13 August, 1974:

$\mathcal{P}$ exact category, T a compact space. I am seeking the "space" of "chains" on T with coefficients in $\mathcal{P}$. In the case when $\mathcal{P}$ is an additive category, Segal has told us what to do more or less.

Segal's process. Let $\mathcal{P}_n$ be the groupoid whose objects are systems $\{P_\sigma,\ \sigma \subset \{1,\dots,n\}\}$ together with transitive maps $P_\sigma \longrightarrow P_\tau$ for $\sigma \subset \tau$ such that

$$P_\sigma \oplus P_{\sigma'} \xrightarrow{\sim} P_\tau \qquad \text{if} \qquad \sigma \amalg \sigma' = \tau$$

Then $\underline{n} \longmapsto \mathcal{P}_n$ is a Γ-category (if $\varphi : \underline{m} \longrightarrow \underline{n}$ is a Γ-map, then $\varphi_*(\{P_\sigma\})(\tau) = P_{\varphi^{-1}(\tau)}$). So $n \mapsto B\mathcal{P}_n$ is a Γ-space. But ~~[illegible]~~ if T is a ptd. space

$$n \longmapsto T^n$$

is a co-Γ-space, so one can contract getting

$$T \otimes \mathcal{P} = \text{contraction of } n \mapsto T^n,\ n \mapsto B\mathcal{P}_n$$

I have seen **before** that things simplify if one works **without** basepoint. In this case what happens is that we ~~can simplify things by g~~ get a decomposition

$$V = \bigoplus_\alpha V_\alpha$$

?

$K_0 A$ groth group of $\mathcal{P}_A$

$K_1 A = GL(A)/E(A)$

$K_2 A =$ Ker $\{St(A) \longrightarrow E(A)\}$

defined alg.

$0 = M_0 \subset M_1 \subset \cdots \subset M_n = P \in \mathcal{P}(A)$

admissible filtration if $M_i/M_{i-1} \in \mathcal{P}(A)$

a subquotient M_2/M_1 of P, $0 \subset M_1 \subset M_2 \subset P$ will be called <u>admiss.</u> if $0 \subset M_1 \subset M_2 \subset P$ is adm.

$\mathrm{Ob}\, Q(\mathcal{P}_A) = \mathrm{Ob}\, \mathcal{P}_A$

$$\mathrm{Hom}_{Q(\mathcal{P}_A)}(P, Q) = \coprod_{\substack{0 \subset M_1 \subset M_2 \subset Q \\ \text{admiss. filtration}}} \mathrm{Isom}(P, M_2/M_1)$$

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classifying space BC = geom. of nerve.

<u>Def:</u> $K_n A = \pi_{n+1}\left(BQ(\mathcal{P}_A)\right)$

$\mathcal{F}$ = fredholm operators ess. spectrum $\{-1, 1\}$

$\mathcal{C}$ = top. cat. objects are unitary vector spaces and
$\overset{\shortparallel}{Q(\mathcal{VR})}$ the maps are:

$$\mathrm{Hom}_{Q(\mathcal{VR})}(V, W) = \coprod_{0 \subset W_1 \subset W_2 \subset V} \mathrm{Isom}(V, W_2 \cap W_1^{\perp})$$

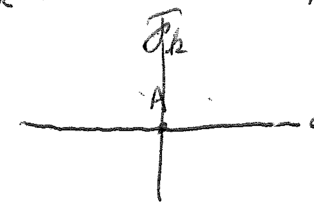
$$\mathrm{Hom}_{Q(\mathcal{VR})}(\mathbb{C}^p, \mathbb{C}^q) = \coprod_{0 \le i \le q-p} U(q) \big/ U(i) \times U(p) \times U(q-p-i)$$

Now over $\mathcal{F}$ I want to understand the ~~bundle~~ thing which associates to a Fredholm op A its Kernel.

$$\mathcal{F} \ni A \longmapsto \mathrm{Ker}\, A.$$

$\mathcal{F}_k = \{A \mid \dim(\mathrm{Ker}\, A) = k\}$. Then ~~to what extent is~~ I want to describe the family $A \mapsto \mathrm{Ker}\, A$. So it is a vector bundle of rank k over the stratum $\mathcal{F}_k$, in fact, it is a universal bundle of rank k, since $\mathcal{F}_k \to G_k(H)$ is a heq by Kuiper's thm. But next I have to describe how to change strata. Here because $\mathcal{F}_k$ sits in the closure of $\mathcal{F}_l$, for $l < k$ I perhaps want to understand specialization i.e. as $A_n \to A$ what happens to $\mathrm{Ker}(A_n)$. $\mathrm{Ker}(A_n)$ as $A_n \to A$. then $\mathrm{Ker}\, A_n \to ?$

Suppose $A \in \mathcal{F}_k$. Choose splitting



normal space to $\mathcal{F}_k$ at A = self adjoint ops. on Ker A.

Title: Finite generation of K-groups in the function field case.

Definition of K-groups. A ring with identity, $\mathcal{P}_A =$ cat. of fin. gen. projective A-modules. By an admissible subquotient of P in $\mathcal{P}_A$, I mean a subquotient of the form M_2/M_1, where $0 = M_0 \subset M_1 \subset M_2 \subset M_3 = P$ is a filtration such that M_i/M_{i-1} is in $\mathcal{P}_A$ for $1 \leq i \leq 3$. Define $Q(\mathcal{P}_A)$.

k finite field, q elements
C curve$/k$, proj. non-sing., $k = H^0(C, \mathcal{O}_C)$
∞ a point of C
$A = \Gamma(C - \infty, \mathcal{O}_C)$

Lichtenbaum has given conjectures as to what $K_i A$ should be. I assume ∞ is a rational point

Conjecture (Lichtenbaum)

$$\begin{cases} K_{2i+1} A \cong K_{2i+1} k \\ K_{2i} A \cong \mathrm{Ker}\,(\mathrm{Id} - q^i \,\mathrm{Frob} \text{ acting on } J(\bar{k})) \end{cases}$$

Theorem: The abelian groups $K_i A$ are finitely generated.

Finite gen. of K-gps in fn. field case

1) Definition of K-groups. The category $Q(P_A)$ and its classifying space.

2) K-groups of [illegible] rings of integers. Borel's theorem

3) function field analogue: [illegible]

C complete non-singular curve over $\mathbb{F}_q$, ∞ a point of C, A = coordinate ring of the affine curve $C-\infty$. The anal of Borel's thm. here is

Conjecture:

~~To a ring A of Given a ring with identity A, one defines a sequence K_iA, $i\geq 0$, of abelian groups [illegible] starting from the category P_A of finitely generated proj A-modules in the following way. [illegible] Denote~~

~~2) One wants to compute the groups K_iA~~

~~b) It would be very inter~~

~~b) One~~

2) It would be very interesting to be able to compute the

2) When A is the ring of integers in a number field, [illegible] the groups seem

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2) A basic problem is to compute these groups when A is the ring of integers in a number field. ~~In this case Lichtenbaum has given a series~~ In this case one has a series of conjectures due to Lichtenbaum relating the K-groups with the ζ function of the number field.

~~2) A basic problem is to calculate these K-groups when A is the ring of integers in a number field. In this case, the groups $K_i A$ are finitely generated, and Borel has determined their ranks. Segal and Harris~~

~~2) Suppose A is the ring of integers in a number field.~~

~~2) When A is the ring of integers in a number field~~

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2) A basic problem is to calculate these K-groups when A is the ring of integers in a number field F. In this case, one has conjectures due to Lichtenbaum relating the K-groups to the zeta function of F and to the tower of cyclotomic extensions of F. Although these conjectures have not been established in dimensions ≥ 3 even for $A = \mathbb{Z}$, one has partial information. ~~One knows that~~ I have shown that the groups $K_i A$ are fin. gen., and Borel has determined their ranks. Also Segal and Harris have shown the odd K-groups contain at least as much torsion as predicted by the Lichtenbaum conjectures. ~~For example~~ For $\mathbb{Z}$, one has

$i =$	0	1	2	3
$K_i\mathbb{Z} =$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\geq \mathbb{Z}/48$

but $K_i\mathbb{Z}$ is unknown for $i \geq 3$.

$C =$ curve over $\mathbb{F}_q$, $\infty =$ point of C, $A = \Gamma(C-\infty, \mathcal{O}_C)$, $F =$ fn. field
$P =$ proj. mod. of rank n over A, $V = F \otimes_A P$

Thm. ~~$H_m(\mathrm{Aut}(P), I(F\otimes_A P))$ finitely generated, and even finite for $m>0$.~~ There exists a normal subgroup Γ' of $\mathrm{Aut}(P)$ of finite index, such that $I(F\otimes_A P)$ is a fin. gen. proj. $\mathbb{Z}[\Gamma']$-module. Consequently $H_m(\mathrm{Aut}(P), I(F\otimes_A P))$ is finite for $m>0$ and fin. gen. for $m=0$.

For the proof of this, we consider the building of V

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1) Let A be a ring with identity, and $\mathcal{P}_A$ the category of fin. gen. projective A-modules. Denote by $Q(\mathcal{P}_A)$ the cat. with the same objects as $\mathcal{P}_A$ in which a morphism from P' to P is defined be an isomorphism $P' \simeq M_2/M_1$, where $0 = M_0 \subset M_1 \subset M_2 \subset M_3 = P$ is a filtration of P whose quotients M_i/M_{i-1} are in $\mathcal{P}_A$. Put

$$K_i A = \pi_{i+1}(BQ(\mathcal{P}_A))$$

where BC is the classifying space of the category C. One can show the groups $K_i A$ agree with those defined algebraically by Bass and Milnor for $i = 0, 1, 2$.

3) The function field analogue of the preceding is as follows. Let C be a comp. n.s. curve $/\mathbb{F}_q$, with fn field F, let ∞ be a point of C, let A be the coordinate ring of $C-\infty$. The Lichtenbaum conjectures state (to simplify assume ∞ is a rational point)

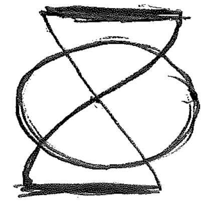
$$K_{2j-1} A = K_{2j-1} \mathbb{F}_q$$

$$K_{2j} A = \operatorname{Ker}(id - q^j \mathrm{Frob} \text{ acting on } J(\overline{\mathbb{F}}_q))$$

where J is the Jacobian of C. One has

<u>Theorem</u>: $K_i A$ is finitely generated for $i \geq 0$.

k finite field

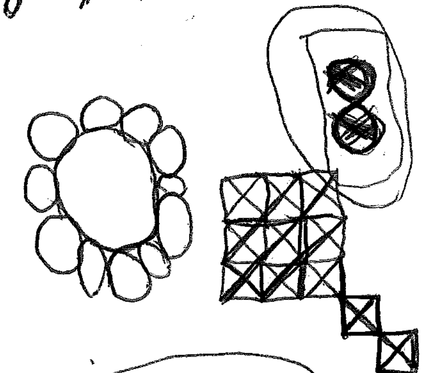
C curve over k compl. n.s. $H^0(C, \mathcal{O}_C) = k$ 

∞ point of C

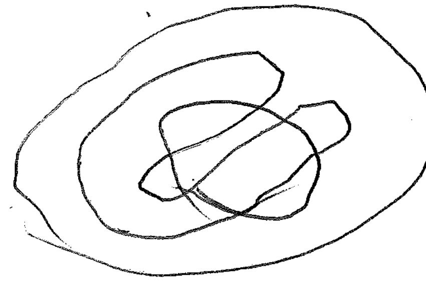
$A = \Gamma(C-\infty, \mathcal{O}_C)$

Thm: K_iA finitely generated abelian group.

Conjectures

① $\tilde{K}_iA$ finite $i \geq 0$, $\tilde{K}_0A$ finite 

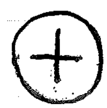
② $$\frac{\#\tilde{K}_{2i-2}A}{\#\tilde{K}_{2i-1}A} = \left|\zeta_{C-\infty}(1-i)\right|$$

$$\zeta_{C-\infty}(s) = \frac{\det\{1-q^{-s}F_{H^1}\}}{(1-q^{1-s})} = \frac{\#\tilde{K}_0A}{1-q}$$ 

$$K_{2i-1}A = K_{2i-1}k \cong \mathbb{Z}/(q^i-1)\mathbb{Z}$$

$$K_{2i-2}A = \mathrm{Ker}$$

$1-q^{i-1}F$ acting on $J(\bar{k})$  

$\pi_*(BQ(\mathcal{P}_A))$ fin gen.   

$\Updownarrow$

$H_*(BQ(\mathcal{P}_A))$ fin. gen.

$\subset Q_{\leq n}(\mathcal{P}_A) \subset \cdots\cdots$

$E_{n,\alpha}$ $\alpha \in \mathrm{Pic}A$

$$\coprod_{\alpha} \mathrm{Aut}(E_{n,\alpha})$$
$$\downarrow$$
$$Q_{\leq n}(A) \subset Q_{\leq n}(\mathcal{P}_A)$$

still don't understand stabilization. Prepare abstract for a talk. Title:

Finite generation of K-groups. 1) Recall definition of K-groups — $Q(P_A)$, $K_n A = \pi_{n+1}(BQ(P_A))$.

Suppose X is a space, what should I mean by a $Q(\mathcal{U})$-bundle over X. At each point x I should get a unitary vector space V_x. How do I want to specify continuity. Presumably if I want to map into F, then if $X_k = \{x \mid \dim V_x = k\}$, then $X_0 \cup \cdots \cup X_k$ is open for each k. So in addition, one would need to know what are

Def. F field, V vec. sp. of dim n over F.
$T(V) =$ simp. complex ass. to ordered set of proper subspaces of V

Thm. (Tits) $T(V) \sim \vee S^{n-2}$

$$\tilde{H}_{n-2}(T(V), \mathbb{Z}) = I(V)$$ free module (of rank $q^{n(n-1)/2}$ if F finite).

Steinberg module (for $\mathrm{Aut}(V)$).

$\boxed{\begin{array}{l}\text{Prop: One has for a Dedekind domain} \\ H_*(BQ_{\leq n}(\mathcal{P}_A), BQ_{<n}(\mathcal{P}_A)) \cong \bigoplus_\alpha H_{*-n}(\mathrm{Aut}(E_{n,\alpha}), I(F \otimes_A E_{n,\alpha}))\end{array}}$

$$H_q(BQ(\mathcal{P}_A)) = \varinjlim H_q(BQ_{\leq n}(\mathcal{P}_A))$$

Prop. E vector bundle $/C$
$\Rightarrow \exists!$ filtration $0 = E_0 < E_1 < \cdots < E_n = E$
$\ni$ E_i/E_{i-1} is semi-stable and
$\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$

Consider now the vertices of X such that

$$\mu_{max}(E \cap W) > \mu_{max}(E/E \cap W) + d_\infty$$

$X_W \qquad X' = \bigcup_{0<W<V} X_W$

1) $X - X'$ finite
2) $X' \sim T(V)$

talk:

$K_i A$

$GL(A) = \bigcup GL_n A$

$K_0 A =$ Groth group of proj. fin gen. A-mods

$K_1 A = GL(A)/E(A) = H_1(GL(A), \mathbb{Z})$

$K_2 A = H_2(E(A), \mathbb{Z})$

$K_i A = \pi_i$ (H-space).

Lichtenbaum conjectures: $[F : \mathbb{Q}] = n = r_1 + 2r_2$

$A =$ integers in F

$$\zeta_F(s) \sim \begin{cases} C_0 s^{r_1+r_2-1} & s \to 0 \\ C_k s^{r_1+r_2} & s \to -k \quad k \text{ even} \geq 2 \\ C_k s^{r_2} & s \to -k \quad k \text{ odd} \geq 1 \end{cases}$$

$$C_0 = \frac{h \cdot R}{w_1}$$

$h = \#(\mathrm{Pic}\,A)$ $\qquad K_0 A = \mathbb{Z} \oplus \mathrm{Pic}\,A$

$K_1 A = A^\times = A^\times_{tors} \times (\mathbb{Z} u_1 + \dots + \mathbb{Z} u_{r_1+r_2-1})$

$w_1 = \#(A^\times_{tors})$.

$F \xrightarrow{\sigma_i} \mathbb{R} \quad 1 \leq i \leq r_1$

$F \xrightarrow{\sigma_j} \mathbb{C} \quad r_1+1 \leq j \leq r_1 + r_2$

$R = \det \log |\sigma_i(u_j)|$

$0 \to \mu(A) \to A^\times \to \mathbb{R}^{r_1+r_2}$

$a \quad \ln|\sigma_1(a)| \cdots \ln|\sigma_{r_1+r_2}(a)|$.

Paper: Brown, Douglas, Fillmore – Unitary equivalence mod compact operators and extensions of $\mathbb{C}^*$-algs.

X compact metric space, $\mathcal{L}$ = bounded op, $\mathcal{K}$ = compact operators on Hilb., $\mathcal{A} = \mathcal{L}/\mathcal{K}$ Calkin alg.

Def: An extension of $C(X)$ by $\mathcal{K}$ is a $*$ mono: $C(X) \hookrightarrow \mathcal{A}$.

One considers two such as being equivalent if they are conjugate wrt a unitary element of $\mathcal{A}$ (which it turns out can be assumed to come from a unitary in $\mathcal{L}$)

Example: $X = S^1$, so that $C(X) =$ $\mathbb{C}^*$-algebra version of the Laurent polynomial ring. Then $\tau: C(X) \longrightarrow \mathcal{A}$ is given by $\tau(z)$ which is a unitary element of $\mathcal{A}$. ?

So let $T \in \mathcal{L}$ be $\ni$ $TT^* \equiv T^*T \equiv 1$ mod $\mathcal{K}$. Then $(T^*T)^{1/2} \equiv 1$, so in the polar decomposition $T = W(T^*T)^{1/2}$ $T \equiv W$. But W is a partial unitary so if $\operatorname{ind} T = 0$, T is a compact perturbation of a unitary operator (W + unitary iso of $\operatorname{Ker}(W)$ with $\operatorname{Coker}(W)$).

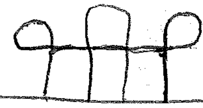
Thm: (Borel – Garland).

$$\dim\{K_i A \otimes \mathbb{R}\} = \begin{cases} 1 & i=0 \\ r_1+r_2-1 & i=1 \\ 0 & i=2 \\ r_2 & i=3 \\ 0 & 4 \\ r_1+r_2 & 5 \\ \dots & \end{cases}$$

period 4

$\exists$ can. map $e_i : K_{2i-1}(\mathbb{C}) \longrightarrow \mathbb{R}$ $\qquad (e_1 = \log|\ |)$
such that

$K_i A \otimes \mathbb{R}$



$V_{\alpha,\beta}$

Fred. operators. A.

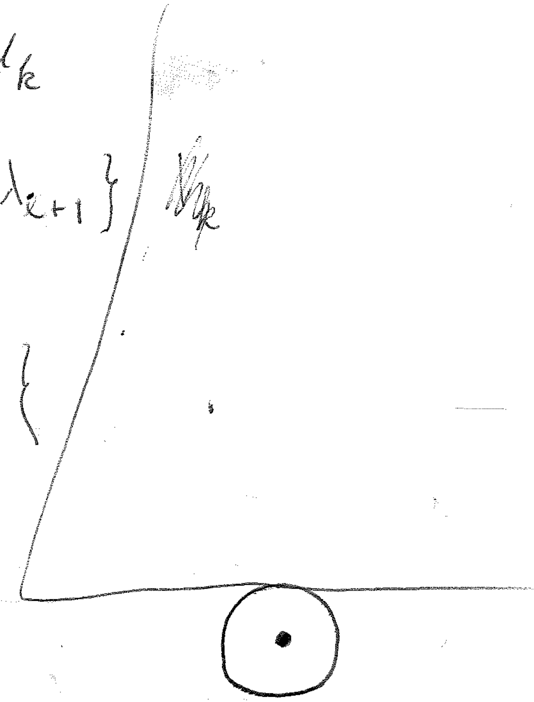
~~$= \{A \mid$ spectrum has k points~~

$V_k = \{A \mid \exists \ \alpha < 0 < \beta \ni \alpha, \beta$ not in spectrum and k eigenvalues are between $\alpha, \beta.\}$

open. $V_k \sim G_k(H) = BU_k$

Q category. Classifying space ~~it~~

$$V_k = \{A \mid |\lambda_k| < |\lambda_{k+1}|\} \sim BU_k$$



$k<l$ $V_k \cap V_l = \{A \mid |\lambda_k| < |\lambda_{k+1}|, |\lambda_l| < \lambda_{l+1}\}$

$$\wr$$

$$\left\{A \;\middle|\; \begin{array}{l}\lambda_1 = \dots = \lambda_k = 0 \\ |\lambda_{k+1}| = \dots = |\lambda_l| = \frac{1}{2}\end{array}\right\}$$

S^1

$$\wr$$

$$\coprod_{i+j=l-k} BU_i \times BU_k \times BU_j$$

~~The classifying space of a category~~

Over V_k we need a U_k-torsor

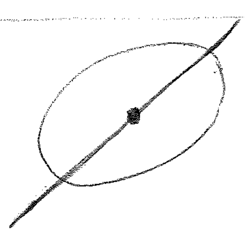
Thus we must give an isom $\mathbb{C}^k \longrightarrow$ eigenspace of

~~dimension~~

locally one gives for each k a sheaf

$$\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \coprod BU_i \times BU_j \times BU_k \rightrightarrows \coprod_k BU_k$$

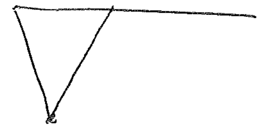
~~BU_k ... BU_l~~ $\coprod_{k,l} U_l / U_i \times U_k \times U_j \;\Big|\; \coprod_k pt$ $U_k / U_i \times U_l \times U_j$



Consider functions $Y \xrightarrow{f} [0,1]$
such that $Y \hookrightarrow \{(y,t) \mid 0 \le t \le f(y)\}$

is good for E.

Replace X/A by $X \cup CA$.



$$SP(X \cup CA) \overset{?}{=} \text{geom. realization of } \coprod SP^m A \text{ acting on } \coprod SP^n X. \quad ?$$

$$SP(X \cup CA) \longrightarrow SP([0,1])$$

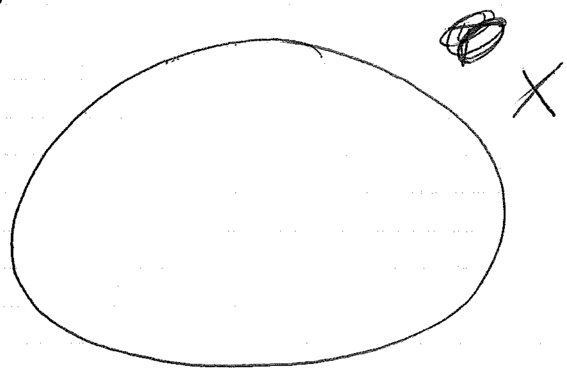
calculate what sits over $(t_0 \leq \ldots \leq t_a) < (\leq \quad) < (\quad)$

~~now one wins the following. Let X be compact.~~

go back to connected K-theory where I consider over X the category of bundles with T-decomposition gives me a monoid $\mathrm{Vect}(X;T)$. Next I have a basepoint t_0 of T and I want to understand

$$\mathrm{Vect}(X;T)/\mathrm{Vect}(X;t_0) = k(X;T,t_0).$$

So I want to have a formula for an element of $k(X;T,t_0)$ ~~which would make it clear that $\mathrm{Vect}(X;T)/\mathrm{Vect}(X;t_0)$~~ and my idea was to consider a topological category consisting of T-bundles ignoring t_0-bundles



~~[scribbled out]~~ $\boxed{\text{Symmetric products}}$

$$SP(X) \xrightarrow{\pi} SP(X/A)$$

given distance from A. function. ~~[crossed out]~~
Then I have $SP(X/A) \longrightarrow SP([0,1])$ namely given $\{x_1, \dots, x_m, *, \dots\}$ I arrange

$$\varphi(x_1) \leq \varphi(x_2) \leq \cdots\cdots \qquad s_0 = 0.$$

and I define $\quad U_i = \{(\underline{x}) \mid s_i < s_{i+1}\}$

$$0 \leq s_1 \leq \cdots\cdots\cdots\cdots$$

And on $U_i \cap U_j \qquad i < j$

$$s_i < s_{i+1} \qquad s_j < s_{j+1}$$

so I can speak of the points which $x_{i+1} \cdots x_j$ and I can project them into A, so on $U_i \cap U_j$ I get something in $SP^{j-i}(A)$. This is my cocycle. Suppose next I consider $\pi^{-1} U_i$ i.e. $s_i(x) < s_{i+1}(x)$, hence I can project onto $x_{i+1}, \cdots$ and project into A. Thus I get

$$p_i : \pi^{-1}(U_i) \longrightarrow SP(A)$$

and on the overlap, I have $\quad c_{ij} : U_i \cap U_j \longrightarrow SP^{j-i}(A)$.
so one has for $i < j$ that

$$\pi^{-1}(U_i \cap U_j)$$

$$c_{ij}\, p_j = p_i$$

and moreover, when I glue I get $SP(X)$ at least set-theoretically.

~~next, one finds out about things.~~ If an

Generalization: Let V_i be a numerable covering of X and $\sigma \longmapsto \begin{array}{c} E_\sigma \\ \downarrow \\ V_\sigma \end{array}$ defined for all $\sigma \subset I$

a contravariant system. Form the space

$$Cyl(\sigma \mapsto E_\sigma) = \bigcup \sigma \times E_\sigma$$

where σ runs over simplices in the nerve of the covering. A map $T \longmapsto Cyl(\sigma \mapsto E_\sigma)$ may be identified with a map $T \xrightarrow{\lambda} K = \bigcup \sigma$ and a natural transf.

$$\lambda^{-1}(U_\sigma) \longrightarrow E_\sigma$$

where U_σ = open star of σ. Thus $Cyl(\sigma \mapsto E_\sigma)$ is the thing I have been thinking of in terms of twisting $\sigma \mapsto E_\sigma$ with respect to the torsor $\sigma \mapsto U_\sigma$ on K.

Now assume that $\exists\, i_0$ such that $V_{i_0} = X$, so that for any σ one has

$$V_{\sigma i_0} = V_\sigma$$

and hence $E_{\sigma i_0} \longrightarrow E_\sigma$ is a heq. Thus

$$\begin{array}{c} Cyl(\sigma \mapsto E_\sigma) \\ \uparrow \text{heq} \\ Cyl(\sigma \mapsto E_{\sigma i_0}) \\ \downarrow \\ Cyl(\sigma \mapsto V_\sigma \times_X E_{i_0}) \\ \| \\ Cyl(\sigma \mapsto V_\sigma) \times_X E_{i_0} \end{array} \qquad \begin{array}{ccc} E_{\sigma i_0} & \longrightarrow & E_{i_0} \\ \downarrow & & \downarrow \\ V_{\sigma i_0} & \longrightarrow & V_{i_0} \end{array}$$

$$Cyl(\sigma \mapsto V_\sigma) \times_X E_{i_0} \underset{f.h./X}{\sim} E_{i_0}$$

V vector space, T compact space

$D(V;T)$ consists of decomp.

$$V = V_{t_1} \oplus \cdots \oplus V_{t_k}$$

indexed by points of T

So when $T = [0,1]$ we can arrange $0 \leq t_1 < \cdots < t_k \leq 1$.

Thus a point of $D(V;T)$ consists of a flag

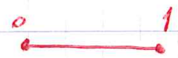
$$0 < V_1 < V_2 < \cdots < V_k = V \qquad \text{length } k$$

and a sequence

$$0 \leq t_1 < \cdots < t_k \leq 1$$

points in $\Delta(k)$ in the open star of the vertices $1, \ldots, k-1$

$k=1$

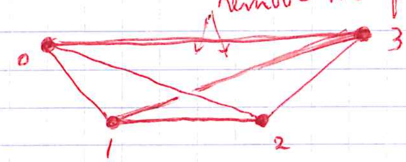


$k=2$

$0 \leq t_1 < t_2 \leq 1$

remove the faces 013, 023

$k=3$



Therefore to get something simplicial maybe I have to ~~remove the~~ add in 0.

$0 < V_1 < \cdots < V_k = V.$

t_1

$\mathcal{P}_n =$ groupoid whose objects are a vector space V decomposed into n pieces, i.e. given

$$id_V = e_1 + \cdots + e_n \qquad \begin{array}{l} e_i^2 = e_i \\ e_i e_j = e_j e_i = 0 \end{array}$$

i.e. a map ~~[illegible]~~ of the Boolean algebra of subsets of $1, \ldots, n$ into the projections in V.

So it appeared before the basic object was $D(V;T)$ = decompositions of V with respect to T.
e.g. if $T = [0,1]$, then a point of $D(V;T)$ is a sequence $\quad 0 \leq t_1 < \cdots < t_k \leq 1 \quad$ + decomp.

$$V = V_1 \oplus \cdots \oplus V_k$$

By mapping $\quad \{t_i\} \longmapsto \{t_2 - t_1, \ldots, t_{k-1} - t_1\} \cdot \frac{1}{t_k - t_1} \qquad$ $k-2$ simp.

one gets a map of $D(V;T)$ ~~to~~ the simp. cx. associated to the ordered set ~~whose simplices are [illegible]~~ whose elements are ~~[illegible]~~ splittings $V = V' \oplus V''$ ~~[illegible]~~, $V' \neq 0$

This poset is reasonably closed to the building. In fact an exact sequence generalization is clear.

~~So now what one [illegible] wants~~

~~X~~

$$X \xrightarrow{g} Y \xrightarrow{f} Z \qquad Y/X \to Z/X \to Z/Y$$

$$C_g \quad C_{fg} \quad C_f \qquad\qquad 0 \quad 0$$

next point

further properties of equivalences I need

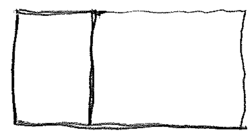
If I have E over $[0,2]$ and I know that

$$\pi^{-1}(0) \subset \pi^{-1}[0,1]$$
$$\pi^{-1}(1) \subset \pi^{-1}[1,2]$$

are h-eq's, then I want to know that

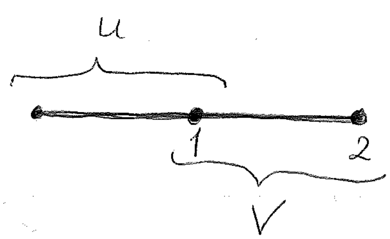
$$\pi^{-1}(0) \subset \pi^{-1}[0,2]$$

is a h-eq.



In fact I can thicken a bit so as to use

$$[A \times I \cup X \times 0] \subset X \times I$$



$\pi^{-1}(U \cap V) \to \pi^{-1}(V)$ is an equiv.

$$\begin{tikzcd}
\pi^{-1}(U \cap V) \arrow[r] \arrow[d] & \pi^{-1}(V) \arrow[d] \\
\pi^{-1}(U) \arrow[r] & \pi^{-1}(U \cup V)
\end{tikzcd}$$

$$X \rightrightarrows X \times I \longrightarrow Y$$

$$\begin{tikzcd}
\lambda^{-1}[0,1) \arrow[d, "p_0"'] & \subset & T \arrow[dd] \\
E_0 \arrow[d] & & \\
U_0 & \subset & X
\end{tikzcd}$$

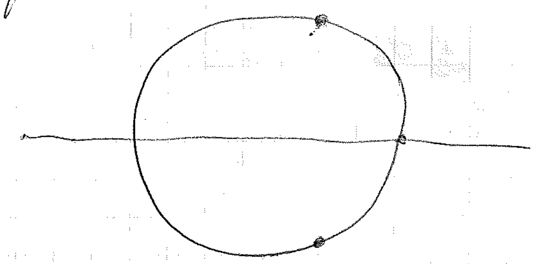
$$\begin{tikzcd}
 & & C \arrow[d] \\
Y \arrow[r] & Z \arrow[r] & X
\end{tikzcd}$$

$$\begin{tikzcd}
Y \times E \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"'] & Y \times C \arrow[d] \\
Z \times E \arrow[r, "\varepsilon"'] & Z \times C
\end{tikzcd}$$

$SP^n(S^1) \longrightarrow S^1$ Given by addition

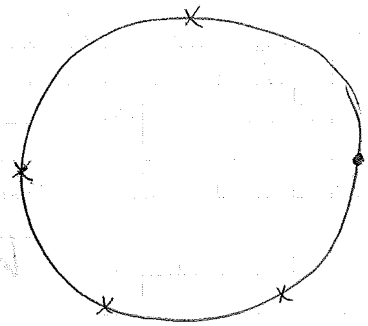
This should be a homotopy $\boxed{\text{equivalence.}}$

$n=2$. fibre is an interval



SP^{n-1} SP^n

V u



SU_n

what is a $k[t]/t^n$ module? P obviously filtered by $t^nP \subset t^{n-1}P \subset \cdots \subset P$

$$P/tP \oplus tP/t^2P \oplus \cdots$$

In a big vector space V, one can consider all subquotients fin. diml. $\Rightarrow$ Ker + Cok are inf. dim. and one can order by inclusion to get a poset.

So at least I understand a category which plays the role of $S^1 \otimes P$, namely Q. Thus I have a varying vector space V_x where the variation is Q-morphisms. And I can think of trying to decompose V_x according to points on S^1.

Think now of a chain on S^1 as being a filtred object

$$P_1 < \cdots < P_p$$

together with $0 < t_1 < \cdots < t_p < 1$. Given another

$$Q_1 < \cdots < Q_q$$

$$0 < u_1 < \cdots < u_q < 1$$

one has $P_i \otimes Q_j$

Now what about exact sequences — ~~suppose that~~ ~~$x_1, x_2, \ldots, x_n \neq 0$~~ $X = [0,1]$ mod $0, 1$ then a chain is

$$t_1 P_1 + \cdots + t_k P_k \qquad k\text{-simplex}$$

$$\text{where } 0 < t_1 < \cdots < t_k < 1 \qquad P_i \neq 0$$

So

Why $SP^n(S^1) \longrightarrow S^1$ (the addition map in the group S^1) is a homotopy equivalence.

Think of S^1 as $\mathbb{R}/\mathbb{Z}$, and lets compute the fibre over 0. This consists of subsets $\{\bar{x}_1, \dots, \bar{x}_n\}$ of S^1 with $\sum \bar{x}_i = 0$. Using a fundamental domain, i.e. $[0,1)$ we can lift the $\bar{x}_i$ to x_i in $\mathbb{R}$, and further by permuting we get a unique lifting of $\bar{x}_1, \dots, \bar{x}_n$ to $x_1, \dots, x_n$ in $[0,1)$ such that $x_1 \leq x_2 \leq \dots \leq x_n$. Then $\sum x_i =$ an integer m, so by shifting x_1 to $x_1 + 1$, or x_n to $x_n - 1$ we can get a lifting $x_1, \dots, x_n$ with $x_1 \leq x_2 \leq \dots \leq x_n \leq x_1 + 1$ and $\sum x_i = 0$. Claim this lifting unique. In effect if $y_i = x_i + n_i$, $\sum n_i = 0$, $n_i \in \mathbb{Z}$, and $\operatorname{diam}(y_i) \leq 1$, then if $n_i \geq 1$, $n_j \leq -1$ one would have

$$x_i - x_j = y_i - y_j + \underbrace{(n_i - n_j)}_{\geq 2}$$

hence $x_i - x_j = 1$, $y_i - y_j = -1$, $n_i = 1, n_j = -1$; so y_i and y_j can be interchanged without affecting the diameter of $\{y_i\}$ whence $x_i = y_i$, $x_j = y_j$. Clear.

But now the fibre is a simplex

$$x_1 \leq x_2 \leq \dots \leq x_n \leq x_1 + 1 \longmapsto 0 \leq (x_2 - x_1) \leq \dots \leq (x_n - x_1) \leq 1$$

$$\longleftarrow\!\!\dashv \; 0 \leq t_1 \leq \dots \leq t_{n-1} \leq 1$$

$$x_1 = -\frac{1}{n} \sum t_i$$

$$x_2 = x_1 + t_1$$

$$x_3 = x_1 + t_2 \qquad \text{etc.}$$

So every fibre of $SP^n(S^1) \longrightarrow S^1$ is a $\overset{n-1}{\text{simplex}}$.

Suppose one takes $X = [0,1]$ modulo endpoints, $X = S^1$. Then a point of $S^1 \otimes P$ is a chain

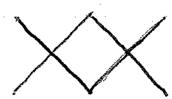
$$t_1 P_1 + \cdots + t_k P_k$$

$$0 < t_1 < \cdots < t_k < 1 \qquad P_i \neq 0$$

hence $S^1 \otimes P$ is [redacted] the realization of the simplicial space with $(P-0)^k$ for non-degenerate k-simplices, i.e. $S^1 \otimes P = BP$.

Now the problem is to find a suitable [redacted] generalization for exact sequence K-theory. The space one starts with is ~~filtered vector spaces~~ the realization of the simplicial category of filtered modules. Hence a point of this would appear as a point of a k-simplex

$$0 < t_1 < \cdots < t_k < 1$$



together with a filtered object

$$0 < Q_1 < \cdots < Q_k$$

of length k. Then we have this peculiar notion of specialization as we go to a face of this simplex.

It is going to be difficult to make this work.

Atiyah's idea: Finite module over $\mathbb{C}^T$

suitable generalization for exact sequence K-theory

~~Suppose that one understands exact sequence K theory~~

[redacted]

//////

I am trying to define what I should mean by a chain on a space T with coefficients in the exact category $\mathcal{P}$. What this means is that I want to describe something called <u>chains</u> and make a space out of them. [redacted] Old idea [redacted] was that you worked with something like

$$\sum x_i P_i$$

Let Γ denote the cat of finite pointed sets, $\underline{\underline{n}}$ the set $\{0, 1, \cdots, n\}$ with basepoint 0. If X is a space

$$\underline{\underline{n}} \longmapsto \text{Maps of ptd. spaces}(\underline{\underline{n}}, X)$$
$$= X^n$$

is a contravariant functor from Γ to spaces (co-Γ-space). (Think of a map $\underline{\underline{n}} \xrightarrow{\varphi} \underline{\underline{m}}$ in Γ as a partially defined map from $\{1, \cdots, n\}$ to $\{1, \cdots, m\}$. Then $\varphi^* : X^m \longrightarrow X^n$ sends $(x_1, \cdots, x_m)$ into $(x_{\varphi(1)}, \cdots, x_{\varphi(n)})$ where if $\varphi(i)$ is undefined, we put $x_{\varphi(i)} =$ basepoint of X).

Let P be a top. abelian monoid. Then

$$\underline{\underline{n}} \longmapsto P^n$$

is a covariant functor from Γ to spaces (a Γ-space). (Here given $\varphi : \underline{\underline{n}} \to \underline{\underline{m}}$, φ_* sends $(P_1, \cdots, P_n)$ into $\left(\sum_{\varphi(j)=i}' P_j \,,\; i = 1, \cdots, m \right)$.)

Denote by $X \otimes P$ the contraction of these two functors. By general nonsense this should be

$$\varinjlim_{\underline{\underline{n}} \to X} P^n .$$

It should be easy to see that a point of $X \otimes P$ is a chain

$$x_1 P_1 + \cdots + x_n P_n$$

where the x_i's are distinct and $P_i \neq 0$, and none of the x_i are at the basepoint

$\boxed{\text{Idea}}$
$$\begin{aligned}
\lambda^2(x+y) &= \lambda^2 x + xy + \lambda^2 y \\
0 = \lambda^2(0) &= \lambda^2(-y) - y^2 + \lambda^2 y \\
\lambda^2(-y) &= y^2 - \lambda^2 y \\
\lambda^2(x-y) &= \lambda^2 x - xy + y^2 - \lambda^2 y
\end{aligned}$$

Using this formula one can extend

$$\lambda^2 : \mathrm{Vect}(X;T) \longrightarrow \mathrm{Vect}(X; SP^2(T))$$

to

$$\lambda^2 : K(X;T) \longrightarrow K(X; SP^2(T)).$$

From

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K(X;T,*) \arrow[d] & K(X; SP^2(T),*) \arrow[d] \\
K(X;T) \arrow[r, "\lambda^2"] \arrow[d] & K(X; SP^2(T)) \arrow[d] \\
K(X;pt) \arrow[r, "\lambda^2"] \arrow[d] & K(X; pt) \arrow[d] \\
0 & 0
\end{tikzcd}$$

one gets an induced map

$$K(X;T,*) \xrightarrow{\lambda^2} K(X; SP^2(T),*)$$

which may be interpreted as follows. Given a vector space decomposed wrt T

$$\sum t_i P_i$$

one writes it as a difference $\sum t_i P_i - \sum P_i$ and applies λ^2. Thus additivety reduces one to

$$\begin{aligned}
\lambda^2(tP - P) &= t^2 \Lambda^2 P - tP^2 + P^2 - \Lambda^2 P \\
&= (t^2-1)\Lambda^2 P - (t-1)P^2 .
\end{aligned}$$

so here is a possibility suppose we have

$$0 < V_1 < \cdots\cdots < V_k$$

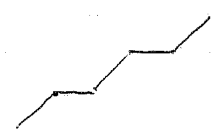
and we think of V_i/V_{i-1} as having eigenvalue x_i

$$x_1 < x_2 < \cdots < x_k$$

Now when I take $\Lambda^2 V$ I get the eigenvalues $x_i + x_j$ $i \leq j$ which I can order lexicographically

$$\{ x_1 + x_1$$

$$\begin{cases} x_1 + x_2 \\ x_2 + x_2 \end{cases}$$



$$\begin{cases} x_1 + x_3 \\ x_2 + x_3 \\ x_3 + x_3 \end{cases}$$

Thus $x_i + x_j < x_{i'} + x_{j'}$ if $j < j'$ or if $j = j'$, $i < i'$.
~~Assume that~~ This will be the case if

$$2x_j \leq x_{j'} \qquad \text{for} \qquad j < j'.$$

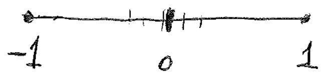
Thus if $x_1 \leq \frac{1}{2} x_2$, $x_2 \leq \frac{1}{2} x_3$, ... these will have the correct ordering.

Thus there should be maps.

$$\lambda_t : K(X; T, *) \longrightarrow 1 + K(X; SP^\infty(T), *)\,[[t]]\,t$$

I have this topological model for $S^1 \otimes bU$, namely it is the relative K-theory of bundles with unitary operator, modulo those with the eigenvalue 1.

Think of



$$\sum x_i P_i \qquad |x_i| < +1$$

$$\sum_{i<j} (x_i + x_j) P_i P_j + \sum_i 2x_i \Lambda^2 P_i$$

$$V(X, S^1) \xrightarrow{\lambda^n} V(X, SP^n(S^1)) \xrightarrow{\sim} V(X, S^1)$$

$$\lambda_t : V(X, S^1) \longrightarrow 1 + V(X, S^1)[[t]]t$$

$$V(X, 1)$$

I want an operator

$$V(X, S^1) \xrightarrow{\lambda^2} V(X, S^1)$$

$$\lambda^2(\alpha + \tau) = \lambda^2\alpha + \alpha\tau + \lambda^2\tau$$

$$V(X; T) \xrightarrow{\lambda_n} V(X, SP^n(T))$$

$$\lambda_t : K(X; T) \longrightarrow 1 + \prod_{n \geq 1} K(X, SP^n(T)).$$

$$\downarrow$$

$$K(X; T, t_0) \qquad 1 + t\, K(X; SP^\infty(T, t_0))[[t]]$$

so given a T-bundle E over X look at ~~$SP^\infty(T, *) = SP^\infty(T)$~~

$\lambda^n(E)$ as a SP^∞ bundle. Now if I take

$$K(X; SP^\infty(T), *)$$

this should be a ring, and it should be so that etc.

Given a vector space V and a "space" T I want to define a space $D(V;T)$ of decompositions with respect to T. In the direct sum situation what I get is a space whose points are direct sum decomp. $V = V_{t_1} \oplus \cdots \oplus V_{t_k}$ indexed by **distinct** points of T.

Now when $T = [0,1]$, I have an ordering on points of T so I can replace a chain

$$V = V_{t_1} \oplus \cdots \oplus V_{t_k} \qquad 0 \le t_1 < \cdots < t_k \le 1$$

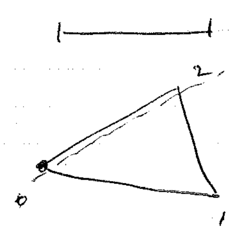
by the flag

$$V_{t_1} < V_{t_1} \oplus V_{t_2} < \cdots < V_{t_1} \oplus \cdots \oplus V_{t_{k-1}}$$

(Somehow the exactness theorem will allow me to subdivide)

Thus what I do is to take the thickened simplex

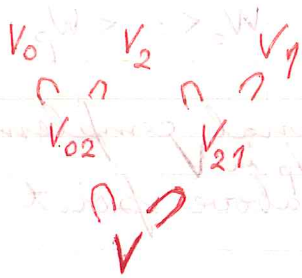
$$0 \le t_1 < \cdots < t_k \le 1$$



$k=1$

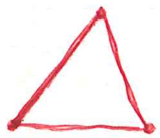
$k=2$

It is clear what the points are and how they specialize

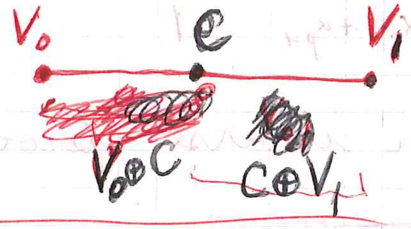


$$0 \to V_2 \to V_{02} \oplus V_{21} \longrightarrow V \longrightarrow 0$$

exact



$$C \oplus V_0 \oplus V_1 = V$$

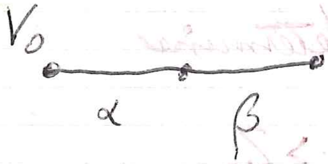


no

~~[illegible]~~ some one gives you $V_0 \subset V \supset V_1$ $\quad V_0 \cap V_1 = 0$
and now you have to tell him where to put the eigenvalues between 0 and 1

eigenvalues are

$V_0 \subset V$



$$V = V_\alpha + V_\beta$$

where $V_0 \cap V_\beta = 0$

$V_\alpha \cap V_1 = 0$

Conversely given a point

$$\sum_{i=0}^{p} \alpha_i W_i \qquad \sum \alpha_i = 1 \qquad W_0 < \cdots < W_p$$

of $B(\{W \leq V\})$ let V_i be the orthogonal complement of W_{i-1} in W_i, $V_0 = W_0$, $V_{p+1} =$ orth comp. of W_p in V. Assoc. to above point the decomposition

$$V = \underset{0}{W_0} \oplus \underset{\alpha_0}{W_1} \oplus \cdots \oplus \underset{\alpha_0 + \cdots + \alpha_{p-1}}{W_p} \oplus \underset{1}{W_{p+1}}$$

Clearly seems to be an inverse procedure.

~~next step is to try to understand the map~~

$$\cancel{D(V, [0,1]) \longrightarrow D(\Lambda^2 V, SP^2[0,1])}$$

$$\cancel{V = \oplus V_{t_i} \longmapsto \Lambda^2 V = \oplus \Lambda^2 V_{t_i} \oplus \bigoplus_{i<j} [V_{t_i} \otimes V_{t_j}]}$$

Piecing together: Try to describe decompositions of V over.

The idea would be that a decomposition would determine flags

$$W_0 < W_1 < \cdots < W_p \,, \qquad Z_q > Z_{q-1} > \cdots > Z_0$$

such that W_p and Z_q would be "orthogonal"

closed

$$Z_{q+1} \geq Z_q > \cdots > Z_0 = W_0 < W_1 < \cdots < W_p \leq W_{p+1}$$

$$Z_{q+1} \cap W_{p+1} = W_0 \qquad Z_{q+1} + W_{p-1} = V.$$

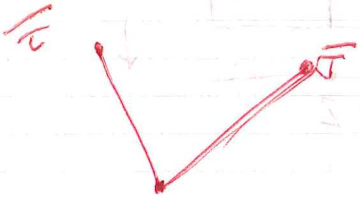
$D^h(V; K)$ consists of $F_2 \to F_2 \perp F_2'$ mod $F_{2 \cap 2'}$.

claim such a ξ is same as an orth. decomp

$$V = \oplus V_\sigma$$

In effect define V_σ = orth. comp. of $V_{\partial\sigma}$ in $V_{\bar\sigma}$

~~To see $V_0 \perp V_2$ induction~~

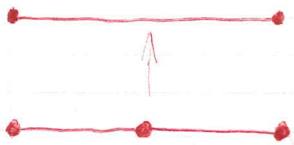


now topologize $D^h(V; K)$ in the obvious way and also define ordering, so that it becomes an ordered space

Subdivision question

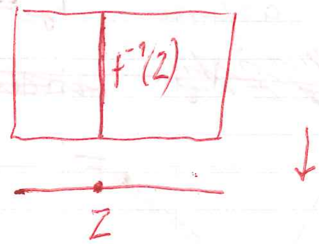
$$L \xrightarrow{f} K, \quad L = K'$$

$$D(V; K') \longrightarrow D(V; K) \quad \text{heq?}$$



$V_0 \subset V_{01} \supset V_1 \qquad V_0 \cap V_1 = 0$

$$f_*(F)(Z) = F_{f^{-1}(Z)}$$



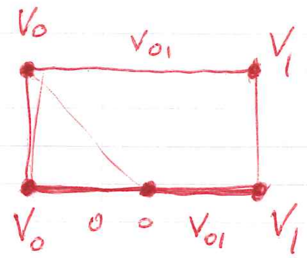
try to do it with a 1-complex

$$K' \longrightarrow K$$

So K' has two kinds of vertices, and the edges are ordered. ~~[illegible] decompos~~

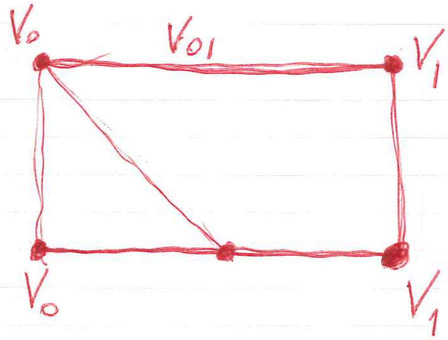
K_e = edges

K_v



Order the vertices in the correct way.

$\cancel{D(V; K') \longrightarrow D(V; K)}$



Question: Modify the definition so that a decomposition of V with respect to K becomes a splitting

$$V = \bigoplus_{\sigma} V_\sigma$$

indexed by the simplices of K. Then put

$$F_Z = \bigoplus_{\sigma \in Z} V_\sigma$$

to get an old style decomposition. ~~[illegible]~~ Not clear how to define the partial ordering except perhaps that we want

$$F_Z \subset F_Z'$$

for each Z and maybe the complements to be in the other direction.

Be more specific. We have an ordered set of ~~[illegible]~~ complemented-suspaces, i.e projectors $E: V \to V$ $E^2 = E$.

Suppose given $\{F_Z\}$, and suppose I am over $\mathbb{C}$, so I can put in a ~~[illegible]~~ metric

---

Suppose V Hilbert space and $D(V;T)$ is as before I will this time consider only the part of ~~$D(V;K)$~~ $D(V;K)$ such that F_Z and $F_{Z'}$ meet orthogonally

~~Example: Let $D(V_3$,~~

Question: Is $D(V; [0,1])$ the simplicial complex **associated to the poset** of subspaces of V?

Map. Given $W \subset V$, send W to the decomposition

$$\begin{array}{ccccc} V = & W & \oplus & (V/W) \\ & 0 & & 1 \end{array}$$

~~[illegible]~~ More generally, given a chain

$$W_0 < \cdots < W_k$$

~~[illegible]~~ map $\Delta(k)$ in by the map

$$0 \le t_1 \le t_2 \le \cdots \le t_k \le 1 \longmapsto \begin{array}{cccccccc} V = & W_0 & + & W_1/W_0 & + \cdots + & W_k/W_{k-1} & + & V/W_k \\ & 0 & & t_1 & & t_k & & 1 \end{array}$$

~~Next suppose one tries to understand what happens~~

Any element of $D(V; [0,1])$ is of the form

$$\begin{array}{cccccccccc} V = & Q_0 & \oplus & Q_1 & \oplus & Q_2 & & \oplus Q_k & \oplus & Q_{(1)} \\ & 0 & < & t_1 & < & t_2 & \cdots & < t_k & < & 1. \end{array}$$

$V = 1.$

$$Q_0 < Q_0 + Q_1 < \cdots < Q_0 + \cdots + Q_k$$

~~So a point of $D(V;[0,1])$ is first a sequence~~

~~$0 \le t_1 < \cdots < t_k \le 1$~~

~~then a flag $0 < V_1 < \cdots < V_k = V$.~~

~~Thus it is a monotone map from $[0,1]$ into subspaces of V which ends at V. $t \mapsto V_t$~~

So I am trying to describe $D(V;T)$ as the realization of a simplicial space. We have a description of a point as a pair

$0 < V_1 \cdots < V_k = V$

$0 \le t_1 < \cdots < t_k \le 1.$ ← break up into

$0 < t_1 < \cdots < t_k < 1$ $\quad \Delta(k)$

$\cup \quad 0 = t_1 < t_2 < \cdots < t_k < 1$ $\quad \Delta(k-1)$

$\cup \quad 0 < t_1 < t_2 < \cdots < t_k = 1$ $\quad \Delta(k-1)$

$\cup \quad 0 = t_1 < \underline{t_2} < \cdots < \underline{t_{k-1}} < t_k = 1$ $\quad \Delta(k-2)$

so therefore a given flag of length k contributes 4 simplices to $D(V;T)$

$a(V_1 < \cdots < V_{k-1})$ $\quad$ k simp.

b $\quad$ $k-1$

c $\quad$ $k-1$

d $\quad$ $k-2$

clear how to define ~~[illegible]~~ $d_i\, a(V_1 < \cdots < V_{k-1})$ $\quad 1 \le i \le k-1$

$d_0\, a(V_1 < \cdots < V_{k-1}) = b(V_1 < \cdots < V_{k-1})$

$d_k\, a(V_1 < \cdots < V_{k-1}) = c(V_1 < \cdots < V_{k-1})$

The reason I wanted this is to decide what is going to happen when I take ~~operator~~ λ- operations on a filtered object. Suppose I have a vector space V decomposed according to S^1, i.e.

$$V = \bigoplus_{z \in S^1} V_z$$

Then $\Lambda^n V$ is decomposed according to $SP^n(S^1)$.

$$\Lambda^n V = \bigoplus \cdots$$

i.e. if
$$V = V_{z_1} \oplus \cdots \oplus V_{z_k}$$

$$\Lambda^2 V = \Lambda^2 V_{z_1} \oplus \cdots \oplus \Lambda^2 V_{z_k} \oplus \bigoplus_{i<j} V_{z_i} \otimes V_{z_j}$$

and for my purposes I will identify this $SP^n(S^1)$ with S^1 via the addition, which means that I get the eigenvalue sequence

$$2z_1, \ldots, 2z_k, \quad z_i + z_j \qquad i<j.$$

Thus if I think of V as having a unitary operator Θ, then $\Lambda^n V$ carries the operator $\Lambda^n \Theta$. But this seems to screw up the eigenvalues

I have the Q-category which maybe classifies thing I can recognize over S^1.

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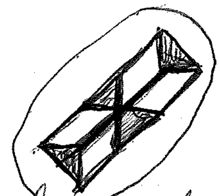
self-adjoint Fredholm operators. — Sequence of decreasing strata. So over a space X we would be looking at a stratification

$$X_0 \cup X_1 \cup X_2 \cup \cdots$$



where maybe $X_p \cup X_{p+1} \cup \cdots$ is closed, and where over X_p we give a bundle E_p of rank p, and on specializing from X_p to X_q $p<q$ one gives a Q-morphism from E_p into E_q.

So think of this thing as arising from a map of X into self-adjoint Fredholm operators, transversal to the strata.

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self-adjoint Fredholm operator. ~~Suppose I have a certain set of sheaves~~

$\mathcal{F}$ = self adj. Fred. operators ess. spec ± 1.
let me look only at the part of the spectrum which sits in $(-\varepsilon, \varepsilon)$, whence one has for every Fredholm operator A a sequence

$$t_1 P_1 + \cdots\cdots + t_q P_q$$

$$-\varepsilon < t_1 < \cdots < t_q < \varepsilon$$

by looking at the eigenspaces between $-\varepsilon$ and ε.
Make this infinitesimal somehow so one can't separate P_1 from $P_1 \oplus P_2$ etc.

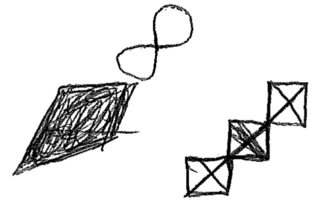
To give a Hilbert vector space V with $\mathbb{C}^{T^*}$ action is same as splitting V up according to the points of T.

But what would be the analogues in the exact case?

~~Suppose then that it doesn't work~~.

E $\quad \mathbb{C}^T$ structure $\quad T$ one-dimensional

$$\boxed{E_0 \to E_1 \to E_2 \to E_3}$$



k-homology of T

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discrete case: Segal's method. $\quad X \wedge P \quad$ ~~generators~~

$X^n \times P^n$ $\quad$ element of this is a sequence $x_1, x_2, \dots, x_n$ and $P_1 \cdots P_n \in P$

think of as a chain $\sum P_i x_i$

so we have

$$X^n \times P^n \longrightarrow X \wedge P$$

and if one has a map $n \xrightarrow{\varphi} m$ then one gets maps

$$X^n \longleftarrow X^m$$

$$\coprod_n P \longrightarrow \coprod_m P$$

$$\varphi_*(P_i)_j = \bigoplus_{i \in \varphi^{-1}(j)} P_i$$

hence one can form the ~~tensor product~~ contraction and one gets a space. For example if P is an abelian group it would seem that we are getting exactly the monoid of chains on X with coeff. in P.

August 15, 1974

Claim $D(V; [0,1])$ = simp. cx. associated to the poset of subspaces of V.

A point of $D(V; [0,1])$ is a decomposition

$$V = \oplus V_\lambda$$

indexed by points λ, $0 \le \lambda \le 1$. Let the points λ such that $V_\lambda \neq 0$ and such that $0 < \lambda < 1$ be arranged in order $- \; 0 < t_1 < \cdots < t_k < 1$, and put

$$W_i = V_0 \oplus V_{t_1} \oplus \cdots \oplus V_{t_i} \qquad i = 0, \ldots, k.$$

where $W_0 = V_0$. Then

$$W_0 < W_1 < \cdots < W_k$$

is a k-simplex in the poset of subspaces of V. ~~Thus to each point of $D(V,[0,1])$ we have associated a k-simplex in the poset with~~ And

$$\sum_{i=0}^{k} (t_{i+1} - t_i) W_i \qquad (t_0 = 0, t_{k+1} = 1).$$

is a point in the simp. complex $B(\{W \le V\})$. Claim the map

$$D(V; [0,1]) \longrightarrow B(\{W \le V\})$$

is continuous. check compatibility of faces: Suppose we let our point approach a boundary, i.e. cases:

d_0: $t_1 \searrow 0$ corres. to $W_0 < \cdots < W_k \longmapsto W_1 < \cdots < W_k$

d_j: $t_{j+1} - t_j \searrow 0$ $\qquad W_0 < \cdots < W_k \longmapsto W_0 < \cdots < \hat{W_j} < \cdots < W_k$

d_k: $t_k \nearrow 1$ $\qquad W_0 < \cdots < W_k \longmapsto W_0 < \cdots < W_{k-1}$.