September 2, 1974. Mumford's view of GL_n "at ∞".

$G = GL(V)$, $\dim V = n$.

A 1-param. subgp $\lambda : \mathbb{G}_m \to G$ may be identified with a grading:

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \qquad \lambda(\alpha)v = \sum \alpha^n v_n \quad \text{if } v = \sum v_n$$

$P(\lambda)$ = subgroup consisting of γ such that $\lambda(\alpha)\cdot\gamma\cdot\lambda(\alpha)^{-1}$ has a specialization in G as $\alpha \to 0$. Thus if we choose a basis with λ diagonalized: $\lambda(\alpha) = \{\alpha^{r_i}\delta_{ij}\}$ $\quad r_1 \geq r_2 \geq \cdots$ then $\lambda(\alpha)\gamma\lambda(\alpha)^{-1} = (\alpha^{r_i - r_j}\gamma_{ij})$ has such a specialization iff $\gamma_{ij} = 0$ for $r_i < r_j$. This means that γ can't carry part of V_m into V_n for $m > n$; i.e. γ increases filtration. Thus $P(\lambda)$ is the parabolic subgroup assoc. to the flag

$$p \longmapsto \bigoplus_{r \geq p} V_r$$

Put another way, we are thinking of behavior of $\lambda(\alpha)$ as $\alpha \mapsto 0$, hence we ought to lump together the part where λ has order $\geq p$.

Next Mumford introduces an equivalence relation on 1 P-S: First λ_1 λ_2 are equivalent if $\lambda_2 = \gamma^{-1}\lambda_1\gamma$ for some γ in $P(\lambda_1)$. This means simply that the flags associated to λ_1 and λ_2 are the same. Secondly λ_1 and λ_2 are equivalent if $\exists$ n_1, n_2 pos. integers $\ni$ $\lambda_2(\alpha^{n_2}) = \lambda_1(\alpha^{n_1})$.

For the first type of equivalence, a complete set of invariants is the flag

$$0 = W_0 < W_1 < \cdots < W_m = V$$

assoc. to λ together with the sequence of exponents:

$$r_1 < r_2 < \cdots < r_m \qquad \begin{matrix} m \geq 1 \\ (r_1 \neq 0 \text{ if } m=1) \end{matrix}$$

where $\lambda(\alpha) = \alpha^{r_i}$ on W_i/W_{i-1} $\quad i = 1, \ldots, m.$

For the second type of equivalence, we must identify two sequences if they differ by rational mult:

$$\{r_i\} \sim \{r_i\} \quad \text{if} \quad n' r_i' = n r_i \qquad \text{some } n, n' > 0$$

For $SL(n)$, one has the condition $\sum r_i d_i = 0$ $\quad d_i = \dim W_i/W_{i-1}$

Over $\mathbb{C}$, given a unitary structure on V, one can consider $\neq 0$ self-adjoint operators A. Each A would give a sequence of eigenvalues $\lambda_1 < \cdots < \lambda_m$ and a corresponding flag $0 < W_1 < \cdots < W_m = V$. To get the corresp gadget to Mumford's equivalence classes, one would want to identify A with tA for $t > 0$. Hence Mumford's gadget is something like the unit sphere in the space of self-adjoint operators. (Better the rational points on the unit sphere).

Similarly for $SL(n)$, one would consider $\neq 0$ self-adjoint operators of trace 0, identifying A and tA for $t > 0$; so one gets rational points on the sphere of s.a. op. of tr 0.

(Maybe this explains Bruno Harris's statement – Yes, the point is that [redacted] the space [redacted]

$$\{ t_1 \leq \cdots \leq t_n \mid \Sigma t_i = 0 \} / [redacted]\ \mathbb{R}^+$$

is homeomorphic to $\Delta(n-1)$. This is the sphere in $\{\Sigma t_i = 0\}$ which inherits a [redacted] simplicial subdivision from the Weyl chambres.)

Problem: In what sense is the [redacted] Mumford gadget like the "rational pts at ∞ of G"?

Take example of $G = GL(V)$ acting in the obvious way on $G_n(W \oplus V)$ $n = \dim V$. As $G_n(W \oplus V)$ is a projective variety Mumford's theory should give a theory of when [redacted] elements of $G_n(W \oplus V)$ are stable, etc. [redacted] Assume $W = V$ so that there is a big open [redacted] orbit of $G_n(W \oplus V)$, namely the graphs of [redacted] isoms. [redacted] $W \xrightarrow{\sim} V$. Probably this open set is the stable set.

Repeat: For GL_n a point of the Mumford space is indexed by a flag $0 < W_1 < \cdots < W_m = V$ and a sequence of rationals $t_1 \leq \cdots \leq t_m$ such that [redacted] some $t_i \neq 0$, modulo homothety relation by positive $\neq 0$ rational nos. Thus it corresponds to self-adjoint ops on V which are non-zero modulo mult. by $\lambda \in \mathbb{R}^+$. Hence one gets the unit sphere in the space of self-adjoint ops.

Now suppose I want to investigate the limit points of $\mathrm{Isom}(W,V) \subset G_n(W\oplus V)$. One takes orbits $\lambda(\alpha)\Theta$, where λ is a 1-parameter subgroup, and lets α approach zero. If $V = \bigoplus_n V_n$ is the grading defined by λ, then

$$\lambda(\alpha)\Theta = \sum \alpha^n \pi_n \Theta \qquad \pi_n = \text{proj on } V_n.$$

As $\alpha \to 0$, $\lambda(\alpha)\Theta$ goes toward the correspondence from W to V which is the comp. of Θ with the corresp. defined by the subquotient $V_0 \simeq \bigoplus_{n\geq 0} V_n \Big/ \bigoplus_{n>0} V_n$.

Point appears to be that if we think of the Mumford complex as being analogous to non-zero self-adj. operators mod pos. homothety, then the subquotient defined by the eigenvalue 0 corresponds to the limit point in $G_n(V\oplus V)$ of the corresponding 1-parameter subgroup.

Another interpretation - GL_n/U_n is the symmetric space of pos. def. self-adjoint operators $\sim$ space of self-adjoint operators. The natural way to put a boundary on this open sell is to use the unit sphere, or to use pos. homothety classes.

Sept 5, 1974. Grassmannians.

Let E, F be two unitary f.d.v.s. and consider $Gr_n(E\oplus F)$ $n = \dim E$. Let $Un(E)\times Un(F)$ act on $Gr_n(E\oplus F)$ and try to find invariants for the action. [redacted] An $A \in Gr_n(E\oplus F)$ can be interpreted as a correspondence from E to F, i.e. a map $\theta : D \longrightarrow F/N$ where $D \subset E$, $N\subset F$ and $\dim(N) + \dim(D) = \dim(A) = n$. – First consider the case of an isomorphism $\theta : E \xrightarrow{\sim} F$, and let $\lambda_1 < \cdots < \lambda_k$ be the eigenvalues of [redacted] $\theta^*\theta$, and $E = E_1\oplus\cdots\oplus E_k$ be the corresponding eigenspace decomposition. Then [redacted] $F = F_1\oplus\cdots\oplus F_k$, where $F_i = \theta E_i$, is an orth. decomposition, and $\lambda_i^{-1}\theta : E_i \to F_i$ is unitary. If θ is not an isomorphism, we get

$$E = E_0\oplus E_1\oplus\cdots\oplus E_k$$

$$F_1\oplus\cdots\oplus F_k\oplus F_0$$

$$\lambda_1<\cdots<\lambda_k \quad + \quad \text{unitary isos. } E_i \xrightarrow{\sim} F_i$$

Finally if D, N are not zero in general one gets

$$E = E_0\oplus E_1\oplus\cdots\oplus E_k\oplus E_\infty \qquad E_\infty = D^\perp$$

$$F = F_0\oplus F_1\oplus\cdots\oplus F_k\oplus F_\infty \qquad F_\infty = N$$

$$0<\lambda_1<\cdots<\lambda_k<\infty \quad + \text{ isos. unitary } E_i \xrightarrow{\sim} F_i$$

Example: $n=1$. Here [redacted] the three cases are

$E = E_0$ $\quad k=0$ $\quad$ stratum = pt because $F = F_0$

$E = E_1$ $\quad k=1$ $\quad$ stratum = [redacted] $SO(1)$ as $F_1 \in PF$, and $E\xrightarrow{\sim}F_1$ has to be given.

$E = E_\infty$ $\quad k=0$ $\quad$ stratum = PF as $F_\infty\in PF$

so

$$\mathbb{P}(\mathbb{C}\oplus F) = pt \cup \{0<\lambda<\infty\}\times SO(-1) \cup \mathbb{P}F.$$

Example: $n=2$. Generic stratum is $0<\lambda_1<\lambda_2<\infty$, over which we have to give $E = E_1 \oplus E_2$, $F = F_0 \oplus F_1 \oplus F_2$ and unit isos. $E_1 \simeq F_1$, $E_2 \simeq F_2$. ?

Goal: I was hoping to realize $Gr_n(V)$ as the geometric realization of some space which I might make discrete. Try to work inductively:

I have seen that

$$Gr_n(E\oplus F)/Aut_{un}(E)\times Aut_{un}(F) \simeq \Delta(n).$$

The map is given by assoc. to $A \subset E\oplus F$ the sequence of its eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \infty$. Over a point

$$0=\lambda_1=\cdots=\lambda_{\mu_0}<\lambda_{\mu_0+1}=\cdots=\lambda_{\mu_0+\mu_1}<\cdots\cdots<\lambda_{\mu_1+\cdots+\mu_k+1}=\cdots=\lambda_n=\infty$$

sits the space of orthogonal decomps.

$$E = \underbrace{E_0}_{\mu_0} \oplus \underbrace{E_1}_{\mu_1} \oplus\cdots\oplus \underbrace{E_k}_{\mu_k} \oplus \underbrace{E_\infty}_{n-\mu_1-\cdots-\mu_k=\mu_\infty}$$

$$F = F_0 \oplus \overset{\mu_1}{F_1} \oplus\cdots\oplus \overset{\mu_\infty}{F_\infty}$$

+ unit isos. $E_i \simeq F_i$

Sept. 7, 1974. Kuiper's theorem.

It says the unitary group $U(H)$ of Hilbert space is contractible.

Proof. 1) $U(H)$ is a def. ret. of $\mathrm{End}(H)^*$ which is an open subset of the Banach space $\mathrm{End}(H)$. Hence general results tell one that $U(H)$ has the h-type of a CW cx. Thus it suffices to show any map $X \longrightarrow \mathrm{End}(H)^*$, $x \mapsto \theta_x$ contracts to a point, X finite s.cx.

2) By simp. approx. thm. can suppose θ piecewise-linear, i.e. if $x = \sum t_i v_i$, then $\theta_x = \sum t_i \theta_{v_i}$, where t_i are the baryc. coords of x. Thus θ is determined by the finitely many θ_{v_i} in $\mathrm{End}(H)^*$, $i \in I = \mathrm{Vert}(X)$

3) Construct a sequence of unit vectors e_n and orth. subspaces V_n, $n \geq 1$ as follows. To start, choose a unit vector e_1 and a unit vector e_1' orth. to $\{\theta_{v_i}(e_1) \mid i \in I\}$, and let V_1 be gen. by $\{\theta_{v_i}(e_1), e_1'\}$. (assume $\theta_{v_i} = id$ some i). Having got to stage $n-1$, observe that

$$\bigcap_{i \in I} \theta_{v_i}^{-1} \{V_1 + \cdots + V_{n-1}\}^{\perp} \neq 0$$

(intersection of fin. many subsp. of finite codim is again of finite codim.). Hence we can choose e_n such that $\theta_{v_i}(e_n)$ is $\perp$ to V_j, $1 \leq j < n$. Can also choose e_n' perp. to V_j, $\theta_{v_i}(e_n)$, $i \in I, 1 \leq j < n$, and put V_n = subspace gen by $\theta_{v_i}(e_n)$ and e_n'.

The result of this construction is a sequence of orth. fin. dim. subspaces V_n and unit vectors $e_n, e_n' \in V_n$ such that $\theta_{v_i}(e_n) \in V_n \cap \langle e_n' \rangle^{\perp}$ for all i.

~~4) In each space V_n we can find a path of autos. α_t^n such that $\alpha_0^n = id$, $\alpha_1^n \theta_{v_i}(e_n) = e_n$ for all i. And we can suppose $\|\alpha_t^n\|$ bdd in n, so that we can take $\oplus_n \alpha_t^n \oplus id$ on comp of $V_1 \oplus$~~ .

~~Changing θ~~

4) For each n we have a map $X \longrightarrow V_n \cap \langle e_n' \rangle^\perp$
$x \longmapsto \theta_x e_n$. Choose a homotopy

$$X \times I \longrightarrow Aut(V_n)$$
$$x, t \longmapsto \alpha^n_{x,t}$$

such that $\alpha^n_{x,0} = id$, $\alpha^n_{x,1}(\theta_x e_n) = e_n$. $t \mapsto \alpha^n_{x,t}$ rotates $\theta_x e_n$ to e_n' then back to e_n.

Being careful about norms the homotopy $\bigoplus_n \alpha^n_{x,t} \oplus id$ on $(V_1 + \cdots + \cdots)^\perp$ is a map $X \times I \longrightarrow Aut(H)$

Composing this homotopy with θ we reduce to the case where $\theta e_n = e_n$ for an orth. sequence $e_1, \cdots$

5)

Kuiper's thm. X finite complex, E Hilbert space bundle over X. To show X is trivial.

$E = \underset{\sim}{H} \oplus E'$

Let $H(X)$ = isom. classes of Hilbert space bundles. This is an abelian monoid with an infinite sum – so one knows the Groth. grp. is trivial. Next one wants to prove $H(X)$ is trivial itself. So let $e \in H(X)$ be the class of the trivial bundle. Clearly $e+e=e$ and e is cofinal in $H(X)$, so elements of the Groth group are of the form $x-e$. ~~To prove also $H(X)$ is trivial one has only to prove that~~ Thus for the Groth. grp to be 0 means that $e+x=e$ (specifically: Given x one finds a y such that $x+y=e$. Then $x+(y+x+y+x+\dots) = (x+y)+(x+y)+\dots$)

So to finish all we have to do is to show that x splits off a copy of e, for then $x = x'+e \Rightarrow$
$e = x+e = x'+e+e = x'+e = x$.

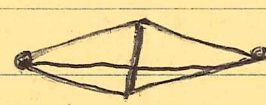
~~Does it follow~~

$\forall x \quad x = x'+e \Rightarrow x+e = x$.

~~Only on putting in inverses that one wins.~~

Use the fact that your bundle is a direct summand of a

E over $X = U \cup V$.



clutching function $U \cap V \to \mathrm{Aut}(H)$

Sept. 8, 1974

Fix a finite cx. T. ~~[illegible]~~ If V is a fin. dim. H.S., let $D(V;T)$ be the space of decompositions of V indexed by T. Thus a point of $D(V;T)$ is an orthogonal decomp.

$$V = \bigoplus_{t\in T} V_t$$

necessarily finite; equivalently it is a $*$-homomorphism $\mathbb{C}(T) \longrightarrow \mathrm{End}(V)$.

If E is a vector bundle over X (say X conn.) with an inner product on the fibres, ~~[illegible]~~ $\dim E = n$, let P be the corresponding ~~[illegible]~~ principal bundle, and form $P\times^{U_n} D(\mathbb{C}^n;T)$. ~~[illegible]~~ A T-decomposition of E is by definition a section of $P\times^{U_n} D(\mathbb{C}^n;T)$. Call two T-decompositions homotopic if the corresp. sections are. Given two bundles with T-decomp. E, E', call them homotopic if there is an isom $E \simeq E'$ with resp. to which the two T-decomp. are homotopic. Let $\mathrm{Vect}(X;T)$ be the homotopy classes of bundles with T-decomp. over X. Clearly

$$\mathrm{Vect}(X;T) = \Big[X, \coprod_n PU_n \times^{U_n} D(\mathbb{C}^n;T)\Big]$$

Let H be a sep. H.S., so that BU_n can be taken as the space of n-planes in H: $Gr_n(H)$. A point of $PU_n \times^{U_n} D(\mathbb{C}^n;T)$ is then an n-plane $V^n \subset H$ together with a decomp. of V^n wrt. T, i.e. we give a finite subset of T: $\{t_1,\dots,t_q\}$, and finite diml. orth subspaces $V_1,\dots,V_q$ of ~~[illegible]~~ which are non-zero and such that $\sum \dim(V_i) = n$. For any finite set S, let $D'(H;S)$ be the space consisting of f.d. orth subspaces V_s of H for each $s\in S$; whence $S \longmapsto D'(H,S)$ is covariant from Γ' = cat of finite sets to spaces. At least set-theoretically sufficiently large finite-dim'l H.S. The point is that if instead of Hilbert space H we take $\bigcup \mathbb{C}^n$, then

$$D(\bigcup \mathbb{C}^n; T) = \varinjlim_n D(\mathbb{C}^n; T)$$

where the basepoint t_0 is used to embed $D(\mathbb{C}^n;T)$ into $D(\mathbb{C}^{n+1};T)$.

<u>Lemma</u>: $\mathrm{Vect}(X;T)/\mathrm{Vect}(X;t_0) \xrightarrow{\sim}$ [redacted] $\varinjlim_n [X, D(\mathbb{C}^n;T)]$

On the left is the quotient of $\mathrm{Vect}(X;T)$ by the equiv. relation $\xi \sim \xi' \iff \exists\, \tau, \tau' \in \mathrm{Vect}(X;t_0) \ni \xi + \tau = \xi' + \tau'$. A map $f: X \longrightarrow D(\mathbb{C}^n;T)$ is the same as a T-decomp. of the trivial bundle $X \times \mathbb{C}^n$, hence [redacted] it gives an element $\gamma(f) \in \mathrm{Vect}(X;T)$. [redacted] If f is the constant map with values the decomp with all eigenvalues at the basept, then $\gamma(f) \in \mathrm{Vect}(X;t_0)$. Thus we get a map γ from the right to the left.

γ is surjective: Given $\xi \in \mathrm{Vect}(X;T)$, we can find a ~~ban~~ vector bundle η on X such that the underlying bundle of $\xi + \eta$ is trivial. Equipping η with the basept eigenvalue $\eta \in \mathrm{Vect}(X;pt)$, and $\xi + \eta$ $\therefore$ comes from a T-decomp f on a trivial bundle, whence $\xi + \eta = \gamma(f)$.

γ is injective: Given f_1 [redacted] $: X \longrightarrow D(\mathbb{C}^n;T)$ suppose $\gamma(f_1) = \gamma(f_2) \bmod \mathrm{Vect}(X;t_0)$ i.e. $\gamma(f_1) + \tau_1 = \gamma(f_2) + \tau_2$ in $\mathrm{Vect}(X;T)$. [redacted] As the underlying bundles of $\gamma(f_1) + \gamma(f_2)$ are trivial, [redacted] the underlying bundles of [redacted] τ_1, τ_2 are stably-trivial, hence adding trivial bundles we can suppose τ_1, τ_2 are trivial with eigenvalue t_0, hence we can suppose $\gamma(f_1) = \gamma(f_2)$ in $\mathrm{Vect}(X;T)$. This means that we have an isom $\alpha: X \times \mathbb{C}^n \circlearrowleft$ wrt which f_1, f_2 are homotopic. Replacing α by $\alpha \oplus \alpha^{-1}$, we can suppose $\alpha \sim id$ whence $f_1 \sim f_2$.

Put $D(\mathbb{C}^\infty;T) = \varinjlim D(\mathbb{C}^n;T)$ so that

$$[X, D(\mathbb{C}^\infty;T)] = k(X;T,t_0).$$

To show that $T \mapsto k(X;T,t_0)$ is a homology theory we must show for A a finite subcx. of T cont. t_0 ~~we have~~

$$D(\mathbb{C}^\infty;A) \longrightarrow D(\mathbb{C}^\infty;T) \xrightarrow{f} D(\mathbb{C}^\infty;T/A)$$

has the h-type of a fibration. Clear that $D(\mathbb{C}^\infty;A) = f^{-1}\{*\}$, so I want to show f is a quasi-fibration. But let $Z_k \subset D(\mathbb{C}^\infty;T/A)$ be the decomps having $\leq k$ eigenvalues different from the basepoint. Over $Z_k - Z_{k-1}$ $D(\mathbb{C}^\infty;T)$ is locally trivial with fibre $D(V^\perp;A)$, $V =$ eigenspace $\neq$ basepoint.

Fix T and filter $D(\mathbb{C}^n;T)$ by the number of eigenvalues $\neq t_0$.

$$pt = Z_0 \subset Z_1 \subset \cdots\cdots \subset Z_n.$$

Then $Z_k - Z_{k-1}$ fibres over $G_k(\mathbb{C}^n)$ with fibre over $W \subset \mathbb{C}^n$ the space $D(W, T - t_0)$.

$$\coprod_n PU_n \times^{U^n} D(\mathbb{C}^n; T) = \text{contraction of } \begin{cases} S \longmapsto D'(H; S) \\ S \longmapsto T^S \end{cases}$$

$\mathrm{Vect}(X;T)$ is an abelian monoid under $\oplus$. Put

$$k(X;T) = \text{assoc. abelian group.}$$

I want to show $k(X;T)$ is a cohom. theory in X and a homology theory in T.

Assume T connected with basepoint t_0. Let E be a bundle with T-decomp. over X; say X conn. To show E is a direct summand of a bundle homotopic to a trivial bundle over X, T-decomposed with all eigenvalues $= t_0$. If U is an open set ~~of~~ of X over which U is trivial, choose $E_U \simeq U \times \mathbb{C}^n$ and transport the T-~~de~~ decomp. via this isom, whence we have a map $U \to D(\mathbb{C}^n; T)$. Since can assume U is contractible this map extends, after shrinking U, to a map $X \to D(\mathbb{C}^n; T)$ which is homotopic to ~~the~~ a constant map; ~~the~~ and ~~[illegible]~~ since T is connected, $D(\mathbb{C}^n; T)$ is also, so we get a map $X \to D(\mathbb{C}^n; T)$ homotopic to the constant map with valued the decomposition with all eigenvalues $= t_0$. So locally we can ~~[illegible]~~ embed E in "trivial" bundles (ones coming from $\mathrm{Vect}(pt; t_0)$). Thus we get $X = \bigcup_i U_i$, a partition $\sum \rho_i = 1$, and $\varphi_i : E|_{U_i} \longrightarrow E_i|_{U_i}$, φ_i commutes with $C(T)$-action, and is unitary. Define

$$E \longrightarrow \bigoplus_i E_i$$

$$e \longmapsto \sum_i \sqrt{\rho_i} \begin{cases} \varphi_i(e) & pr(e) \in U_i \\ 0 & \text{otherwise} \end{cases}$$

$$\Big\| \sum_i \sqrt{\rho_i}\, \varphi_i(e) \Big\|^2 = \sum_i \rho_i \|\varphi_i(e)\|^2 = \|e\|^2$$

so you get a unitary embedding.

So what the preceding argument shows is that the map

$$N = \mathrm{Vect}(pt; t_0) \longrightarrow \mathrm{Vect}(X; T)$$

is cofinal. This implies that on inverting the image of N one gets $k(X;T)$, hence $k(X;T)$ is a limit of rep. functors of X, hence it will be representable.

<u>Theory with basepoint:</u>

Here I want a formula for

$$k(X;T,t_0) = \mathrm{Vect}(X;T)/\mathrm{Vect}(X;t_0).$$
$$= k(X;T)/k(X;t_0) = K(X).$$

We will follow Segal's ideas: Γ will be the category of finite ptd sets, $\underline{n} = \{0, 1, \ldots, n\}$ with basepoint zero. For each S in Γ, let $D(H;S)$ be the space consisting of fin. diml. orth subspaces V_s for $s \in S - \{0\}$, or equivalently of orth decompositions:

$$H = \bigoplus_{s \in S} V_s$$
$$\dim(V_s) < \infty \quad \text{if } s \neq 0.$$

Thus an element of $D(H;S)$ is a family of projectors $\{e_s, s \in S\}$ satisfying:

$$e_s = e_s^*$$
$$e_s \text{ compact} \qquad s \neq 0$$
$$e_s e_{s'} = e_{s'} e_s = 0 \qquad s \neq s'$$
$$1 = \sum_{s \in S} e_s$$

and the topology is clear – $\{e_s\}$ is close to $\{e'_s\}$ if e_s is close to e'_s.

~~Homotopy type of $D(H;S)$~~: Given integers $n_1, \ldots, n_q \geq 0$ let $D_{n_1,\ldots,n_q}(H)$ be the space of cons. orth. subspaces $V_1, \ldots, V_q$ of H with $\dim V_i = n_i$. Then

$$D_{n_1,\ldots,n_q}(H) = \frac{UN(H)}{U_{n_1} \times \cdots \times U_{n_q} \times UN(H \ominus \mathbb{C}^{n_1+\cdots+n_q})}$$

$$\sim BU_{n_1} \times \cdots \times BU_{n_q}$$

and

$$D(H, \{0, \ldots, q\}) \simeq \coprod_{n_1,\ldots,n_q \geq 0} D_{n_1,\ldots,n_q}(H).$$

$$\sim \left(\coprod BU_n\right)^q$$

Thus $S \longmapsto D(H;S)$ is a "special" Γ-space.

Now contract:

$$D(H;T) = \text{contraction of } \begin{matrix} S \mapsto D(H;S) \\ S \mapsto T^S \end{matrix}$$

where $T^S = \mathrm{Hom}_{pt}(S, T) = T^n$ if $S = \underline{n}$.

A point of $D(H;T)$ is a decomposition

$$H = \bigoplus_{t \in T} V_t$$

such that V_{t_0} is of finite codimension in H. Example: A point of $D(H;S^1)$ is a unitary operator Θ such that $\Theta - 1$ has finite rank.

I would like to show that

$$\mathrm{Vect}(X;T)/\mathrm{Vect}(X;t_0) \simeq [X, D(H;T)]$$

and in fact I want an unstable result somehow saying that this remains true with H replaced by a

<u>Stability question</u> $[X, D(\mathbb{C}^n; T)] \xrightarrow{\sim} [X, D(\mathbb{C}^{n+1}; T)]$
n large?

Suppose to start that E is a bundle over X with T decomposition. What will be the obstruction to removing a given point t_1 of T as an eigenvalue? Wrong question. The right question is to consider the number of eigenvalues not at the basepoint and to investigate the obstruction to lowering this number. Thus I would be interested in stratifying X according to the ~~number~~ of eigenvalues away from the basepoint

$$X_0 \subset X_1 \subset X_2 \subset \cdots\cdots$$

and using the barycentric coord. of the basept I have tubular nbds of these strata.

Variant Question: Are the following of the same homotopy type

$$\varinjlim_n D(\mathbb{C}^n; T)$$

$D(H; T)$ = space of $C(T)$-* actions on H $\ni \forall f$ vanishing at t_0, f has finite rank on H.

$\bar{D}(H; T)$ = space of $C(T)$-* actions $\ni \forall f$, $f(t_0) = 0$, f acts compactly on H

I want to formulate a variant of Kuiper's thm. to use in understanding Douglas + company.

~~Beautifully~~

Conjecture: Let E be a Hilbert space bundle over X (finite complex) together with a $\mathbb{C}(T)$-$*$action such that $\mathbb{C}(T) \longrightarrow \mathrm{End}(E_x)/\text{compacts}$ is injective for each x. Let $C(T) \hookrightarrow \mathrm{End}(H)$ be a $*$-action which is injective mod compacts. Thus $\exists$ continuous family of unitaries $u_x : E_x \longrightarrow H$ $x \in X$ such that (f, u_x) is compact $\forall f \in C(T)$.

Special cases:

1. $T = pt$ Kuiper's thm.
2. $X = pt.$ Douglas thm. that two embeddings $C(T) \hookrightarrow A(H)$ lifting to $\mathrm{End}(H)$ are unitarily conjugate.

How this result could be used. Consider the cat whose objects are the different $C(T)$ $*$ actions on H suitably topologized (if $f_1, \ldots, f_k$ generate $C(T)$, then one uses the norms of the associated operators to topologize). Space: $\mu(T)$. Morphisms are unitary operators commuting with the the $C(T)$-action modulo compacts. One has a direct sum

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parameterized version of Douglas thm. Suppose we have a family $\alpha_x : C(T) \longrightarrow \mathrm{End}(H)$, each injective mod compacts. Say we have a family of self-adjoint operators A_x with same spectrum, e.g. a Cantor set Λ, and we have selected a nice dense sequence $\lambda_1, \lambda_2, \ldots$ in Λ. I would like to construct an orth sequence $e_1^{(x)}, e_2^{(x)}, \ldots$ such that $A_x e_i^{(x)} - \lambda_i e_i(x) \longrightarrow 0$ unif. in X. Thus if we conjugate A_x $A_x e_i - \lambda_i e_i \longrightarrow 0$

<u>September 14, 1974:</u>

Finite sets can be realized as follows. Given a finite set S with basepoint, put

$A[S]$ = finite subsets of Hilbert space equipped with a map to $S - \{*\}$.

Thus $$A[S] \sim \coprod_{f: S-\{*\}\to \mathbb{N}} \prod_{s\in S-\{*\}} B\Sigma_{f(s)} . \quad = \coprod_n P\Sigma_n \times^{\Sigma_n} (S-*)^n$$

And $S \longmapsto A[S]$ is a "special" Γ-space. Now if T is a ptd. space (connected), we extend this as usual:

$$A[T] = \text{contraction of } \begin{matrix} S \mapsto A[S] \\ S \longmapsto T^S \end{matrix}$$

so that $A[T]$ = finite subsets of Hilbert space equipped with a map to T topologized so that points indexed by the basepoint of T disappear.

Try to describe the natural stratification of $A[T]$. Put $A[T]_k$ = subspace of configurations where there are $\leq k$ points sitting over $T - \{*\}$. Then

$$A[T]_k - A[T]_{k-1} \sim P\Sigma_k \times^{\Sigma_k} (T - t_0)^k$$

(in fact equal if I take $P\Sigma_k = \mathrm{Inj}(\{1,\ldots,k\}, H)$).

I should be able to describe [redacted] over a smooth manifold X what a map $X \to A[T]$ should be. Choose a function $\rho : T \to [0,1]$ with $\rho^{-1}(1) = t_0$. ~~Since [illegible] Then A[T]~~ Since $A[T]$ maps to $SP(T)$, we get a map

$$A[T] \longrightarrow SP(T) \longrightarrow SP[0,1] = \Delta(\infty)$$

which associates to a set $\begin{array}{c}\sigma \subset H \\ \downarrow u \\ T\end{array}$

the image of $\sigma \longrightarrow T \xrightarrow{\rho} [0,1]$ arranged in order. Call these $s_1(\sigma,u) \leq s_2(\sigma,u) \leq \cdots\cdots$ so that $s_n(\sigma) = 1$ for n large. Define

$$\mathcal{U}_k = \{(\sigma,u) \mid s_k(\sigma,u) < s_{k+1}(\sigma,u)\}. \qquad k \geq 0$$

open set in $A[T]$ = inverse image of open star of k-th vertex. Over $\mathcal{U}_k$ we can shove to the basepoint those points of (σ,u) with value $> s_k(\sigma,u)$. Hence [redacted] $\mathcal{U}_k$ is homotopic to $P\Sigma_k \times^{\Sigma_k} (T-t_0)^k$; it is [redacted] a normal tube about the stratum $A[T]_k - A[T]_{k-1}$.

Next we can see $\mathcal{U}_k \cap \mathcal{U}_\ell$ is homotopy equivalent to

$$\left[P\Sigma_k \times^{\Sigma_k} (T-t_0)^k\right] \times [redacted] \left[P\Sigma_{k-\ell} \times^{\Sigma_{k-\ell}} (U-t_0)\right]$$

U = open star of t_0.

So it is now clear how we should think of a map $X \longrightarrow A[T]$. We have first a map $X \longrightarrow \Delta(\infty)$ given by functions $0 \leq s_1 \leq s_2 \leq \cdots\cdots$, then over $U_k = \{x \mid s_k(x) < s_{k+1}(x)\}$ we give a k-fold covering $E_k \longrightarrow U_k$ together with a map from E_k to T. Over $U_k \cap U_\ell$ we give a covering of degree $k-\ell$, $E_{k\ell}$ indexed by the basepoint of T and an isomorphism

$$\left(E_k \middle| U_k \cap U_\ell\right) \underset{U_k \cap U_\ell}{\dot{\times}} E_{k\ell} \xrightarrow{\sim} \left(E_\ell \middle| U_k \cap U_\ell\right).$$

But now granted that you have this description of maps $X \longrightarrow A[T]$, how are you going to use it.

Task lemma: Take the analogous thing for the unitary groups, where you know bundles have negatives, and show you can replace one of these bundles with varying fibre dimensions by a trivial bundle of large rank.

Example of two strata – suppose $X = U_k \cup U_\ell$ with $k < \ell$. Thus we have bundles E_k over U_k, E_ℓ over U_ℓ, and over $U_{k\ell}$ we will give an embedding $E_k \hookrightarrow E_\ell$ such that the orth

<u>September 19, 1974</u> The Grassmannian.

Let V be a vector space over $\mathbb{C}$ of dimension $p+q$. For any q-plane Q in V, let $U(Q) = \{A \in G_p(V) \mid A \oplus Q = V\}$; $U(Q)$ is an open affine of $G_p(V)$ isom. to the affine space $\mathrm{Hom}(Q, Q/V)$. The complement $G_p(V) - U(Q) = \{A \mid A \cap Q \neq 0\}$ is a divisor. Hence if $Q_1, \ldots, Q_m$ are q-planes in general position, then

$$G_p(V) - \bigcup_{i=1}^{m} U(Q_i) = \{A \mid A \cap Q_i \neq 0 \quad 1 \leq i \leq m\}$$

will be a subvariety of (complex) codimension m. For example if $Q_1, \ldots, Q_m$ are independent, this subvariety is the image of

$$\{(A, L_1, \ldots, L_m) \mid L_i \in \mathbb{P}Q_i,\ L_1 \oplus \cdots \oplus L_m \subset A\}$$

which has dimension

$$m(q-1) + (p-m)q = pq - m$$

and $G_p(V)$ has dimension pq. By general position, we know that any map $K \longrightarrow G_p(V)$, where K is a polyhedron of dimension $\leq 2m-1$, can be moved off this subvariety; hence

$$\bigcup_{i=1}^{m} U(Q_i) \subset G_p(V)$$

will induce isos on homology in degrees $\leq 2m-2$, and a surjection in degree $2m-1$.

So suppose now that $V = Q_0 \oplus \cdots \oplus Q_m$ with $\dim Q_i = q$. Let U_i = open set of $G_p(V)$ consisting of A which are complements to Q_i. Then for m large we have seen that

$$\bigcup U_i \subset G_p(V)$$

is a good approximation. But suppose we compute the nerve of this covering up to homotopy type. Each U_i is contractible. Given $0 \le i < j \le m$ one has a map

$$U_i \cap U_j \longrightarrow \mathrm{Isom}(Q_i, Q_j)$$

$$A \longmapsto Q_i \underset{\sim}{\xleftarrow{pr_i}} A \underset{\sim}{\xrightarrow{pr_j}} Q_j$$

whose fibres are affine spaces. Similarly one has

$$U_{i_0} \cap \cdots \cap U_{i_j} \longrightarrow \mathrm{Isom}(Q_{i_0}, Q_{i_1}) \times \cdots \times \mathrm{Isom}(Q_{i_{j-1}}, Q_{i_j})$$

with affine spaces for fibres.

Observe that if we take the limit as $m \longrightarrow \infty$, then the approximation becomes better, but it is still not true that $\bigcup_{i=0}^{\infty} U_i = G_p(Q_0 \oplus \cdots\cdots)$. For, an element of $G_p(Q_0 \oplus \cdots)$ is a q-dimensional quotient E of $Q_0 \oplus \cdots$ in which $Q_{m+1} \oplus \cdots$ dies for some m, and it need not be the case that $\exists\, i$ such that $Q_i \xrightarrow{\sim} E$, (once $q \geq 2$).

Suppose that in [redacted] V we have a set of q planes in general position, $Q_i \quad i = 1, 2, \dots$ meaning that any subset of $[\frac{n}{q}]$ [redacted] planes ($n = \dim V$) is independent. Then is it true that the dimension of $G_p(V) - \bigcup_{i=1}^{m} U(Q_i)$ goes down as m increases?

Observe the following: [redacted] If S is any finite subset of $G_p(V)$, then $S \subset U(Q)$ for some Q. In effect, for each $A \in S$ the open set $U(A) \subset G_q(V)$ is open & dense, so these have a non-empty intersection.

Thus [redacted] we [redacted] can construct $Q_1, Q_2, \dots$ inductively as follows. Having obtained Q_s, choose Q_{s+1} so that $U(Q_{s+1})$ contains one point from each component of $G_p(V) - \bigcup_{i \leq s} U(Q_i)$.

For example $G_2(V^4)$. $V = Q_1 \oplus Q_2$. Then $\{A \mid A \cap Q_1 \neq 0,\ A \cap Q_2 \neq 0\}$ is $\mathbb{P}Q_1 \times \mathbb{P}Q_2$. So let ~~$L_1 \oplus L_2$ with $L_i \in \mathbb{P}Q_i$. Then $\{A \mid A \cap Q_1 \neq 0, A \cap Q_2 \neq 0, A \cap Q_3 \neq 0\}$~~ Q_3 be the graph of an isom. $\theta : Q_1 \xrightarrow{\sim} Q_2$. Then $\{A \mid A \cap Q_i \neq 0 \quad i = 1, 2, 3\} = \{L \oplus \theta L \mid L \in \mathbb{P}Q_1\}. \simeq \mathbb{P}Q_1$

Next let Q_4 be the graph of an isom $\theta' : Q_1 \xrightarrow{\sim} Q_2$. Here $\{A \mid A \cap Q_i \neq 0\} = \{L \in \mathbb{P}Q_1 \mid (\theta')^{-1}\theta L = L\}$ which will