0 \arrow[r] & P' \arrow[r, Rightarrow] \arrow[d] & P' \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

so the index is well-defined.

~~Next suppose $u, u' : E \to F$ are two Fredholm maps such that $u - u'$ is compact. Ind u = Ind u' Choose $P \xrightarrow{r} F$ so that $u+r_0$, $u+r'$ are surjective. Reduces to case where both are surjective~~

Next show that Index adds for composition

$$\begin{tikzcd}
E \arrow[r, "u"] \arrow[d] & F \arrow[r, "v"] \arrow[d] & G \arrow[d, equal] \\
E \oplus P \oplus Q \arrow[r, "(u+r)\oplus id_Q", "\underset{u''}{=}"'] & F \oplus Q \arrow[r, "v+s", "\underset{v'}{=}"'] & G
\end{tikzcd}$$

$v' = v+s$

$u' = u+r$

$u'' = (u+r) \oplus id_Q$

Then
$$\begin{aligned}
\mathrm{Ind}(vu) &= [\mathrm{Ker}\, v'u''] - [P \oplus Q] \\
&= [\mathrm{Ker}\, u''] + [\mathrm{Ker}\, v'] - [P] - [Q] \\
&= [\mathrm{Ker}\, u'] - [P] + [\mathrm{Ker}\, v'] - [Q] \\
&= \mathrm{Ind}\, u + \mathrm{Ind}\, v.
\end{aligned}$$

Next suppose $u, u' : E \rightrightarrows F$ are two Fredholm operators with $u-u'$ compact. To show: $\mathrm{Ind}(u) = \mathrm{Ind}(u')$. Suppose $z = u'-u$ factors $E \xrightarrow{i} P \xrightarrow{j} F$. Then

$$E \xrightarrow{\Gamma_i} E\oplus P \xrightarrow{u+j} F \qquad \text{gives } u'$$

whereas $(u+j)\Gamma_0 = u$. Using the fact that Index adds, we reduce to showing Γ_i, Γ_0 have same index, this being clear as $pr_1 \Gamma_i = pr_1 \Gamma_0$.

Thus we have defined

$$\mathrm{Ind} : K_0(\pi) \longrightarrow K_0 A$$

If $\gamma : K_0 A \to K_0(\pi)$ sends P to $[P, 0, 0]$, then it is clear that $\mathrm{Ind}\circ\gamma = \mathrm{id}$. On the other hand, given $[E, F, \pi(u)]$ in $K_0(\pi)$, choosing $r : P \to F$ so that $u+r : E\oplus P \twoheadrightarrow F$, then we have in $K_0(\pi)$

$$[E,F,\pi(u)] = [E, E\oplus P, \pi(in_1)] + [E\oplus P, F, \pi(u+r)]$$

$$= [0, P, 0] + [\mathrm{Ker}\,\pi(u+r), 0, 0]$$

$$= \gamma\ \mathrm{Ind}\,[E, F, u].$$

Thus we win.

You have used the following:

<u>Lemma</u>: If $\pi : C \twoheadrightarrow S$, then in $K_0(\pi)$ we have
$$[E,F,\alpha] + [F,G,\beta] = [E,G,\beta\alpha].$$

$$\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}$$

<u>Proof</u>: $$[E,F,\alpha] + [F,G,\beta] = [E\oplus F, F\oplus G, \alpha\oplus\beta]$$
and $$[E,G,\alpha] + [F,F,id] = \left[E\oplus F, F\oplus G, \begin{pmatrix}0 & 1\\ \beta\alpha & 0\end{pmatrix}\right]$$

But $$\begin{pmatrix} & \beta^{-1}\\ \beta & \end{pmatrix}\begin{pmatrix}0 & 1\\ \beta\alpha & 0\end{pmatrix} = \begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}$$
and $\begin{pmatrix} & \beta^{-1}\\ \beta & \end{pmatrix} \in \mathrm{Aut}(\pi(F\oplus G))$ lifts to $\mathrm{Aut}(F\oplus G)$. QED.

So thus one proves Karoubi's theorem:

$$K_1(SA) \xrightarrow{\sim} K_0(A)$$
$$(E,\alpha) \longmapsto \mathrm{Ind}(\alpha).$$

An application is to define the residue
$$K_1(A[t,t^{-1}]) \longrightarrow K_0 A$$

For one can embed $A[t,t^{-1}] \subset SA$ by sending t to the shift operator.

What's involved in getting an exact sequence for K-groups is an ideal generated by idempotents. Suppose $C \xrightarrow{\pi} B$ is a surjective ring homomorphism such that the kernel is generated by idempotents in C. Then we might be able to generalize the preceding. Put $\mathcal{P}'$ for the full subcategory of $\mathcal{P}(C)$ consisting of P such that $\pi(P) = 0$. Now let $E \xrightarrow{u} F$ be a map in $\mathcal{P}(C)$ which becomes zero in $\mathcal{P}(B)$. I would like to show that u factors thru a $P \in \mathcal{P}'$. Since one has $E \xrightarrow{i} C^m \xrightarrow{p} E$ with $pi = \mathrm{id}_E$, it is clear that if up factors thru a P so does $upi = u$. Thus we can suppose $E = C^m$ and $F = C^n$ whence u is given by a matrix with coefficients in I. If $m = n = 1$, $u : C \to C$ is left multiplication by an element of I and so I want uC to be contained in an ideal eC where e is an idempotent killed by π. The condition seems to be

(*) Any finite subset of I is contained in an ideal of the form eC, e an idempotent in I

October 29, 1974 Periodicity modulo m.

Work over an alg. closed field F, where we know $K_2(F, \mathbb{Z}/m\mathbb{Z}) = \mu_m$ $\quad (m, \text{ch } F) = 1$. Thus if ζ generates μ_m, multiplication by ζ gives a map $K_i(F, \mathbb{Z}/m\mathbb{Z}) \longrightarrow K_{i+2}(F, \mathbb{Z}/m\mathbb{Z})$ which I want to show is an isomorphism.

In top. K-theory one has the Bott map

$$\mathbb{Z} \times BU \longrightarrow \Omega(U)$$

which ought to be able to be interpreted as cupping with $\beta \in K^{-2}(pt)$. In alg. K-theory since $K_2 F$ is a $\mathbb{Q}$-vector space, it will be possible to define this map only modulo m.

The procedure should go somewhat as follows. First of all multiplication by $\zeta \in K_1 F$ defines a map $K_i F \longrightarrow K_{i+1} F$ which I should be able to see as a map either

$$Q \longrightarrow \mathbb{Z} \times BGL(F)^+$$

or $\quad \mathbb{Z} \times BGL(F)^+ \longrightarrow \widetilde{GL(F)}$

(Introduce notation $U = \widetilde{GL(F)}$, $B = \mathbb{Z} \times BGL(F)^+$). Now this should induce a map

$$\begin{tikzcd} Q \arrow[r, "m"] & Q \arrow[r] \arrow[d] & Q/mQ \arrow[dll, dashed] \\ \Omega(B/mB) \arrow[r] & B \arrow[r, "m"] & B \end{tikzcd}$$

which is the map we are interested in.

Bott map topologically $G_n \longrightarrow \Omega(U_{2n})$ works by associating to an n-plane W in $\mathbb{C}^{2n}$ the path $t \mapsto e^{2\pi i t} E_W \oplus E_{W^\perp}$ in U_{2n}.

Idea concerning this map: One can classify the [redacted] homomorphisms $G_m \longrightarrow GL_n$. One could try to construct a [redacted] model for [redacted] $\Omega(GL_n)$ using 1-parameter subgroups. [redacted]

November 2, 1974. Period theorem (continued)

I think it is a reasonable goal to prove the Bott periodicity theorem ~~[illegible]~~ in the context of algebraic geometry + etale homotopy. What I mean is that I should be able to show an equivalence between $\Omega(GL_n)_{et}$ and $G_n(F^N)_{et}$ away from the characteristic, without having to use the comparison with classical topology.

I think I already understand the Atiyah-Singer proof of periodicity which goes as follows. Let $\mathcal{U}$ = unitary operators $\equiv 1$ mod compacts, $\mathcal{T} = \{$self adj A, $0 \leq A \leq 1$, ess. spec. $\{0,1\}\}$, $\mathcal{P}$ = orth projectors E in H, $E \not\equiv 0, 1$ mod compacts, $\bar{\mathcal{P}}$ = orth proj. e in $\mathcal{L}/\mathcal{K}$, $e \neq 0,1$. Then $\mathcal{T} \longrightarrow \bar{\mathcal{P}}$ is a heq since the fibres are convex, and $\exp 2\pi i : \mathcal{T} \longrightarrow \mathcal{U}$ is a heq by stratification + the Kuiper theorem. Next let $\mathcal{G}$ = unitary group of $\mathcal{L}/\mathcal{K}$. One has fibrations

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{U}N \longrightarrow \mathcal{G}_{(0)} \longrightarrow 1$$

$$\mathcal{G} \longrightarrow \left\{ \begin{matrix} a \in \mathcal{L}/\mathcal{K} \\ a^*a = 1 \\ aa^* \neq 0,1 \end{matrix} \right\} \longrightarrow \bar{\mathcal{P}} \qquad \bar{\mathcal{P}} = \mathcal{G}/\mathcal{G}\times\mathcal{G}$$

which establish $\Omega\bar{\mathcal{P}} = \mathcal{G}$, $\Omega\mathcal{G} = \mathcal{U}$. Or I could use the fibrations

$$\mathcal{P}_{e_0} \longrightarrow \mathcal{P} \longrightarrow \overline{\mathcal{P}}$$

$$\mathcal{U}/\mathcal{U}(\underset{\mathrm{Im}}{E_0})\times\mathcal{U}(\mathrm{Im}\,1-E_0) = \mathcal{U}/\mathcal{U}^{E_0} = (\mathcal{P}_{e_0})_{E_0}$$

$$\mathcal{U}(\mathrm{Im}\,E_0) \longrightarrow \mathcal{U}/\mathcal{U}(\mathrm{Im}\,1-E_0) \longrightarrow (\mathcal{P}_{e_0})_{E_0}$$

to get

$$\Omega\overline{\mathcal{P}} = \mathcal{P}_e \qquad \Omega\mathcal{P}_e \simeq \mathcal{U}.$$

Atiyah - Bott proof goes as follows. One defines an explicit map

$$\Omega\, GL_n \longrightarrow \mathcal{F} = \text{Fredholm operators}$$

as follows. Given $S^1 \xrightarrow{\theta} GL_n$, interpret Hilbert space as $(H^2)^n$ $\quad H^2 =$ subspace of $L^2(S^1)$ gen. by z^n, $n \geq 0$. Then mult. by θ in $(L^2)^n$ followed by projection onto $(H^2)^n$ gives the required Fredholm operator.

<u>Question</u>: $GL_n(\mathbb{C}[z,z^{-1}]) = \mathrm{Maps}(\mathbb{G}_m, GL_n)$ is an inductive limit of affine varieties over $\mathbb{C}$. Is there some sense in which this can be interpreted as the loop space of GL_n?

Suppose I take

$$\alpha(z)=\sum a_n z^n \qquad \alpha_n \in M_r(\mathbb{C})$$

such that $\alpha(z)$ is invertible for $|z|=1$. Suppose $\alpha(z)$ is a polynomial of degree $\leq d$. Then to α I can associate over $\mathbb{P}^1$ a map

$$\mathcal{O}^r \xrightarrow{A} \mathcal{O}(d)^r$$

where I homogenize α according to $z \mapsto \frac{z_1}{z_0}$

$$z_0^d \, \alpha\left(\frac{z_1}{z_0}\right) = \sum a_n z_0^{d-n} z_1^n$$

The cokernel M of A is a torsion module over $\mathbb{P}^1\mathbb{C}$ with support not meeting $|z|=1$. We can write

$$M = M_0 \oplus M_\infty$$

where M_0 has support in $|z|<1$, and M_1 has support in $|z|>1$. Then put

$$E = \text{Kernel } \mathcal{O}(d)^r \twoheadrightarrow M \twoheadrightarrow M_\infty$$

whence we have factored A into

$$\mathcal{O}^r \xrightarrow{A_0} E \xrightarrow{A_\infty} \mathcal{O}(d)^r$$

where $\mathrm{Coker}(A_\infty) = M_\infty$. In fact we have the picture

$$\begin{tikzcd}
 & \mathcal{O}(d)^r & \\
E' \arrow[ur, hook, "M_0"] & & E \arrow[ul, hook', "M_\infty"'] \\
 & \mathcal{O}^r \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Now $\quad E = \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_r) \quad$ where

$$0 \leq p_1 \leq \cdots \leq p_r \leq d$$

Choosing such an isomorphism

$$A_0 : \mathcal{O}^r \longrightarrow \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_r)$$

becomes a matrix $\quad a_{ij}(z) \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(p_i))$
of polynomials with a_{ij} of degree p_i.
Better

$$A_0 = \left(a_{ij}(z_0, z_1) \right) \qquad a_{ij} \text{ homog. of degree } p_i$$

$$A_\infty = \left(b_{ij}(z_0, z_1) \right) \qquad b_{ij} \text{ ——— } d - p_j$$

Basic question is now whether I can dehomogenize so that A_0 is not singular for $|z| \geq 1$ including ∞ and, A_∞ is non-sing for $|z| \leq 1$.

Review a little prediction theory:

One is given a Hilbert space H with a unitary operator U and a vector v_0 which one might as well suppose to be cyclic. Then we get a measure μ on S^1 such that

$$H \cong L^2(S^1, \mu)$$
$$v \longmapsto 1$$
$$U \longmapsto \text{mult. by } z^{-1}$$

The "prediction" problem is to construct the orthogonal complement of $V_1 = \langle U^{-1}v_0, U^{-2}v_1, \dots \rangle$ in $V_0 = \langle v_0, U^{-1}v_0, \dots \rangle$. Let e_0 generate this complement, so that in the $L^2(S^1, \mu)$ description

$$e_0(z) = a_0 + a_1 z + a_2 z^{+2} + \cdots \qquad a_0 \neq 0$$

$$(*) \qquad \big(e_0, \underset{z^i}{\underbrace{U^{-i}v_0}}\big) = \int e_0(z)\, \bar{z}^i \, d\mu \qquad i \geq 1.$$

Let me suppose $d\mu = f\, dz$ with f smooth > 0. Then $(*) \Rightarrow$

$$e_0(z) f(z) = \sum_{n \geq 0} b_n z^{-n} = c_1\left(\tfrac{1}{z}\right)$$

Thus being given $f(z)$ smooth > 0, the problem of finding $e_0(z)$ is the same as factoring

f into the product of $e_0(z)^{-1}$ which is holomorphic in $|z| \leq 1$ and $e_1(\frac{1}{z})$ which is holomorphic in $|z| \geq 1$. The uniqueness of this factorization is clear for if

$$\frac{e_1(\frac{1}{z})}{e_0(z)} = \frac{\tilde{e}_1(\frac{1}{z})}{\tilde{e}_0(z)}$$

then $\dfrac{\tilde{e}_0(z)}{e_0(z)} = \dfrac{\tilde{e}_1(\frac{1}{z})}{e_1(\frac{1}{z})}$ would be entire on $P^1(\mathbb{C})$ hence a constant.

Method for constructing the factorization of f:
Take $\log f(z)$ and expand into a Fourier series and split into positive + negative powers of z

$$\log f(z) = g_0(z) + g_1(\tfrac{1}{z}).$$

$$f(z) = e^{g_0(z)}\, e^{g_0(1/z)}.$$

In fact because $\log f(z)$ is real $g_1(\frac{1}{z}) = \overline{g_0(z)}$ determines $g_0(z)$ up to ia, $a \in \mathbb{R}$, hence $f(z) = |e^{g_0(z)}|^2$ where $e^{g_0(z)}$ is unique up to a scalar in S^1.

Suppose a smooth function $\alpha : S^1 \longrightarrow \mathbb{C}^*$ is given. It has a winding number m and the function $z \longmapsto z^{-m}\alpha(z)$ has winding number zero, hence it lifts to

$$z \mapsto \log\left(z^{-m}\alpha(z)\right) \qquad S^1 \longrightarrow \mathbb{C}$$

unique up to adding $2\pi i n$ $\quad n \in \mathbb{Z}$. Now

$$\log\left(z^{-m}\alpha(z)\right) = \sum_{n\in\mathbb{Z}} a_n z^n$$

(Fourier expansion)

$$= \sum_{n\geq 0} a_n z^n + \Big(\sum_{n<0} a_n z^{1-n}\Big) z^{-1}$$

$$= f_0(z) + \frac{1}{z} f_1\left(\frac{1}{z}\right)$$

where f_0, f_1 have analytic extensions to $|z|<1$. Thus

$$z^{-m}\alpha(z) = e^{f_0(z)}\, e^{\frac{1}{z} f_1\left(\frac{1}{z}\right)}$$

$$= \alpha_0(z)\, \alpha_\infty(z)$$

where $\alpha_0(z)$ extends to a holom. map $|z|<1 \longrightarrow \mathbb{C}^*$ and $\alpha_\infty(z)$ extends to a holom. map $|z|>1$ to $\mathbb{C}^*$ (including ∞). Hence

$$\alpha(z) = z^m \alpha_0(z)\, \alpha_\infty(z)$$

and this factorization has been normalized by requiring $\alpha_\infty(\infty) = 1$.

<u>Example:</u> If $\alpha(z)$ is a polynomial, or more generally a rational function, then this factorization gets done as follows. $\alpha(z)$ is a product of the following sorts of degree 1 things:

i) $\alpha(z) = z - \lambda \qquad 1 < |\lambda| < \infty$

$$\alpha(z) = (z)^0 (z-\lambda)(1)$$

ii) $\alpha(z) = z - \lambda \qquad |\lambda| < 1$

$$\alpha(z) = z^1 (1)\left(1 - \frac{\lambda}{z}\right)$$

Thus if $\alpha(z) = \prod (z - \lambda_i)$, the factorization is

$$\alpha(z) = z^m \prod_{|\lambda_i| > 1} (z - \lambda_i) \prod_{|\lambda_i| < 1}\left(1 - \frac{\lambda_i}{z}\right) \qquad m = \text{no. of } |\lambda_i| < 1$$

Question: Given $\underset{z}{\overset{S^1}{\cup\!\!\shortmid}} \longmapsto f(z)$ (strictly) positive self-adjoint operators on W, can I find $g(z) \in \mathrm{End}(W)$ holomorphic $|z| < 1$ such that $f(z) = g^*(z)\, g(z)$.

Idea is to consider the set of Laurent polynomials $W[z, z^{-1}]$ with coefficients in W, and to make this into a pre-Hilbert space by setting

$$\langle z^i w, z^j w' \rangle_H = \int z^{i-j} \left(f(z) w', w\right)_W dz$$

Then one completes to get a Hilbert space H, a unitary operator U given by multiplying by z^{-1}. Let V be the closure of $W[z]$ in H. Then $U^{-1}V$ is the closure of $zW[z]$ so it is clear that $V = W + U^{-1}V$, at least if W is finite-dimensional.

~~Assume that $V = W \oplus U^{-1}V$~~

~~Then for each $w \in W$ we have a unique vector $e(w) \in V$ such that $e(w) - w \in U^{-1}V$ and $e(w)$ is $\perp$ to $U^{-1}V$. Presumably $e(w) = \sum_{n \geq 0} a_n(w) z^n$ with $a_0(w) = w$. Then~~

Let X be the orthogonal complement of $U^{-1}V$ in V. Then $L^2(S^1; X) \xrightarrow{\sim} H$. (?) Precisely any vector in H can be described as a series $\sum x_n z^n$ with $\sum |x_n|^2 < \infty$. So for each $w \in W$ we get a series

$$w = \sum_{n \geq 0} x_n^{(w)} z^n$$

i.e. we get a function $w, z \longmapsto \sum x_n(w) z^n$ which we denote $z \longmapsto g(z) \in \mathrm{Hom}(W, X)$. This function has the property that

$$\langle P(z) w, w' \rangle_H = \int P(z) \big(f(z) w, w' \big)_W \, dz$$

$$= \int \left(P(z)\, g(z) w,\ g(z) w'\right)_X dz$$

$$= \int P(z) \left(g(z)^* g(z) w,\ w'\right)_X dz$$

for any Laurent poly. $P(z)$. Thus $\forall w, w'$

$$\left(f(z)\, w, w'\right)_W = \left(g(z)^* g(z)\, w, w'\right)_X$$

so indeed
$$f(z) = g(z)^* g(z)$$
as was to be shown.

In the preceding there is the technical problem of whether $L^2(S^1; X)$ in fact gives all of H. (In the one-dimensional case one ~~[illegible]~~ encounters this problem with absolute continuity of μ. The point is that when one has the measure $\lambda \mapsto \int \lambda\, \overline{e_0(z)}\, d\mu(z)$ on S^1 annihilating z^i, $i \geq 1$, there is a theorem telling you that the measure $\overline{e_0(z)}\, d\mu(z)$ is absolutely continuous with respect to Lebesgue measure and extends holom. in $\frac{1}{z}$ for $|\frac{1}{z}| < 1$.

Factoring $\alpha : S^1 \longrightarrow GL_n(\mathbb{C})$.

<u>Assertion</u>: Let $\alpha(z) = \sum \alpha_\nu z^\nu$ $\alpha_n \in \mathrm{End}(W)$ be a Laurent polynomial with matrix coefficients such that $\alpha(z)$ is invertible for $|z| = 1$. Then α can be factored

$$\alpha(z) = \alpha_0(z) \begin{pmatrix} z^{r_1} & & \\ & \ddots & \\ & & z^{r_n} \end{pmatrix} \alpha_\infty(z)$$

where $r_1 \geq \cdots \geq r_n$, where $\alpha_0(z), \alpha_\infty(z)$ are matrix valued Laurent polys, α_0 holom for $|z| \leq 1$, α_∞ holom for $|\frac{1}{z}| \leq 1$. (Hence α_0 is a polynomial in z and α_∞ is a polynomial in z^{-1})

Proof. Put $z = \frac{z_1}{z_0}$ and form $\alpha(z) = \alpha\left(\frac{z_1}{z_0}\right)$ and clear denominators by multiplying by $z_0^a z_1^b$; one gets

$$z_0^a z_1^b \, \alpha\left(\frac{z_1}{z_0}\right) = A(z_0, z_1) = \sum \alpha_\nu z_0^{a-\nu} z_1^{b+\nu}$$

a matrix of homogeneous polynomials in z_0, z_1 of degree $d = a+b$. I can interpret A as a map of vector bundles over $\mathbb{P}^1$

$$\mathcal{O}\cancel{(+d)} \otimes W \xrightarrow{A} \mathcal{O}(d) \otimes W$$

whose cokernel is a torsion module F with support outside $|z| = 1$. Then $F = F_+ \oplus F_-$

where $\operatorname{Supp} F_+ \subset \{|z|<1\}$, $\operatorname{Supp} F_- \subset \{|z|>1\}$.
If $\mathcal{E} = \operatorname{Ker}\{\mathcal{O}\otimes W \twoheadrightarrow F \twoheadrightarrow F_-\}$, then we have a factorization of A

$$\mathcal{O} \otimes W \xrightarrow{A_+} \mathcal{E} \xrightarrow{A_-} \mathcal{O}^{(d)} \otimes W$$

where $\operatorname{Coker} A_- = F_-$, $\operatorname{Coker} A_+ \cong F_+$.

One knows from the structure of vector bundles on $\mathbb{P}^1$ that

$$\mathcal{E} \cong \mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n)$$
$$d \geq p_1 \geq \cdots \geq p_n \geq 0$$

Choose a basis $e_1, \dots, e_n$ for W whence

$$\mathcal{O}(p_1) \oplus \cdots \oplus \mathcal{O}(p_n) \xrightarrow{A_-} \mathcal{O}^{(d)} \oplus \cdots \oplus \mathcal{O}(d)$$

is a matrix with entries

$$(A_-)_{ij} : \mathcal{O}(p_j) \longrightarrow \mathcal{O}(d).$$

Thus (A_-) is a matrix of forms $((A_-)_{ij})$ with $\deg (A_-)_{ij} = d - p_j$. Improve notation $A_- \mapsto A^-$.

Therefore we have a factorization

$$A = A^- A^+$$

$$\deg (A_{ij}) = d \;, \quad \deg (A^-_{ij}) = d - p_j \;, \quad \deg (A^+_{jk}) = p_j$$

$$A_{ij}(z_0,z_1) = \sum_k A^-_{ik}(z_0,z_1) \quad A^+_{kj}(z_0,z_1)$$

Have to dehomogenize somehow. Take the case ~~when~~ $n=1$ whence $A = A^- A^+$ where $\deg A^- = d-p$, $\deg A^+ = p$. A^- is a polynomial having no zeroes for $|z| \leq 1$. If we put $z_1 = z z_0$ then $A^-(z_0, z z_0) = z_0^{d-p} A^-(1,z)$, so $A^-(1,z)$ is a polynomial in z of degree $\leq d-p$ not vanishing in $|z| \leq 1$. A^+ is a polynomial having no zeroes for $|z| \geq 1$, i.e. $A^+(\frac{1}{z},1) = z^{-p} A^+(1,z)$ is a polynomial in $\frac{1}{z}$ having no zeroes for $|\frac{1}{z}| \leq 1$. Thus in the general case

$$\cancel{z^d A_{ij}(1,z) = \sum_k z^{d-p_k} A^-_{ik}(1,z)\, z^{p_k} A^+_{kj}(1,z)}$$

$$A_{ij}(1,z) = \sum_k \quad A^-_{ik}(1,z)\, z^{p_k}\, A^+_{kj}(\tfrac{1}{z},1)$$

so

$$\boxed{\alpha(z)_{ij} = z^{-b} A_{ij}(1,z) = \sum_k A^-_{ik}(1,z)\, z^{-b+p_k} A^+_{kj}(\tfrac{1}{z},1)}$$

hence we have our factorization

$$\alpha(z) = \alpha_0(z) \begin{pmatrix} z^{r_1} & & \\ & \ddots & \\ & & z^{r_n} \end{pmatrix} \alpha_\infty(\tfrac{1}{z})$$

as desired.

Another proof of this factorization. To simplify suppose $\alpha \in GL_n(F[z,z^{-1}])$. Then I will consider $F[z]$ lattices in $F[z,z^{-1}]^n$. $GL_n(F[z,z^{-1}])$ acts transitively on these lattices, so

$$GL_n(F[z,z^{-1}])\big/ GL_n(F[z]) = \text{set of lattices } \mathcal{L}.$$

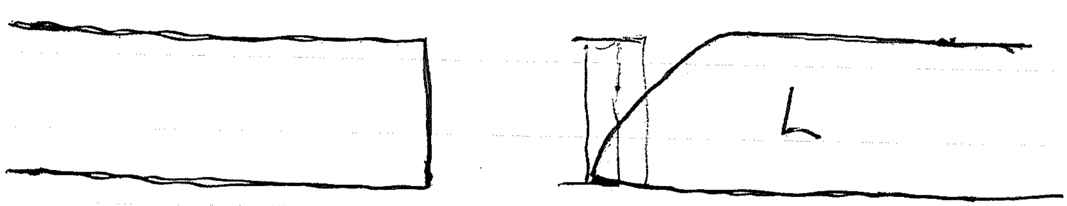
But now I want to let $GL_n(F[z^{-1}])$ act on $\mathcal{L}$.

Change notation. Let $\mathcal{L}$ be the set of $F[z^{-1}]$ lattices in $F[z,z^{-1}]^n$. Then

$$GL_n(F[z,z^{-1}])\big/ GL_n(F[z^{-1}]) \xrightarrow{\sim} \mathcal{L}$$

$$g\, GL_n(F[z^{-1}]) \longmapsto g\, F[z^{-1}]^n$$

Now let $GL_n(F[z])$ act on $\mathcal{L}$. $GL_n(F[z])$ is the stabilizer of $F[z]^n = \Lambda_0$. Given $L \in \mathcal{L}$ one considers the intersections $L \cap z^{-n}\Lambda_0$.



Suppose r such that $L \cap z^{-r+1}\Lambda_0 = 0$, $L \cap z^{-r}\Lambda_0 \neq 0$.

Then let $Z = L \cap z^{-r}\Lambda_0 \neq 0$. It should be clear that Z generates a direct summand of L, $z^r Z$ generates a direct summand of Λ_0, etc. This follows because the pair (Λ_0, L) determines a vector bundle E on $\mathbb{P}^1$, and the pair $(z^{-r}\Lambda_0, L)$ determines the bundle $E(r)$, etc. The net result is that under the action of $GL_n F[z]$, L has the canonical form

$$L = z^{p_1} F[z^{-1}] \oplus \cdots \oplus z^{p_n} F[z^{-1}]$$

with $p_1 \geq p_2 \geq \cdots \geq p_n$. If these integers are arranged $p_1 = \cdots = p_{a_1} > p_{a_1+1} = \cdots = p_{a_1+a_2} > \cdots\cdots$

then the stabilizer of L is the subgroup

$$\begin{pmatrix} GL_{a_1}(F) & \begin{pmatrix} \text{polys} \\ \deg \leq p_{a_1} - p_{a_1+a_2} \end{pmatrix} & \begin{pmatrix} \quad \end{pmatrix} \\ & GL_{a_2}(F) & \begin{pmatrix} \deg \leq \\ p_{a_1+a_2+a_3} - p_{a_1+a_2} \end{pmatrix} \\ & & GL_{a_3}(F) \end{pmatrix}$$

Scattering theory version: We are given a Hilbert space H with a unitary operator V.
By an outgoing subspace one means one D_+ closed under V such that $0 = \bigcap V^{+n} D_+$, $H = \overline{\bigcup V^n D_+}$.
One sees that if $N = D_+ \ominus V D_+$, then

$$H = L^2(S^1; N)$$
$$V = \text{mult. by } z$$
$$D_+ = H^2(S^1; N).$$

(Thus H, D_+ are analogous to a $F[z,z^{-1}]$ module, free of f.t., and a $F[z]$-lattice)

Now let there be given an incoming subspace D_- orthogonal to D_+, or what amounts to the same thing, an outgoing subspace $(D_-)^\perp$ containing D_+. Then we have

$$\begin{array}{ccccccc} L^2(S^1,N') & \simeq & H & & \simeq & L^2(S^1;N) \\ \cup & & \cup & \supset & & \cup \\ H^2(S^1;N') & \simeq & (D_-)^\perp & & D_+ \simeq & H^2(S^1;N) \end{array}$$

and this isomorphism $L^2(S^1,N') \xleftarrow{\sim} L^2(S^1;N)$ has to be given by multiplying by a function

$$S^1 \ni z \longmapsto S(z) \in \text{Unitary maps}(N, N')$$

which extends to a holomorphic invertible matrix for $|z|>1$
Then $S(z)$ is the scattering matrix.

Given ~~[illegible]~~ S^{-1} $z \longmapsto \alpha(z) \in \mathrm{Isom}(N,N)$,
one gets a new incoming space $\alpha(D_+)^\perp$ ~~[illegible]~~
which if α is Laurent polynomial will be such that

$$z^{-N}\alpha(z)(D_+)^\perp \supset (D_+)^\perp$$?

hence

$$z^{-N}\alpha(z)(D_+)^\perp = S(z)(D_+)^\perp$$

where $S(z)$ is holom. for $1<|z|<\infty$. Then

$$S(z)^{-1}z^{-N}\alpha(z)(D_+)^\perp = (D_+)^\perp$$

so

$$S(z)^{-1}z^{-N}\alpha(z) = \alpha_0(z) \quad \text{holom. for } |z|<1.$$

hence

$$\alpha(z) = \alpha_0(z)\, z^N S(z)$$

and this is the factorization you want, except for the fact that $S(z)$ has poles at $z=\infty$, and that to take these out one has to remove the diagonal matrix $\begin{pmatrix} z^{-\Delta_1} & & \\ & \ddots & \\ & & z^{-\Delta_k}\end{pmatrix}$.

Let $z \longmapsto \alpha(z)$ be a map $S^1 \longrightarrow U_n = \mathrm{Isom}$ unitary (N, N), whence multiplication by α is a unitary operator on $H = L^2(S^1; N)$. (In effect $\int |\alpha(z) f(z)|^2 dz = \int |f(z)|^2 dz$.) Seems to be enough that α be measurable for this operator to exist.) Recall $D_+ = H^2(S^1; N)$. αD_+ is another outgoing subspace. Actually any (not necessarily unitary) automorphism of H commuting with V acts on the set of outgoing subspaces, because these are defined in terms of the topology + linear structure of H.

Suppose $\alpha(z)$ is a Laurent polynomial in z. Then $z^\nu \alpha(z)$ is a polynomial in z, hence $z^\nu \alpha(z) D_+ \subset D_+$.

Next suppose $\alpha(z) D_+ \subset D_+$. Let $e_1, \dots, e_n$ be a basis for N so that $\alpha(z)$ is a matrix of Laurent polynomials. Then $\alpha_{ij}(z) \cdot 1 \in H^2(S^1; N)$, hence $\alpha_{ij}(z)$ is analytic for $|z| < 1$, hence ~~also for $|z| \le 1$, as it is a Laurent polynomial without pole on $|z| = 1$. Thus $\alpha(z) \in Gl_n(\mathbb{C}[z])$~~ $\alpha(z)$ is a polynomial in z.

If $\alpha(z) D_+ = D_+$, then $\alpha(z)^{-1}$ is analytic for $|z| = 1$ (because I am assuming $\alpha(z)$ is invertible for $|z| = 1$) and analytic for $|z| < 1$, hence

$\alpha(z)$ is a matrix polynomial invertible for $|z| \leq 1$.

Conversely if $\alpha : S^1 \longrightarrow GL_n$ is analytic for $|z| \leq 1$, then $\alpha D_+ = D_+$. In particular this holds if α is a matrix polynomial which is invertible for $|z| \leq 1$.

So if $\alpha(z) = \sum \alpha_k z^k$ is a matrix of Laurent polynomials invertible for $|z| = 1$, then I recall one knows that the operator

$$f \longmapsto P(\alpha f) \qquad\qquad f \in H^2(S^1)^n = D_+$$

$P =$ projection on $D_+ = H^2(S^1)$, is Fredholm.* Hence if $\alpha(z)$ is a polynomial, we have $P(\alpha f) = \alpha f$, so αD_+ is of finite index in D_+.

~~(* Start with the operators $M_n(\mathbb{C}[z])$ acting on D_+ by multiplication. Then the operator Pz^{-1} is an inverse mod compacts for z, hence $\alpha \mapsto P\alpha$ is going to be a homomorphism from $M_n(\mathbb{C}[z, z^{-1}])$ into the Calkin algebra.~~ ?

(* The point is that the map $\alpha \mapsto P\alpha$ into the Calkin algebra is a ring homomorphism from functions on S^1. Norm decreasing hence the continuous extension of what happens [redacted] for [redacted] Laurent polynomials.)

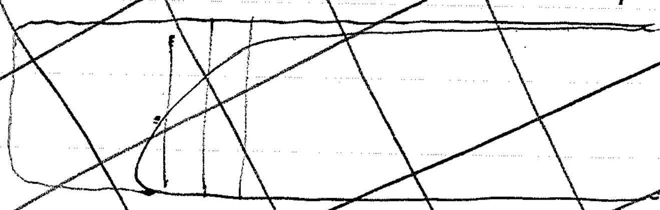
So at the moment I have a Laurent polynomial matrix $\alpha \in M_n(\mathbb{C}[z, z^{-1}])$ such that $\alpha(z)$ is invertible for $|z| = 1$. Assume to begin with that $\alpha \in M_n(\mathbb{C}[z])$. Then I know that αD_+ is of finite codimension in D_+, whence $D_+/\alpha D_+$ is a $\mathbb{C}[z]$-module of finite length with support in $|z| < 1$. Hence

$$D_+/\alpha D_+ = \bigoplus_{i=1}^{n} \mathbb{C}[z]/(z-\lambda_i)^{p_i} \qquad |\lambda_i| < 1$$

where $\quad p_1 \geq p_2 \geq \cdots \geq p_n$

~~Better perhaps to proceed to filter αD_+ by $z^i D_+$.~~

~~αD_+~~



~~If $f(z)$ kills $D_+/\alpha D_+$ where $f(z) = \prod (z-\lambda_i)$ $|\lambda_i| < 1$. But~~

Repeat: I know that on D_+ multiplication by $z-\lambda$ for $|\lambda| > 1$ is an isomorphism, since $\frac{1}{z-\lambda}$ is holom. for $|z| \leq 1$

hence carries D_+ into itself. And I also know that for $|\lambda| < 1$ $\frac{z-\lambda}{z} = 1 - \frac{\lambda}{z}$ is an isomorphism. NO.

αD_+ is an outgoing subspace contained in D_+. According to scattering theory we therefore have a scattering matrix $S(z)$ unitary for $|z| = 1$ and holomorphic for $|z| > 1$ such that

$$S(z)\, \alpha D_+ = D_+ \quad ?$$

whence $S(z)\, \alpha(z) = \beta(z)$ $\qquad$ β holom. inv. for $|z| \leq 1$

$$\alpha(z) = S^{-1}(z)\, \beta(z)$$

Finally the polar behavior of $S(z)$ should give the exponents I want.

Bott's remark: Consider the Bott map $\mathbb{P}(\mathbb{C}^n) \to \Omega Gl_n$ which associates to a line L the path $zP + (1-P)$ where $P =$ orth. projection on L. Bott has proved that $H_*(\Omega U_n) = \mathbb{Z}[t_0^{-1}, t_0, t_1, \ldots, t_{n-1}]$ where t_i is the image of the generator of $H_{2i}\,\mathbb{P}(\mathbb{C}^n)$. Here $H_*(\Omega U_n)$ is a ring with the Pontryagin product obtained from the product on ΩU_n, which one can take to be pointwise product of loops in U_n. Since the ~~map~~

$$\widetilde{GL}_n(\mathbb{C}[z,z^{-1}]) \longrightarrow \Omega U_n$$

($\sim$ denotes paths $\alpha(z)$ such that $\alpha(1) = 1$) is a group homomorphism, it follows from Bott's thm. that this map is at least surjective on homology.

Thus the conjecture that $\widetilde{GL}_n(\mathbb{C}[z,z^{-1}]) \to \Omega U_n$ be a homotopy equivalence is apt to be true. What will we be able to say about $\widetilde{GL}_n(\mathbb{C}[z,z^{-1}])$. Recall that

$$GL_n(\mathbb{C}[z,z^{-1}]) / GL_n(\mathbb{C}[z^{-1}])$$

can be identified with the set $\mathcal{L}$ of $\mathbb{C}[z^{-1}]$ lattices in $\mathbb{C}[z,z^{-1}]^n$. Note that $GL_n(\mathbb{C}[z,z^{-1}]) = \widetilde{GL}_n(\mathbb{C}[z,z^{-1}])\,GL_n$ so

$$\widetilde{GL}_n(\mathbb{C}[z,z^{-1}]) / \widetilde{GL}_n(\mathbb{C}[z^{-1}]) = \mathcal{L}.$$

Notice that $\mathcal{L}$ can be viewed as the set of lattices in $\mathbb{C}[[z^{-1}]][z]^n$ for $\mathbb{C}[[z^{-1}]]$, i.e. it is local for $z=\infty$. Also, if I fix Λ a $\mathbb{C}[z]$-lattice in $\mathbb{C}[z,z^{-1}]^n$, then I can view $\mathcal{L}$ as the set of extensions of Λ to a v.b. over $\mathbb{P}_1$.

Since $\widetilde{GL}_n(\mathbb{C}[z,z^{-1}])/\widetilde{GL}_n(\mathbb{C}[z^{-1}]) = \mathcal{L}$ and $\widetilde{GL}_n(\mathbb{C}[z])$ is evidently contractible, it follows from the conjecture that $\mathcal{L}$ has the homotopy type of ΩU_n. Topology on $\mathcal{L}$. Fix a lattice L_0. Then any other L is sandwiched $z^m L_0 \supset L \supset z^{-m} L_0$, for some m. Thus it seems that if we let $\mathcal{L}_m$ be the subset of those L, then $\mathcal{L}_m$ is the subspace of $\coprod_\nu G_\nu(z^m L_0/z^{-m}L_0)$ consisting of subspaces invariant under multiplication by z^{-1}.

Obvious question: Before we use as a model for $\mathbb{Z}\times BU$ the limit of $\coprod_\nu G_\nu(z^m L_0/z^{-m}L_0)$ as $m\to\infty$, that is, the set of subspaces W commensurate with L_0. Thus we get a map from $\mathcal{L}$ to $\mathbb{Z}\times BU$. The obvious question is to relate this to the Wiener-Hopf operator constructed by Atiyah.

Recall that construction. Start with $\alpha(z) \in GL_n(\mathbb{C}[z,z^{-1}])$, or more generally with $z \mapsto \alpha(z)$, $S^1 \longrightarrow Gl_n$. Then one has $D_+ = H^2(S^1)^n \subset L^2(S^1)^n$. Multiplication by α is an operator on $L^2(S^1)^n$, and so one gets the operator $f \longmapsto P(\alpha f)$, $P =$ proj. on D_+.

~~To $\alpha \in Gl_n(\mathbb{C}[z,z^{-1}])$ we associate to lattice $\alpha\,\mathbb{C}[z]^n \subset \mathcal{L}$~~

Start with $\alpha : S^1 \longrightarrow Gl_n$ given by a Laurent polynomial. Make α act by mult. on $L^2(S^1)^n$. Let $D_+ = H^2(S^1)^n$ be ~~[illegible]~~ our "basic lattice", so that we are interested in the map $\alpha \longmapsto \alpha D_+$. (Here we will be changing notation - now $\mathcal{L}$ will be the set of $\mathbb{C}[z]$-lattices in $\mathbb{C}[z,z^{-1}]$. Topologically $\mathcal{L}$ will roughly be "outgoing subspaces". Recall we want somehow for $\mathcal{L}$ to have the homotopy type $\Omega(Gl_n)$)

~~[illegible]~~ To α we have thus associated two things: ~~[illegible]~~ an outgoing subspace αD_+, and a Fredholm operator $T_\alpha = P\alpha$ on D_+, $P =$ projection on D_+. (Such an operator T_α is called a Toeplitz operator)

I want to interpret αD_+ as an element in a kind of Grassmannian of the form $\mathcal{P}_e$. Here is a way of "mapping" $\mathcal{P}_e$ into Fredholm operators so that the lattice αD_+ maps to the Fred. op. T_α.

Given a projector E congruent to P modulo compacts, choose an isom $\theta : D_+ = \operatorname{Im} P \xrightarrow{\sim} \operatorname{Im} E$. The choice of such a θ is innocuous by Kuiper's thm. Then one sends E to the operator $P\theta$ on D_+. Thus we have

$$\begin{array}{ccc} \mathcal{P}_e & \xleftarrow{\log} & \left\{ (E,\theta) \;\middle|\; \begin{array}{l} \theta : D_+ \xrightarrow{\sim} \operatorname{Im} E \\ P \equiv E \bmod K \end{array} \right\} \\ & & \downarrow \\ & & \text{Fredholm operators on } D_+ \end{array}$$

November 7, 1974. On ΩGL_n.

Let $\widetilde{GL}_n(\mathbb{C}[z,z^{-1}])$ denote the subgp of $GL_n(\mathbb{C}[z,z^{-1}])$ consisting of $\alpha(z)$ such that $\alpha(1)=1$. One has an evident map

$$(1) \qquad \widetilde{GL}_n(\mathbb{C}[z,z^{-1}]) \longrightarrow \Omega GL_n(\mathbb{C})$$

which is a group homomorphism if one uses the pointwise product of loops. I want to prove this map is a homotopy equivalence. ($\widetilde{GL}_n(\mathbb{C}[t,t^{-1}])$ is naturally an inductive limit of affine varieties.)

Bott's remark: We have a map

$$\mathbb{P}(\mathbb{C}^n) \longrightarrow \widetilde{GL}_n(\mathbb{C}[z,z^{-1}])$$

sending a line L into $zP_L + (1-P_L)$ where P_L is orth. proj. on L. If $b_i \in H_{2i}(\Omega GL_n)$ is the image of the gen. of $H_{2i}\mathbb{P}(\mathbb{C}^n)$ $0 \le i < n$, a theorem of Bott says that

$$H_*(\Omega GL_n) = \mathbb{Z}[b_0, .., b_{n-1}, b_0^{-1}].$$

Thus the map (1) induces a surjection on homology.

Let $\mathcal{L}$ be the set of lattices for $\mathbb{C}[z]$

inside of $\mathbb{C}[z,z^{-1}]^n$. More precisely, I mean those $\mathbb{C}[z]$-submodules L such that for some ν

$$z^{-\nu}\mathbb{C}[z]^n \supset L \supset z^{\nu}\mathbb{C}[z]^n .$$

If $\mathcal{L}_\nu$ denotes the lattices sandwiched between $z^{-\nu}\mathbb{C}[z]^n$ and $z^{\nu}\mathbb{C}[z]^n$, then $\mathcal{L}_\nu$ is a union of closed subsets of Grassmannians, hence $\mathcal{L} = \bigcup \mathcal{L}_\nu$ is a limit of projective varieties.

Make $GL_n(\mathbb{C}[z,z^{-1}])$ act on $\mathcal{L}$; the action is transitive so

$$\widetilde{GL}_n(\mathbb{C}[z,z^{-1}])/\widetilde{GL}_n(\mathbb{C}[z]) = GL_n(\mathbb{C}[z,z^{-1}])/GL_n(\mathbb{C}[z]) = \mathcal{L}$$

Since $\widetilde{GL}_n(\mathbb{C}[z])$ is contractible, it follows modulo topology questions that $\widetilde{GL}_n(\mathbb{C}[z,z^{-1}])$ has the same homotopy type as $\mathcal{L}$.

But now let ~~me~~ analyze the space $\mathcal{L}_\nu$ and see if I can produce a cell decomposition for it. $\mathcal{L}_\nu$ = $\mathbb{C}[z]$-submodules L with $\mathbb{C}[z]^n \supset L \supset z^{2\nu}\mathbb{C}[z]^n$. Let $e_1,\dots,e_n$ be the standard basis for $\mathbb{C}[z]^n$. Then the standard thing one does is to consider the induced filtration

$$0 < L_1 < L_2 < \cdots\cdots < L_n = L$$

$$L_i = L \cap \left(\mathbb{C}[z]e_1 + \cdots + \mathbb{C}[z]e_i\right)$$

Now $\quad L_i/L_{i-1} \subset \mathbb{C}[z]e_1 + \cdots + \mathbb{C}[z]e_i \big/ \mathbb{C}[z]e_1 + \cdots + \mathbb{C}[z]e_{i-1}$
so we get a unique generator for L_i/L_{i-1}
mapping to $z^{r_i}e_i$. Lift this to
$x_i \in L_i$. Then

$$x_1 = z^{r_1}e_1$$

$$x_2 = a_{21}e_1 + z^{r_2}e_2$$

$$x_3 = a_{31}e_1 + a_{32}e_2 + z^{r_3}e_3$$

where a_{21} is unique mod z^{r_1}, etc.
Better one can normalize a_{21} by requiring $\deg(a_{21}) < r_1$.
Thus we get a unique basis for L with

$$\deg(a_{ij}) < r_i \qquad (i > j).$$

It would seem therefore that we get a
decomposition of $\mathcal{L}_\nu$ into cells, one cell
for each set $r_1, r_2, \ldots, r_n$ such that
$0 \le r_1, \ldots, r_n \le 2\nu$, this cell
being of dimension

$$r_1 + (r_1 + r_2) + \cdots + (r_1 + \cdots + r_{n-1})$$

The index of L in $\mathbb{C}[z]^n$ is $r_1+\dots+r_n$.

<u>Example $n=2$.</u> Consider L in $\mathcal{L}_\nu$ of index 2ν. Thus $r_1+r_2=2\nu$, $0\le r_1, r_2\le 2\nu$ and we get cells of dimension r_1. Thus we get one cell of dimension $0, 1, \dots, 2\nu$.

Poincare series for $\mathcal{L}_\nu$ is

$$\left(\sum_{0\le r_1,\dots,r_n\le 2\nu} t^{(n-1)r_1+\dots+r_{n-1}}\right) u^{r_1+\dots+r_n}$$

$$\lim_{\nu\to\infty} = \left(1+ut^{n-1}+u^2t^{2(n-1)}+\dots\right)\left(1+ut^{n-2}+\dots\right)$$

$$= \frac{1}{1-ut^{n-1}}\,\frac{1}{1-ut^{n-2}}\cdots\frac{1}{1-ut}$$

I want to compute the Poincare series for the component of $\mathcal{L}$ of degree 0. In embedding $\mathcal{L}_\nu$ into $\mathcal{L}_{\nu+1}$, I keep track of index $r_1+\dots+r_n$. Poincare series for $\mathcal{L}_\nu(2\nu)$ is

$$\sum_{\substack{0\le r_1,\dots,r_n\le 2\nu \\ r_1+\dots+r_n=2\nu}} t^{(n-1)r_1+\dots+r_{n-1}}$$

If I let $\nu \longrightarrow \infty$ then this approaches

$$\sum_{0\le r_1,\dots,r_{n-1}} t^{(n-1)r_1+\dots+r_{n-1}}$$

$$= \frac{1}{1-t}\,\frac{1}{1-t^2}\cdots\frac{1}{1-t^{n-1}}$$

On the other hand, according to Bott the Poincare series of $H_*(\Omega SU_n) = \mathbb{Z}[\underset{2}{b_1},\dots,\underset{2(n-1)}{b_{n-1}}]$ is the same (using $\mathbb{C}$ dim).

It seems that I have now proved the conjecture. Next step would be to remove the use of Bott's theorem and to prove directly that $\mathcal{L}$ has the homotopy type of ΩGL_n.

Start with $\alpha: S^1 \longrightarrow GL_n$ given by a Laurent polynomial, choose N so that $z^N\alpha$ is a polynomial. Make α act by multiplication on $L^2(S^1)^n$, and let $D_+ = H^2(S^1)^n$ be the "outgoing" subspace of holomorphic stuff. Then $z^N\alpha D_+ \subset D_+$ and $D_+/z^N\alpha D_+$ is a finite $\mathbb{C}[z]$-module with support inside $|z|<1$, because $z-\lambda$, $|\lambda|>1$, acts invertibly on D_+ ($(z-\lambda)^{-1}$ is holom. inside).

Make a space out of the set of finite quotients of $\mathbb{C}[z]^n$. The idea is that if we have such a finite quotient M of length p, then we get a point of the p-fold symmetric product of the affine line which is affine space of dim p (symm. polys), and the fibre will be finite dimensional. Anyway the set of finite quotients of $\mathbb{C}[z]^n$ of a fixed length is a nice algebraic variety. I would like to be able to say that I can ~~want~~ contract this variety into the subvariety consisting of M with support at 0, using the radial deformation $z \mapsto \varepsilon z$.

So what I am now working with is the space Δ of outgoing subspaces contained in D_+ of fixed index. I want to show that Δ deforms to subspaces in $\mathcal{L}$.

~~Call such an outgoing space~~

So given D in Δ I can filter it with respect to the ~~lattice~~ filtration $\mathbb{C}[z]e_1 + \cdots + \mathbb{C}[z]e_i$ getting $0 < D_1 < \cdots < D_n = D$, with $D_i/D_{i-1} \hookrightarrow \mathbb{C}[z]$. Thus we get a unique basis for D of the form

$$x_1 = f_1(z)e_1$$
$$x_2 = a_{21}^{(z)}e_1 + f_2(z)e_2$$
etc.

where $f_i(z)$ is a monic polynomial in z having its roots inside $|z|=1$, and where $a_{ij}(z)$ is a polynomial of degree $< r_i = \deg f_i$. So now replace z by $\frac{z}{\varepsilon}$ and the new canonical form is

$$\varepsilon^{r_1}x_1 = \varepsilon^{r_1}f_1\left(\frac{z}{\varepsilon}\right)e_1$$

$$\varepsilon^{r_2}x_2 = \varepsilon^{r_2}a_{21}\left(\frac{z}{\varepsilon}\right) + \varepsilon^{r_2}f_2\left(\frac{z}{\varepsilon}\right)e_2$$

Evidently if $r_2 < r_1$ this needn't approach a limit as $\varepsilon \to 0$. So we have to be careful because the cells we might like to put in Δ are not stable under the radial contraction to $\mathcal{L}$.

However all we have to do is define the radial deformation once and for all.

New idea: Start with α as Laurent poly matrix invertible for $|z|=1$. Then we replace $z^N\alpha$ by $z^N\alpha D_+$. Now $D_+/z^N\alpha D_+$ is a finite $\mathbb{C}[z]$-module with support in $|z|<1$. Thus I ought to be able to find a polynomial matrix δ invertible for $1\le|z|$, unique up to right multiplication by an invertible polynomial matrix, such that

$$\delta D_+ = z^N\alpha D_+$$

Then $\delta^{-1}z^N\alpha D_+ = D_+$ so $\delta^{-1}z^N\alpha$ would be a rational matrix function β holom. + invertible for $|z|<1$. Thus $z^N\alpha = \delta\beta$. Now β can be deformed to a point (normalize α, δ, β by requiring them to have the value 1 when $z=1$. Then β deforms to $\beta(0)$ which deforms to $\beta(1)=1$.)

Thus up to homotopy-trivial choices, we have replaced [redacted] $z^N\alpha$ by δ which is invertible for $|z|\ge 1$. Reversing z to z^{-1}, this means that $z^N\alpha$ could be deformed into the subspace of polys. invertible for $0<|z|\le 1$. If $z^N\alpha$ is of this type, then applying the preceding reasoning, we ought to be able to deform it to an δ which now is singular only at [redacted] 0.

The way I want to proceed is as follows. Start with α a poly. matrix invertible for $|z| = 1$. Then I can factor α

$$\mathbb{C}[z]^n \xrightarrow{\beta} E \xrightarrow{\gamma} \mathbb{C}[z]^n$$

where E is a free module of rank n, and Coker γ has support in $|z| < 1$ and Coker β has support in $|z| > 1$. If one chooses an isomorphism $E \simeq \mathbb{C}[z]^n$, then γ, β become polynomial matrices unique up to elements of $GL_n \mathbb{C}[z]$. In fact if we start with α such that $\alpha(1) = 1$, then we can suppose $\beta(1) = \gamma(1) = 1$, in which case γ, β are unique up to elements of the contractible set $\widetilde{GL}_n(\mathbb{C}[z])$.

Since the set of polynomial matrices β invertible for $|z| \leq 1$ is contractible, it should be that we have homotoped α to γ, that is, down to the set of matrices invertible for $|z| \geq 1$.

Next assuming α invertible for $|z| \geq 1$, we consider $z^N \alpha(\frac{1}{z})$ which will be invertible for $0 < |z| \leq 1$, so if we factor as above

$$z^N \alpha\left(\frac{1}{z}\right) = \gamma(z)\beta(z) \qquad \begin{array}{l} \beta \text{ inv. } |z| \leq 1 \\ \gamma \text{ inv. } |z| \geq 1 \end{array}$$

it follows that $\gamma(z)$ is invertible for $z \neq 0$. Then we will get a deformation of $z^N\alpha(\frac{-1}{z})$ to an invertible Laurent polynomial, hence a deformation of the original α to an inv. Laurent polynomial.

The basic analytic problem here goes as follows. If $x \longmapsto \alpha_x$ is a continuous family of Laurent polynomial matrices all invertible for $|z|=1$, then factoring as before we get

$$\mathbb{C}[z]^n \xhookrightarrow{\beta_x} E_x \xhookrightarrow{\gamma_x} \mathbb{C}[z]^n$$

hence we get a continuous family E_x of $\mathbb{C}[z]$-modules free of rank n. It is not clear how to trivialize such a family.

Put another way, if I homogenize α, then I get a continuous family of maps

$$\begin{tikzcd} \mathcal{O}^n \arrow[rr, hook, "\alpha_x"] \arrow[dr, hook, "\beta_x"'] & & \mathcal{O}(d)^n \\ & \mathcal{E}_x \arrow[ur, hook, "\gamma_x"'] & \end{tikzcd}$$

and I know for each x that $\mathcal{E}_x$ restricted to $z \neq \infty$ is trivial, but I have to see how to effect this trivialization continuously in x.

More heuristics. Following Disney's paper, let G be the group of continuous $\alpha : S^1 \longrightarrow GL_n$ having L^1 Fourier series, G_0, G_∞ the subgroups admitting holom. invertible extensions to $|z| \leq 1$, $|z| \geq 1$ resp. ~~Precisely~~ Have Banach alg. R of cont. fns. on S^1 with L^1 Fourier series, and the subalgebras R_0, R_∞ consisting of ones with F. series $\sum_{n\geq 0} a_n z^n$ {resp. $n \leq 0$}. Then G, G_0, G_∞ are the units in $M_n(R)$, $M_n(R_0)$, $M_n(R_\infty)$ respectively. Analogously I have

$$GL_n\,\mathbb{C}[z,z^{-1}]\ ,\ GL_n\,\mathbb{C}[z],\ GL_n\,\mathbb{C}[z^{-1}].$$

$$\begin{array}{ccc} \cap & \cap & \cap \\ G & G_0 & G_\infty \end{array}$$

what's more, I have isos.

$$\begin{array}{ccc} G_\infty \backslash G / G_0 & = & \{ (r_1, \dots, r_n) \mid r_1 \leq \dots \leq r_n \in \mathbb{Z} \} \\ \wr\uparrow & & \\ GL_n(\mathbb{C}[z^{-1}]) \backslash GL_n\,\mathbb{C}[z,z^{-1}] / GL_n\,\mathbb{C}[z]. & & \end{array}$$

and I know the stabilizers $G_\infty \cap d G_0 d^{-1}$ are the same. Since $GL_n\,\mathbb{C}[z] \longrightarrow G_0$, $G_n\,\mathbb{C}[z^{-1}] \longrightarrow G_\infty$ is a homotopy equivalence, I would like to be able to conclude $GL_n\,\mathbb{C}[z,z^{-1}] \longrightarrow G$ is also one. But all I can conclude is that this holds after stratifying G according to these double cosets.

Suppose $\alpha = \alpha_\infty d \alpha_0$ is a standard factorization. $\alpha_\infty = \sum_{n\le 0} b_n z^n \Longrightarrow \alpha_\infty^* = \sum_{n\le 0} b_n^* z^{-n}$ is holom. inv. for $|z|\le 1$. Thus $\alpha_\infty^* D_+ = D_+$ and more generally this holds for any outgoing subspace D. Thus for $f,g \in D_+$

$$(T_\alpha f, g)_{D_+} = (\alpha f, g) = (\alpha_\infty d\alpha_0 f, g) = (d\alpha_0 f, \alpha_\infty^* g)$$

which implies that d is the full invariant of T_α modulo autos. of D_+.

α_∞ preserves incoming subspaces, α_0 preserves outgoing subspaces. Thus

$$H = (dD_+)^\perp \oplus (dD_+) \qquad \text{direct sum alg.}$$

$$\Longrightarrow H = \alpha_\infty (dD_+)^\perp \oplus \alpha_\infty d D_+$$
$$= (dD_+)^\perp \oplus \alpha D_+$$

This leads one to suspect that maybe we can recover the good lattice dD_+ by this algebraic property.

Conversely if D is outgoing with $H = D^\perp \oplus \alpha D_+$

then multiplying by α_∞^{-1} gives $H = D^\perp \oplus dD_+$. Assume D equivalent to D_+ in the sense that $z^{-\nu}D_+ \supset D \supset z^{\nu}D_+$ for some ν; does this force $D = dD_+$. Since d is unitary I can suppose ~~[illegible]~~ $d = 1$. Seems ~~[illegible]~~ unlikely.

Scattering matrix: Let D_1, D_2 be two outgoing subspaces in $L^2(S^1)^n$. Choose ~~[illegible]~~ a unitary isom between $D_1/zD_1 \simeq D_2/zD_2$, whence we get a unitary operator

$$\mathcal{S}: L^2(S^1; D_1/zD_1) \simeq L^2(S^1; D_2/zD_2)$$

which is given by $f \longmapsto \mathcal{S}(z)f(z)$ where $\mathcal{S}(z) = \sum z^n c_n$ $\quad c_n \in \mathrm{Hom}(D_1/zD_1, D_2/zD_2)$. $\mathcal{S}(z)$ is a unitary matrix for $|z| = 1$.

Conjecture: Let Δ = outgoing subspaces D such that $z^n D \subset D_+$ for some n. Then the scattering matrix ~~carrying~~ $\mathcal{S}$ carrying D to D_+

Scattering matrix. Let D be an outgoing subspace of $L^2(S^1)^n$, whence if $N = D \ominus zN$ we have an isom.

$$\mathcal{S} : L^2(S^1; N) \xrightarrow{\sim} L^2(S^1)^n$$

unitary commuting with z. By multiplicity theory $\dim N = n$. For each $n \in N$, $\mathcal{S}n$ is an L^2 function on S^1. Thus

$$\mathcal{S} = \sum z^\nu a_\nu$$

where $a_\nu \in \mathrm{Hom}(N, \mathbb{C}^n)$. Moreover $\mathcal{S} H_2(S^1; N) = D$. Now if I choose an isom of $N \simeq \mathbb{C}^n$, then I can view $\mathcal{S}$ as an operator on $L^2(S^1)^n$ carrying D_+ to D. So we get:

<u>Assertion</u>: If D is an outgoing subspace of $L^2(S^1)^n$, and if $x_1, \ldots, x_n$ is a unitary basis for $D \ominus zD$, then there is a unique operator $\mathcal{S} : L^2(S^1)^n \longrightarrow L^2(S^1)^n$, which is unitary, which commutes with z, and which carries e_i to x_i, hence D_+ to D.

It would be nice to know that if D is of finite codimension in D_+, then $\mathcal{S}(z)$ is a polynomial in z which could then be normalized so that $\delta(1) = 1$.

Idea: Then sending D to its scattering matrix is the Bott map I want.

Example: $n = 1$. $D \subset D_+$ is of finite codimension, hence $D \supset g D_+$ for some $g \in \mathbb{C}[z]$, and I can suppose that $g(z)$ has no zeroes outside of $|z| = 1$. Also I can suppose g has no zeroes on the circle for $(z-\lambda) D_+ = D_+$ if $|\lambda| = 1$. (?).

$D = g D_+$ which we want to write δD^+, δ orthogonal to $z^i D$ $i \geq 1$. Thus I want to find $h \in D_+$ with

$$g = \delta h \qquad (\delta, g z^i) = 0 \qquad i \geq 1$$

$$\text{or} \qquad (g^* g\, h^{-1}, z^i) = 0 \qquad i \geq 1$$

Thus I want $h \in D_+$, $k \in D_-$ such that

$$g^* g = h k$$

$$\log(g^* g) = f + \bar{f}$$

$$g^* g = e^f \, e^{\bar{f}}$$

f holom. unique up to $i\mathbb{R}$.

e^f holom + invertible $|z| \leq 1$.

If $\quad g(z) = \sum a_n z^n, \quad \overset{*}{g}(z) = \sum \bar{a}_n z^{-n}$. Thus if g is a poly with roots inside the circle, g^* is a polynomial in $\frac{1}{z}$ with roots outside the circle, hence it would seems that if $m =$ order of poly of $\overset{*}{g}(z)$ at $z=0$, then

$$e^f = z^m \overset{*}{g}(z) = h$$

$$\overline{e^f} = z^{-m} g(z) = k$$

Thus $\quad s = g/h = g/z^m g^*$.

Summary: Let g be a polynomial in z having all its zeroes in $|z| < 1$, and suppose g has degree m. If $g(z) = a_0 + \dots + a_m z^m$, put $g^*(z) = \bar{a}_0 + \bar{a}_1 \frac{1}{z} + \dots + \bar{a}_m \frac{1}{z^m}$, so that $g^* = \bar{g}$ on S^1.

$$g^*(z) = \overline{g\left(\frac{1}{\bar{z}}\right)}$$

so that if λ is a root of g, $\frac{1}{\bar{\lambda}}$ is a root of g^*. Then $z^m g^*(z) = \bar{a}_0 z^m + \dots + \bar{a}_m$ is a polynomial vanishing outside $|z| = 1$. Put

$$s = g/z^m g^* \qquad \text{holom for } |z| \leq 1$$

It is a rational function of z with zeroes inside S^1, poles outside S^1, and of absolute value 1 on S^1. s is the scattering operator. $\quad sD_+ = g D_+$.

Classify outgoing subspaces D of $L^2(S^1)$. Such a D determines a vector S up to a scalar of absolute value 1, namely, a basis for $D \ominus zD$. Since $(S, Sz^i) = 0$ $i \neq 0$, we have

$|S|^2 = 1$. Therefore it is clear that outgoing subspaces D are in one-one correspondence with measurable functions $S: S^1 \longrightarrow S^1$ modulo multiplication by S^1.

Similarly outgoing spaces D of $L^2(S^1)^n$ may be identified with measurable functions $S: S^1 \longrightarrow U_n$ modulo right multiplication by elements of U_n, the correspondence being given by $D = S D_+$.

Now it is clear that $S \longmapsto S P_{D_+} S^{-1}$ is a continuous map from meas. functions S to projectors, and the map in the other direction should also be continuous. In effect given D, the subspace $D \ominus zD$ should depend continuously on D, etc.

The problem for me now is to narrow down the class of possible D's so that the resulting S is at least continuous.

First examine the case where $D \subset D_+$. Suppose [redacted] that S is smooth. Now I know that if $\alpha : S^1 \longrightarrow GL_n$ is continuous, then $T_\alpha = P_{D_+}\alpha$ is a Fredholm operator on D_+. Thus if $D = S D_+$ should be contained in D and if S is continuous, then D must be of finite codimension in D_+. Hence D_+/D is a $\mathbb{C}[z]$-module of finite length, hence it has a composition series $D = D_0 < D_1 < \cdots < D_n = D_+$ [redacted] $D_i/D_{i-1} \simeq \mathbb{C}[z]/(z-\lambda_i)$. Claim each λ_i is such that $|\lambda_i| < 1$. Can suppose $D_+/D \cong \mathbb{C}[z]/(z-\lambda)$ whence $\overline{(z-\lambda)D_+} \subset D < D_+$. Now if $f \in D_+$ is perpendicular to $(z-\lambda)D_+$, then writing

$$f(z) = \sum a_n z^n$$

$$0 = \left(f(z), (z-\lambda)z^i\right) = a_{i+1} - \lambda a_i$$

so $\quad f(z) = a_0 + \lambda a_0 z + \lambda^2 a_0 z^2 + \cdots$ $\quad$ works for vectors also with a change of notation.

[redacted] and for this to converge in L^2 one must have $|\lambda| < 1$. Thus if $D \subset D_+$ is of finite [redacted] codimension, the support of D_+/D is a finite subset of $|z| < 1$.

Let $g(z)$ be the monic polynomial of smallest degree such that $g(D_+/D) = 0$, i.e. $\mathbb{C}[z]\, g = \mathrm{Ann}(D_+/D)$. Then

$$D_+ \supset D \supset g D_+$$

so knowing about the scattering operator for gD_+, I ought to be able to say something about the one for D. If g has degree m, then its scattering operator is

$$\frac{g}{z^m g^*} \qquad\qquad g(z) = a_0 + \cdots + a_m z^m$$
$$g^*(z) = \bar{a}_0 + \cdots + \bar{a}_m \frac{1}{z^m}$$

Now if $SD_+ = D \subset D_+$, then we know that the matrix S extends holomorphically to the unit disk.

Now because $\mathbb{C}[z]^n / g\,\mathbb{C}[z]^n \twoheadrightarrow D_+/D$ we get certainly a matrix $\alpha(z)$ of polynomials such that $\mathbb{C}[z]/\alpha\mathbb{C}[z]^n = D_+/D$. If $\beta\alpha = g$ with β a matrix of polynomials, then we see α is invertible for $|z| \geq 1$. Moreover it is clear that $D = \alpha D_+$. Let S be the scattering matrix for D so that

$$\alpha D_+ = S D_+$$

$$S^{-1}\alpha = h \qquad \text{is holom. invertible } |z| < 1$$

$$\alpha = Sh$$

$$\alpha^*\alpha = h^* S^* S h = h^* h$$

However I already know that because α is a polynomial matrix, h must also be one. Assume this to be so for the moment. Then

$$\delta = \alpha h^{-1}$$

is a rational function of z.

Summary: I know that outgoing spaces D in $L^2(S^1)^n$ are in one-one correspondence with measurable functions $\delta : S^1 \longrightarrow U_n$ modulo right multiplication by elements of U_n. + (agreeing off null sets)

In more detail, any bdd operator $L^2(S^1) \longrightarrow L^2(S^1)$ commuting with multiplication by z is given by multiplication by a bdd measurable function in $L_\infty(S^1)$. Thus any bdd operator $L^2(S^1)^n \longrightarrow L^2(S^1)^n$ commuting with z is given by a matrix δ with coeffs. in $L_\infty(S^1)$, and for this operator to be unitary means the matrix $\delta(z)$ is unitary. If D is outgoing in $L^2(S^1)^n$ then one can prove $D \ominus zD \simeq \mathbb{C}^n$, hence choosing such an isom. one gets a unitary matrix δ in $M_n(L_\infty(S^1))$ such that $\delta D_+ = D$.

Now $GL_n(L_\infty(S^1))$ acts on the set of outgoing spaces. The stabilizer of D_+ is $GL_n(H_\infty(S^1))$

where $H_\infty(S^1)$ = subring of $L_\infty(S^1)$ extending holomorphically inside the unit disk. Hence we seem to obtain:

$$GL_n(L_\infty(S^1))/GL_n(H_\infty(S^1)) \simeq U_n(L_\infty(S^1))/U_n$$

What is perhaps significant is that we have a different way of describing GL_n/U_n using a subgroup of GL_n. Normally the Borel subgroup is used. Thus we have a kind of Tits gadget with $GL_n(H_\infty(S^1))$ playing the role of the Borel subgroup.

I want to use the scattering operator to identify ΩU_n with the space of outgoing subspaces D commensurate with D_+.

<u>Question:</u> Under the correspondence between D and $U_n(L_\infty(S^1))$, do the continuous elements correspond exactly to the $D \equiv D_+$ modulo $\mathcal{K}$?

I claim that if S is the scattering operator associated to D, then S is a rational function of z iff D is commensurable with D_+, i.e. $D \cap D_+$ is of finite codimension in both D and D_+. To prove $\Leftarrow$ we reduce immediately to the case where D is of finite codimension in D_+, and then to the case where D is of codimension 1 in D_+. Let $(z-\lambda)D_+ \subset D$, and let e generate $D_+ \ominus D$. If

$$e(z) = \sum_{\nu \geq 0} z^{\nu} v_{\nu} \qquad v_\nu \in \mathbb{C}^n$$

then for $e(z)$ to be $\perp$ to $(z-\lambda) D_+$ means that

$$\left(\sum z^{\nu} v_{\nu} \, , z^{i}(z-\lambda) v\right) = (v_{i+1}, v) - (v_i, \lambda v) = 0$$

or that $v_{i+1} = \bar{\lambda} v_i$. Hence $|\lambda| < 1$

$$e(z) = v_0 \left(\sum z^{\nu} \bar{\lambda}^{\nu}\right) = \left(\frac{1}{1-\bar{\lambda} z}\right) v_0 .$$

So I can arrange that $v_0 = e_1$ and so

$$D = (z-\lambda) H^2(S^1) \oplus H^2(S^1)^{n-1} \subset H^2(S^1)^n$$

which reduces us to calculating the scattering matrix for $n=1$ and showing it is rational.

If $D = (z-\lambda) H^2(S^1)$, let s generate $D \ominus zD$.

$s = (z-\lambda) h \qquad |s|^2 = (z-\lambda) h \bar{h} (\bar{z} - \bar{\lambda}) = 1.$

$$h(z)\overline{h(z)} = \frac{z}{(z-\lambda)(1-\bar{\lambda} z)}$$

$$\therefore \quad h(z) = \frac{1}{1-\bar{\lambda} z}$$

$$S = \frac{z-\lambda}{1-\bar\lambda z} \quad \text{for } |z|=1 \qquad S = \frac{z-\lambda}{1-\bar\lambda/\bar z} = \frac{z-\lambda}{\bar z-\bar\lambda}\,\bar z$$

is of abs. 1.

Thus it is clear that when D is commensurate with D_+, then S is rational.

Conversely suppose α is a rational matrix function invertible for $|z|=1$. Let g be a polynomial having sufficiently high order at the poles of α so that $g\alpha$ is a polynomial matrix. Then αD_+ is commensurate with $g\alpha D_+$, so we can suppose α is a polynomial matrix, whence $\alpha D_+ \subset D_+$. Because α is continuous we know that $T_\alpha = P_{D_+}\alpha : D_+ \to D_+$ is Fredholm, which implies in this case that αD_+ is of finite codim. in D. done.

Can one see in the case $n=1$ that $\alpha D_+ \equiv D_+ \mod \mathcal{K}$ implies α is continuous, when α is unitary.

Suppose $\alpha : S^1 \to GL_n$ is holomorphic in an annulus $r<|z|<R$, where $r<1<R$. Then α is a clutching fn. for a holom. bundle and the factorization $\alpha = \alpha_\infty d\, \alpha_0$, $\alpha_0 :$ $\{|z|<R\} \to GL_n$, $\{|z|>r\} \to GL_n$ results from the fact this bundle has the standard structure. A section of this bundle is a pair (f_0, f_∞) where $f_0 : \{|z|<R\} \to \mathbb{C}^n$, $f_\infty : \{|z|>r\} \to \mathbb{C}^n$ are holomorphic and $f_0 = \alpha f_\infty$ on $r<|z|<R$. For example if $\alpha(z) = z$, then we get the sections $(z,1)$, and $(1,\frac{1}{z})$, so we get the line bundle $\mathcal{O}(1)$.

We can interpret f_0 as an element of D_+ and f_∞ as an element of D_-, hence sections of the bundle E_α are elements of $D_+ \cap \alpha D_-$, or better $\alpha^{-1} D_+ \cap D_-$, which also are elements $f_0 \in D_+$ killed by the Toeplitz operator $T_{z^{-1}\alpha^{-1}} = P_{D_+}(z^{-1}\alpha^{-1})$. Similarly

$$\Gamma(E_\alpha(n)) = \Gamma(E_{z^n\alpha}) =$$

$$= z^{-n}\alpha^{-1} D_+ \cap D_-$$

Question: For any outgoing subspace D we can consider the intersections $z^{-n}D_+ \cap D_-$ as well as $H / z^{-n} D_+ + D_-$, which should be nice if $D \equiv D_+ \bmod \mathcal{K}$. Do they have the right dimension always or does this imply something special about D?

For example if $\alpha = \alpha_0 d \alpha_\infty$, then

$$z^{-n}\alpha^{-1} D_+ \cap D_- = z^{-n} \alpha_\infty^{-1} d^{-1} D_+ \cap D_-$$

$$\cong z^{-n} d^{-1} D_+ \cap D_-$$

So now maybe I want to think of D as being sections of the bundle over $\mathbb{P}^1$ holomorphic over the unit disk.

At the moment I have a Bott map given by the scattering operator, which goes roughly as follows. Given an outgoing subspace D commensurable with D_+, I know the associated scattering matrix S_D is a rational fn. of z invertible for $|z|=1$, hence it can be normalized so that $S_D(z)=1$. Then

$$S_D : S^1 \longrightarrow U_n$$

is the desired path. I still have to straighten out the topology, however notice that in terms of the dimensions of $D/D\cap D_+$ and $D_+/D\cap D_+$ I get a bound on the number ~~[illegible]~~ of Möbius factors involved in S_D.

<u>November 10, 1974</u> On ΩU_n

The Bott map for ΩU_n.

Let $\mathcal{L}$ be the set of $\mathbb{C}[z]$-submodules L of $\mathbb{C}[z,z^{-1}]^n$ such that for some m

$$z^{-m} L_0 \supset L \supset z^m L_0$$

where $L_0 = \mathbb{C}[z]^n$. Equip $\mathbb{C}[z,z^{-1}]^n$ with the inner product such that $z^i e_j$, $i \in \mathbb{Z}$, $1 \le j \le n$, is an orthonormal basis.

Given L I can associate to L the space $W = L \ominus zL$ which is an n-dimensional subspace of $\mathbb{C}[z,z^{-1}]^n$ generating $\mathbb{C}[z,z^{-1}]^n$ and such that $(W, z^i W) = 0$ for all $i \neq 0$. In this way I can identify $\mathcal{L}$ with the set of such W. (Note: since $z^{-m}L_0 \supset L \supset z^m L_0$ it follows that W is naturally the orthogonal complement of $zL/z^{m+1}L_0$ in $L/z^{m+1}L_0$) ~~in essence you have an element~~ It is clear that W is a subspace of $z^{-m}L_0 \cap (z^{m+1}L_0)^\perp$ = Laurent polyn. vectors $\sum_{-m \le i \le m} v_i z^i$.

Since $\mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}} W \xrightarrow{\sim} \mathbb{C}[z,z^{-1}]^n$, it is clear that by evaluating at $z=1$ we get an isomorphism $W \xrightarrow{\sim} \mathbb{C}^n$. Therefore we get a unique basis $w_1, \dots, w_n$ for W such that $w_i(1) = e_i$, hence we get a matrix S_W such that $S_W e_i = w_i$. Thus if

$$w_i = \sum a_{ji}(z) e_j$$

then $S_W = (a_{ij}(z))$. Thus if $z^{-m}L_0 \supset L \supset z^m L$ the matrix S_W involves monomials z^i, $|i| \leq m$. Conversely if W is of "degree" $\leq m$, whence $W \subset z^{-m}L_0 \cap z^{m+1}L_0^{\perp}$ then I know that $z^{-1}W + z^{-2}W + z^{-3}W + \cdots \subset z^{m-1}W_0 + z^{m-2}W_0 + \cdots$, hence taking orth. complements, I get

$$L = W + zW + \cdots\cdots \supset z^m W_0 + z^{m+1} W_0 + \cdots\cdots$$
$$= z^m L_0 .$$

Claim for $z = 1$ that $S_W(z)$ is unitary.

$$\delta_{0k}\delta_{ij} = (z^k w_i, w_j) = \left(z^k \sum_\nu a_{\nu i}(z) e_\nu , \sum_\nu a_{\nu j}(z) e_\nu \right)$$

$$= \sum_\nu \int_{S^1} z^k a_{\nu i}(z) \overline{a_{\nu j}(z)} \, dz$$

Since this is zero for $k=0$, it follows that for any z

$$\sum_\nu a_{\nu i}(z) \overline{a_{\nu j}(z)} = \delta_{ij}$$

i.e. $\qquad (a_{\nu j}(z))^* (a_{\nu i}(z)) = (\delta_{ij})$

<u>Proposition</u>: 1-1 correspondence between

i) $\mathbb{C}[z]$-submodules L such that $z^{-m}L_0 \supset L \supset z^m L_0$

ii) Subspaces W of $z^{-m}W_0 + \cdots + z^m W_0$ such that $\dim(W) = n$ and such that $z^k W \perp W$ for $k \neq 1$.

iii) Elements s of $U_n(\mathbb{C}[z, z^{-1}])$ (here $\mathbb{C}[z, z^{-1}]$ has the involution $\lambda^* = \bar{\lambda}$ $z^* = \frac{1}{z}$) having "degree" $\leq m$ (monomials in z^i, $-m \leq i \leq m$.) such that $s(z) = 1$

In fact these form a compact space, which is a disjoint union of closed subvarieties of Grassmannian varieties.

$\mathcal{L}_m$ = space of L as in (i). $\mathcal{L} = \bigcup \mathcal{L}_m$.

The correspondence is

$$\tilde{U}_n(\mathbb{C}[z, z^{-1}]) \xrightarrow{\sim} \mathcal{L}$$

$$s \longmapsto sL_0$$

$\sim$ means $s(z) = 1$

Bott map:

$$\mathcal{L} \xleftarrow{\sim} \tilde{U}_n(\mathbb{C}[z, z^{-1}]) \hookrightarrow \Omega U_n$$

Claim this is a homotopy equivalence. Proof involves following steps.

1) Bott's computation that $H_*(\Omega U_n) = \mathbb{Z}[b_0, \ldots, b_{n-1}, b_0^{-1}]$

where b_i is the image of the gen. of $H_{2i}($[redacted]$\mathbb{P}(\mathbb{C}^n))$, under the [redacted] Bott map from

$$\mathbb{P}(\mathbb{C}^n) \subset \mathcal{L}_1$$

$$\ell \longmapsto \text{[redacted]} \left(z\ell \oplus \ell^{\perp}\right)\cdot\mathbb{C}[z].$$

This shows $H_*(\mathcal{L}) \twoheadrightarrow H_*(\Omega U_n)$.

2) cell decomposition of $\mathcal{L}$ which computes the homology $H_*(\mathcal{L})$ and its Poincare series.
This shows $H_*(\mathcal{L}) \xrightarrow{\sim} H_*(\Omega U_n)$.