October 11, 1974:

Given $U_1 < U_2 \subset U_3 \subset U_4$ with $\dim U_2/U_1 = \dim U_4/U_3 = 1$, we associate a Schubert cell

$$C(U_1, \cdots, U_4) = \{A \in G_2 V \mid 0 = A \cap U_1 < A \cap U_2 = A \cap U_3 < A \cap U_4 = A\}$$

If $U_2 = U_3$, then this cell depends only on (U_1, U_4) and we use the notation

$$C(U_1, U_4) = \{A \in G_2 V \mid A \oplus U_1 = U_4\}.$$

I want to determine the various inclusions which hold between these cells.

Suppose $(U_1, \cdots, U_4) \in L_{11}(V)$ and $(W_1, W_2) \in L_2(V)$ and $C(U_1, \cdots, U_4) \subset C(W_1, W_2)$. $C(U_1, \cdots, U_4)$ = set of planes of the form $A = L_1 \oplus L_2$, where $L_1 \in \mathbb{P}U_2 - \mathbb{P}U_1$ and $L_2 \in \mathbb{P}U_4 - \mathbb{P}U_3$. Such lines L_2 ~~span~~ span U_4 so $U_4 \subset W_2$. Since any A in $C(W_1, W_2)$ contained in U_4 belongs to $C(U_4 \cap W_1, U_4)$, we have

$$C(U_1, \cdots, U_4) \subset C(U_4 \cap W_1, U_4)$$

Put $V = U_4 \cap W_1$.

If $V \not\subset U_3$ then we can find $L_2 \in \mathbb{P}V - \mathbb{P}U_3$ whence $A = L_1 \oplus L_2$ (any L_1) is not ind. of V. Thus $V \subset U_3$.

$(\mathbb{P}U_2 - \mathbb{P}U_1) \cap \mathbb{P}V = \phi \Rightarrow \mathbb{P}U_1 \supset \mathbb{P}U_2 \cap \mathbb{P}V$
$\Longrightarrow U_1 \supset U_2 \cap V$, hence $U_1 = U_2 \cap W$ as ~~[illegible]~~ the intersection is at most of codim 1. Thus we get the picture

$$
\begin{array}{ccccc}
 & & U_4 & — & W_2 \\
 & & \vee & & \\
U_2 & — & U_3 & & \\
\vee & & \vee & & \\
U_1 & — & U_4 \cap W_1 & — & W_1
\end{array}
$$

<u>Prop</u>: If $C(U_1, \ldots, U_4) \subset C(W_1, W_2)$, then $U_1 \subset W_1$, $U_4 \subset W_2$, and $U_4 \cap W_1 \subset U_3$

$$(U_1, U_2) \leq (U_4 \cap W_1, U_3) .$$

Suppose now that $C(W_1, W_2) \subset C(U_1, \ldots, U_4)$. Then for every choice of V.

$$
\begin{array}{ccccc}
W_2 & & & & U_4 \\
 & & & & \vee \\
 & & U_2 & — & U_3 \\
 & & \vee & & \vee \\
W_1 & & U_1 & — & V
\end{array}
$$

we have $(W_1, W_2) \subset (V, U_4)$. It follows that $W_1 \subset U_1$.

Thus what's happening it that we take (W_1, W_2) and break it into $(W_1, W_2 \cap U_3)$ and $(W_2 \cap U_3, W_2)$ and map these to (U_1, U_2) and (U_3, U_4). So

<u>Prop</u>: If $C(W_1, W_2) \subset C(U_1, \ldots, U_4)$, then there is a unique $W_1 < H < W_2$ such that $(W_1, H, H, W_2) \leq (U_1, \ldots, U_4)$ in $L_{1,1}(V)$.

Finally suppose $C(V_1, V_2, V_3, V_4) \subset C(U_1, \ldots, U_4)$. Assuming $V_2 < V_3$, $U_2 < U_3$ I want to prove $(V_1 \cdots V_4) \leq (U_1 \cdots U_4)$ in $L_{1,1}(V)$. We know already that $V_4 \subset U_4$, $V_1 \subset U_1$ so we can assume $V_1 = 0$, $U_4 = V$. This means that
$$C(V_1, V_2, V_3, V_4) = \{ V_2 \oplus L_2 \mid L_2 \in \mathbb{P}(V_4) - \mathbb{P}(V_3) \}.$$

~~Now for any such thing $V_2 \oplus L_2 \twoheadrightarrow U_4/U_3$. I want to show $V_2 \subset U_3$. Assume not, i.e. $V_2 \twoheadrightarrow U_4/U_3$. [illegible] Then $(V_2 \oplus L_2) \cap U_3 \subset U_2$~~

Try to show $V_3 \subset U_3$. Note that $V_4 \twoheadrightarrow U_4/U_3$ so $V_3, V_4 \cap U_3$ are hyperplanes in V_4. If $V_3 \neq V_4 \cap U_3$ can find $L_2 \in \mathbb{P}(V_4 \cap U_3) - \mathbb{P} V_3$, whence $V_2 \oplus L_2 \twoheadrightarrow U_4/U_3$

showing that $V_2 \xrightarrow{\sim} U_4/U_3$. But then we know $(V_2 \oplus L_2) \cap U_3 = L_2$, so L_2 must be a complement for U_1 in U_2. Thus

$$\mathbb{P}(V_4 \cap U_3) - \mathbb{P}V_3 \subset \mathbb{P}U_2 - \mathbb{P}U_1$$

$$\cancel{\Rightarrow (V_3, V_4 \cap U_3) \leq (U_1, U_2)}$$

$\cancel{\Rightarrow V_2 \subset V_3 \subset U_1 \subset U_3}$ ~~which is non-sense~~.

<u>Check:</u> $V_4 \twoheadrightarrow U_4/U_3 \Rightarrow V_4 \cap U_3$ is a hyperplane in V_4. If this hyperplane differs from V_3, then I can find $L_2 \in \mathbb{P}(V_4 \cap U_3) - \mathbb{P}V_3$. Then taking $A = L_1 \oplus L_2$, any $L_1 \in \mathbb{P}V_2 - \mathbb{P}V_1$ I get

$$L_1 \oplus L_2 \twoheadrightarrow U_4/U_3$$

so $L_1 \xrightarrow{\sim} U_4/U_3 \Rightarrow \mathbb{P}V_2 - \mathbb{P}V_1 \subset \mathbb{P}U_4 - \mathbb{P}U_3$

$$\Rightarrow (V_1, V_2) \leq (U_3, U_4)$$

But more: I know $A \cap U_2 = A \cap U_3 = L_2$ is a complement for U_1 in U_2. Thus

$$\mathbb{P}(V_4 \cap U_3) - \mathbb{P}V_3 \subset \mathbb{P}U_2 - \mathbb{P}U_1$$

$$\cancel{\Rightarrow (V_3, V_4 \cap U_3) \leq (U_1, U_2)}.$$

~~[crossed out]~~ so

$$\left(V_3 \cap U_3, V_4 \cap U_3\right) \leq \left(U_1, U_2\right)$$

$$\Longrightarrow \quad V_4 \cap U_3 = V_4 \cap U_2 \qquad V_3 \cap U_3 = V_3 \cap U_1 = V_4 \cap U_1$$

Somehow (assume $V_1 = 0$ again, so $L_1 = V_2$), the point is that if $V_2 \xrightarrow{\sim} U_4/U_3$, then any plane $A = V_2 \oplus L_2$ has a canonical choice for L_2, namely $A \cap U_3 = A \cap U_2$ which is a line in $V_4 \cap U_2$.

~~[crossed out]~~

Normally on $C(V_1, .., V_4)$ there is no canon. way of splitting the exact sequence

$$0 \to V_2/V_1 \longrightarrow A \longrightarrow V_4/V_3 \longrightarrow 0.$$

But the complement U_3 for V_1 does this. But ~~[crossed out]~~ then the A's I get will be in $V_2 \oplus V_4 \cap U_2$ so we will have to have

$$V_2 \oplus (V_4 \cap U_2) = V_4$$

$$1$$

$$V_4 \cap U_2 = V_4 \cap U_3 \qquad \text{codim } 1$$

So what I have managed to find is

$$C(V_1, V_2, V_3, V_4) \subset C(V_4 \cap U_1 < V_4 \cap U_2 = V_4 \cap U_3 < V_4)$$

So $\quad C(V_1, V_2, V_3, V_4) \subset C(V_4 \cap U_1, V_4) \subset C(U_1, \dots, U_4)$

this inclusion results from

$$\begin{array}{cc} & V_4 \\ V_2 & V_3 \\ V_1 & V_4 \cap U_1 \end{array}$$

this inclusion results from

$V_4 \cap U_1 < V_4 \cap U_2 = V_4 \cap U_3 < V_4$

so the first inclusion implies $V_4 \cap U_1$ is comp. to V_2 in V_3. Can this happen. Seems so.

Example of the inclusion.

$$\begin{array}{cc} & V_4 \\ V_2 & V_3 \\ V_1 & \Gamma \end{array} \qquad \begin{array}{ccccc} V_4 & & & & U_4 \\ & & & & | \\ & & U_2 & - & U_3 \\ & & | & & \\ \Gamma & = & U_1 & & \end{array}$$

Start again: Assume $C(V_1, \dots, V_4) \subset C(U_1, \dots, U_4)$. Then choosing $T \subset V_4$ suitably I have $C(V_1, T) \subset C(V_1, \dots, V_4)$ showing by my earlier work that $V_1 \subset U_1$. Similarly choosing $W \subset U_3$ suitably I have $C(U_1, \dots, U_4) \subset C(W, U_4)$, so again by earlier work I will have $V_4 \subset U_4$.

Since $V_4 \twoheadrightarrow U_4/U_3$, $V_4 \cap U_3$ is a hyperplane in V_4.

Case 1: $V_3 \neq V_4 \cap U_3$.

Then we can choose $L_2 \in \mathbb{P}(V_4 \cap U_3) - \mathbb{P}V_3$. If L_1 in any elt of $\mathbb{P}V_2 - \mathbb{P}V_1$, then $L_1 + L_2 \in C(V_1, \dots, V_4)$, hence $L_1 + L_2 \twoheadrightarrow U_4/U_3 \Rightarrow L_1 \twoheadrightarrow U_4/U_3 \Rightarrow V_2/V_1 \simeq U_4/U_3$. Thus one has

$$\begin{array}{ccccc} V_2 & - & V_4 & - & U_4 \\ | & & | & & | \\ V_1 & - & V_4 \cap U_3 & - & U_3 \end{array}$$

~~In addition, we know $L_2 = (L_1 + L_2) \cap U_3 = (L_1 + L_2) \cap U_2$~~

Thus mod V_1, V_2 and $V_4 \cap V_3$ are complementary in V_4. This means that for every A in $C(V_1, \dots, V_4)$

$$A = (V_1 \cap A) \oplus (U_3 \cap A) = (V_1 \cap A) \oplus (U_2 \cap A)$$

where $U_2 \cap A \in (\mathbb{P}U_2 - \mathbb{P}U_1) \cap \mathbb{P}(V_4 \cap U_3) = \mathbb{P}(U_2 \cap V_4) - \mathbb{P}(U_1 \cap V_4)$

?

<u>Summary</u>: Suppose that

$$C(V_1, \dots, V_4) \subset C(U_1, \dots, U_4).$$

Then $V_1 \subset U_1$, $V_4 \subset U_4$.

<u>Case 1</u>. $V_3 \neq V_4 \cap U_3$. In this case the inclusion factors

$$C(V_1, \dots, V_4) \subset C(V_4 \cap U_1, V_4) \subset C(U_1, \dots, U_4)$$

$$\begin{pmatrix} & V_4 \\ V_2 & V_3 \\ V_1 & \end{pmatrix} \leq \begin{pmatrix} V_4 \\ \\ V_4 \cap U_1 \end{pmatrix} = \begin{pmatrix} V_4 \\ V_4 \cap U_2 \\ V_4 \cap U_3 \\ V_4 \cap U_1 \end{pmatrix} \leq \begin{pmatrix} & U_4 \\ & \vert \\ U_2 - & U_3 \\ \vert & \\ U_1 & \end{pmatrix}$$

So what I would like to say is that in this case, there is a ~~unique layer (W_1, W_2)~~ unique interval in $L_2(V)$ consisting of layers (W_1, W_2) such that

$$C(V_1, \dots, V_4) \subset C(W_1, W_2) \subset C(U_1, \dots, U_4).$$

The least layer is $(V_4 \cap U_1, V_4)$, the largest $(U_1, V_4 + U_1)$.

<u>Case 2</u>: $V_3 = V_4 \cap U_3$. In this case one should have also that $V_2 + U_1 = U_2$, so that $(V_1, \dots, V_4) \leq (U_1, \dots, U_4)$ in $L_{1,1}(V)$.

I further hope that the poset of Schubert cells described the following category. Objects are of two kinds:

i) two lines L_1, L_2

ii) a 2-diml space M

Following maps:

isoms. $M \xrightarrow{\sim} M'$ $\quad (L_1, L_2) \xrightarrow{\sim} (L_1', L_2')$

maps $(L_1, L_2) \leftrightarrows M$ for any exact sequence

$$0 \longrightarrow L_1 \longrightarrow M \longrightarrow L_2 \longrightarrow 0.$$

finally a map $(L_1, L_2) \longrightarrow (L_1', L_2')$ for an isomorphism $L_1 \oplus L_2 \longrightarrow L_1' \oplus L_2'$.

October 13, 1974.

Idea: The poset $Sh_2(V)$ of Schubert cells in $G_2(V)$ should be the classifying space of some category made up out of vector spaces, which I could construct as follows.

$$C(W_1, W_2) \longmapsto W_2/W_1 \qquad \text{2 dim'l v.s.}$$

$$C(U_1, U_2, U_3, U_4) \longmapsto (U_2/U_1, U_4/U_3) \qquad \text{pair of lines}$$

Then I want to define morphism so that I get a functor.

$$C(W_1', W_2') \le C(W_1, W_2) \longmapsto W_2'/W_1' \xrightarrow{\sim} W_2/W_1$$

$$C(U_1, \dots, U_4) \le C(W_1, W_2) \longmapsto 0 \to U_2/U_1 \to W_2/W_1 \to U_4/U_3 \to 0$$

$$C(W_1, W_2) \le C(U_1, \dots, U_4) \longmapsto 0 \to U_2/U_1 \to W_2/W_1 \to U_4/U_3 \to 0$$

$$C(V_1, \dots, V_4) \le C(U_1, \dots, U_4) \longmapsto \text{either } (V_2/V_1, V_4/V_3) \simeq (U_2/U_1, U_4/U_3)$$
$$\text{or } (V_2/V_1, V_4/V_3) \simeq (U_4/U_3, U_2/U_1)$$

A problem is that if I have $C(W_1', W_2') \subset C(U_1, U_2, \dots, U_4) \subset C(W_1, W_2)$, I then get

$$\begin{tikzcd}[column sep=small] & & W_2'/W_1' \arrow[dr] & & \\ 0 \arrow[r] & U_2/U_1 \arrow[ur] \arrow[dr] & & U_4/U_3 \arrow[r] & 0 \\ & & W_2/W_1 \arrow[ur] & & \end{tikzcd}$$

and I can't seem to deduce the isom $W_2'/W_1' \simeq W_2/W_1$. So something here doesn't work.

<u>Question:</u> I know that the subset L_2V of $Sh_2(V)$ is a classifying space for Gl_2. Is the complementary subset consisting of cells $C(U_1, \cdots, U_4)$ with $U_1 < U_2 < U_3 < U_4$ a classifying space for $\Sigma_2 \ltimes (F^*)^2$?

Alternative approach.

Try classifying Shubert cells close to a fixed one.

Every Schubert cell gives us a pair of integers (i,j) $0 \le i \le j$. The integers associated to $U_1 < U_2 \subset U_3 < U_4$ are

$$i = \dim U_1 \qquad = \dim \mathbb{P}U_2 - \mathbb{P}U_1$$

$$j = \dim U_3 - 1 \qquad = \dim \mathbb{P}U_4 - \mathbb{P}U_3 - 1$$

and the dimension of the Schubert cell is $i+j$. We have seen that $C(U_1 \cdots U_4) \subset C(V_1, \cdots, V_4) \Rightarrow$ $U_1 \subset V_1$, $U_4 \subset V_4$

hence $(i,j) \leq (i',j')$ for the product ordering.

It is natural to ask if by using simplices $c_0 < \cdots < c_k$ with small distance we get a homotopy equivalent complex.

October 15, 1974:

Still trying to prove the following conjecture:

true
see p. 9

$$\begin{tikzcd} L'_{1,1}(V) \arrow[r] \arrow[d, hook] & L_2(V) \arrow[d] \\ L_{1,1}(V) \arrow[r] & Sh_2(V) \end{tikzcd} \qquad \dim V = \infty$$

is homotopy-cocartesean. Here $L_2(V)$ = poset of layers (W_1, W_2) in V with $\dim(W_2/W_1) = 2$. $L_{1,1}(V)$ = poset of layers (U_1, U_2, U_3, U_4) with $\dim U_2/U_1 = \dim U_4/U_3 = 1$, and $L'_{1,1}(V)$ = subposet with $U_2 = U_3$.

What this conjecture says is

$$B\,Sh_2(V) = B\,T_2(k) \underset{B\,B_2(k)}{\cup} B\,GL_2(k)$$

(k = field under consideration).

So to prove this conjecture, it is undoubtedly necessary to understand more about inclusions between Schubert cells.

Case 1: $C(W_1', W_2') \subset C(W_1, W_2)$. Then $W_2' \subset W_2$. ~~If $W_1' \neq W_1$ then one could find~~ ~~$\subset PW_1$~~ since $W_2' \twoheadrightarrow W_2/W_1$, W_1' and $W_2' \cap W_1$

are of codim 2 in W_2'. If $W_1' \neq W_2' \cap W_1$, then $\exists\, L \in (\mathbb{P}W_2' \cap \mathbb{P}W_1) - \mathbb{P}W_1'$. L can be extended to $A \in C(W_1', W_2')$ not in $C(W_1, W_2)$. Thus $W_1' = W_2' \cap W_1$.

Therefore

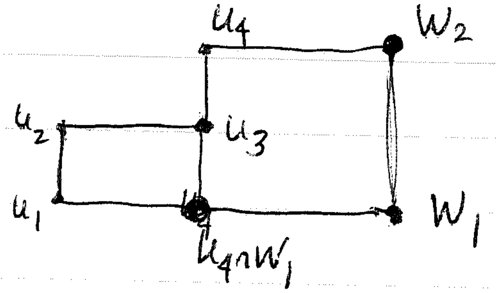
$$C(W_1', W_2') \subset C(W_1, W_2) \iff (W_1', W_2) \leq (W_1, W_2)$$

<u>Case 2</u>: $C(U_1, U_2, U_3, U_4) \subset C(W_1, W_2)$. Again $U_4 \subset W_2$. Since $U_4 \twoheadrightarrow W_2/W_1$, one has $(U_4 \cap W_1, U_4) \leq (W_1, W_2)$ and $C(U_1, U_2, U_3, U_4) \subset C(U_4 \cap W_1, U_4)$.

If $U_4 \cap W_1 \not\subset U_3$, then can find $L_2 \in \mathbb{P}U_4 \cap \mathbb{P}W_1 - \mathbb{P}U_3$; so for any $L_1 \in \mathbb{P}U_2 - \mathbb{P}U_1$, we have $L_1 \oplus L_2 \in C(U_1, \ldots, U_4)$ but $L_1 \oplus L_2 \notin C(U_4 \cap W_1, U_4)$. Thus $U_4 \cap W_1 \subset U_3$.

~~so $U_4 \cap W_1$ is a hyperplane in U_3. If $U_2 \not\subset U_4 \cap W_1$, then taking any L_1 and L_2 we would have $L_1 \oplus L_2 \notin C(U_4 \cap W_1, U_4)$. Thus $U_2 \not\subset U_4 \cap W_1$, so $U_2 \cap W_1$ is a hyperplane in U_2. If $U_1 \neq U_2 \cap W_1$ there exists $L_1 \in \mathbb{P}(U_2 \cap W_1) - \mathbb{P}U_1$~~

If $U_2 \cap W_1 \not\subset U_1$, then $\exists\, L_1 \in \mathbb{P}U_2 \cap \mathbb{P}W_1 - \mathbb{P}U_1$, so for any L_2, we have $L_1 \oplus L_2 \in C(U_1, \ldots, U_4) - C(W_1, W_2)$. Thus $U_2 \cap W_1 \subset U_1$. Since $U_2 \cap W_1 = U_2 \cap (U_4 \cap W_1)$ and $U_4 \cap W_1$ is a hyperplane in U_3, it follows that $\operatorname{cod}(U_2 \cap W_1 \text{ in } U_1) \leq 1$. $\therefore U_2 \cap W_1 = U_1$. So we find the picture



<u>Case 3</u>: $C(W_1, W_2) \subset C(U_1, U_2, U_3, U_4)$. Again $W_2 \subset U_4$. Consider the filtration $W_2 \cap U_1 \subset W_2 \cap U_2 \subset W_2 \cap U_3 \subset W_2$. As $W_2 \twoheadrightarrow U_4/U_3$, $W_2 \cap U_3$ is a hyperplane in W_2. Since for $A \in C(W_1, W_2)$, $A \cap U_2 \to W_2 \cap U_2 / W_2 \cap U_1 \hookrightarrow U_2/U_1$ is an isomorphism, it follows $W_2 \cap U_1$ is a hyperplane in $W_2 \cap U_2$.

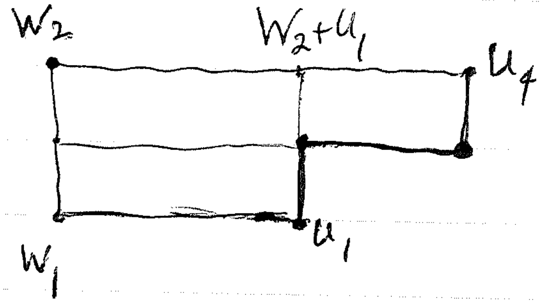
As $W_2 \cap U_3$ is a hyperplane in W_2, if $W_1 \not\subset W_2 \cap U_3$ then we can find $A \in C(W_1 \cap U_3, W_2 \cap U_3) \leq C(W_1, W_2)$, and then $A \to U_4/U_3$ is zero, contradiction. Thus $W_1 \subset W_2 \cap U_3$, so W_1 is a hyperplane in $W_2 \cap U_3$. Fix $L' \in \mathbb{P}W_2 - \mathbb{P}(W_2 \cap U_3)$; then if $L \in \mathbb{P}(W_2 \cap U_3) - \mathbb{P}W_1$, $L \oplus L' \in C(U_1, \ldots, U_4) \Rightarrow$

$$(L \oplus L') \cap U_2 = (L \oplus L') \cap U_3 = L$$

$$\Rightarrow \quad \mathbb{P}(W_2 \cap U_3) - \mathbb{P}W_1 \subset \mathbb{P}U_2 - \mathbb{P}U_1$$

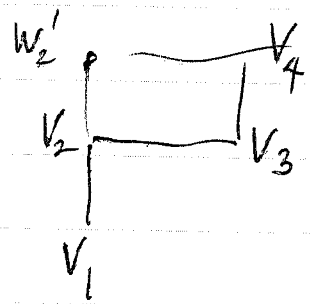
$$\Rightarrow \quad (W_1, W_2 \cap U_3) \leq (U_1, U_2).$$

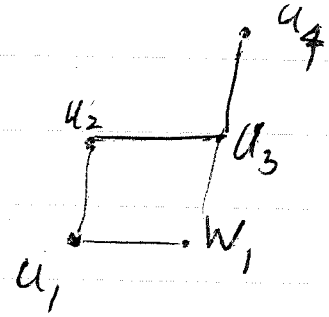
Thus W_1 is a hyperplane in $W_2 \cap U_3$, so for dimensional reasons $W_2 \cap U_2 = W_2 \cap U_3$. Thus we get the picture



<u>Case 4[redacted]:</u> $C(V_1, V_2, V_3, V_4) \subset C(U_1, U_2, U_3, U_4)$

[redacted] If one chooses a W_2' and W_1 so that





then we have $C(V_1, W_2') \subset C(V_1, \cdots, V_4) \subset C(U_1, \cdots, U_4) \subset C(W_1, U_4)$, hence $(V_1, W_2') \leq (W_1, U_4)$. Thus we will get by using all possible choices for W_2' that $V_4 \subset U_4$, and by using all possible choices for W_1 that $V_1 \subset U_1$.

Consider the induced map

$$V_2/V_1 \longrightarrow U_4/U_3$$

<u>Case 4a:</u> This map is zero, i.e. $V_2 \subset U_3$.

If $L_1 \in \mathbb{P}V_2 - \mathbb{P}V_1$, $L_2 \in \mathbb{P}V_4 - \mathbb{P}V_3$, then $L_1 \oplus L_2 \in C(V_1, \dots, V_4) \subset C(U_1, \dots, U_4)$, hence $L_1 \oplus L_2 \twoheadrightarrow U_4/U_3$ $\Longrightarrow$ (as $L_1 \in V_2 \subset U_3$) $L_2 \xrightarrow{\sim} U_4/U_3 \Longrightarrow (L_1 \oplus L_2) \cap U_3 = L_1$. But $(L_1 \oplus L_2) \cap U_3 = (L_1 \oplus L_2) \cap U_2$, so we have

$$\mathbb{P}V_2 - \mathbb{P}V_1 \subset \mathbb{P}U_2 - \mathbb{P}U_1$$

$$\mathbb{P}V_4 - \mathbb{P}V_3 \subset \mathbb{P}U_4 - \mathbb{P}U_3$$

$$\Longrightarrow (V_1, V_2) \le (U_1, U_2) \quad \text{and} \quad (V_3, V_4) \le (U_3, U_4).$$

Case 4b: $V_2/V_1 \xrightarrow{\sim} U_4/U_3$. Consider the filtration $V_4 \cap U_1 \subset V_4 \cap U_2 \subset V_4 \cap U_3 \subset V_4$. I know $V_4 \cap U_3$ is a hyperplane in V_4 different from V_3.

Take any $A \in C(V_1, \dots, V_4)$.

$$\begin{tikzcd} A \cap V_2 \arrow[r] \arrow[d, "\wr"] & U_4/U_3 \\ U_2/U_1 \arrow[ur, "\sim"'] & \end{tikzcd}$$

hence we can write A ~~uniquely~~ in the form $A = L_1 \oplus L_2$ where $\underset{A \cap V_2}{\underset{\shortparallel}{L_1}} \in \mathbb{P}V_2 - \mathbb{P}V_1$ and $\underset{A \cap U_3}{\underset{\shortparallel}{L_2}} \in \mathbb{P}(U_3 \cap V_4) - \mathbb{P}V_3$ where any such L_1, L_2 occur. As

$$L_2 = (L_1 \oplus L_2) \cap U_3 = (L_1 \oplus L_2) \cap U_2$$

we see that

$$\mathbb{P}(U_3 \cap V_4) - \mathbb{P}(U_3 \cap V_3) \subset \mathbb{P}U_2 - \mathbb{P}U_1$$

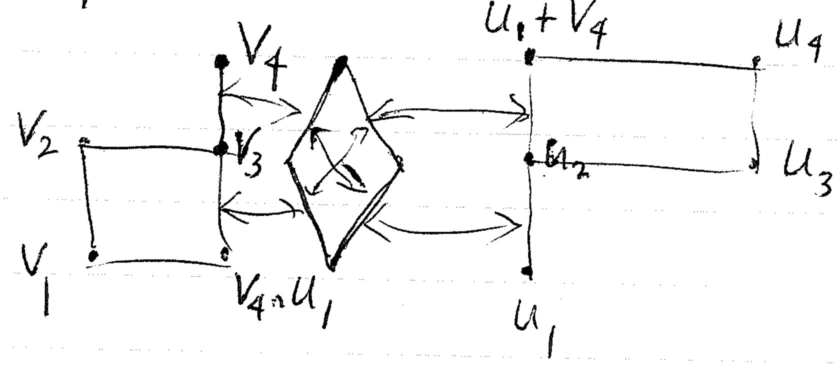
$$\Longrightarrow (U_3 \cap V_3, U_3 \cap V_4) \le (U_1, U_2)$$

Thus $U_3 \cap V_4 \subset U_2 \implies V_4 \cap U_2 = V_4 \cap U_3$.
Since L_2 $\in V_4 \cap U_2$ not in $V_4 \cap U_1$, it follows that $(V_4 \cap U_1, V_4 \cap U_2) \leq (U_1, U_2)$.

So we have $V_4 \cap U_1 < V_4 \cap U_2 = V_4 \cap U_3 < V_4$, and any A in V_4 and $A \in C(U_1, \dots, U_4)$, is in $C(V_4 \cap U_1, V_4)$. Thus we have

$$C(V_1, \dots, V_4) \subset C(V_4 \cap U_1, V_4) \subset C(U_1, \dots, U_4)$$

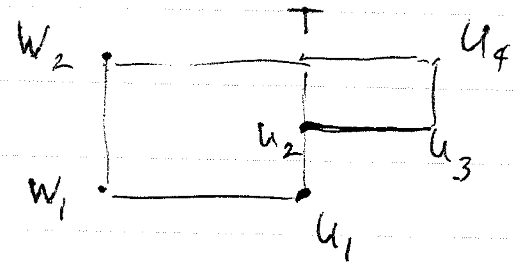
so what we have found above shows us that we get the picture



Thus in this case we factor the inclusion into

$$C(V_1, \dots, V_4) \subset C(V_4 \cap U_1, V_4) \subset C(U_1, V_4 + U_1) < C(U_1, \dots, U_4)$$

Take functor $j : L_2(V) \hookrightarrow Sh_2(V)$. Given $C(U_1, \cdot\cdot, U_4)$ in $Sh_2(V)$, we have seen that $C(W_1, W_2) \subset C(U_1, \cdot\cdot, U_4) \iff (W_1, W_2) \leq (U_1, T)$, where $T/U_2 \oplus U_3/U_2 = U_4/U_2$:



Therefore $j/C(U_1, \cdot\cdot, U_4)$ is homotopy equivalent to the set of such T.

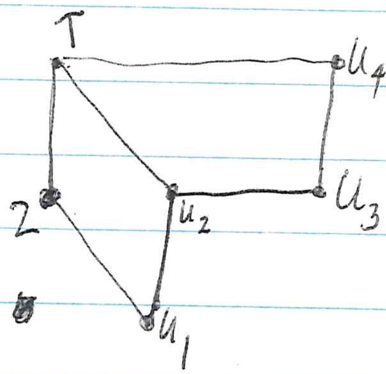
Let $F(U_1, \cdot\cdot, U_4) = \{ T \mid T/U_2 \text{ comp. to } U_3/U_2 \text{ in } U_4/U_2 \}$

$$= \{ T \mid (U_2, T) \leq (U_3, U_4) \}$$

Then F is a covariant functor from $Sh_2(V)$ to sets, and $L_2(V)$ is homotopy equivalent to the cofibred category $Sh_2(V)_F$.

Similarly $C(U_1, \cdot\cdot, U_4) \backslash j$ is heq to the set of complements to U_2/U_1 in U_3/U_1.

Take $j' : L_{1,1}(V) \longrightarrow Sh_2(V)$. Given $C(V_1, \cdot\cdot, V_4) \subset C(U_1, \cdot\cdot, U_4)$, we have either $U_2/V_1 \to U_4/U_3$ is zero or an isom., thus distinguishing components of $j'/C(U_1, \cdot\cdot, U_4)$. If this map is zero, there is a unique arrow $(V_1, \cdot\cdot, V_4) \longrightarrow (U_1, \cdot\cdot, U_4)$ in $L_{1,1}(V)$. If this arrow is $\neq 0$, then one has a unique arrow [redacted] in $L_{1,1}(V)$ to an object (U_1, Z, Z, T) where T/U_2 is comp. to U_3/U_2 in U_4/U_2, and Z/U_1 is comp. to U_2/U_1 in T/U_1. i.e. $Z/U_1 \oplus U_3/U_1 = U_4/U_1$ and $T = Z + U_2$



[redacted] Thus we get the functor which assigns to (U_1, U_2, U_3, U_4) the union of a point and the set of lines in U_4/U_1 complementary to U_3/U_1, (except where $U_2 = U_3$, when we get the set of lines in U_4/U_1.

Similarly, $L'_{1,1}(V)$ is homotopy equivalent to the cofibred cat. over $Sh_2(V)$ defined by the functor assigning to (U_1, U_2, U_3, U_4) the pairs (L, T) consisting of a $T \ni T/U_2 \oplus U_3/U_2 = U_4/U_2$ and over a line L in T/U_1. The map $L'_{1,1}(V) \xrightarrow{a} L_2(V)$ forgets L; the map $L'_{1,1}(V) \xrightarrow{b} L_{1,1}(V)$ collapses all (L, T) with $L = U_2/U_1$ to a point.

Now let me fix $(U_1, \ldots, U_4)$ and compute the map

$$\mathbb{Z}\{(T,L)\} \longrightarrow \mathbb{Z}\left\{\begin{matrix} pt \cup (T,L) \\ L \neq U_2/U_1 \end{matrix}\right\} \overset{\times}{\oplus} \mathbb{Z}\{T\}.$$

Here (T,L) runs over all pairs: $T/U_2 \oplus U_3/U_2 = U_4/U_2$, L/U_1 = a line in T/U_1. The map goes onto the first factor. The kernel is freely generated by elements of the form $(T, U_2/U_1) - (T_0, U_2/U_1)$, as T ranges over the complements to U_3/U_2 in U_4/U_2. This hits exactly the augmentation zero part of $\mathbb{Z}[T]$. Hence we can conclude working with covariant functors

(*)
$$\begin{tikzcd} L'_{1,1}(V) \arrow[r, "a"] \arrow[d, "b"'] & L_2(V) \arrow[d, "j"] \\ L_{1,1}(V) \arrow[r, "j'"'] & Sh_2(V) \end{tikzcd}$$

that $L_+ j'_! \mathbb{Z} = L_+ j_! \mathbb{Z} = L_+ (ja)_! \mathbb{Z} = 0$ and that

$$0 \to (ja)_! \mathbb{Z} \longrightarrow j_! \mathbb{Z} \oplus j'_! \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact, whence (*) is homotopy-cocartesian.

October 18, 1974.

I now want to generalize the preceding in order to understand the poset $Sh_p(V)$ of Schubert cells in $G_p(V)$.

What is a Schubert cell? Take a full flag $0 = V_0 < V_1 < V_2 < \cdots\cdots$ in V, $\dim V_i = i$, and a sequence $0 < i_1 < i_2 < \cdots < i_p$. The corresponding Schubert cell is

$$\left\{ A \in G_p(V) \;\middle|\; \begin{array}{l} \dim(A \cap V_{i_j}) = j \qquad j = 1, \ldots, p \\ \dim(A \cap V_{i_j - 1}) = j-1 \end{array} \right\}$$

The cell is perhaps best described using filtrations

$$U_1 \subset U_2 \subset \cdots\cdots \subset U_{2p}$$

such that $\dim(U_{2i}/U_{2i-1}) = 1$ for $1 \leq i \leq p$. The corresponding Schubert cell is

$$C(U_1, \ldots, U_{2p}) = \{ A \in G_p(V) \mid 0 = A \cap U_1 < A \cap U_2 = A \cap U_3 < A \cap U_4 = \cdots\cdots \}$$

Observe that if $U_{2i} = U_{2i+1}$ for some i then the cell $C(U_1, \ldots, U_{2p})$ depends only on $U_1, \ldots, U_{2i-1}, U_{2i+2}, \ldots, U_{2p}$. So therefore it would be better to give a filtration

$$U_1 < U_2 \subset \cdots < U_{2q}$$

with $\displaystyle p = \sum \dim U_{2i}/U_{2i-1} \qquad 1 \leq i \leq q$

and to define $C(U_1, \ldots, U_{2q})$ as the set of A in $G_p(V)$ such that

$$A \cap U_{2i} + U_{2i-1} = U_{2i} \qquad 1 \le i \le q$$

(Note this implies $A \cap U_{2i} / A \cap U_{2i-1} = U_{2i}/U_{2i-1}$, hence by dimensional considerations that $A \cap U_{2i-1} = A \cap U_{2i}$).

Suppose we have $a_1 + \cdots + a_q = p$ with $a_i > 0$. Then we define $L_{a_1, \ldots, a_q}(V)$ to be the subset of $L_{a_1}(V) \times \cdots \times L_{a_q}(V)$ consisting of

$$(U_1, U_2), (U_3, U_4), \ldots, (U_{2q-1}, U_{2q})$$

such that $U_2 \subset U_3$, $U_4 \subset U_5, \ldots$. [redacted] Previous argument should generalize to show that $L_{a_1, \ldots, a_q}(V)$ is a classifying space for $BGL_{a_1} \times \cdots \times BGL_{a_q}$.

It might be better to think of $a_1 + \cdots + a_q = p$ as a subset σ of simple roots. (The simple roots for SL_p are pair $(i, i+1)$ $1 \le i \le p-1$. So here the simple roots are $(a_1, a_1+1), (a_1+a_2, a_1+a_2+1), \ldots, (a_1 + \cdots + a_{p-1}, \cdots + 1)$. Thus σ is a subset of $1, \ldots, p-1$. Use the notation $L_\sigma(V)$ for $L_{a_1, \ldots, a_q}(V)$. σ is allowed to be the empty subset, whence we get $L_p(V)$.

If now $\tau \subset \sigma \subset \{1, \dots, p-1\}$, then we let $L_{\sigma,\tau}(V)$ be the subset of $L_\sigma(V)$ consisting of $U_1 \subset \cdots \subset U_{2k}$ such that $U_{2i} = U_{2i+1}$ for each element of σ not in τ.

Maybe a better notation would be to label σ as $1 \le i_1 < \dots < i_{q-1} < i_q = p$. Then the filtration is

$$U_{2i_1-1} \underbrace{<}_{i_1} U_{2i_1} \subset \underbrace{U_{2i_2-1} < U_{2i_2}}_{i_2 - i_1} \subset \cdots\cdots < U_{2i_q}$$

$\sigma \subset \{1, \dots, p-1\}$. $\sigma = \{i_1 < i_2 < \dots < i_{q-1}\}$. $L_\sigma(V)$ consists of flags

$$U'_{i_1} < U''_{i_1} \subset U'_{i_2} < U''_{i_2} \subset \cdots \subset U'_{i_{q-1}} < U''_{i_{q-1}} \subset U'_p < U''_p$$

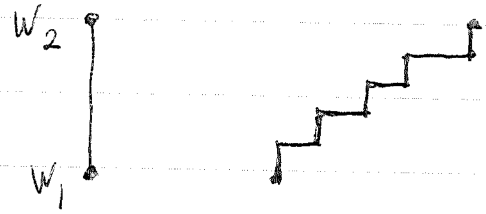
i.e. a succession of layers of dimensions $i_1, i_2 - i_1, \dots, p - i_{q-1}$

If $\tau \subset \sigma$ it is clear what I mean by $L_{\tau<\sigma}(V)$ namely the subset of $L_\sigma(V)$ such that for each minimal interval $j < j'$ in τ, the refining σ layers are squeezed together.

Start by trying to understand inclusions. Given $(U_1, \dots, U_{2p})$, $\dim U_{2i}/U_{2i-1} = 1$, $1 \le i \le p$, one first wants to understand an inclusion

$$C(W_1, W_2) \subset C(U_1, \dots, U_{2p})$$

The conjecture is that one has the picture:



So we consider the induced filtration $W_2 \cap U_j$ $1 \le j \le 2p$. Fixing $A \in C(W_1, W_2)$, we know

$$\begin{tikzcd}[column sep=small]
A \cap U_{2i}/A \cap U_{2i-1} \arrow[rr, "\sim"] \arrow[dr] & & U_{2i}/U_{2i-1} \quad \dim 1 \\
& W_2 \cap U_{2i}/W_2 \cap U_{2i-1} \arrow[ur] &
\end{tikzcd}$$

hence $W_2 \cap U_{2i}/W_2 \cap U_{2i-1} \xrightarrow{\sim} U_{2i}/U_{2i-1}$. ~~What I have to show is that $W_2 \cap U_1 = W_1$~~

Claim $W_1 \supset W_2 \cap U_1$. If not choose $L_1 \in \mathbb{P}(W_2 \cap U_1) - \mathbb{P}W_1$ and extend L_1 to $A \in C(W_1, W_2)$. Then $A \cap U_1 \neq 0$ contradiction.

Then $W_1 + W_2 \cap U_2 > W_1$ is of codim 1, since any $A \in C(W_1, W_2)$ has $A \cap U_2 = L_1 \not\subset W_1$.

Claim $W_1 + W_2 \cap U_2 = W_1 + W_2 \cap U_3$. If not $\exists$

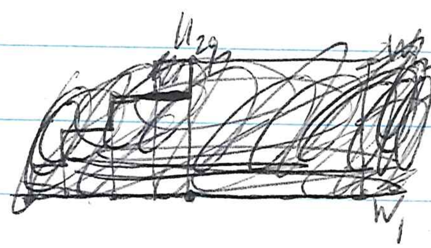
$L_2 \in \mathbb{P}(W_2 \cap U_3) - \mathbb{P}(W_1 + W_2 \cap U_2)$. Fix $L_1 \in \mathbb{P}(W_2 \cap U_2) - \mathbb{P}W_1$ and extend $L_1 + L_2$ to an $A \in C(W_1, W_2)$. Then $L_1 + L_2 \subset A \cap U_3$ contradiction.

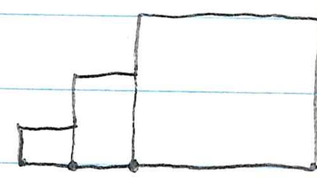
Continuing, one sees that

$$W_1 = W_1 + W_2 \cap U_1 < W_1 + W_2 \cap U_2 = W_1 + W_2 \cap U_3 <$$

etc. Counting dimensions, it follows that $W_2 \cap U_1$ has the same codim in W_2 as does W_1, thus $W_1 = W_2 \cap U_1$.

Next consider an inclusion $C(U_1, \dots, U_{2p}) \subset C(W_1, W_2)$, where we want the picture





Here we have $U_{2p} \twoheadrightarrow W_2/W_1$ and so we can as well suppose $U_{2q} = W_2$. What I want to show is that

$$U_1 \subset W_1$$

$$W_1 \cap U_{2i-1} = W_1 \cap U_{2i}$$

$$W_1 \cap U_{2i+1} / W_1 \cap U_{2i} \xrightarrow{\sim} U_{2i+1}/U_{2i}$$

So I will consider the filtration

$$W_1 \subset U_1 + W_1 \subset U_2 + W_1 \subset \cdots \subset U_{2g} + W_1 = W_2.$$

If $U_{2g-1} + W_1 = W_2$, $\exists\, L_g \in \mathbb{P}W_1 - \mathbb{P}U_{2g-1}$, so if $L_i \in \mathbb{P}U_{2i} - \mathbb{P}U_{2i-1}$ $i = 1, \ldots, g-1$, then $A = L_1 \oplus \cdots \oplus L_g \in C(U_1, \ldots, U_{2g})$ but $A \notin C(W_1, W_2)$. This contradiction shows $U_{2g-1} \supset W_1$.

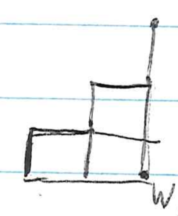
If $U_{2g-3} + W_1 = U_{2g-2} + W_1$, then $\exists\, L_{g-1} \in \mathbb{P}U_{2g-2} \cap \mathbb{P}W_1 - \mathbb{P}U_{2g-3}$.

If for some i, $U_{2i-1} + W_1 = U_{2i} + W_1$ then $\exists\, L_i \in \mathbb{P}U_{2i} \cap \mathbb{P}W_1 - \mathbb{P}U_{2i-1}$. Then with any other $L_1, \ldots, \hat{L}_i, \ldots, L_g$ we have $A = L_1 + \cdots + L_g \in C(U_1, \ldots, U_{2g})$, but $A \notin C(W_1, W_2)$. Thus conclude

$$W_1 \subset U_1 + W_1 < U_2 + W_1 < \cdots \subset U_{2g-1} + W_1 < U_{2g} + W_1 = U_{2g}$$

So dimension-counting shows that $W_1 = U_1 + W_1$ $\Longrightarrow U_1 \subset W_1$ and that $U_{2i} + W_1 = U_{2i+1} + W_1$. So we do get the picture

~~$U_{2i} \subset W_1 \subset U_{2i+1}$~~



that we expected.

Suppose $C(W_1, W_2) \subset C(U_1, \dots, U_{2p})$, where $\dim(U_{2i}/U_{2i-1}) = 1$ $i = 1, \dots, p$. I consider the filtration of W_2/W_1 induced by $U_1 \subset \dots \subset U_{2p}$. Recall

$$\frac{W_1 + (W_2 \cap U_j)}{W_1 + (W_2 \cap U_{j-1})} \cong \frac{W_2 \cap U_j}{W_1 \cap U_j + W_2 \cap U_{j-1}}$$

Choose $A \in C(W_1, W_2)$, so that

$$0 = A \cap U_1 < A \cap U_2 = A \cap U_3 < A \cap U_4 = \cdots$$

Then

$$\begin{tikzcd}
\dfrac{A \cap U_{2i}}{A \cap U_{2i-1}} \arrow[rr, "\sim"] \arrow[dr] & & U_{2i}/U_{2i-1} \\
& W_2 \cap U_{2i}/W_2 \cap U_{2i-1} \arrow[ur, hook] &
\end{tikzcd}$$

so $W_2 \cap U_{2i-1} < W_2 \cap U_{2i}$.

October 20, 1974. Schubert cells

Given a flag of fin. diml. subspaces of V

$$U_1 \subset U_2 \subset \cdots \subset U_{2p}$$

with $\dim U_{2i}/U_{2i-1} = 1$ for $1 \le i \le p$, we put

$$C(U_1, \ldots, U_{2p}) = \{ A \in G_p(V) \mid 0 = A \cap U_1 < A \cap U_2 = A \cap U_3 < \cdots \}$$

such a subset of $G_p(V)$ we call a Schubert cell, and we let $Sh_p(V)$ be the poset of Schubert cells, ordered by inclusion.

Notice that if $U_{2j} = U_{2j+1}$, then $C(U_1, \ldots, U_{2p})$ doesn't depend upon $U_{2j} = U_{2j+1}$, in fact we have

$$C(U_1, \ldots, U_{2p}) = \left\{ A \in G_p(V) \,\middle|\, \begin{array}{l} \dim (A \cap U_{2i-1}) = \dim (A \cap U_{2i}) = i \\ \text{for } 1 \le i \le p, \ i \ne j \end{array} \right\}$$

(The point is that the conditions $\dim A \cap U_{2i-2} = i-1$, $\dim A \cap U_{2i+2} = i+1$, $\dim U_{2i+2}/U_{2i-2} = 2$ force $\dim A \cap U_{2i} = i$.) When this happens, we write $C(U_1, \ldots, \hat{U}_{2j}, \hat{U}_{2j+1}, \ldots, U_{2p})$ for $C(U_1, \ldots, U_{2p})$. In this way we define $C(W_1, \ldots, W_{2k})$ for any flag $W_1 \subset \cdots \subset W_{2k}$ such that $W_{2i-1} < W_{2i}$, $1 \le i \le k$, and

$$\sum_{i=1}^{k} \dim (W_{2i}/W_{2i-1}) = p$$

Suppose $C(W_1, W_2) \subset C(U_1, \ldots, U_{2p})$. If $W_1 \not\supset W_2 \cap U_1$, $\exists\ L_1 \in \mathbb{P}(W_2 \cap U_1) - \mathbb{P}W_1$, which can be extended to $A \in C(W_1, W_2)$; but $A \cap U_1 \supset L_1 \neq 0$ so $A \notin C(U_1, \ldots, U_{2p})$, a contradiction. Thus $W_1 \supset W_2 \cap U_1$, i.e. $W_1 = W_1 + (W_2 \cap U_1)$.

If $W_1 + W_2 \cap U_{2i-2} \lneq W_1 + W_2 \cap U_{2i-1}$, $\exists\ L_i \in \mathbb{P}(W_2 \cap U_{2i-1}) - \mathbb{P}(W_1 + W_2 \cap U_{2i-2})$. Choose $A \in C(W_1, W_2)$ so that $A \cap U_{2i-2}$ has dim $i-1$. Then $L_i + A \cap U_{2i-2}$ can be extended to $A' \in C(W_1, W_2)$; but $A' \cap U_{2i-1} \supset L_i + A \cap U_{2i-2}$ which has dim i, a contradiction. Thus $W_1 + W_2 \cap U_{2i-2} = W_1 + W_2 \cap U_{2i-1}$ for $i = 1, \ldots, p$.

By dimensional considerations, it follows that $W_1 + W_2 \cap U_{2i-1}$ is a hyperplane in $W_1 + W_2 \cap U_{2i}$, $1 \le i \le p$.

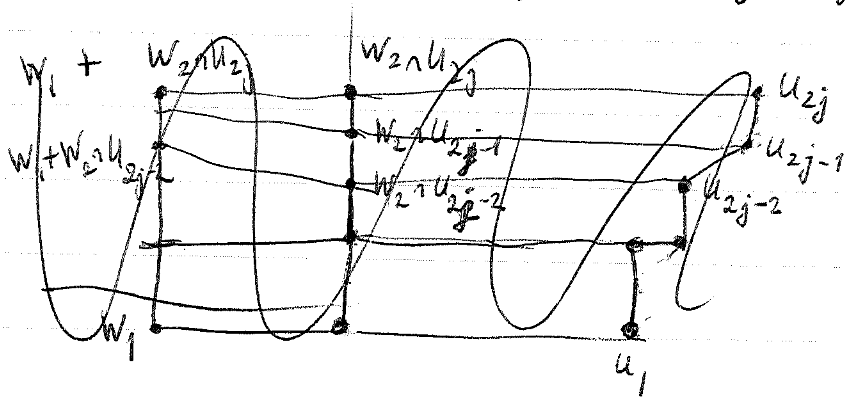
~~If $W_2 \cap U_{2i} < W_2 \cap U_{2i+1}$ for some $1 \le i < p$, then $\exists\ L_{i+1} \in \mathbb{P}(W_2 \cap U_{2i+1}) - \mathbb{P}(W_2 \cap U_{2i})$, hence~~

Can I show $W_2 \cap U_2 = W_2 \cap U_3$?

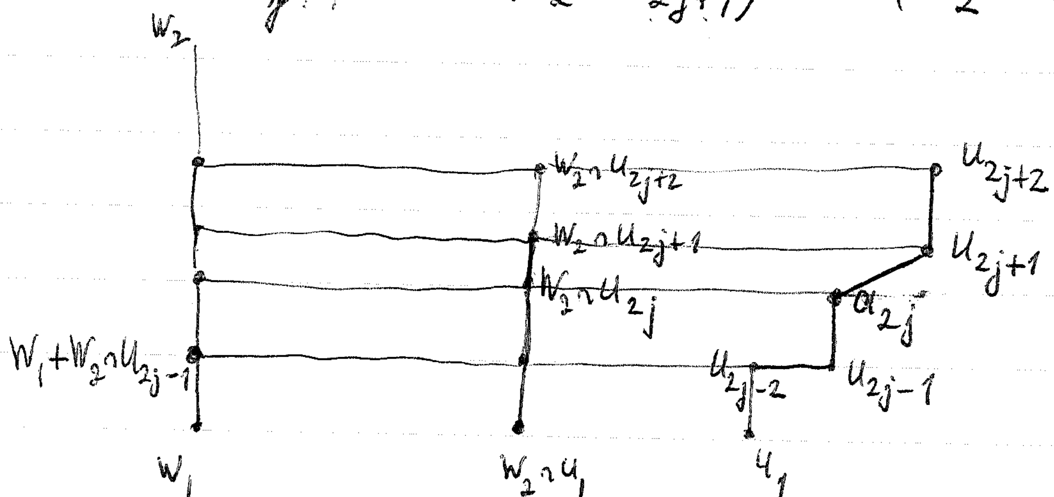
$$\begin{tikzcd} 0 \arrow[r] & \dfrac{W_1 \cap U_3}{W_2 \cap U_1} \arrow[r] & \dfrac{W_2 \cap U_3}{W_2 \cap U_1} \arrow[r, two heads] & \dfrac{W_1 + W_2 \cap U_3}{W_1} \arrow[r] & 0 \\ & & \dfrac{W_2 \cap U_2}{W_2 \cap U_1} \arrow[u] \arrow[ur, "\sim"'] & & \end{tikzcd}$$

Thus you see that if $W_2 \cap U_2 < W_2 \cap U_3$, then we can find a line in $W_2 \cap U_3$ not contained in the hyperplanes $W_2 \cap U_2$, $W_1 \cap U_3$. (Use the fact that projective space has ≥ 3 elements).

Continue: Assuming we know $W_2 \cap U_{2i} = W_2 \cap U_{2i+1}$ for $i < j$, assume $W_2 \cap U_{2j} < W_2 \cap U_{2j+1}$ for some $j < p$.



Choose $L_j \in \mathbb{P}(W_2 \cap U_{2j+1}) - \mathbb{P}(W_2 \cap U_{2j}) - \mathbb{P}(W_1 + W_2 \cap U_{2j-1})$



This is possible. Combining this with $A \cap U_{2j-1}$ to get $L_j + A \cap U_{2j-1}$, we can extend this to $A' \in C(W_1, W_2)$. ~~But then $A' \cap U_{2j+1} \supseteq L_j + A \cap U_{2j-1}$ which has dimension j, hence $A' \cap U_{2j+1} = L_j + A \cap U_{2j-1}$.~~

~~Then~~ $\cancel{A\cap U_{2j} = (L_j + A\cap U_{2j-1})\cap U_{2j} = L_j \cap U_{2j} + A \cap U_{2j-1}}$

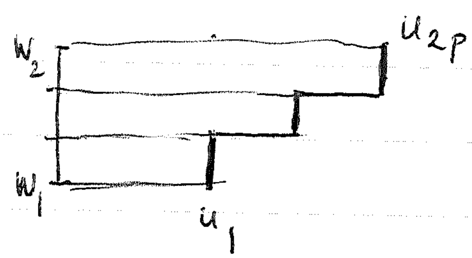
~~But then~~ $\cancel{A'\cap U_{2j+1}}$ ~~contains~~ $\cancel{L_j}$~~, which is not~~

But then $A'\cap U_{2j+1} \supset L_j + A\cap U_{2j-1}$ which has dimension $j \implies A'\cap U_{2j+1} = L_j + A\cap U_{2j-1}$. $\Longrightarrow$ ~~But~~ $A'\cap U_{2j} \neq A'\cap U_{2j+1}$ as $L_j \notin A'\cap U_{2j}$. Contradiction, so we find $W_2 \cap U_{2i} = W_2\cap U_{2i+1}$ for $1 \leq i \leq p-1$. Now by counting we find that $W_1 = W_2 \cap U_1$.

Thus we have proved.

<u>Prop</u>: If $C(W_1, W_2) \subset C(U_1, \dots, U_{2p})$, then $(W_1, W_2) \leq (U_1, W_2 + U_1)$, where $W_2 + U_1/U_1 \in C(U_1/U_1, \dots, U_{2p}/U_1)$.

Have picture:



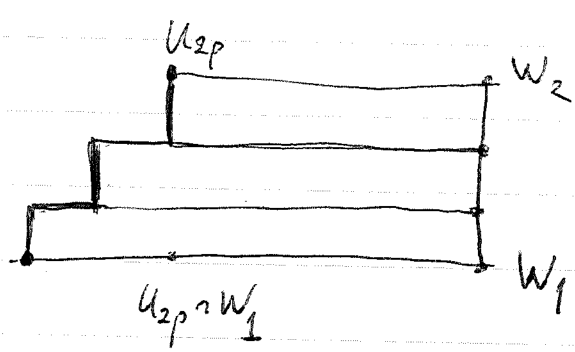
~~Suppose now $C(U_1,\dots,U_{2p}) \subset C(W_1,W_2)$. Consider the filtration Since $U_{2p} \to W_2/W_1$, it follows $W_1 \subset U_1+W_1 \subset U_2+W_1$ that this inclusion factors~~

~~$C(U_1,\dots,U_{2p}) \subset C(U_{2p}\cap W_1, U_{2p})$.~~

Suppose now that $C(U_1,\dots,U_{2p}) \subset C(W_1,W_2)$ and consider the flag

$$W_1 \subset U_1+W_1 \subset U_2+W_2 \subset \cdots \subset U_{2p}+W_1 = W_2$$

Suppose $U_{2i-1}+W_1 = U_{2i}+W_1$ for some i, $1 \le i \le p$. Then $U_{2i-1} + U_{2i}\cap W_1 = U_{2i}$, hence $\exists\, L_i \in \mathbb{P}U_{2i}\cap W_1 - \mathbb{P}U_{2i-1}$. Choosing $L_j \in \mathbb{P}U_{2j} - \mathbb{P}U_{2j-1}$ for $1 \le j \le p$, $j \ne i$, we have $A = L_1 + \cdots + L_p \in C(U_1,\dots,U_{2p})$, but $A \notin C(W_1,W_2)$, a contradiction. Thus $U_{2i-1}+W_1 < U_{2i}+W_1$ for $1 \le i \le p$. Since W_1 is of codim p in W_2, this forces $W_1 = U_1+W_1$ (hence $U_1 \subset W_1$) and $U_{2i}+W_1 = U_{2i+1}+W_1$ $1 \le i \le p-1$, so we get the picture:



<u>Prop</u>: If $C(U_1, \ldots, U_{2p}) \subset C(W_1, W_2)$, then one has

$$U_1 \subset W_1, \quad U_{2p} \subset W_2$$

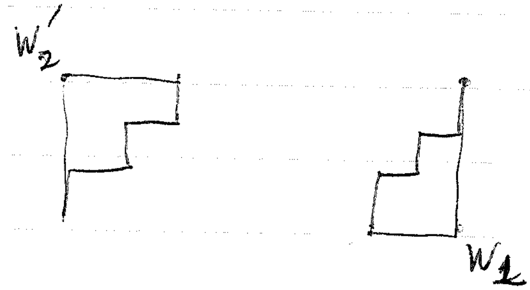
$$U_{2i} + W_1 = U_{2i+1} + W_1 \qquad 1 \le i \le p-1$$

$$U_{2i} \cap W_1 = U_{2i-1} \cap W_1 \qquad 1 \le i \le p.$$

Next let us consider an inclusion

$$C(V_1, \ldots, V_{2p}) \subset C(U_1, \ldots, U_{2p})$$

Choose $C(W_1', W_2') \subset C(V_1, \ldots, V_{2p})$ with $W_1' = V_1$
and $C(U_1, \ldots, U_{2p}) \subset C(W_1, W_2)$ with $W_2 = U_{2p}$



One then has $(W_1', W_2') \le (W_1, W_2)$ showing that $V_1 \subset U_1$, $V_{2p} \subset U_{2p}$.

Review the Bruhat decomposition. Let Z be a vector space with a full flag $0 = Z_0 < Z_1 < \cdots < Z_p = Z$, and let $0 = F_0 < F_1 < \cdots < F_p = Z$ be another flag. Recall the Schreier isom

$$gr_i(F_j/F_{j-1}) = \frac{Z_i \cap F_j + F_{j-1}}{Z_{i-1}\cap F_j + F_{j-1}} = \frac{Z_i \cap F_j}{Z_{i-1}\cap F_j + Z_i \cap F_{j-1}}$$

$$\|$$

$$gr^F_j(Z_i/Z_{i-1}) = \frac{Z_i \cap F_j + Z_{i-1}}{Z_i \cap F_{j-1} + Z_{i-1}} = \frac{Z_i \cap F_j}{Z_i \cap F_{j-1} + Z_{i-1}\cap F_j}$$

Therefore we get a unique permutation σ of $\{1,\dots,p\}$ such that $gr_i(F_{\sigma i}/F_{\sigma i-1}) \neq 0$ for $1 \le i \le p$. Better, for each i, $1 \le i \le p$, σi is the unique index such that

$$F_{\sigma i}/F_{\sigma i-1} \xleftarrow{\sim} \frac{Z_i \cap F_{\sigma i}}{Z_{i-1}\cap F_{\sigma i} + Z_i \cap F_{\sigma i-1}} \xrightarrow{\sim} Z_i/Z_{i-1}$$

So now given $C(V_1,\dots,V_{2p}) \subset C(U_1,\dots,U_{2p})$, I want to associate a permutation to this inclusion. Choose $A \in C(V_1,\dots,V_p)$ and consider the permutation associated to the two flags

$$A \cap V_{2i} \quad , \quad A \cap U_{2i}$$

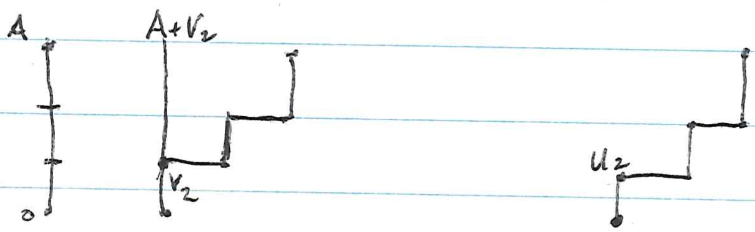
On the other hand we can choose $C(U_1,\dots,U_{2p}) \subset C(W_1,W_2)$, whence we get two flags in W_2/W_1

$$\frac{V_{2i}+W_1}{W_1} \quad , \quad \frac{U_{2i}+W_1}{W_1}$$

Since under the isomorphism $A \xrightarrow{\sim} W_2/W_1$, one has $A \cap V_{2i} \xrightarrow{\sim} V_{2i} + W_1/W_1$, it follows that the permutations obtained by either choosing A or (W_1, W_2) are the same, hence independent of these choices.

Let us consider the two extreme cases.

First suppose the permutation is the identity, i.e. $A \cap V_{2i} = A \cap U_{2i}$, $1 \le i \le p$. Since we know $U_1 \subset V_1$, $U_2 = A \cap U_2 \oplus U_1$, $V_2 = A \cap V_2 \oplus U_2$, it follows that $U_2 \subset V_2$. Note that if $A \in C(V_1, .., V_{2p})$ then $A + V_2/V_2 \in C(V_3/V_2, V_4/V_2, .., V_{2p}/V_2)$:



Moreover the resulting map $C(V_1, .., V_{2p}) \to C(V_3/V_2, .., V_{2p}/V_2)$ is onto. Claim $C(V_3/V_2, .., V_{2p}/V_2) \subset C(U_3/V_2, .., U_{2p}/V_2)$. Indeed take B in the former and lift it to $A \in C(V_1, .., V_{2p})$, so $B = A + V_2/V_2$. Then clearly $A + V_2/V_2 \in C(U_3/V_2, .., U_{2p}/V_2)$.

So we get $V_3/V_2 \subset U_3/V_2$, whence $V_4 = V_3 + V_4 \cap A \subset U_3 + U_4 \cap A = U_4$ and we can continue

<u>Prop.</u> If the permutation assoc. to $C(V_1, \dots, V_{2p}) \subset C(U_1, \dots, U_{2p})$ is the identity, then $(V_{2i-1}, V_{2i}) \leq (U_{2i-1}, U_{2i})$ for $1 \leq i \leq p$.

Here's another proof. Let $L_i \in \mathbb{P}V_{2i} - \mathbb{P}V_{2i-1}$, so that $A = L_1 + \dots + L_p \in C(V_1, \dots, V_{2p}) \subset C(U_1, \dots, U_{2p})$. We know $A \cap U_{2i} = A \cap V_{2i} = L_1 + \dots + L_i$, hence $L_i \in \mathbb{P}U_{2i}$. And we know $A \cap U_{2i-1} = A \cap U_{2i-2} = L_1 + \dots + L_{i-1}$ so $L_i \notin \mathbb{P}U_{2i-1}$. Thus

$$\mathbb{P}V_{2i} - \mathbb{P}V_{2i-1} \subset \mathbb{P}U_{2i} - \mathbb{P}U_{2i-1}$$

so $(V_{2i-1}, V_{2i}) \leq (U_{2i-1}, U_{2i})$.

Next extreme case is where the permutation σ reverses the order: $\sigma(i) = p - i + 1$. In this case we know that the filtrations $V_{2i} \cap A$ and $U_{2i} \cap A$ are complementary, i.e.

$$V_{2i} \cap A \oplus U_{2(p-i)} \cap A = A.$$

Or put another way, we have unique lines $L_1, \dots, L_p$ such that

$$V_{2i} \cap A = L_1 + \dots + L_i$$

$$U_{2i} \cap A = L_{p-i+1} + \dots + L_p$$

L_1 can be arbitrary in $\mathbb{P}V_2 - \mathbb{P}V_1$

L_2 ———————— $(\mathbb{P}V_4 - \mathbb{P}V_3) \cap \mathbb{P}U_{2p-2}$

L_3 ———————— $(\mathbb{P}V_6 - \mathbb{P}V_5) \cap \mathbb{P}U_{2p-4}$

$$(\mathbb{P}V_4 - \mathbb{P}V_3) \cap \mathbb{P}U_{2p-2} = (\mathbb{P}V_4 - \mathbb{P}V_3) \cap \mathbb{P}U_{2p-1} \subset \mathbb{P}U_{2p-2} - \mathbb{P}U_{2p-3}$$

so $$(V_3 \cap U_{2p-2}, V_4 \cap U_{2p-2}) = \underline{(V_3 \cap U_{2p-1}, V_4 \cap U_{2p-1}) \leq (U_{2p-3}, U_{2p-2})}$$

In general $L_i \in (\mathbb{P}V_{2i} - \mathbb{P}V_{2i-1}) \cap \mathbb{P}U_{2(p-i)+3} \subset \mathbb{P}U_{2(p-i)+2} - \mathbb{P}U_{2(p-i)+1}$

so

$$(V_{2i-1} \cap U_{2p-2i+3}, V_{2i} \cap U_{2p-2i+3}) \leq (U_{2p-2i+1}, U_{2p-2i+2})$$

First case would be for L_p

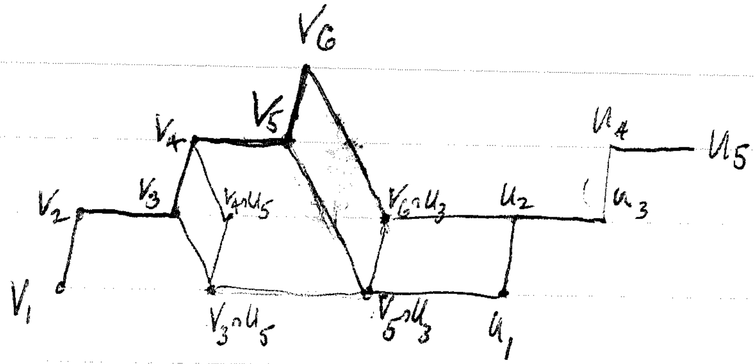
$$(V_{2p-1} \cap U_3, V_{2p} \cap U_3) \leq (U_1, U_2)$$

Write this for $p=3$.

$$(V_5 \cap U_3, V_6 \cap U_3) \leq (U_1, U_2)$$

$$(V_3 \cap U_5, V_4 \cap U_5) \leq (U_3, U_4)$$

$$(V_1, V_2) \leq (U_5, U_6)$$



~~Analogously if you work with~~ The basic question ~~is~~ whether $C(V_1, \ldots, V_6) < C(V_5 \cap U_3, V_6)$ I need $V_6 \cap U_4 = V_6 \cap U_5$?

Generalize & consider also flag manifolds instead of Grassmannians.

$$\underbrace{G_{(1,\dots,1)}}_{p \text{ times}}(V) = \text{set of flags } 0 < A_1 < \cdots < A_p \text{ in } V \text{ with } \dim A_i/A_{i-1} = 1.$$

Let $0 < V_1 < V_2 < \cdots\cdots$ be a full flag in V.
For each i, $1 \leq i \leq p$ there is an unique s_i such that $A_i/A_{i-1} \simeq V_{s(i)}/V_{s(i)-1}$ in the Schreier isomorphism. Thus $s(i)$ is the unique integer ≥ 1 such that

$$\frac{A_i \cap V_{s(i)} + A_{i-1}}{A_i \cap V_{s(i)-1} + A_{i-1}} \neq 0$$

$$\|$$

$$\frac{A_i}{A_{i-1}} \xleftarrow{\text{Qmap}} \frac{A_i \cap V_{s(i)}}{A_i \cap V_{s(i)-1} + A_{i-1} \cap V_{s(i)}} \xrightarrow{\text{Qmap}} \frac{V_{s(i)}}{V_{s(i)-1}}$$

Here $s : \{1, \dots, p\} \hookrightarrow \{1, 2, \dots, \dim V\}$. Question: Fixing s, does the set of $(0 < A_1 < \dots < A_p)$ belonging to s form a cell?

Suppose $V = \{ke_1 + \dots + ke_i\}$. $e_1, \dots$ basis for V. Choosing a basis x_i for A_p with $A_i = kx_1 + \dots + kx_p$ one gets a matrix [redacted] of size $p \times \dim(V)$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{matrix} \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \end{matrix}$$

we are permitted to add multiples of x_i to x_j for $i < j$, [redacted] Resulting canonical form for the matrix is

$$\begin{matrix} * & * & * & * & * & * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & * & * & * & 0 & * & \underline{1} & 0 & 0 & 0 \end{matrix}$$

The 1's occur in positions $s_1, s_2, \dots, s_p$. Below any [redacted] 1 [redacted] are zeroes.

Notice that this cell depends only [redacted] on the layers $(V_{s(1)-1}, V_{s(1)}) \dots\dots (V_{s(p)-1}, V_{s(p)})$. So what seems to happen is that we have a flag $(U_1, \dots, U_{2p})$ as before together with a permutation σ of $\{1, \dots, p\}$. The corresponding cell consists of pairs (A, f) where $A \in C(U_1, \dots, U_{2p})$, and where f is a full flag in A bearing the relation σ to the flag $0 < U_2 \cap A < \dots < U_{2p} \cap A = A$.

Infinite Grassmannian: Let V contain V_0 such that V_0 and V/V_0 are of infinite dimension. Then we can consider $A \subset V$ commensurable with V_0, meaning that $A/A \cap V_0$, $V_0/A \cap V_0$ are finite dimensional. Call this set $G(V, V_0)$. Each such A has an index $= \dim(A/A\cap V_0) - \dim(V_0/A \cap V_0)$, so

$$G(V, V_0) = \coprod_n G_n(V, V_0)$$

where $G_n(V, V_0)$ consists of those A of index n. Clearly

$$G_p(V, V_0) = \bigcup_{V_1 \subset V_0 \subset V_2} G_{p+\dim(V_0/V_1)}(V_2/V_1)$$

where V_1, V_2 run over subspaces such that V_2/V_0, V_0/V_1 are finite dimensional. From now on concentrate on index 0.

What is a Schubert cell in $G_0(V, V_0)$? Suppose we have a full flag which I will suppose to pass through

V_0. Call it $\cdots\cdots < V_{-1} < V_0 < V_1 < \cdots\cdots$ If I have [redacted] $A \in G_0(V,V_0)$, then we get a set of n such that [redacted] V_n/V_{n-1} appears in A, meaning that

$$V_{n-1} \cap A < V_n \cap A.$$

It seems reasonable [redacted] to suppose A contains V_{-N} for some [redacted] N; this would certainly be the case if I just took A with this property, which would still give me an infinite Grassmannian. [redacted] Can suppose $A \subset V_N$, whence $\{n \mid V_{n-1} \cap A < V_n \cap A\} \subset [-N, N]$. So it is clear that fixing the flag $\{V_n\}$ and this finite set S of integers, [redacted] the Schubert cell I am considering is just the image of a cell in $G_0(V_N/V_{-N})$.

These cells can be described as follows: One gives [redacted] $U_1 \subset \cdots \subset U_{2k}$, [redacted] $\dim U_{2i}/U_{2i-1} = 1$ commensurable with V_0 and defines

$$C(U_1, U_2, \ldots, U_{2k}) = \left\{ A \in G_0(V,V_0) \;\middle|\; \begin{array}{l} \dim U_{2i} \cap A / U_{2i-1} \cap A = 1 \\ U_1 \subset A \subset U_{2k} \end{array} \right\}$$

U_1 will have to have index $-k$.

<u>October 24, 1974</u>

Recall how one constructs BU classically. $G_p(\mathbb{C}^n) \subset G_p(\mathbb{C}^{n+1}) \subset \cdots G_p(\mathbb{C}^{(\infty)})$ is a classifying space for U_p. Then one realizes $U_p \subset U_{p+1} \subset \cdots$ by $G_p(\mathbb{C}^n) \subset G_{p+1}(\mathbb{C} + \mathbb{C}^n)$ etc. One obtains in the limit the set of subspaces A of $\bigoplus_{\mathbb{N}} \mathbb{C}e_n$ such that $\bigoplus_{n \le n_0} \mathbb{C}e_n \subset A$ and such that the codimension of this inclusion is $-n_0$.

What seems to be at stake is that we have a space V with a flag $V_{n-1} \subset V_n \subset V_{n+1}$ $\quad n \in \mathbb{Z}$ and we are taking

$$\bigcup G_p(V_p / V_{-p})$$

Variant: We have a Hilbert space V, and a splitting $V = V_0 \oplus V_0^{\perp}$ into two infinite pieces. Then we consider closed subspaces A which essentially "coincide with A mod finite dimensional subspaces." Here are various possible meanings:

i) codim of $A \cap V_0$ in A, V_0 is finite.

ii) If E_A and E_{V_0} are the orth projectors, then $E_A - E_{V_0}$ is compact

Concentrate on this: In the Calkin alg. C we fix a projector e, not equal to 1. Then we can consider the inverse image of e in [redacted] $\mathrm{End}(V)$, and inside of this we can consider the space of orthogonal projectors E in $\mathrm{End}(V)$ such that $E \mapsto e$. It is this space of projectors which is the infinite Grassmannian. [redacted] It should be possible to construct a contractible space over the set of projectors in C. Clear.

[redacted]

So the fibration I want is

$$\left\{\begin{matrix}\text{infinite Grass} \\ \text{of } E \mapsto e\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{orth. proj. } E \\ \text{such that } \mathrm{Im}\,E \\ \mathrm{Im}\,(1-E) \text{ inf. dim}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} e \text{ orth proj} \\ \text{in } C \ni e \neq 0,1\end{matrix}\right\}$$

$$\mathbb{Z} \times U/U \times U \qquad\qquad \cong \ UN/UN \times UN$$

contractible

Next let us consider Fredholm operators. [redacted] Consider the fibration

$$C^* \ [\text{redacted}] \longrightarrow \left\{\begin{matrix}\text{embeddings } C \xrightarrow{x} C \\ \text{with image a} \\ \text{direct summand } \cancel{\text{and}} \\ \text{cokernel} \simeq C.\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{projectors } e \text{ in } C \\ e \neq 0,1.\end{matrix}\right\}$$

Lifted version

$$\left\{\begin{array}{l} \theta : V \to V \\ \ni \theta^*\theta \equiv 1 \\ \quad \theta\theta^* \equiv \lambda \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \text{bounded operators} \\ \theta : V \longrightarrow V \\ \quad \theta^*\theta \equiv 1 \quad \text{mod } K. \\ \text{Coker } \theta \text{ infinite} \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \text{selfadjoint} \\ \lambda \text{ on } V \\ 0 \leq \lambda \leq 1 \\ \text{ess. spectrum } 0,1 \end{array}\right\}$$

$\sim$ unitary Fredholm operators

Can one relate the Grassmannian + Fredholm operators directly? For example I can form over the space of E (orth proj. $\ni$ Im E, Im $1-E$ inf.), the space of pairs consisting of E, θ where $\theta : V \xrightarrow{\sim} \operatorname{Im} E$ is a unitary isomorphism. Thus I could consider the space of $\theta : V \longrightarrow V$ such that $\theta^*\theta = id$, which should be contractible.