October 4, 1974. On SA.

Suppose A is a field; let's study the suspension SA, which can probably be thought of as endos. of an inf. vector space V over A modulo endos. of finite rank.

First some general comments about a ring B such that $B \simeq B \oplus B$ as left B-modules. I claim SA has this property. In effect this amounts to the existence of elements $\iota_1, \iota_2, p_1, p_2$ in B such that $p_1\iota_1 = p_2\iota_2 = 1$, $p_1\iota_2 = p_2\iota_1 = 0$, $\iota_1 p_1 + \iota_2 p_2 = 1$. Since such elements exist in CA, they also exist in SA.

So assume $B \oplus B \simeq B$ in $\mathcal{P}(B)$; ~~That [B] is~~ let $S = \pi_0 \underline{\mathrm{Iso}}\, \mathcal{P}(B)$. Then $[B]$ is idempotent in S, and $\{0, [B]\} \subset S$ is cofinal. It follows that $K_0 B = \mathrm{Im}\{ S \xrightarrow{+[B]} S \}$. Also

$$H_*(GL(B)) \simeq \mathrm{Im}\left\{ H_*(B^*) \xrightarrow{\cdot [B]} H_*(B^*) \right\}.$$

Iterate

$$\begin{aligned} B \simeq\ & B \oplus B \\ & \| \\ & B \oplus B \oplus B \\ & \| \\ & B \oplus B \oplus B \oplus B \end{aligned}$$

so that we get:

$$* \qquad \begin{tikzcd} \bigoplus_{n\in\mathbb{N}} B \arrow[r, hook] \arrow[rr, bend left, "\text{standard embedding}"] & B \arrow[r] & \prod_{n\in\mathbb{N}} B \end{tikzcd}.$$

In any case we get compatible isoms

$$B \simeq B^n \oplus P_n \simeq B^n \oplus B \oplus P_{n+1} \simeq B^{n+1} \oplus P_{n+1}$$

and so we get an embedding of $GL(B)$ into B^*. The composite

$$GL(B) \subset B^* = GL_1(B) \subset GL(B)$$

induces on H_* the projection $H_*(B^*) \longrightarrow H_*(GL(B))$.

<u>Question:</u> Does the embedding $B^* \hookrightarrow B^*$ coming from an isom $B \oplus B \simeq B$ induce isomorphisms on homology, say for $B = SA$?

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* Observe that it is never possible for $\bigoplus_{\mathbb{N}} B$ to be a direct summand of B, for then $\bigoplus_{\mathbb{N}} B$ would be finitely generated.

Even in the case of a flask ring such as CA one does not have that the infinite direct sum within $Mod(B)$ carries $\mathcal{P}(B)$ into itself.

Unimodular complex + Stability. The question about whether $B^* \hookrightarrow B^*$ is a homology isomorphism, is the stability question in this setting.

We will only consider unimodular vectors $B \xhookrightarrow{v} B$ such that B/Bv is "stable" i.e. $B/Bv \simeq B \oplus K$ ($\Rightarrow$ $B \oplus B/Bv \simeq B \oplus B \oplus K \simeq B \oplus K$ so can suppose $K = B/Bv$). More generally we consider only unimodular sequences $B^n \hookrightarrow B$ whose complement is stable. The question arises as to whether the simplicial complex whose simplices are such unimodular sequences is ~~stable~~ contractible.

I have to recall what I **did** for countable sets. Given such an X I form a poset J whose elements are the inf. subsets $S \subset X$ with infinite complement; I call $S' < S$ if $S' \subset S$ and $S - S'$ is infinite. I showed this poset J is contractible, arguing that if

K is a finite subset of J, then one can find a vertex v such that for any $w \in K$ either $v < w$ or $v \cap w = \phi$ and $v \cup w \in K$. Then

$$w \leq w + v \geq w$$

contracts K to a point. (To get v, first let K' be a maximal subset of K such that $a = \bigcap_{w \in K'} w$ is infinite. Then $a \cap w$ is finite for $w \in K - K'$ so we can shrink a to a_1 such that $a_1 \subset w$ for $w \in K'$, $a_1 \cap w = \phi$ $w \in K - K'$. Cutting a_1 in half, we get v such that $v < w$ for $w \in K'$ and $w \cap v = \phi$, $w \cup v$ has inf. comp. for $w \in K - K'$.)

Analogous to the special unimodular complex would be the simplicial complex whose vertices are embeddings $u : N \hookrightarrow X$ with inf. complement, in which the simplices are independent embeddings $(u_0, \ldots, u_p)$ such that $u_0 N, \ldots, u_p N$ are disjoint with infinite complement.

Grassmannian: partial category consisting of submodules P of B such that P, B/P are isom. to B. Morphism occurs when P, P' are independent + one gives an isomorphism.

Let V be a vector space. Consider the set $L_q(V)$ consisting of layers (W_1, W_2) in V such that $\dim(W_2/W_1) = q$. Say $(W_1, W_2) \leq (W_1', W_2')$ if

$$\begin{tikzcd} W_2 \arrow[r, dash] \arrow[d, dash] & W_2' \arrow[d, dash] \\ W_1 \arrow[r, dash] & W_1' \end{tikzcd} \qquad \begin{aligned} W_2 + W_1' &= W_2' \\ W_2 \cap W_1' &= W_1 \end{aligned}$$

i.e. if $W_i \subset W_i'$ and $W_2/W_1 \xrightarrow{\sim} W_2'/W_1'$. Clearly this makes $L_q(V)$ into a partially ordered set.

<u>Question</u>: If V is of infinite dimension, is $L_q(V)$ a classifying space for $GL_q(F)$?

October 6, 1974.

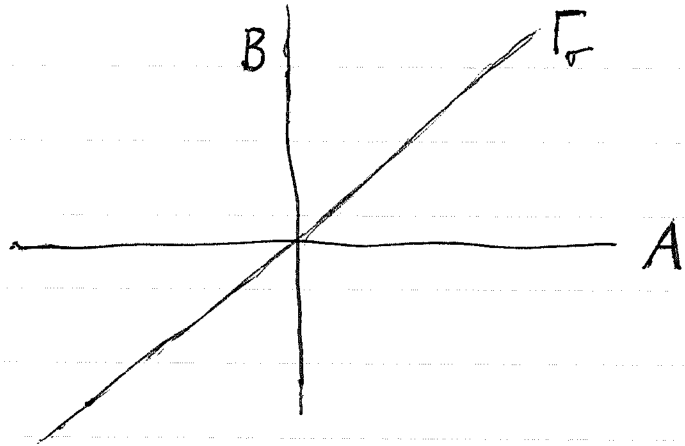
Conjecture: $L_q(V)$ = ordered set of layers of dim q in the vector space V of inf. diml. Then $L_q(V)$ is a classifying space for GL_q.

Suppose we try to embed GL_q in $\pi_1 L_q(V)$, taking for basepoint a layer $(0, A)$, A a fixed q-dimensional subspace. Let B be complementary to A, and fix an isomorphism $\sigma: A \xrightarrow{\sim} B$, whence

$$\sigma \cdot \mathrm{Aut}(A) = \mathrm{Isom}(A, B)$$

Now σ gives rise to a path from $(0,A)$ to $(0,B)$ in $L_q(V)$. Put $\Gamma_\sigma = \{\sigma a \; a \mid a \in A\}$.

$$(0, A) \leq (\Gamma_\sigma, A \oplus B) \geq (0, B)$$



(Maybe you should remark first that if A, C are complementary q-planes, then $(0, A) \leq (C, A \oplus C)$. In fact, a basic idea here seems to involve three q-planes A, C, B each pair spanning the same $2q$-plane. Then C is the graph of an isomorphism of A and B.)

So thus for each $\theta \in \mathrm{Aut}(A)$ we get a path

$$(0,A) \leq (\Gamma_\sigma, A+B) \geq (0,B)$$
$$\leq (\Gamma_{\sigma\theta}, A+B) \geq$$

To prove this is a homomorphism, let C be complementary to A, B, and let $\tau : B \xrightarrow{\sim} C$. Then I want to show that

$$(0,A) \leq (\Gamma_\sigma, A+B) \geq (0,B) \leq (\Gamma_\tau, B+C) \geq (0,C)$$

is homotopic to $\quad (0,A) \leq (\Gamma_{\tau\sigma}, A+C) \geq (0,C)$.

But
$$\begin{tikzcd}
(0,A) \arrow[r] \arrow[dd] & (\Gamma_\sigma, A+B) \arrow[d] & (0,B) \arrow[l] \arrow[d] \\
 & (\Gamma_\sigma + \Gamma_\tau, A+B+C) & (\Gamma_\tau, B+C) \arrow[l] \\
(\Gamma_{\tau\sigma}, A+C) \arrow[ur] & & \\
 & (0,C) \arrow[ul] \arrow[uur] &
\end{tikzcd}$$

$$\tau\sigma a - a = \underset{\uparrow\, \Gamma_\tau}{(\tau\sigma a - \sigma a)} + \underset{\uparrow\, \Gamma_\sigma}{(\sigma a - a)}$$

Proof of the conjecture — First we have to make precise that an object of $L_q(V)$ is a layer (W_1, W_2) where W_2 is finite dimensional.

The principal GL_q bundle over $L_q(V)$ consists of layers (W_1, W_2) together with an isomorphism $F^q \simeq W_2/W_1$. Call this poset $\widetilde{L_q}(V)$ and note that if $V = \varinjlim V_i$ with V_i finite dimensional, then

$$\widetilde{L_q}(V) = \varinjlim \widetilde{L_q}(V_i).$$

Hence it will be enough to show that for W finitedimensional, one has that the inclusion

$$\widetilde{L_q}(W) \subset \widetilde{L}_q(V \oplus A) \qquad A = F^q$$

is null-homotopic. But given $W_1 \subset W_2 \subset V$ and $\theta : F^q \xleftarrow{\sim} W_2/W_1$, I will produce a path to $(0, A)$ together with the given isom. $id: A = F^q$.

$$\cancel{(W_1, W_2, \theta) \leq (W_1 + A, W_2 + A, \theta')}$$

$$(W_1, W_2, \theta) \quad \leq \quad (\mathrm{Ker}\,\theta', W_2 + A, \theta') \quad \geq \quad (0, A, id)$$

Here $\theta'(x + a) = \theta x + a \qquad x \in W_2$

(Improvement: $\widetilde{L}_q V$ consists of (W, θ) where $\theta : W \twoheadrightarrow F^q$, so that the layer is $(\mathrm{Ker}\,\theta, W)$.)

Define $\tilde{L}_q V$ to be the set of pairs (W, θ), where W is a f.d. subspace of V and $\theta : W \twoheadrightarrow F^q$. Partially order these pairs by saying $(W_1, \theta_1) \leq (W_2, \theta_2)$ if $W_1 \subset W_2$ and if $\theta_1 =$ restriction of θ_2. Then $\tilde{L}_q V$ is a $GL_q F$-torsor over $L_q V$.

Question: What is the connectivity of $\tilde{L}_q V$?

Have seen $\tilde{L}_q V$ is contractible if V is infinite dimensional. Put $\tilde{L}_q V' =$ bigger set of pairs (W, θ) where W can be infinite dimensional. Claim $\tilde{L}_q V \subset \tilde{L}_q V'$ is a h.eq. In effect if i denotes this inclusion, then $i/(W, \theta)$ will be a $\neq \phi$ directed set, hence contractible.

Take $q = 1$, say $\dim V = n$. Fix a line A in V and an isom $\varepsilon : A \xrightarrow{\sim} F$. Try to contract $\tilde{L}_1 V$ by

$$(W, \theta) \leq (W + A, \theta') \geq (A, \varepsilon)$$

$$\theta'(w + a) = \theta w + \varepsilon a$$

This works if $W \cap A = 0$ or if $W \supset A$ and θ restricts to ε. Let $\mathcal{H}$ be the bad set $= \{(W, \theta) \mid W \supset A,\ \theta \neq \varepsilon \text{ on } A\}$. What is the link of $(W, \theta) \in \mathcal{H}$. No $\mathcal{H}$ isn't discrete.

Given $(W,\Theta) \in \mathcal{H}$. Then W splits canonically $W = A \oplus \operatorname{Ker}\Theta$. Thus the components of $\mathcal{H}$ are contractible and indexed by the possible isos. $A \xrightarrow{\sim} F$ different from ε. No go.

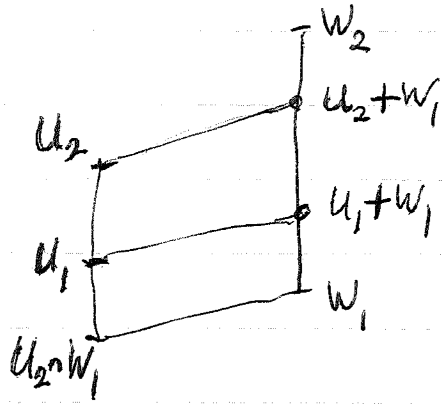
<u>October 7, 1974</u>

We have found that if V is an infinite dimensional v.s. then $L_q V$ = poset of layers (W_1, W_2) with $\dim W_2/W_1 = q$ and $\dim W_i < \infty$ is a classifying space for GL_q. I want to see if this can be generalized to the Q-category.

Suppose we consider all layers (W_1, W_2) of finite-dimensional subspaces of V, and define $(U_1, U_2) \leq (W_1, W_2)$ if $U_2 \subset W_2$ and $U_1 \supset U_2 \cap W_1$. anti-symm. ✓ trans:

$$U_2 \subset W_2 \subset Z_2 \implies U_2 \subset Z_2$$

$$U_1 \supset U_2 \cap W_1, \quad W_1 \supset W_2 \cap Z_1 \implies U_2 \cap Z_1 \subset U_2 \cap W_2 \cap Z_1$$
$$\subset U_2 \cap W_1 \subset U_1 \quad ✓$$



Note that $(U_1, U_2) \leq (W_1, W_2)$

$$\implies U_2+W_1 / U_1+W_1 \simeq U_2 / U_2 \cap (U_1+W_1)$$
$$= U_2 / U_1 + U_2 \cap W_1$$
$$= U_2 / U_1$$

Thus $(U_1, U_2) \leq (W_1, W_2) \implies$ there is a layer (U_1+W_1, U_2+W_1) in (W_1, W_2) such that $(U_1, U_2) \leq$ this layer in L_q where

$q = \dim(U_2/U_1)$. Conversely if $W_1 \subset V_1 \subset V_2 \subset W_2$ and $U_1 = U_2 \cap V_1$, $V_2 = U_2 + V_1$ then

$$V_1 \supset U_1 + W_1 \qquad V_2 \supset U_2 + W_1$$

and

$$V_2/V_1 \simeq U_2/U_1 \simeq \frac{U_2+W_1}{U_1+W_1}.$$

There we see that when $(U_1, U_2) \leq (W_1, W_2)$ there is a smallest layer inside of (W_1, W_2) covering (U_2, U_1) in the layer space of fixed dimension.

Question: Is the poset just defined a classifying space for Q? Call this poset $L(V)$, so that $L_q(V)$ is the sub-set of layers of dimension q.

Suppose we consider the functor(?) $L(V) \to Q$ sending (W_1, W_2) to W_2/W_1. If $(U_1, U_2) \leq (W_1, W_2)$, then

$$U_2/U_1 \simeq U_2+W_1/U_1+W_1$$

which is a subquotient of W_2/W_1. Check trans.

~~This doesn't work because amongst layers sandwiched inside (W_1, W_2) U_2 can move around, but not in Q.~~

Suppose $(U_1, U_2) \leq (V_1, V_2) \leq (W_1, W_2)$. To the first $\leq$ we get

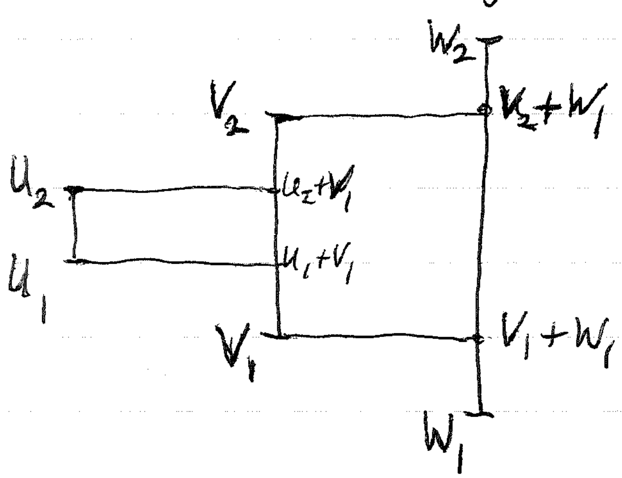
$$U_2/U_1 \simeq U_2+V_1/U_1+V_1 = \text{a subq. of}$$

$$V_2/V_1 \simeq V_2+W_1/V_1+W_1 \quad \text{as subq. of } W_2/W_1$$

The composition is the subquotient

$$U_2+V_1+W_1/U_1+V_1+W_1$$

which is not the subq. U_2+W_1/U_1+W_1 in general



$U_1+V_1 \overset{?}{\supseteq} W_1$

NO

So this doesn't work.

Suppose we fix a layer (W_1, W_2) of dimension n and consider the ordered set of layers under it. If $W_1 = 0$, this is exactly the poset of subquotients of W_2 not W_2.

Recapitulate: We have this poset $L(V)$ stratified by dimension

$$\phi \subset L_{\leq 0}V \subset L_{\leq 1}V \subset L_{\leq 2}V \subset \cdots\cdots$$

Each of these are closed, and $L_{\leq n}V - L_{<n}V$ is $L_n V$ which I know is a classifying space for GL_n. So I want to understand the attaching maps.

So let $w : L_{<n}V \subset L_{\leq n}V$ be the inclusion functor and consider the spect. sequence

$$E^2_{pq} = H_p(L_{\leq n}V, L_q w_! \mathbb{Z}) \Rightarrow H_{p+q}(L_{<n}V)$$

$$(L_q w_! \mathbb{Z})(x) \simeq H_q(w/x, \mathbb{Z}).$$

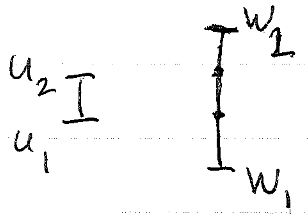
Thus for each $x = (W_1, W_2)$ in $L_n V$, I want to consider $L_{<n}V/x$ which is the poset of layers $(U_1, U_2) \leq (W_1, W_2)$.

<u>Question</u>: Given a pair (W, V) with $\dim(V/W) = n$, consider the poset consisting of layers $(U_1, U_2) \leq (W, V)$ with $\dim(U_2/U_1) < n$. Does this have the homotopy type of the suspended Tits complex?

Certainly we want for each map $x \longrightarrow x'$ in $L_n V$ that $L_{<n}V/x \longrightarrow L_{<n}V/x'$ be a homotopy equivalence. Since for $W=0$, $\dim V = n$ it is clear that $L_{<n}V/x$ is the right this, what we have to do is to show that if we pick a complement: $A \oplus W = V$, then maybe we will win.

Too fact: I think that $L_{<n}V/(0,V)$ where $\dim V = n$ is not quite the poset of proper subquotients of V.

<u>Problem</u>: Given V of dimension n, let $L_{<n}(V)$ be the poset consisting of layers (W_1, W_2) in V which are proper i.e. $\dim(W_2/W_1) < n$, with the ordering $(U_1, U_2) \leq (W_1, W_2)$ if $U_2 \subset W_2$, $U_1 \supset U_2 \cap W_1$.



Does this have the same homotopy type as $\Sigma T(V)$?

Possible approach: suppose V infinite-dimensional, let $L(V)$ be the set of layers in V, i.e. (W_1, W_2) $\ni$ $\dim(W_2) < \infty$. Then we have two partial orderings on $L(V)$:

1) usual sandwiching between layers
$$(W_1', W_2') \leq (W_1, W_2) \quad \text{iff} \quad W_1 \subset W_1' \subset W_2' \subset W_2$$

2) projectivity
$$(W_1', W_2') \prec (W_1, W_2) \quad \text{if} \quad \begin{array}{l} W_1' \subset W_1,\ W_2' \subset W_2 \\ W_2'/W_1' \simeq W_2/W_1. \end{array}$$

Does this lead to a bicategory structure on $L(V)$. Define bimorphism to be a square

$$\begin{array}{ccc} x & \prec & y \\ \vee\!| & & \vee\!| \\ x' & \prec & y' \end{array}$$

such that under the projectivity isomorphism between x and y, the sublayers x' and y' coincide. Clearly these compose in the way they should.

Let us take now the nerve with respect to $\leq$. In dimension zero we get the category of layers and projectivities which I know is the same as $\coprod_n BGL_n$. In dimension 1 I get the category consisting of pairs $(x' \leq x)$ and projectivities.

It seems clear that what I am getting is the following variant. $L(V)$ is now to be the category in which a map from (U_1, U_2) to (W_1, W_2) is a subquotient $(V_1, V_2) \leq (U_1, V_2)$ such that $(W_1, W_2) \prec (V_1, V_2)$. This version of $L(V)$ has the homotopy type of Q, but it is not a poset.

<u>Problem</u>: Inside of V we consider proper layers with ordering $x \ll x'$ if x is projective with a subquotient of x'. Better if $x \prec y$ with $y \leq x'$. Show this poset $\sim \Sigma T(V)$.

$$x = (U_1, U_2) \quad , \quad x' = (W_1, W_2), \quad y = (V_1, V_2)$$

$$W_1 \subset V_1 \subset V_2 \subset W_2$$

$$U_1 = U_2 \cap V_1 \qquad U_2 + V_1 = V_2$$

implies $\quad U_1 \supset U_2 + W_1 \qquad U_2 \subset W_2$

and $\quad V_1 \supset U_1 + W_1 \qquad V_2 \supset U_2 + W_1.$

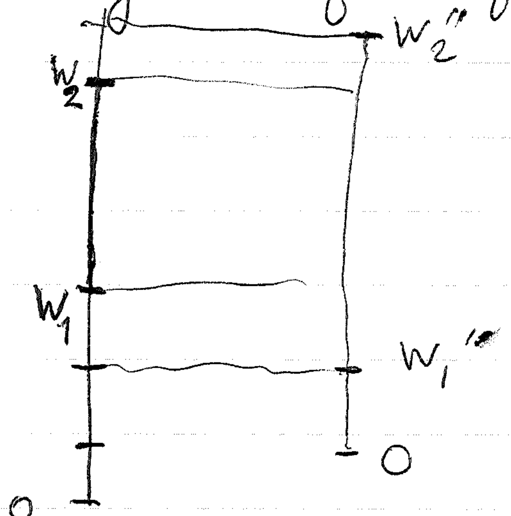
so if $x \prec y$ with $y \leq x'$, there is a least such y.

Try: $J(V)$ = poset of proper layers of V with $\leq$ so that we have a functor $f: J(V) \longrightarrow L'(V)$ = proper layers with $\ll$.

<u>Example:</u> V unitary v.s. $U(V)$ = unitary group of V, V finite-dimensional, or else $U(V)$ = unitaries $\equiv 1$ mod finite rank if V is Hilbert space. Suppose I fix the subspace where -1 has multip. q, and denote this X_q. Stratify by counting the number of eigenvalues $\exp 2\pi i t$ $0 < t \leq \frac{1}{2}$. This gives us a subspace W_1, and then to get the -1 eigenvalue we get to a W_2. In this way to a unitary $\theta \in X_q$ we have assigned an element of $L_q V$. Suppose now that one has θ specializing to θ'. Then some of the eigenvalues in the interval $\exp 2\pi i t$ $0 < t < 1$ could have moved to zero, in which case a piece of W_1 will be forgotten both from W_1 and W_2. Thus we will have a perspectivity: $(W_1, W_2) \succ (W_1', W_2')$.

This leads one to feel that the poset of q-dimensional layers and perspectivities is analogous to X_q which has the homotopy type of $G_q(V)$.

If we allow the -1 eigenvalue to vary we get specializations of the form



in other words $(W_1, W_2) \succ (W_1', W_2') \leq (W_1'', W_2'')$. This looks suspicious, so instead maybe one should ~~have~~ count eigenvalue in some open interval about -1. In this case the variation is

$$(W_1, W_2) \succ (W_1', W_2') \geq (W_1'', W_2'')$$

~~Another variant: Count eigenvalues in a half open interval~~

~~and the eigenspace varies~~

Problem: Homotopy type of $L_q(V)$.

Dimension of $L_q(V)$. To each layer (W_1, W_2) in $L_q(V)$, consider $\dim W_1$ which can vary between 0 and $n-q$. One gets maximal chains

$$(0, A) < (L_1, L_1 \oplus A) < \cdots < (L_{n-q}, L_{n-q} \oplus A)$$

so the dimension of $L_q(V)$ is $n-q$, where $n = \dim V$.

I would like to prove $\tilde{L}_q(V)$ = poset of $(W, \theta : W \twoheadrightarrow F^q)$ is spherical, if this is true.

$\tilde{L}_1(V) = \{(W, W \overset{\theta}{\twoheadrightarrow} F)\}$. Such a pair (W, θ) can be identified with a subspace $Z \subset V \oplus F$ which is not contained in V (hence it maps onto F), and does not contain F (so that Z projects 1-1 to V). Recall that subspaces Z mapping onto F are the same as affine subspaces, because they are not contained in the hyperplane V. F gives a point in the affine spaces, so Z is an affine subspace of V not containing the origin. But Lusztig has shown that the poset of affine subspaces not going through zero is spherical, so we have:

<u>Prop</u>: $\tilde{L}_1(\bar{V})$ is spherical of dimension $(n-1)$.

Improvement: Assign to $(W, W \xrightarrow{\theta} F)$ the affine space $\theta^{-1}(1)$. This gives the correspondence between $\tilde{L}_1(V)$ and affine subspaces not passing through 0.

Suppose next we try to make an induction for $\tilde{L}_q(V)$. [redacted]
Given $(W, \theta : W \twoheadrightarrow F^q)$ we can associate the hyperplane $\theta^{-1}(F^{q-1})$ and the restriction of θ. This gives a map

$$\tilde{L}_q(V) \longrightarrow \tilde{L}_{q-1}(V)$$

and the fibre over [redacted] $U \twoheadrightarrow F^{q-1}$ is the poset of extensions

$$\begin{array}{ccc} U & \twoheadrightarrow & F^{q-1} \\ \cap & & \cap \\ W & \twoheadrightarrow & F^q \end{array}$$

?

Review the Kervaire - Lusztig contraction argument. Take the unimodular complex X of indep. vectors in V. $V = ke_1 + \cdots + ke_n$. Let F_i = subcomplex consisting of σ such that $e_i \notin k\sigma + ke_1 + \cdots + ke_{i-1}$. Then F_i contracts in X to the point e_i. But these contractions of $F_i \cap F_j$ to e_i and e_j are homotopic since $\{e_i, e_j\}$ is independent of any σ in $F_i \cap F_j$. Hence one argues that $F_1 \cup \cdots \cup F_i$ contracts to a point in X. So $F_1 \cup \cdots \cup F_n$ contracts to a point in X. But any σ such that $k\sigma < V$ is in $F_1 \cup \cdots \cup F_n$. Thus the $(n-2)$-skeleton of X contracts to a point in X.

(Point is contraction was this: Assume $A, B \subset X$ ~~[illegible]~~ are such that i) A contracts to a pt. in X, ii) B contracts in X to a point in A, the contraction keeping $A \cap B$ inside A. Then $A \cup B$ contracts a point in X. Proof. The identity map $A \to A$ when restricted to $A \cap B$ has a homotopy to a point; ~~then~~ by HET $\Rightarrow$ get a homot of id_A ~~ending~~ ~~with~~ compatible with given homotopy on $A \cap B$ $\Rightarrow$ get a homotopy of $A \cup B$ starting from id and pulling into A; now use i)).

Volodin approach to the Grassmannians.

Consider the set $G_p(V)$ of p-diml subspaces of V. There are some obvious subsets which should be considered contractible, e.g. if B is of complementary dimension $q = n-p$, then $\{A \mid A \oplus B = V\}$ is contractible. More generally, if B is of codimension p in a subspace W, then $\{A \mid A \oplus B = W\}$ is contractible.

Volodin's idea it seems is this. Suppose X is a set and $\mathcal{F}$ is a family of non-empty subsets of X. Define a finite non-empty subset σ of X to be a simplex if it is contained in some member of $\mathcal{F}$. Claim that the resulting simplicial complex $K_{\mathcal{F}}$ is a classifying space for the poset $\mathcal{F}$. In effect, we can form the ~~category~~ ordered set of pairs (σ, F), $\sigma \subset F$ with $(\sigma, F) \leq (\sigma', F')$ if $\sigma' \subset \sigma \subset F \subset F'$.

$$\mathrm{Simp}\, K_{\mathcal{F}} \longleftarrow \{(\sigma, F)\} \longrightarrow \mathcal{F}$$

Not quite – it seems we need to know that for each σ, the poset of F containing σ is contractible, which would be the case if the family $\mathcal{F}$ is closed under finite intersection. A better hypothesis is to assume that for any σ contained in a member of $\mathcal{F}$ there is a least such.

Now given a finite subset σ of $G_p(V)$ such that $\sigma \subset \{A \mid A \oplus W_1 = W_2\}$ for some (W_1, W_2), I would like to know there is a least W_1, W_2 with this property. Clearly $W_2 = \sum_{A \in \sigma} A$ but it is clear that there are many possibilities for W_1.

Conclude that K_σ is not the correct gadget.

So what we are doing is to visualize the poset $L_q(V)$ as a kind of topology on $G_q(V)$.

For projective space $L_1(V)$ we get cells of the form $\mathbb{P}W_1 - \mathbb{P}W_1$, where (W_1, W_2) is a layer of dim. 1.

For $L_2(V)$ we get big open cells for big Grassmannians — these are Schubert cells of type

$$\begin{matrix} 0 & 1 & * & \cdots & * \\ & 1 & * & & * \end{matrix}$$

but in this formulation, we do not seem to have the cells of the form

$$\begin{pmatrix} 0 & 0 & 1 & * \cdots * & 0 & * & * & \cdots\cdots & * \\ & & & & 1 & * & * & \cdots\cdots & * \end{pmatrix}$$

I think it is essential to form a new category with these type of cells, before I can take the limit.

October 8, 1974

$L_p V$ = poset of layers (W_1, W_2) in V with $\dim(W_2/W_1) = p$, where $(W_1', W_2') \prec (W_1, W_2)$ iff $W_1' \subset W_1$, $W_2' \subset W_2$ and $W_2'/W_1' \simeq W_2/W_1$.

To each (W_1, W_2) we can associate the subset $\{A \mid A \oplus W_1 = W_2\}$ in $G_p V$. From this subset we can recover W_2 by

$$W_2 = \sum_{A \oplus W_1 = W_2} A$$

and W_1 is the unique codim p subspace of W_2 such that

$$\{A \mid A \oplus W_1 = W_2\} = G_p W_2 - \{A \mid A \cap W_1 \neq 0\}$$

[The point is that if $W_1' \neq W_1$, then one can find an A comp. to W_1 such that $A \cap W_1' \neq 0$.] Thus this subset determines the layer (W_1, W_2). Call this subset $C(W_1, W_2)$

If $(U_1, U_2) \prec (W_1, W_2)$, then $C(U_1, U_2) \leq C(W_1, W_2)$. In effect $C(W_1, W_2) = \{A \mid (0, A) \prec (W_1, W_2)\}$. Conversely if $C(U_1, U_2) \subset C(W_1, W_2)$, then clearly $U_2 \subset W_2$. Picking $A \in C(U_1, U_2)$, one has $W_2 = A + W_1 \Rightarrow W_2 = U_2 + W_1$. If $W_1 \neq W_1 \cap U_2$, then as these are of the same dimension one has $W_1 \cap U_2 \not\subset U_1$, so ~~starting~~ with

$v \in W_1 \cap U_2 - U_1$ one can enlarge v to $A \in C(U_1, U_2)$ which is not in $C(W_1, W_2)$. Conclude that $C(U_1, U_2) \leq C(W_1, W_2) \iff (U_1, U_2) < (W_1, W_2)$.

Therefore, we can identify L_pV with the ordered set of affine subspaces of G_pV of the form $C(W_1, W_2)$.

However $C(W_1, W_2)$ is a Shubert cell of type

$$\begin{pmatrix} 0 \cdots 0 & 1 & 0 & * & \cdots & * \\ 0 \cdots 0 & 0 & 1 & * & \cdots & * \end{pmatrix}$$

corresponding to consecutive indices. In general one has Shubert cells

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix}.$$

This suggests I consider the enlarged poset of all Shubert cells of $G_p(V)$.

Go over the definition. Fix a flag

$$0 < ke_1 < ke_1 + ke_2 < \cdots < ke_1 + \cdots + ke_n = V$$
$$0 < F_1 < F_2 < \cdots < F_n = V.$$

Then given integers $1 \leq i < j \leq n$ I can consider the set of all $A \in G_2(V)$ such that the induced filtration

$$0 \subset F_1 \cap A \subset \cdots \subset F_n \cap A$$

has jumps

$$F_{i-1} \cap A < F_i \cap A, \quad F_{j-1} A < F_j A$$

This set is described by matrices

$$\begin{array}{c} \begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & & & & \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots \end{pmatrix} \\ \quad\quad\quad\quad i \quad\quad\quad\quad j \end{array}$$

So in $G_2(V)$ a Shubert cell of the type not considered before is given by a chain

$$W_1 < W_2 < W_3 < W_4$$

$$\dim W_2/W_1 = \dim W_4/W_3 = 1$$

and is

$$C'(W_1, W_2, W_3, W_4) = \{A \mid A \cap W_1 < A \cap W_2, \ A \cap W_3 < A \cap W_4\}.$$

To this cell we associate the pair (i,j), $1 \le i < j \le n$, $i+1 < j$ given by $i = \dim W_2$, $j = \dim W_4$.

To the cells $C(W_1, W_2)$ considered before we associate the pair $i, i+1$ where $i = \dim W_1 + 1$.

On $C(W_1, W_2) = \{A \mid A \oplus W_1 \simeq W_2\}$ all the planes in this cell are canonically isomorphic to W_2/W_1 which gives us the map $L_p V \to BGL_p$. However on $C'(W_1, W_2, W_3, W_4)$ I see no canonical trivialization of the subbundle, rather we have a canonical line sub-bundle $A \mapsto A \cap W_2$ and canonical trivializations of the sub- and quotient-bundles

$$A \cap W_2 \simeq W_2/W_1$$

$$A/A \cap W_2 \simeq W_4/W_3$$

Thus the poset of all Schubert cells is apt to be a better Grassmannian than BGL_p.

Possible directions of investigation

α) understand the poset of Shubert cells

β) stratification of the Grassmannian and the pair categories.

γ) infinite Grassmannian (somehow form a good limit of finite Grassmannians)

δ) If the poset of Schubert cells is a good model for G_p, what about "generic" sections.

<u>October 9, 1974</u> Volodin model for Grassmannians.

Recall that [redacted] I already [redacted] have a geometric picture for the Milnor model for BGL_p, which goes as follows.

Let $V = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, where $\dim A_i = p$. ~~Then consider all homos. $\theta: V \to F^p$ such that $\theta: A_i \simeq F^p$ and make these into a simplicial simplex~~

Simplicial complex whose vertices are pairs $(i, \theta : A_i \simeq F^p)$ in which a simplex is a [redacted] set of pairs

$$(i_0, \theta_0) \cdots\cdots (i_m, \theta_m)$$

with $i_0 < i_1 < \cdots < i_m$. This is the principal bundle.

Alternative description: Let me cover $G_p(V)$ by the open sets $U_i = \{ A \mid A \xrightarrow{\sim} A_i \}$, and consider the nerve of this covering. The intersection

$$U_{i_0} \cap \cdots \cap U_{i_k}$$

splits into a disjoint union of affine spaces. In effect if $A \in U_{i_0} \cap \cdots \cap U_{i_k}$, then the image $\bar{A}$ of A in $A_{i_0} \oplus \cdots \oplus A_{i_k}$ is a subspace projecting isom. onto each factor. The map $A \mapsto \bar{A}$ has affine spaces for fibres. So I will form a simplicial set whose k-simplices are components of $U_{i_0} \cap \dots \cap U_{i_k}$ for $0 \le i_0 < \dots < i_k < \infty$. This is the Milnor model.

But I can identify this model with a sub-poset of $L_p V$. Thus to a sequence $0 \le i_0 < \dots < i_k < \infty$ and $A' \subset A_{i_0} \oplus \dots \oplus A_{i_k}$ projecting isom. to each factor I will associate the cell

$$\{A \mid \operatorname{Im}\{A \xrightarrow{p_{i_0 \dots i_k}} A_{i_0} \oplus \dots \oplus A_{i_k}\} = A'\}$$

Thus if I put $W_2 = p_{i_0 \dots i_k}{}^{-1}(A')$, $W_1 = \operatorname{Ker} p_{i_0 \dots i_k}$

this is the cell belonging to the layer (W_1, W_2).

Take $p = 1$. Then we get all layers (W_1, W_2) with $W_1 =$ sum of A_i.

Only suspicious thing is that these ~~cells~~ layers are all infinite dimensional in contrast to the ones in $L_p V$. ~~So you must dualize and work with~~ ~~p-dim quotients of $V = A_0 \oplus A_1 \oplus \dots$ this~~ ~~this $U_i = \{B \mid A_i \oplus B = V\}$.~~

~~$U_{i_0} \cap \dots \cap U_{i_q} = \{V \to V/B \mid A_{i_j} \xrightarrow{\sim} V/B\}$~~

So proceed as follows to realize the joins of GL_p. Start with $V_1 = A_1$ with the principal bundle of all isos. $A_1 \simeq F^p$. To contract this we embed in $V_2 = A_1 \oplus A_2$ and use the contraction

$$(A_1, \theta) \leq (A_1 \oplus A_2, \theta + \varepsilon) \geq (A_2, \varepsilon)$$

Thus it's clear that at the n-th step, we have just the pairs $A_{i_0} \oplus \cdots \oplus A_{i_k}$ $\quad 0 \leq i_0 < \cdots < i_k \leq n$

(start with $V_0 = A_0$, $V_1 \simeq A_0 \oplus A_1$), together with $\theta : A_{i_0} \oplus \cdots \oplus A_{i_k} \twoheadrightarrow F^p$ such that $\theta | A_{i_j}$ is an iso for $0 \leq j \leq k$.

<u>Conclusion</u>: If $V = A_0 \oplus A_1 \oplus \cdots\cdots$ all $A_i \simeq F^p$, then the Milnor model for $BGL_p(F)$ can be identified with the sub-poset of $L_p(V)$ consisting of layers (W_1, W_2) of the form

$$W_2 = A_{i_0} \oplus \cdots \oplus A_{i_k} \quad \text{for some } 0 \leq i_0 < \cdots < i_k$$

$$W_1 \ \text{common complement of } A_{i_0}, \ldots, A_{i_k} \text{ in } W_2.$$

Guess: [redacted] Given a [redacted] basis in [redacted] V there should be [redacted] an interesting poset of Shubert cells associated to all the Borel subgroups consistent with this basis. This is probably Volodin's model for the Grassmannian.

Connectivity results using general position. Suppose our field F is infinite. Can we show $\tilde{L}_q(V)$ is spherical of dimension $n-q$. By general position arguments.

First show the link of a maximal element [redacted] of $\tilde{L}_q(V)$ is spherical of dim. $n-q-1$. $\theta : V \twoheadrightarrow F^q$. This link is the set of subspaces $W < V$ mapped onto F^q by θ. It is the set of subspaces $W < V$ [redacted] such that $W + \operatorname{Ker}\theta = V$, and if I recollect carefully this is spherical of dim. $n-q-1$.

Thus I know that any map $X \longrightarrow \tilde{L}_q(V)$ with $\dim X < n-q$ can be pushed off the maximal elements. So I am reduced to contracting any finite [redacted] subset of $(W_i, \theta_i : W_i \longrightarrow F^q)$ of $\tilde{L}_q(V)$ such that $W_i < V$. But then I can find a line L independent of all W_i. Don't see what to do next.

Look: If $q=1$, then once we find L ind. of all W_i, we can contract $(W,\theta) \leq (W+L, \theta+\varepsilon) \geq (L,\varepsilon)$

where $\varepsilon : L \xrightarrow{\sim} F$.

For larger q we can argue that if $\dim X$

$<n-q$ can push X off (W,θ) $\dim W = n$

$< n-q-1$ ——— (W,θ) $\dim W = n-1, n$

$< n-q-q+1 = n-2q+1$ $\dim W = n-q+1$

Thus we can prove

<u>Prop:</u> $\tilde{L}_q(V)$ "begins" in $\dim \geq n-2q+1$. (F infinite)

<u>Critical case $q=2$.</u> Here I want to show $\tilde{L}_2(V)$ begins in dimension $n-2$, so I have to contract any finite complex K of $\dim < n-2$, which I can assume doesn't contain any maximal elements $(V, \theta : V \to F^2)$. The problem comes with hyperplanes $(H, H \twoheadrightarrow F^2)$ in K.

For example if $n=3$, I want to prove $\tilde{L}_2(V)$ is connected. Thus I am considering two embeddings $F^2 \hookrightarrow V$ as related if they are congruent mod some complementary line. And I want to know if any two embeddings can be joined this way. ?

Let L be a line in V. $\{A \in G_p V \mid A \supset L\}$ may be identified with $G_{p-1}(V/L)$. The complement

$$G_p(V) - G_{p-1}(V/L) = \{A \mid A \cap L = 0\}$$

maps via $A \longmapsto A+L/L \subset V/L$ to $G_p(V/L)$. This is a homotopy equivalence, a section being obtained by choosing a complement H for L and using the embedding $G_p(H) \subset G_p(V)$.

Next consider $L_p(V)$ and let Y denote those layers such that $W_1 < W_1 + L \subset W_2$ and $\mathcal{U}$ those layers such that $W_1 < W_1 + L$, $W_2 < W_2 + L$ or such that $L \in W_1$. Thus on $\mathcal{U}$ we have a map

$$\mathcal{U} \longrightarrow L_p(V/L)$$
$$(W_1, W_2) \longmapsto (W_1 + L/L,\ W_2 + L/L)$$

Since $(W_1, W_2) \leq (W_1 + L, W_2 + L)$ on $\mathcal{U}$, this map is a homotopy equivalence. Note that

$$\text{if} \quad (W_1, W_2) \leq (W_1', W_2') \quad \left\{ \begin{array}{l} (W_1', W_2') \in \mathcal{U} \Longrightarrow (W_1, W_2) \in \mathcal{U} \\ \qquad\qquad\qquad \Longleftarrow \qquad\qquad\quad Y \end{array} \right.$$

Because $W_1 < L + W_1 \subset W_2 \Longrightarrow W_1' \in L + W_1' \subset W_2'$ and $L + W_1' > W_1'$ for $W_2 \cap (L + W_1') = W_2 \cap (\underline{L + W_1} + W_1') = L + W_1$.

Similarly if $(W_1', W_2') \in Y$

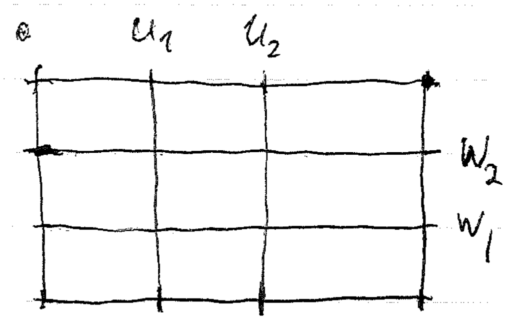
then $(W_1, W_2) \leq (W_1', W_2') \leq (W_1'+L, W_2'+L) \Rightarrow$ [redacted]
$W_2/W_1 \twoheadrightarrow W_2+L/W_1+L \rightarrow W_1'+L/W_2'+L$ isom $\Rightarrow$
$(W_1, W_2) \leq (W_1+L, W_2+L)$. $\therefore$ $\mathcal{U}$ closed under specializing, Y closed under generalizing.

October 10, 1974.

Suppose we consider the problem of two varying layers in a vector space.

Let (U_1, U_2) and (W_1, W_2) be two layers in V. One gets the filtration

$$\begin{array}{ccccccc} W_1 & \subset & (U_1+W_1)\cap W_2 & \subset & (U_2+W_1)\cap W_2 & \subset & W_2 \\ & & \| & & \| & & \\ & & W_1 + U_1\cap W_2 & \subset & W_1 + U_2\cap W_2 & & \end{array}$$



One says the layers are independent if no part of U_2/U_1 is to be seen in W_2/W_1, i.e. if

$$(U_1+W_1)\cap W_2 = (U_2+W_1)\cap W_2$$

$$gr_{1,1} = \frac{(U_2+W_1)\cap W_2}{(U_1+W_1)\cap W_2} = \frac{W_1+U_2\cap W_2}{W_1+U_1\cap W_2} = \frac{U_2\cap W_2}{U_2\cap W_1 + U_1\cap W_2} = 0$$

Now consider the sub-poset of $L_p V \times L_q V$ consisting of pairs of independent layers. Is this a classifying space for $BGL_p \times BGL_q$?

Call this poset of independent layers $L'_{p,q}(V)$. Then $\tilde{L}'_{p,q}(V)$ is the poset consisting of pairs

$$\left(U \xrightarrow{\theta}\!\!\!\!\rightarrow F^p,\ W \xrightarrow{\varphi}\!\!\!\!\rightarrow F^q\right)$$

such that

$$U \cap \operatorname{Ker}\varphi + \operatorname{Ker}\theta \cap W = U \cap W$$

Is this the same as saying θ, φ are induced from a map $U + W \longrightarrow F^p \oplus F^q$?

Simpler question: I considered yesterday a layer $(0, L)$, L a line in V, and denoted by U the subset of (W_1, W_2) in $L_p V$ such that $(0, L)$ is independent of (W_1, W_2), i.e.

$$(W_1, W_2) \leq (L + W_1, L + W_2).$$

Then I saw that U is heq to $L_q(V/L)$.

Suppose that I generalize this to (Z_1, Z_2) instead of $(0, L)$.

Let Γ = set of (W_1, W_2) independent of (Z_1, Z_2). So Γ breaks up into pieces according as how much of W_2/W_1 sits in Z_1 and the rest in V/Z_2.

Suppose $(Z_1, Z_2) = (H, V)$ so that Γ consists of $(W_1, W_2) \in L_p(V)$ with $(H \cap W_1, H \cap W_2) \leq (W_1, W_2)$. Then this Γ is stable under generalization whereas before with $(W_1, W_2) \leq (L+W_1, L+W_2)$ it was stable under specialization. Thus in the general case Γ will be only a hybrid, i.e. locally closed.

Fix H hyperplane in V, and consider the bad set: those layers (W_1, W_2) such that $W_1 \subset W_2 \cap H < W_2$. Call this $T_H \subset L_p(V)$. Basic question is what is the homotopy type of T_H when V is infinite, e.g. is it the classifying space of some subgroup of GL_p?

Induced principal bundle is $(W, \theta : W \twoheadrightarrow F^p)$ such that $\operatorname{Ker} \theta \subset H \cap W < W$.

Obvious map

$$T_H \longrightarrow L_{p-1}(H)$$

$$(W_1, W_2) \longmapsto (W_1, H \cap W_2).$$

discrete fibres of varying sizes.

Denote by $L_{p,q}(V)$ the poset consisting of $U_1 \subset U_2 \subset U_3 \subset U_4$, $\dim(U_2/U_1) = p$ $\dim(U_4/U_3) = q$, with $(U_i) \leq (U'_i)$ if $U_i \subset U'_i$ $i = 1, \ldots, 4$ and $U_2/U_1 \xrightarrow{\sim} U'_2/U'_1$, $U_4/U_3 \xrightarrow{\sim} U'_4/U'_3$. We have a functor

$$L_{p,q}(V) \longrightarrow L_p(V) \times L_q(V)$$
$$(U_1 \subset \cdots \subset U_4) \longmapsto (U_1, U_2), (U_3, U_4)$$

which we claim is a homotopy equivalence.

1) $L_{p,q}(V) \xrightarrow{f} L_p(V)$ is fibred

Fibre $f^{-1}(U_1, U_2)$ is set of $(U_3, U_4) \in L_q(V)$ such that $U_2 \subset U_3$ with induced ordering. This fibre can be identified with $L_q(V/U_3)$.

~~Given $(U'_1, U'_2) \leq (U_1, U_2)$ in $L_p(V)$~~

~~Given $(U'_1 \subset \cdots \subset U'_4) \leq (U_1 \subset \cdots \subset U_4)$, this map factors~~

~~$(U'_1 \subset U'_2 \subset U'_3 \subset U'_4) \leq (U'_1 \subset U'_2 \subset U_3 \subset U_4) \leq (U_1 \subset \cdots \subset U_4)$~~

Given $(U'_1, U'_2) \leq (U_1, U_2)$ in $L_p(V)$ and $(U_3, U_4) \in f^{-1}(U_1, U_2)$, one associates the pair $(U_3, U_4) \in f^{-1}(U'_1, U'_2)$.

Thus to $(U_1', U_2') \leq (U_1, U_2)$ we have associated the functor

$$\begin{array}{ccc} L_q(V/U_2) & \longrightarrow & L_q(V/U_2') \\ \| & & \| \\ f^{-1}(U_1, U_2) & \longrightarrow & f^{-1}(U_1', U_2'). \end{array}$$

This is a base-change functor for given $(U_1' \subset \cdots \subset U_4') \leq (U_1 \subset \cdots \subset U_4)$ it factors uniquely $(U_1' \subset U_2' \subset U_3' \subset U_4') \leq (U_1' \subset U_2' \subset U_3 \subset U_4) \leq (U_1 \subset \cdots \subset U_4)$, the first map being in the fibre over $(U_1', U_2') \leq (U_1, U_2)$.

2) $L_q(V/U) \longrightarrow L_q(V)$ is a homotopy equivalence

This is obvious because the diagram

$$\begin{tikzcd} L_q(V/U) \arrow[rr] \arrow[dr] & & L_q(V) \arrow[dl] \\ & \text{groupoid of vector spaces of dim } q. & \end{tikzcd}$$

← quotient of layer

commutes, and we know already that the vertical arrows are heq's.

Notice that inside of $L_{p,q}(V)$ we have the sub-set of $(U_1 \subset \cdots \subset U_4)$ such that $U_2 = U_3$. This may be identified with a layer in $L_{p+q}(V)$ together with a p-dimensional subspace of the quotient. Hence this subset is of the homotopy type of the group of autos of

$$0 \to F^p \to F^{p+q} \to F^q \to 0$$

So now let us return to the Grassmannian of 2-planes in V and all Shubert cells. The basic problem is to determine the homotopy type of the poset of Shubert cells. Call this $Sh_2(V)$. Then we have maps

$$\begin{tikzcd}
L'_{1,1}(V) \arrow[r, two heads] \arrow[d, hook] & L_2(V) \arrow[dr] & \\
L_{1,1}(V) \arrow[rr] & & Sh_2(V)
\end{tikzcd}$$

I have to understand better what a Shubert cell in $G_2(V)$ is.

Question: Take $(U_1 \subset U_2 \subset U_3 \subset U_4)$ in $L_{1,1}(V)$ and assign to it the cell

$$\mathcal{C}(U_1,U_2,U_3,U_4) = \{A \in G_2(V) \mid 0 = A\cap U_1 < A\cap U_2 = A\cap U_3 < A\cap U_4 = A\}$$

I want to know ~~claim that~~ if $U_2 < U_3$, then this cell determines (U_1,U_2,U_3,U_4).

~~First of all [illegible] U_4 is the subspace of V spanned by the the A in this cycle. Next note that if W is a subspace of U_4 and $W \not\subset U_1$, then I can find an A in this cycle such that $A \cap W \neq 0$. For if $W \not\subset U_2$ we can find L_2 a line [illegible] in W~~

Observe that if W refines U_1, U_2, U_3, U_4 then for all $A \in C(U_1,U_2,U_3,U_4)$, $A \cap W$ has the same dimension.

$$W \subset U_1 \Rightarrow A \cap W = 0$$

$$U_1 < W = U_2 \Rightarrow \dim(A\cap W) = 1$$

$$U_2 < W \subset U_3 \Rightarrow \dim(A \cap W) = 1$$

$$U_4 \subset W \Rightarrow \dim(A\cap W) = 2.$$

I want to prove the converse if I can. Thus suppose $A \cap W = 0$ for all A in $C(U_1,U_2,U_3,U_4)$. It could happen

Suppose $C(U_1, U_2, U_3, U_4) \subset C(W_1, W_2)$ where $(U_1 \subset \cdots \subset U_4) \in L_{11}(V)$, $(W_1, W_2) \in L_2(V)$. Then $U_4 \twoheadrightarrow W_2/W_1$ so replacing (W_1, W_2) by $(U_4 \cap W_1, U_4)$, we can suppose $U_4 = W_2$.

We can describe $C(U_1, U_2, U_3, U_4)$ as the set of planes of the form $L_1 \oplus L_2$ where $L_1 \in \mathbb{P}U_2 - \mathbb{P}U_1$ and $L_2 \in \mathbb{P}U_4 - \mathbb{P}U_3$. If $W_1 \not\subset U_3$ we could find $L_2 \in \mathbb{P}W_1 - \mathbb{P}U_3$, so $A = L_1 \oplus L_2$ (any L_1) would intersect W_1 in L_2 which contradicts $A \in C(W_1, W_2)$. Thus $W_1 \subset U_3$ is of codim. 1. $\therefore$ $A \cap W_1 \subseteq A \cap U_3$ has dim 1.

Consider U_2 and W_1 in U_3. ~~Since~~ If I could find $L_1 \in (\mathbb{P}U_2 - \mathbb{P}U_1) \cap \mathbb{P}W_1$, then for any L_2, I get an A not complementary to W. Thus

$$(\mathbb{P}U_2 - \mathbb{P}U_1) \cap \mathbb{P}W_1 = \phi$$

i.e. $\mathbb{P}U_2 \cap \mathbb{P}W_1 \subset \mathbb{P}U_1 \implies U_2 \cap W_1 \subset U_1$. Thus we have

$$\begin{tikzcd}[column sep=small]
 & U_4 \arrow[d, phantom, "\cup\ 1"] & \\
 & U_3 \arrow[dl, dash] \arrow[dr, phantom, "\cup\ 1"] & \\
U_2 \arrow[dr, phantom, "\cup\ 1"] & & W_1 \arrow[dl, dash] \\
 & U_1 &
\end{tikzcd}$$

note $U_2 \cap W_1$ is at most of codim 1 in U_2, hence equal to U_1.

So we have proved:

<u>Prop</u>: If $\overset{\in L_{11}(V)}{C(U_1,U_2,U_3,U_4)} \subset \overset{\in L_2(V)}{C(W_1,W_2)}$, then one has bicart square

$$\begin{tikzcd}[column sep=small]
 & U_3 \arrow[dl, dash] \arrow[dr, phantom, "\supset"] & \\
U_2 \arrow[dr, phantom, "\supset"] & & U_4\cap W_1 \arrow[dl, dash] \\
 & U_1 &
\end{tikzcd}$$

Consequently one gets a canonical exact sequence

$$\begin{tikzcd}
 & W_2/W_1 & \\
0 \arrow[r] & U_2/U_1 \arrow[r] & U_4/U_4\cap W_1 \arrow[u, "\wr"'] \arrow[r] & U_4/U_3 \arrow[r] & 0.
\end{tikzcd}$$

Suppose $C(W_1,W_2) \subset C(U_1,U_2,U_3,U_4)$. Then for any $V_1 \ni (U_1,U_2) \leq (V_1,U_3)$, I have $C(W_1,W_2) \subset C(V_1,U_4)$, hence $(W_1,W_2) \leq (V_1,U_4)$. In particular $W_1 \subset V_1$ for all such $V_1 \implies W_1 \subset U_1$. Also $(W_2\cap U_3, W_2) \leq (U_3,U_4)$

$(W_1, W_2\cap U_3) \leq (V_1,U_3)$ for any such V_1

$$\begin{tikzcd}[column sep=small]
U_4 \arrow[d, phantom, "\vee"] & & \\
U_3 \arrow[d, dash] \arrow[dr, phantom, "\vee"] & & \\
U_2 \arrow[dr, phantom, "\vee"] & V_1 & W_2 \\
 & U_1 \arrow[dr, dash] & W_2\cap U_3 \\
 & & W_1
\end{tikzcd}$$

$(U_1, U_1 + W_2\cap U_3)$

$(U_1+W_2)\cap U_3$

say $C(W_1, W_2) \subset C(U_1, U_2, U_3, U_4)$. For any hyper-plane V/U_1 in U_3/U_1 not containing U_2/U_1 I have $(W_1, W_2) \leq (V, U_4)$, hence $W_1 + U_1$ must be contained in any such $V \Longrightarrow W_1 \subset U_1$.

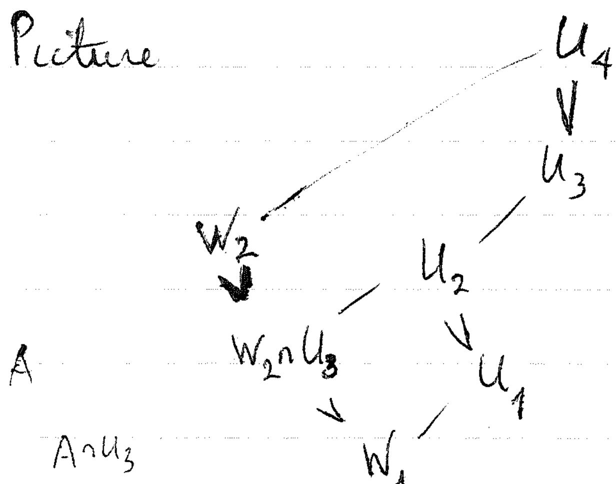
Also $(W_1, W_2 \cap U_3) \leq (V, U_3)$ for any such V

$$\Longrightarrow \quad (W_1 + U_1/U_1,\ W_2 \cap U_3 + U_1/U_1) \leq (V/U_1,\ U_3/U_1)$$

for all hyperplanes V/U_1 not containing U_2/U_1. Thus it is clear that $W_2 \cap U_3 + U_1 = U_2$.

$$(W_1, W_2 \cap U_3) \leq (U_1, U_2)$$
$$(W_2 \cap U_3, W_2) \leq (U_3, U_4)$$

Picture



Now if this holds any $A \in C(W_1, W_2)$ is in $C(U_1, U_2, U_3, U_4)$