JAN 74

The Stability thm. for $GL(k)$, $k = \mathbb{F}_q$

1st two pages bleached.

Cor: $|P(V,V_0)|$ is spherical of dim $= \dim(V_0)-1$.

Proof: Take Z to be a line L in V, where $P(V,L)$ is the set of hyperplanes complementary to L. Then $|P(V,V_0)|$ is the join of $|P(V/L, V_0/L)|$ and a set, so one wins by induction.

Put
$$J(V,V_0) = \tilde{H}_{s-1}(P(V,V_0)) \qquad s = \dim(V_0).$$

It is a free abelian group. One gets from the lemma a can. isom
$$J(V,V_0) \cong J(V/Z, V/Z_0) \otimes J(V,Z).$$

When $k = \mathbb{F}_q$, the number of spheres in $P(V,V_0)$ is $(q^{n-1}-1)\cdots(q^{n-s}-1)$, $n=\dim(V)$, $s=\dim(V_0)$.

$$\{W \leq V \mid W+V_0 = V\} = \bar{P}(V,V_0)$$

Now filter ~~$\bar{P}(V,V_0)$~~ according to $d(W) = \dim(W) - \dim(V/V_0) = \dim(W \cap V_0)$. Since for $Z \in P(V,V_0)$ one has

$$\cancel{\{W < Z \mid W \in P(V,V_0)\}}$$
$$\|$$
$$\{W<Z \mid W+V_0 = V\} \cong \{W<Z \mid W+(Z\cap V_0) = Z\} = P(Z, Z\cap V_0).$$

which is a bouquet of spheres of dimension $\dim(Z\cap V_0)-1 = d(Z)-1$. Thus if we set $F_p\{\bar{P}(V,V_0)\} = \{W \mid \dim(W\cap V_0) \leq p\}$ we have that $H_*(F_p\bar{P}/F_{p-1}\bar{P})$ will be concentrated in degree p.

In fact

$$H_{p+q}(F_p\bar{P}/F_{p-1}\bar{P}) = \bigoplus_{d(Z)=p} H_{p+q}\left(\{W\leq Z \mid W+\overset{Z_1}{V_0}=Z\}, \{W<Z \mid W+Z_1V_0=Z\}\right)$$

$$= \begin{cases} 0 & q\neq 0 \\ \bigoplus_{d(Z)=p} J(Z, Z\cap V_0) & \end{cases}.$$

Since $\bar{P}$ is contractible, the spec. sequence

$$E^1_{pq} = H_{p+q}(F_p\bar{P}/F_{p-1}\bar{P}) \Rightarrow H_{p+q}(\bar{F})$$

degenerates into an exact sequence

$$\cdots \longrightarrow E^1_{p0} \longrightarrow E^1_{p-1,0} \longrightarrow \cdots\cdots \longrightarrow E^1_{0,0} \longrightarrow \mathbb{Z} \longrightarrow 0$$

i.e. we get an exact sequence naturally assoc. to (V, V_0):

$$\cdots \longrightarrow \bigoplus_{\substack{Z+V_0=V \\ \dim(Z\cap V_0)=1}} J(Z, Z\cap V_0) \longrightarrow \bigoplus_{\substack{Z+V_0=V \\ \dim(Z\cap V_0)=0}} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Now we use the above sequence to get a spectral sequence. Let $GL''(V, V_0)$ ~~denote the~~ $= \{\alpha\in GL(V) \mid \alpha(V_0)=V_0$ and $\alpha = id$ on $V/V_0\}$ act on the above sequence. Then $GL''(V,V_0)$ acts transitively on the element of $\bar{P}(V,V_0)$ of the same height: In effect if $W+V_0=V$, $\dim(W\cap V_0)=p$, then we can choose a complement C to V_0 in V $\ni$ $C\subset W$, whence $W = C\oplus W\cap V_0$; since $GL''(V,V_0) = GL(V_0)\tilde{\times}\operatorname{Hom}(C,V_0)$ ~~we can move~~ any two complements of V_0 in V are $GL''(V,V_0)$-conjugate and the stabilizer of any of them which is isom. to $GL(V_0)$ acts transitively on the subspaces of dimension p.

Suppose $V = k^\alpha + k^s = (ke_1 + \cdots + ke_\alpha) + (ke_{\alpha+1} + \cdots + ke_{\alpha+s})$

$V_0 = 0 + k^s$

so that

$$GL''(V,V_0) = \left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_s \end{array}\right)$$

Now take $Z = k^\alpha + k^p = ke_1 + \cdots + ke_{\alpha+p}$. The stabilizer of Z in $GL''(V,V_0)$ is

$$\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_s \end{array}\right) \cap \left(\begin{array}{c|c} GL_{\alpha+p} & * \\ \hline 0 & GL_{s-p} \end{array}\right) = \left(\begin{array}{c|c|c} 1_\alpha & 0 & 0 \\ \hline * & GL_p & * \\ \hline 0 & 0 & GL_{s-p} \end{array}\right)$$

~~and since~~ so

$$\bigoplus_{\substack{Z+V_0=V \\ \dim(Z\cap V_0)=p}} J(Z, Z\cap V_0) = \mathbb{Z}\begin{bmatrix} 1_\alpha & \\ * & GL_s \end{bmatrix} \otimes_{\mathbb{Z}\begin{bmatrix} 1_\alpha & 0 & 0 \\ * & GL_p & * \\ 0 & 0 & GL_{s-p} \end{bmatrix}} J(k^\alpha \oplus k^p, k^p).$$

Thus I get a spectral sequence

$$E^1_{pq} = H_q\left(\begin{pmatrix} 1_\alpha & 0 & 0 \\ * & GL_p & * \\ 0 & 0 & GL_{s-p} \end{pmatrix}, J(k^\alpha \oplus k^p, 0 \oplus k^p)\right) \Longrightarrow H_{p+q}\begin{pmatrix} 1_\alpha & 0 \\ * & GL_s \end{pmatrix}$$

whose edge homo $E^1_{0q} \to H_q$ is the map

$$H_q\begin{pmatrix} 1_\alpha & \\ & GL_s \end{pmatrix} \longrightarrow H_q\begin{pmatrix} 1_\alpha & \\ * & GL_s \end{pmatrix}$$

which is always injective as there is an evident retraction.

$s=1$. Here one has the exact sequence

$\cancel{0 \longrightarrow J(k^\alpha + k^1, 0+k^1) \longrightarrow \oplus}$

$$0 \longrightarrow J(V, L) \longrightarrow \bigoplus_{H \oplus L = V} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and the spectral sequence in question is the long exact sequence

$$H_{q+1}\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_1 \end{array}\right) \longrightarrow H_q\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_1 \end{array}, J(k^\alpha + k^1, 0+k^1)\right) \longrightarrow H_q\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline 0 & GL_1 \end{array}\right) \longrightarrow H_q\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_1 \end{array}\right)$$

<u>Prop</u>: If card$(k) > 2$, then $H_0\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_1 \end{array}, J(k^\alpha + k^1, 0+k^1)\right) = 0$.

<u>Proof</u>: We have

$$H_1\left(\begin{array}{c|c} 1_\alpha & \\ \hline & GL_1 \end{array}\right) \longrightarrow H_1\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_1 \end{array}\right) \longrightarrow H_0\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_1 \end{array}, J(k^\alpha + k^1, 0+k^1)\right) \to 0$$

Now from the extension

$$1 \longrightarrow \underset{k^\alpha}{\overset{*}{\shortparallel}} \longrightarrow \begin{pmatrix} 1_\alpha & 0 \\ * & GL_1 \end{pmatrix} \longrightarrow GL_1 \longrightarrow 1$$

One gets $H_1\begin{pmatrix} 1_\alpha & \\ * & GL_1 \end{pmatrix} = H_0(GL_1, k^\alpha) \oplus H_1(GL_1)$.

But $H_0(GL_1, k^\alpha) = k^\alpha \Big/ \{\sum_i (\lambda_i - 1) v_i \mid \lambda_i \in k^*, v_i \in k^\alpha\}$

$$= \begin{cases} k^\alpha & \text{if } k = \mathbb{F}_2 \\ 0 & \text{if } k \neq \mathbb{F}_2. \end{cases}$$

<u>Prop:</u> If $\mathrm{card}(k) > 2$, then $H_0\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_s \end{array}, J(k^\alpha + k^s, 0 \oplus k^s)\right) = 0$, $s \geq 1$

<u>Proof:</u> Use ~~induction on s, and~~ the isomorphism
$$J(V, V_0) = J(V/L, V_0/L) \otimes J(V, L)$$
Then $H_0(GL''(V,V_0), J(V,V_0))$ is a quotient of $H_0(G, J(V,V_0))$ for any subgroup $G \subset GL''(V,V_0)$. So take $G = GL''(V,L) \subset GL''(V,V_0)$:
$$\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_s \end{array}\right) \supset \left(\begin{array}{c|c} 1_{\alpha+s-1} & 0 \\ \hline * & 1 \end{array}\right)$$
Then $GL''(V,L)$ acts trivially on $J(V/L, V_0/L)$, so
$$H_0(GL''(V,L), J(V,V_0)) = J(V/L, V_0/L) \otimes H_0(GL''(V,L), J(V,L))$$
and $H_0(GL''(V,L), J(V,L))$ is zero as we have already shown.

Suppose now that we assume k is such that
$$H_*\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_1 \end{array}\right) = H_*\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline 0 & GL_1 \end{array}\right)$$
for example

$\mathrm{char}(k) = 0$ or

$\mathrm{char}(k) = p > 0$ and
i) ignore p-torsion
ii) k contains arbitrarily large finite subfield.

Then I want to show that

<u>Thm:</u> $H_*\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_s \end{array}, J(k^\alpha + k^s, 0 + k^s)\right) = 0$ for $s \geq 1$.

Proof: ~~Consider the~~ Proceed by induction on s. For $s=1$ we have the long exact sequence on page 6 which gives the result. Now note that since we know for any k-vector space V that

$$H_*(k^* \tilde{\times} V) \xrightarrow{\sim} H_*(k^*)$$

it follows that

$$H_*\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_s \end{array}\right) = H_*\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline 0 & GL_s \end{array}\right)$$

for any α, s. So when I consider the spectral sequence

$$E^1_{pq} = H_q\left(\left(\begin{array}{c|c|c} 1_\alpha & 0 & 0 \\ \hline * & GL_p & * \\ \hline 0 & 0 & GL_{n-p} \end{array}\right), J(k^\alpha + k^p, 0 + k^p)\right) \Rightarrow H_{p+q}\left(\begin{array}{c|c} 1_\alpha & 0 \\ \hline * & GL_n \end{array}\right)$$

we can write the E^1 term as

$$H_q\left(\begin{pmatrix} 1_\alpha & 0 & \\ * & GL_p & \\ & & GL_{n-p} \end{pmatrix}, J(k^\alpha + k^p, 0 + k^p)\right)$$

$$\Uparrow$$

$$H_*\left(GL_{n-p}, H_*\left(\begin{pmatrix} 1_\alpha & 0 \\ * & GL_p \end{pmatrix}, J(k^{\alpha+p}, k^p)\right)\right)$$

so arguing by induction this will be zero for $1 \leq p < s$. Now take $n = s$ and then

$$E^1_{s*} = H_*\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_s \end{array}, J(k^{\alpha+s}, k^s)\right)$$

$$E^1_{0*} = H_*(GL_s)$$

$$\text{abut: } H_* = H_*\left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_s \end{array}\right)$$

and the other columns are zero. Since $E^1_{0*} \simeq$ abut done.

Now observe that we have a map

$$P(V, V_0) \longrightarrow P(V \oplus k, V_0 \oplus k)$$
$$W \longmapsto W$$

compatible with filtration:

$$W \cap (V_0 \oplus k) = W \cap V_0 \qquad \text{if} \quad W \subset V$$

hence this map will induce a map of exact sequences

$$\begin{array}{c}
\cdots \longrightarrow \displaystyle\bigoplus_{\substack{W + V_0 = V \\ \dim(W \cap V_0) = p}} J(W, W \cap V_0) \longrightarrow \cdots\cdots \\
\Big\downarrow \\
\longrightarrow \displaystyle\bigoplus_{\substack{W + (V_0 \oplus k) = V \oplus k \\ \dim(W \cap (V_0 \oplus k)) = p}} J(W, W \cap (V_0 + k)) \longrightarrow \cdots
\end{array}$$

and this is evidently compatible with the homomorphism $GL''(V, V_0) \longrightarrow GL''(V \oplus k, V_0 \oplus k)$.

If $V = k^\alpha + k^n$, $V_0 = 0 + k^n$

$$GL''(V, V_0) = \begin{pmatrix} 1_\alpha & \\ * & GL_n \end{pmatrix}$$

$$GL''(V \oplus k, V_0 \oplus k) = \left(\begin{array}{c|c} 1_\alpha & \\ \hline * & GL_{n+1} \end{array}\right).$$

Representative for W of height p is

$$k^\alpha + k^p = ke_1 + \cdots + ke_{\alpha+p}$$

works for both the n, $n+1$ cases. Thus I will have a relative spectral sequence:

$$E^1_{st}(n) = \begin{cases} H_t\left(\begin{vmatrix} 1_\alpha & & \\ * & GL_s & * \\ & & GL_{n+1-s} \end{vmatrix}, \begin{vmatrix} 1_\alpha & & \\ * & GL_s & * \\ & & GL_{n-s} \end{vmatrix} ; J(\alpha+s, s) \right) & \text{for } 0 \leq s \leq n \\ H_t\left(\begin{vmatrix} 1_\alpha & 0 \\ * & GL_{n+1} \end{vmatrix}, J(\alpha+n+1, n+1) \right) & s = n+1 \\ 0 & s > n+1. \end{cases}$$

and this converges to $H_*\left(\begin{vmatrix} 1_\alpha & 0 \\ * & GL_{n+1} \end{vmatrix}, \begin{vmatrix} 1_\alpha & 0 \\ * & GL_n \end{vmatrix} \right)$

Statements to be proved by induction:

$$I_i : \forall \alpha, \; H_i\left(\begin{vmatrix} 1_\alpha & 0 \\ 0 & GL_{n+1} \end{vmatrix}, \begin{vmatrix} 1_\alpha & 0 \\ 0 & GL_n \end{vmatrix} \right) \xrightarrow{\sim} H_i\left(\begin{vmatrix} 1_\alpha & 0 \\ * & GL_{n+1} \end{vmatrix}, \begin{vmatrix} 1_\alpha & 0 \\ * & GL_n \end{vmatrix} \right)$$

for $n+1 \geq i$

$$II_i : \forall \alpha \geq 0, \; H_i\left(\begin{vmatrix} 1_\alpha & 0 \\ * & GL_{n+1} \end{vmatrix}, \begin{vmatrix} 1_\alpha & 0 \\ * & GL_n \end{vmatrix} \right) = 0 \qquad n \geq i$$

Clearly II_0 is true, and I_0 provided that when $n=0$ we interpret the empty group in the obvious way.

Now let me check I_1. Using the spectral sequence, I have to show that

$$E^1_{01} = H_1\left(\begin{vmatrix} 1_\alpha & 0 \\ * & GL_{n+1}\end{vmatrix}, \begin{vmatrix} 1_\alpha & 0 \\ * & GL_n\end{vmatrix}\right)$$

maps onto the abutment in degree 1:

$$H_1\left(\begin{vmatrix} 1_\alpha & \\ * & GL_{n+1}\end{vmatrix}, \begin{vmatrix} 1 & \\ * & GL_n\end{vmatrix}\right).$$

for $n \geq 0$. It suffices to show that E^1_{10} is zero. One has

$$E^1_{10} = H_0\left(\begin{vmatrix} 1_\alpha & 0 & 0 \\ * & GL_1 & * \\ & & GL_n\end{vmatrix}, \begin{vmatrix} 1_\alpha & 0 & 0 \\ * & GL_1 & * \\ & & GL_{n-1}\end{vmatrix}, J(\alpha+1, 1)\right)$$

for $n \geq 1$. To show 0, one examines

$$H_i\left(\begin{vmatrix} 1_\alpha & 0 \\ * & GL_1\end{vmatrix}, J(\alpha+1,1) \otimes \underbrace{H_j\left(\begin{vmatrix} 1_\alpha & 1_1 & * \\ & & GL_n\end{vmatrix}, \begin{vmatrix} 1_\alpha & 1_1 & * \\ & & GL_{n-1}\end{vmatrix}\right)}\right)$$

for $i+j=0$. For ~~[illegible]~~ $j=0$ the term ↑ is zero.

For $n=0$,

$$E^1_{10} = H_0\left(\begin{matrix} 1_\alpha & \\ * & GL_1\end{matrix}, J(\alpha+1, 1)\right)$$

and we've seen this is zero. Thus we have proved I_1.

Now I want to show that

$$I_i, II_i \quad \text{for } i < q \implies I_q.$$

In the spectral sequence if $n+1 \geq q$ I have to show $E^1_{0q} = H_q\left(\begin{matrix}1_\alpha \\ & GL_{n+1}\end{matrix}, \begin{matrix}1_\alpha \\ & GL_n\end{matrix}\right)$ goes onto abut $H_q\left(\begin{matrix}1_\alpha \\ * & GL_{n+1}\end{matrix}, \begin{matrix}1_\alpha \\ * & GL_n\end{matrix}\right)$. It's enough to show $E^1_{st} = 0$ for $s+t = q$, $s \geq 1$.

$$E^1_{st}(n) = H_t\left(\begin{vmatrix} 1_\alpha & & \\ * & GL_s & * \\ & & GL_{n+1-s}\end{vmatrix}, \begin{vmatrix} 1_\alpha & & \\ * & GL_s & * \\ & & GL_{n-s}\end{vmatrix}, J(\alpha+s, s)\right)$$

for $1 \leq s \leq n$. If $s+t = q \leq n+1$, then $s \leq n$ unless $q = n+1$, $t=0$ in which case $E^1_{n+1,0} = H_0\left(\begin{matrix}1_\alpha \\ * & GL_{n+1}\end{matrix}, J(\alpha+n+1, n+1)\right)$ and we've seen this is zero for $n \geq 0$ (OKAY). So can forget this case

To show $E^1_{st}(n) = 0$ for $s+t = q \leq n+1$, $1 \leq s \leq n$, ~~[illegible]~~ use spec. seq. which tells us it is enough to show

$$H_i\left(\left|\begin{matrix}1_\alpha & \\ * & Gl_s\end{matrix}\right|, J(\alpha+s,s) \otimes H_j\left(\left|\begin{matrix}1_\alpha & & \\ & 1_s & * \\ & & Gl_{n+1-s}\end{matrix}\right|, \left|\begin{matrix}1_\alpha & & \\ & 1_s & * \\ & & Gl_{n-s}\end{matrix}\right|\right)\right)$$

is zero for $i+j = t$.

If $i > 0$, then $j < t \leq n+1-s$, and also $j < t \leq q$ so I_j, II_j give

$$H_j\left(\left|\begin{matrix}1_\alpha & & \\ & 1_s & * \\ & & Gl_{n+1-s}\end{matrix}\right|, \left|\begin{matrix}1_\alpha & & \\ & 1_s & * \\ & & Gl_{n-s}\end{matrix}\right|\right) = H_j(Gl_{n+1-s}, Gl_{n-s})$$
$$= 0$$

(Here I use the fact that ~~[illegible]~~ inverse transpose ~~[illegible]~~ is an isom of

$$\left|\begin{matrix}1_\alpha & \\ * & Gl_{n+1}\end{matrix}\right|, \left|\begin{matrix}1_\alpha & \\ * & Gl_n\end{matrix}\right| \simeq \left|\begin{matrix}1_\alpha & * \\ & Gl_{n+1}\end{matrix}\right|, \left|\begin{matrix}1_\alpha & * \\ & Gl_n\end{matrix}\right|.\ )$$

If $i = 0$, then $j = t = n+1-s$, ~~[illegible]~~ and $j = t < q$ (as $s \geq 1$). Thus I_j gives

$$H_j\left(\begin{matrix}1_\alpha & & \\ & 1_s & * \\ & & Gl_{n+1-s}\end{matrix}\ \ \begin{matrix}1_\alpha & & \\ & 1_s & * \\ & & Gl_{n-s}\end{matrix}\right) \cong H_j(Gl_{n+1-s}, Gl_{n-s})$$

and ~~[illegible]~~ so we see that $\left(\begin{matrix}1_\alpha & \\ * & Gl_s\end{matrix}\right)$ acts trivially on this. So now done because

$$H_0\left(\begin{matrix}1_\alpha & \\ * & Gl_s\end{matrix}, J(\alpha+s,s) \otimes A\right) = H_0\left(\begin{matrix}1_\alpha & \\ * & Gl_s\end{matrix}, J(\alpha+s,s)\right) \otimes A$$
$$= 0,$$

if A has trivial action.

Next I want to show

$$\mathrm{I}_i \text{ for } i \le g,\ \mathrm{II}_i \text{ for } i < g \implies \mathrm{II}_g$$

Here use spec. sequence

$$E^1_{st} = H_t\left(\left(\begin{array}{c} Gl_s * \\ Gl_{n+1-s} \end{array} \right) \Big| \begin{array}{c} Gl_s * \\ Gl_{n-s} \end{array} \Big| ; I(s) \right) \qquad 0 \le s \le n$$

$$= H_t\left(Gl_{n+1}, I(n+1) \right) \qquad s = n+1$$

$$= 0 \qquad s > n+1$$

which converges to zero. I will first show that $E^1_{1g} \twoheadrightarrow E^1_{0g}$ assuming that $n \ge g+1$. As the spectral seq. converges to zero it suffices to show that $E^1_{st} = 0$ for $s+t = g+2$, $t < g$. As $s \le g+2 \le n+1$, we can have $s = n+1$ only if $t = 0$, ~~n+1 = g+2~~ and $s = g+2 = n+1$, in which case $E'_{0,n+1} = H_0(Gl_{n+1}, I(n+1)) = 0$, ~~because suppose 2 < s ≤~~ (as $n+1 = g+2 \ge 3$).

So can assume $2 \le s \le n$. By spec. seq have to show

$$H_i\left(Gl_s, I(s) \otimes H_j\left(\begin{array}{c} 1_s * \\ Gl_{n+1-s} \end{array}, \begin{array}{c} 1_s * \\ Gl_{n-s} \end{array} \right) \right) = 0$$

for $i+j = t$. If $i > 0$, then $j = t - i < t = g+2-s \le g$ as $s \ge 2$, and $j < g+2-s \le n+1-s$. Thus $j < g$, $j \le n-s$ so

$$H_j\left(\begin{array}{c} 1_s * \\ Gl_{n+1-s} \end{array}, \begin{array}{c} 1_s * \\ Gl_{n-s} \end{array} \right) \overset{\mathrm{I}_j}{=} H_j(Gl_{n+1-s}, Gl_{n-s}) \overset{\mathrm{II}_j}{=} 0$$

If $i = 0$, then $j \le g$, $j \le n+1-s$ so I_j say

$$H_j\left(\begin{array}{c} 1_s * \\ Gl_{n+1-s} \end{array} \Big| \begin{array}{c} 1_s * \\ Gl_{n-s} \end{array} \right) = H_j(Gl_{n+1-s}, Gl_{n-s})$$

and GL_s acts trivially, so again get 0, as

$$H_0(GL_s, I(s) \otimes A) = 0 \qquad s \geq 2.$$

Now we have proved:

$$\begin{array}{ccc} E^1_{1q} & \hookrightarrow & E^1_{0q} \\ \| & & \| \\ H_q\left(\begin{pmatrix} GL_1 & * \\ & GL_n \end{pmatrix}, \begin{pmatrix} GL_1 & * \\ & GL_{n-1} \end{pmatrix}\right) & & H_q(GL_{n+1}, GL_n) \end{array}$$

From the spec. seq.

$$E_{ij} = H_i\left(GL_1, H_j\left(\begin{pmatrix} 1 & * \\ & GL_n \end{pmatrix}, \begin{pmatrix} 1 & * \\ & GL_{n-1} \end{pmatrix}\right)\right) \Rightarrow E^1_{1q}$$

$$\underset{j<q \text{ and } j \leq n-1}{\neq 0} \qquad I_{<q}, II_{<q}$$

the "fibre" is zero in degrees $< q$, so

$$\begin{array}{c} H_0\left(GL_1, H_q\left(\begin{pmatrix} 1 & * \\ & GL_n \end{pmatrix}, \begin{pmatrix} 1 & * \\ & GL_{n-1} \end{pmatrix}\right)\right) \xrightarrow{\sim} H_q\left(\begin{pmatrix} GL_1 & * \\ & GL_n \end{pmatrix}, \begin{pmatrix} GL_1 & * \\ & GL_{n-1} \end{pmatrix}\right) \\ \| \; II_q \\ H_0\left(GL_1, H_q\left(\begin{pmatrix} 1 & 0 \\ 0 & GL_n \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & GL_{n-1} \end{pmatrix}\right)\right) \\ \| \\ H_q\left(\begin{pmatrix} 1 & 0 \\ 0 & GL_n \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & GL_{n-1} \end{pmatrix}\right). \end{array}$$

Thus we find that

$$H_q\left(\begin{pmatrix} 1 & 0 \\ 0 & GL_n \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & GL_{n-1} \end{pmatrix}\right) \hookrightarrow H_q(GL_{n+1}, GL_n)$$

for $q \leq n-1$. Conclude

$$H_q(GL_n, GL_{n-1}) \hookrightarrow H_q(GL_{n+1}, GL_n)$$

where the embedding is given by $A \mapsto 1 \oplus A$.

Now one uses ~~the diagram~~

<u>Lemma:</u> $H_q(Gl_n, Gl_{n-1}) \to H_q(Gl_{n+2}, Gl_{n+1})$ is zero, the map being induced by $A \mapsto I_2 \oplus A$.

<u>Proof:</u>

$$\begin{tikzcd}
 & H_q(Gl_n, Gl_{n-1}) \arrow[r, "\partial"] \arrow[d] \arrow[dl, dotted] & H_{q-1}(Gl_{n-1}) \arrow[r, "(A\mapsto A\oplus 1)_*"] \arrow[d, "(A\mapsto 1\oplus A)_*"] & H_{q-1}(Gl_n) \arrow[d] \\
H_q(Gl_{n+1}) \arrow[r] & H_q(Gl_{n+1}, Gl_n) \arrow[r, "\partial"] & H_{q-1}(Gl_n) \arrow[r] & {}
\end{tikzcd}$$

Since $A \mapsto A \oplus 1$ is conjugate to $A \mapsto 1 \oplus A$, these induce the same map on H_{q-1}. $\therefore$ dotted arrow exists, and going one more step one gets zero.

$\therefore$ We get $H_q(Gl_n, Gl_{n-1}) = 0$ $\quad q \leq n-1$
which is II_q. And we have proved

<u>Thm:</u> k field $\neq \mathbb{F}_2$ then

$$H_q(Gl_q) \twoheadrightarrow H_q(Gl_{q+1}) \xrightarrow{\sim} H_q(Gl_{q+2}) \xrightarrow{\sim} \cdots$$

<u>January 3, 1974</u>

<u>Stability problem</u>. It is now necessary to write out proofs for those results which you think you can prove.

1). A finite commutative ring $\Longrightarrow$ $K_i(A)$ finite

2). A finite R-algebra where R is a finitely generated comm. alg. over an infinite field k, then $H_i(GL_n(A), \mathbb{Z})$ stabilizes as a function of n.

3). Let A be a Dedekind domain, and let p be a prime number which is a unit in A. Then $H_i\left(GL_n(A), \begin{Bmatrix}\mathbb{F}_p\\ \mathbb{Q}\end{Bmatrix}\right)$ stabilizes.

<u>January 3, 1974</u> Stability.

What is the stability problem in alg. K-theory?

The K-theory of a ring A is constructed out of the category $P(A)$ of fin. gen. proj. A-modules, ~~their isomorphisms, and their exact sequences~~ and their isomorphisms. The classifying space for the theory, namely $K_0A \times BGL(A)^+$, is built up from an infinite number of objects. The stability problem in general terms consists in showing that on restricting the K-theory to spaces of dim $\leq m$, one needs only use projective modules of rank $\leq \varphi(m)$.

<u>Example</u> of K_0A: In what sense does every element of K_0A come from projective modules of rank $\leq N$? Possible interpretations:

(i) $\{[P] \mid rg(P) \leq N\}$ is a set of generators for K_0A.

(ii) Every element of $\tilde{K}_0A = \operatorname{Ker}\{K_0A \xrightarrow{rg} H^0(\operatorname{Sp} A, \mathbb{Z})\}$ is of the form $[P] - [A^n]$ with $rg(P) = n \leq N$.

Formulation (i) is perhaps natural from the point of view of the Q-category and schemes.

However it is (ii) that I should concentrate on now, for the following reasons:

1) Ultimately I think you want to construct

the γ-filtration of the K-theory, and especially the theory of a fixed weight. Therefore as a start you must be able to construct the part of filtration ≥ 1 which means we must kill the rank. This seems to mean that the objects of interest are things of the form $[P]-[A^n]$, $rg(P)=n$.

2) The proof of Serre's theorem tends to show that the basic geometric part of stability is splitting off a trivial line bundle. For example: if I_n is the set of iso classes of P of rank n, then one has an inductive system

$$I_n \longrightarrow I_{n+1} \longrightarrow \cdots\cdots$$

with limit $\tilde{K}_0(A)$, and Serre's thm. says $I_n \to \tilde{K}_0(A)$ is onto, while Bass's thm. says $I_n \to \tilde{K}_0(A)$ is injective for n large.

Let us do this discussion a bit more carefully. Suppose that I let S = set of iso. classes of $P(A)$. Assume A connected (commutative) so that one has then homos. of monoids

$$\mathbb{N} \longrightarrow S \xrightarrow{rg} \mathbb{N}$$
$$n \longmapsto [A^n]$$

with composition = the identity. If $S_n = rg^{-1}(n)$, then we have the inductive system

$$\longrightarrow S_n \longrightarrow S_{n+1} \longrightarrow \cdots\cdots$$

whose limit is $\tilde{K}_0(A)$. It is clear that

$$S_d \twoheadrightarrow S_{d+1} \twoheadrightarrow \cdots\cdots \longrightarrow \tilde{K}_0(A)$$

iff one has <u>Serre's thm:</u> $rg(P) > d \Rightarrow P \simeq A \oplus P'$
and that one has

$$S_{d+1} \hookrightarrow \text{------}$$

iff one has <u>Cancellation</u>: $A \oplus P' \simeq A \oplus P''$ $rg(P') \geq d$ $\Rightarrow P' \simeq P''$. In practice one has these results with $d = \dim(\mathrm{Max}(A))$. For example:

A local : $S_0 \xrightarrow{\sim} S_1 \xrightarrow{\sim} S_2 \xrightarrow{\sim} \cdots\cdots = pt$

A Dedekind : $\underset{\substack{\| \\ pt}}{S_0} \hookrightarrow S_1 \xrightarrow{\sim} S_2 \xrightarrow{\sim} \cdots\cdots = \mathrm{Pic}(A)$

(This actually tends to be better than one would expect because in the Dedekind situation one has $S_1 \xrightarrow{\sim} S_2$ instead of just $S_1 \twoheadrightarrow S_2$. But perhaps this is exceptional and there are problems with cancellation for general dim 1 rings. No - because of the determinant which says $\exists\ S_1 \rightleftarrows S_k$ given by ″ .)

Now cancellation is exactly the injectivity of the maps $S_n \longrightarrow S_{n+1} \longrightarrow \cdots$. But in topology, when you prove something is injective you often prove that some relative group is zero, and this gives some extra information. Thus any really good proof of cancellation should also prove a surjectivity result in dimension 1.

Now let's discuss carefully the stability situation for manifolds. Let E be a vector bundle over a manifold X which is connected of dimension $= d$. To split off a trivial bundle, one chooses a section s of E transversal to the zero section. Recall the proof that this can be done: Let V be a space of sections generating E

$$\begin{tikzcd}
Z_{s_v} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & X \arrow[d, "0"] \\
X \arrow[r] \arrow[d] & X \times V \arrow[r] \arrow[d, "pr_2"] & E \\
pt \arrow[r, "v"] & V &
\end{tikzcd}$$

One chooses $v \in V$ to be a regular value of the map $S \longrightarrow V$, where $S = \mathrm{Ker}\{V_X \twoheadrightarrow E\}$. The corresp. section s_v of E is then transversal to the zero section and conversely.

When $rg(E) > d$, then the section s is nowhere-vanishing, hence gives an exact sequence

$$0 \longrightarrow 1 \longrightarrow E \longrightarrow E/s \longrightarrow 0$$

The uniqueness version of this proof goes as follows. One has two regular values $v_1, v_2 \in V$ for the map $S \longrightarrow V$, one joins them by a path and moves the path transversal to $S \longrightarrow V$, keeping the endpoints fixed. Then one obtains a family s_t of non-vanishing sections (if $rg > d+1$), and one argues that the bundles E/s_t are homotopic, hence isomorphic.

The preceding argument makes sense for affine varieties over a field of char. zero. Precisely, suppose I write E as a quotient of a trivial bundle generated by a k-vector space V. Then in the case where $rg(E) > d$, S is of $\dim < \dim(V)$ and so $\exists$ non-vanishing sections. Notice that $S \longrightarrow V$ is homogeneous of degree one, hence the image of S is a cone in V.

$$\begin{aligned} \dim(S) &= \dim X + rg(S) \\ &= \dim(X) + \dim(V) - rg(E) \\ &= \dim(V) - [rg(E) - \dim(X)] \end{aligned}$$

Assuming now that $rg(E) \geq d+2$, then I ought to be able to argue that the complement of S in V is connected by lines. Replaces the image of S by its closure $\bar{S}$. Then $\bar{S}$ gives a subvariety Z of codim ≥ 2 in $\mathbb{P}V$. Now because Z has codim 2 most lines do not intersect Z. Thus if I have a point $l \in \mathbb{P}V - Z$, then the set of $l' \neq l$ such that the line (l, l') doesn't meet Z is open and dense. Thus we see that given $l_1, l_2 \in \mathbb{P}V - Z$ we can find an l_3 such that the lines $(l_1, l_3), (l_2, l_3)$ do not meet Z. In terms of our original vector bundle, this means that given two non-vanishing sections

s_1, s_2 we can find a third s_3 independent of both s_1 and s_2. (This argument is slightly incomplete in that S is not closed. One will eventually have to beef it up by showing that any non-vanishing section s can be made part of a generating family V such that s is not the limit of somewhere-vanishing sections.)

Philosophy: All this discussion involving transversality, even if we can make it work for varieties over finite fields, it does not help us over a finite field. What I want to do is to somehow get at the higher connectivity structure even over a point.

January 4, 1974

Let V be a vector space over a field k, let W be a subspace. Denote by $Y_2(V,W)$ the simpl. cx. whose simplices are finite non-empty subsets $\sigma = \{L_1, \ldots, L_q\}$ of 2-planes such that the sum $L_1 + \cdots + L_q + W$ is direct. If $\operatorname{cod}(W) = 2m + \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$, then $Y_2(V,W)$ is of dimension $m-1$. I want to prove it is a bouquet of $(m-1)$-spheres by induction on the codim of W.

Let e be a vector not in W. Then $Y_2(V, ke+W)$ is a subcomplex of $Y_2(V,W)$. Let $\sigma = \{L_1, \ldots, L_q\}$ be a simplex of $Y_2(V,W)$ which is not in $Y_2(V, ke+W)$, that is, $L_1 + \cdots + L_q + ke + W$ is not a direct sum, although $L_1 + \cdots + L_q + W$ is. Hence $e \in L_1 \oplus \cdots \oplus L_q \oplus W$. Writing $e = \sum e_i + w$ with $e_i \in L_i$ one sees that a face $\sigma' = \{L_{i_1}, \ldots, L_{i_p}\}$ of σ has the property that $e \in L_{i_1} \oplus \cdots \oplus L_{i_q} \oplus W$ iff $e_j \neq 0 \Rightarrow j =$ one of the i_p. Thus there is a least face τ of σ such that $e \in k\sigma' + W$ iff $\sigma' \supset \tau$; τ is the face consisting of the L_i in σ such that $e_i \neq 0$. Denote by Γ the set of simplices τ of $Y_2(V,W)$ such that $e \in k\tau + W$ and such that no proper face of τ has this property. Then we have shown

$$Y_2(V,W) - Y_2(V, ke+W) = \coprod_{\tau \in \Gamma} \operatorname{Openstar}(\tau)$$

and so we know that $Y_2(V,W)$ is obtained from $Y_2(V, ke+W)$ by attaching a cone on $\partial\tau * \operatorname{Link}(\tau)$ for every τ in Γ. Better:

$$Y_2(V,W) = \bigcup_{\tau\in\Gamma} \underbrace{\Big[Y_2(V, ke+W) \bigcup_{\partial\tau * \mathrm{Link}(\tau)} \tau * \mathrm{Link}(\tau) \Big]}_{Z_\tau}$$

where $Z_\tau \cap Z_{\tau'} = Y_2(V, ke+W)$ if τ, τ' are distinct elements of Γ. Now

$$\mathrm{Link}(\tau) = Y_2(V, k\tau + W)$$

so we have control over this by induction.

Suppose codimension of $W = n$, and assume known that $Y_2(V,W)$ is quasi-spherical ~~for lower codimension~~ of dimension $\left[\frac{\mathrm{cod}(W)}{2}\right] - 1$ for lower codimension. Put $m = [\frac{n}{2}]$, $n = 2m+\varepsilon$. If $\tau\in\Gamma$ and $\mathrm{card}(\tau) = q$, then $\mathrm{Link}(\tau) = Y_2(V, k\tau+W)$ is quasi-spherical of dim $\left[\frac{n-2q}{2}\right] - 1 = m-q-1$, so $\partial\tau * \mathrm{Link}(\tau)$ is quasi-spherical of dimension $(m-q-1)+(q-2)+1 = m-2$. ($\partial\tau = S^{q-2}$) If $\varepsilon = 1$, then $Y_2(V, ke+W)$ is a bouquet of $(m-1)$-spheres, so $Y_2(V,W)$ is obtained by attaching cones on bouquets on $(m-2)$-spheres in a bouquet of $(m-1)$-spheres, hence $Y_2(V,W)$ is a bouquet of $(m-1)$-spheres.

But if $\varepsilon = 0$, i.e. $\mathrm{cod}(W)$ is even, then $Y_2(V, ke+W)$ is a bouquet of $(m-2)$-spheres, and we must prove that

$$\Big\{\; Y_2(V, ke+W) \longrightarrow Y_2(V,W) \quad \text{is null-homotopic when } \dim(V/W) \text{ is even}$$

Let e' be ind. of $ke+W$, and let Γ' be the set of simplices $(L_1, \ldots, L_q)$ in $Y_2(V, ke+W)$ such that $e' \in L_1 \oplus \cdots \oplus L_q \oplus (ke \oplus W)$ and such that the component e'_i of e' in L_i is non-zero. Then I know from the above that

$$Y_2^{(V,ke+W)} = \bigcup^{Y_2(V,ke'+ke+W)}_{\tau\in\Gamma'} \left(Y_2(V,ke'+ke+W) \cup_{\partial\tau * Y_2(V,k\tau+ke+W)} \tau * Y_2(V,k\tau+ke+W)\right)$$

On the other hand, the subcomplex $Y_2(V,ke'+ke+W)$ in $Y_2(V,W)$ is ~~contained in~~ the Link of the vertex $ke'+ke$, hence it contracts to a point.

Lemma: Let $Z = \bigcup\limits^{A}_{i\in I} Z_i$ (i.e. $\begin{array}{ccc} \coprod A & \hookrightarrow & \coprod Z_i \\ \downarrow{\scriptstyle \text{fold}} & & \downarrow \\ A & \hookrightarrow & Z \end{array}$ cocart)

be a subspace of X. Assume that A contracts to a point in X, and that this contraction extends to Z_i. Then Z contracts to a point in X.

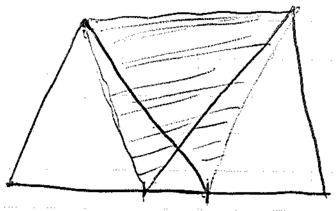
This is obvious. We will apply it to

$$Z_\tau = Y_2(V,ke'+ke+W) \cup_{\partial\tau * Y_2(V,k\tau+ke+W)} \tau * Y_2(V,k\tau+ke+W)$$

$$A = Y_2(V,ke'+ke+W) = \text{Link}(ke'+ke)$$

and $X = Y_2(V,W)$, and the conical contraction of A to $ke'+ke$.

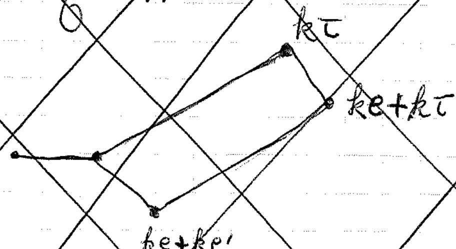
Lemma: Let $Z = A\cup B \subset X$, and suppose given contractions of A to a point and B to a point in X. Assume the two retractions restricted to $A\cap B$ are homotopic. Then the given contraction of A to a point extends to a contraction of B to a point.



This is clear more or less.

Now I apply this lemma to $Z = Z_\tau$, $A = Y_2(V, ke' + ke + W)$ and with $B = \tau * Y_2(V, k\tau + ke + W)$ which contracts to the barycenter of τ. Now I have to show these two contractions on $A \cap B = \partial\tau * Y_2(V, k\tau + ke + W)$ are consistent.

By assumption $\dim(V/W)$ is even, hence $k\tau \oplus ke \oplus W$ is not all of V.



Thus I should be able to find a 2-plane M independent of $ke \oplus ke' \oplus W$, and also of $k\tau \oplus W$. This is clear — given two hyperplanes H, H' in a vector space one can always find a vector outside of both. (Note — this fails for $(q+1)$-hyperplanes $\operatorname{card}(k) = q$.)

But now it is clear that $\operatorname{Link}(M) = Y_2(V, M \oplus W)$ contains $\partial\tau * Y_2(V, k\tau + ke + W)$

I want to find a 2-plane M of $Y(V, W)$ such that M is independent of

$$k\tau \oplus W = L_1 \oplus \cdots \oplus L_q \oplus W \quad \Rightarrow \tau \in \operatorname{Link}(M)$$

and

$$(ke' \oplus ke) \oplus k\partial_j\tau \oplus W = ke' \oplus ke \oplus L_1 \oplus \cdots \hat{L}_j \oplus \cdots \oplus L_q \Rightarrow (ke' + ke) \cup \partial_j\tau \in \operatorname{Link}(M)$$

Total of $(q+1)$-hyperplanes in a space of dimension $2q + 1 + \dim(W)$ namely $ke \oplus L_1 \oplus \cdots \oplus L_q \oplus W$

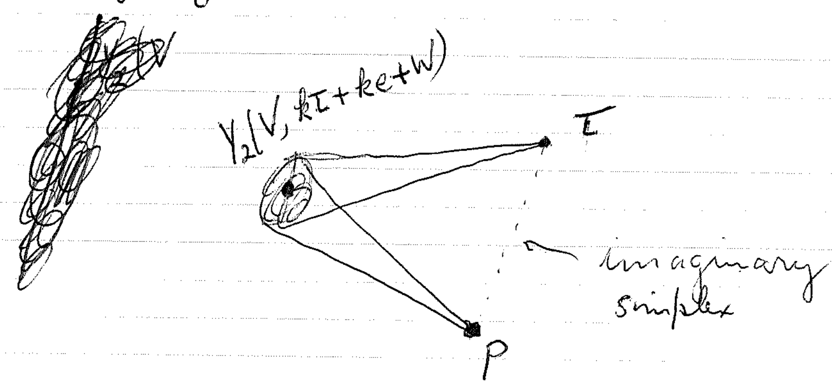
Now here's the idea. What's happening is that I have $Y_2(V, k\tau + ke + W)$ which can be joined to each of the hyperplanes

$$k\tau: \qquad L_1 \oplus \cdots \oplus L_q$$

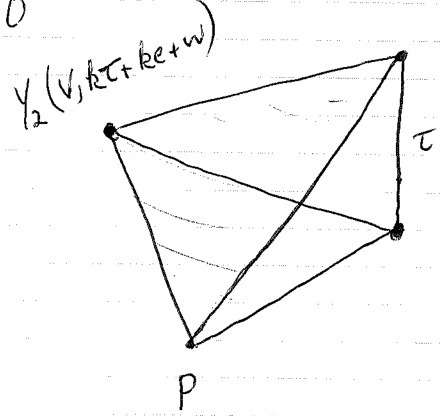
$$ke' + ke + k\partial_j\tau: \qquad L_1 \oplus \cdots \oplus \hat{L}_j \oplus \cdots \oplus L_q \oplus ke \oplus ke'$$

in $k\tau + ke$. These hyperplanes, actually, they are $(q+1)$-
$(q-1)$-Simplices which form the boundary of a q-simplex, which would be the simplex with vertices $L_1, \ldots, L_q, ke + ke'$ if this existed. (Everything would be trivial if this simplex existed, for then it would be clear that the barycenter of τ contraction agrees with the contraction to $P = ke + ke'$

Picture if $q = 1$:

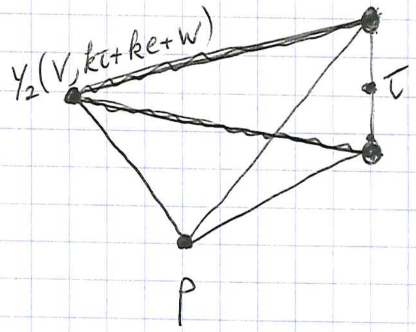


Picture if $q = 2$.



The imaginary simplex would fill in $P * \partial\tau$ and would show the consistency of the contractions of $\partial\tau * Y_2(V, k\tau + ke + W)$ to P and to the barycenter of τ.

The problem is therefore summarized by the picture



One has 2 contractions of $\partial\tau * Y_2(V, k\tau + ke + W)$ to a point - one to the barycenter of τ and the other to p. The problem is to show the consistency of these two contractions, i.e. that the inclusion

$$\boxed{(p * \partial\tau \cup \tau) * Y_2(V, k\tau + ke + W) \subset Y_2(V, W)}$$

is null-homotopic.

Here's the hope. Suppose τ is so large that $k\tau \oplus ke \oplus W$ is a hyperplane in V, whence $Y_2(V, k\tau + ke + W) = \phi$, hence what we have to show is that $p * \partial\tau \cup \tau$ is null-homotopic in $Y_2(V, W)$. I will do this by producing a 2 plane M in $Y_2(V, W)$ independent of the faces of $p * \partial\tau \cup \tau$.

$$\dim(V/W) = 2g + 2$$

$$\dim(k\tau \oplus e) = 2g + 1$$

One vector of M we find outside of $k\tau + ke + W$. For the other we need a vector of $k\tau + ke$ not contained in the span of any of the faces of $p * \partial\tau \cup \tau$. So write

$$k\tau \oplus ke = \underbrace{ke_1 + ke_2}_{L_1} + \cdots + \underbrace{ke_{2g-1} + ke_{2g}}_{L_g} + \underset{ke}{\overset{\parallel}{ke_{2g+1}}}$$

$e = e_{2g+1}$

where $\quad e' = e_1 + e_3 + \cdots + e_{2g-1} + \lambda e + w$

so put $\quad e'' = e_2 + \cdots + e_{2g} + e$. Then e'' is not in

$$\begin{cases} L_1 \oplus \cdots \oplus L_g \oplus W \\ L_1 + \cdots + \hat{L}_j + \cdots + L_g + ke' + ke + W \end{cases}$$

and so it works. (Do this in $L_1 \oplus \cdots \oplus L_g \oplus ke \oplus W$.
$\quad ** \; (\cancel{*}0) \; ** \quad * \quad * \;)$

<u>Problem</u>: If $W \subset W' \subset V$, then have an inclusion

$$Y_2(W', W) * Y_2(V, W') \hookrightarrow Y_2(V, W)$$

In effect given $L_1, \ldots, L_g \subset W'$ independent of W
and $M_1, \ldots, M_p \subset V$
independent of W', then $L_1, \ldots, L_g, M_1, \ldots, M_p$ is independent of W. The problem is to show this is null-homotopic when $\dim(V/W)$ is even and $\dim(W'/W)$ is odd.

So what we have shown is this. I can find a vector $e'' \in k\tau + ke$ which is independent of the planes

$$\begin{cases} k\tau + W \\ k\partial_i\tau + P + W \qquad i = 1, \ldots, g \end{cases}$$

belonging to the faces of $P * \partial\tau \cup \tau$. This means that if e'' is a vector outside of $k\tau + ke + W$, then

$$(P * \partial\tau \cup \tau) * Y_2(V, ke''' + k\tau + ke + W)$$

contracts to the plane $ke'' + ke'''$.

But now take apart

$$Y_2(V, k\tau + ke + W) / Y_2(V, ke''' + k\tau + ke + W)$$

into pieces indexed by those simplices $\xi \in Y_2(V, k\tau + ke + W)$ which are minimal such that

$$e''' \in k\xi + k\tau + ke + W$$

The piece I must worry about is then

$$\partial\xi * Y_2(V, k\xi + k\tau + ke + W)$$

which I will contract to the barycenter of ξ and to the plane $Q = ke'' + ke'''$. Thus I must worry about

$$(Q * \partial\xi \cup \xi) * Y_2(V, k\xi + k\tau + ke + W)$$

I think that everything needed should now follow from the fact that

$$(P * \partial\tau \cup \tau) * (Q * \partial\xi \cup \xi) * Y_2(V, k\xi \oplus k\tau \oplus ke \oplus W)$$

is a well-defined subcomplex of $Y_2(V, W)$, plus some sort of explicit contraction of the first two factors, or maybe induction.

January 6, 1974 (More stability)

In the following A will be a <u>semi-local</u> (not necessarily commutative) ring. This means $A/\mathrm{rad}(A)$ is semi-simple (finite product of matrix rings over skew-fields; equivalently, the category of modules is semi-simple), where $\mathrm{rad}(A)$ = Jacobson radical = an ideal $\ni$ a unit mod $\mathrm{rad}(A) \Rightarrow a$ unit.

Let $X(A^n)$ be the unimodular complex of A^n. It is a simplicial complex of dimension $n-1$.

<u>Hypothesis 1:</u> $X(A^n)$ is a bouquet of $(n-1)$-spheres up to homotopy for every n.

It follows from this that we have an exact sequence

$$0 \longrightarrow H_{n-1}(X(A^n)) \longrightarrow C_{n-1}(X(A^n)) \longrightarrow \cdots\cdots \longrightarrow C_0(X(A^n)) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Put $X_p(A^n)$ = set of p-frames of A^n. Then

$$C_{p-1}(X(A^n)) = \mathbb{Z}[X_p(A^n)] \otimes_{\mathbb{Z}[\Sigma_n]} \mathbb{Z}^{sgn}$$

<u>Lemma 1:</u> A semi-local $\Rightarrow GL_n(A)$ acts transitively on $X_p(A^n)$, $p = 1, \ldots, n$.

<u>Proof:</u> Suppose we have $u: A^p \longrightarrow A^n$, $v: A^p \longrightarrow A^n$ injections onto direct summand. Then

$$A^n \simeq u(A^p) \oplus \mathrm{Cok}(u) \simeq v(A^p) \oplus \mathrm{Cok}(v).$$

Cancellation $\Rightarrow \mathrm{Cok}(u) \simeq \mathrm{Cok}(v)$, hence we get an isom. of A^n

transforming u to v.

Thus

$$X_p(A^n) = GL_n(A) \big/ \text{stabilizer of } e_1, \dots, e_p$$

$$= GL_n(A) \Big/ \left(\begin{array}{c|c} 1 \,\ddots\, 1 & * \\ \hline 0 & GL_{n-p}(A) \end{array}\right)$$

~~For later purposes~~

Put $\quad P_p = C_{p-1}(X(A^n)) \qquad J_n = H_{n-1}(X(A^n))$

$$0 \to J_n \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$$

so we get a spectral sequence

$$E^1_{pq} = H_q(GL_n, P_p) \Longrightarrow H_{p+q}(GL_n, J_n[n]) = \begin{cases} 0 & p+q<n \\ ? & \end{cases}$$

Now

$$H_*\left(GL_n, \mathbb{Z}[X_p] \otimes_{\mathbb{Z}[\Sigma_p]} \mathbb{Z}^{sgn}\right) = H_*\left(GL_n, \mathbb{Z}[GL_n] \otimes_{\mathbb{Z}\left[\begin{smallmatrix}\Sigma_p & * \\ & GL_{n-p}\end{smallmatrix}\right]} \mathbb{Z}^{sgn}\right)$$

$$= H_*\left(\left[\begin{array}{c|c} \Sigma_p & * \\ \hline 0 & GL_{n-p} \end{array}\right], \mathbb{Z}^{sgn}\right)$$

by the Shapiro lemma.

Let G be the affine group of a vector space $\bar{V}$; $G = GL(\bar{V}) \tilde{\times} \bar{V}$, where $\bar{V}$ is interpreted as translations. Let $Y(\bar{V})$ be the affine building of $\bar{V}$, that is, the poset of affine subspaces $W < \bar{V}$. One knows (Lusztig) that if $\dim(\bar{V}) = n$, then $Y(\bar{V})$ is q-spherical of dim. $n-1$, and that it is a bouquet of $(q^n-1)\cdots\cdots(q-1)$ spheres of dim. $(n-1)$. What sort of spectral sequence do we get from these buildings?

Set $J(\bar{V}) = \tilde{H}_{n-1}(Y(\bar{V}), \mathbb{Z})$. Then we get an acyclic complex

$$0 \to J(\bar{V}) \to \bigoplus_{\mathrm{cod}(H)=1} J(H) \to \cdots\cdots \to \bigoplus_{P \in \bar{V}} \mathbb{Z} \to \mathbb{Z} \to 0$$

~~Now the stabilizer of an affine subspace W~~

Now suppose $V = ke_1 + \cdots + ke_n + e_{n+1}$. Then

$$G = \left(\begin{array}{c|c} GL_n & * \\ \hline 0 & 1 \end{array}\right)$$

Now if $W = ke_1 + \cdots + ke_p + e_{n+1}$, then the stabilizer G_W is

$$G_W = \begin{pmatrix} GL_p & * & * \\ 0 & GL_{n-p} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear that a better notation would be this: $V = e_0 + ke_1 + \cdots + ke_n$ $\quad W = e_0 + ke_1 + \cdots + ke_p$

$$G = \begin{pmatrix} 1 & 0 \\ * & GL_n \end{pmatrix} \qquad G_W = \begin{pmatrix} 1 & 0 & 0 \\ * & GL_p & * \\ 0 & 0 & GL_{n-p} \end{pmatrix}$$

<u>January 7, 1974.</u>

V vector space, V' subspace, $G = \{g \in GL(V) \mid (g-1)V \subset V'\}$

$T(V,V')$ = subspaces $W < V \ni V' + W = V$.

If $m = \dim(V/V')$, then $T(V,V')$ has dimension $m-1$, and one knows from Lusztig that it is a bouquet of $(q^n-1)\cdots\cdots(q^m-1)$ spheres

Example to keep in mind: V' a hyperplane - here G is the affine group of V'. If $V = ke_0 + ke_1 + \cdots + ke_m$ and $V' = ke_1 + \cdots + ke_m$, then

$$G = \begin{pmatrix} 1 & 0 \\ * & GL_m \end{pmatrix}$$

Now if $W_0 \in T(V,V')$, then its "boundary" $= \{W \in T(V,V') \mid W < W_0\}$ is equal to $T(W_0, W_0 \cap V')$. In effect

$$W + V' = V \iff$$

$$W + W_0 \cap V' = W_0 \cap (W + V') = W_0$$

~~Acts transitively on the W of~~

Set
$$J(V,V') = \tilde{H}_{m-1}\big(T(V,V')\big).$$

Then we have the following exact sequence

$$0 \to J(V,V') \to \cdots \to \bigoplus_{W \in T_1(V,V')} J(W, W \cap V') \longrightarrow \bigoplus_{W \in T_0(V,V')} \mathbb{Z} \longrightarrow \mathbb{Z} \to 0$$

by Lusztig's theory.

On the other hand, the group G acts transitively on $T_p(V,V') = \{W \in T(V,V') \mid \dim(W) = m+p\}$.

Put $V = \underbrace{Q}_{\dim(m)} \oplus \underbrace{ke_1 \oplus \cdots \oplus ke_m}_{V'}$ and compute the stabilizer G_W where

$$W = Q \oplus ke_1 \oplus \cdots \oplus ke_p.$$

$$G = \left[\begin{array}{c|c} id_Q & 0 \\ \hline * & Gl_m \end{array}\right]$$

$$G_W = \left[\begin{array}{c|c|c} id_Q & 0 & 0 \\ \hline * & Gl_p & * \\ \hline 0 & 0 & Gl_{m-p} \end{array}\right]$$

Notation: $G(V, id_{V/V'})$ for G. Then given W its stabilizer G_W is

$$G_W = G(W, id_{W/W\cap V'}) \underset{G(W\cap V')}{\times} G(V', W\cap V')$$

and so we have an exact sequence

$$1 \longrightarrow G(V', id_{W\cap V'}) \longrightarrow G_W \longrightarrow G(W, id_{W/W\cap V'}) \longrightarrow 1$$

In the case where V' is a hyperplane, W corresp. to the affine subspace $e_0 + W\cap V'$ of $e_0 + V'$, so this corresponds to the exact sequence which one obtains by restricting an affine transformation preserving $e_0+W\cap V'$ to $e_0+W\cap V'$; one maps onto the affine group of $W\cap V'$, and the kernel is those linear transformations of V' which induce the identity on $W\cap V'$.

Thus one ends with a spectral sequence

$$E^1_{pq} = H_q\left(\begin{pmatrix} id_m & & \\ * & GL_p & * \\ & & GL_{n-p}\end{pmatrix}, J(k^m+ke_1+\dots+ke_p,\ ke_1+\dots+ke_p)\right)$$

converging to the homology of $G = \begin{bmatrix} id_m & 0 \\ * & GL_n\end{bmatrix}$

I want to use this to prove a stability theorem for the mod p homology of $GL_n(\mathbb{F}_q)$. For this it would appear necessary to know something about the groups

$$H_*\left(\begin{pmatrix} id_m & \\ * & GL_p\end{pmatrix}, J(k^{m+p}, k^p)\right).$$

For example take $p=1$. Here $T(k^{m+1}, k^1)$ is the set of hyperplanes tranversal to the line k^1, i.e. the splittings of

$$0 \longrightarrow k^1 \longrightarrow k^{m+1} \longrightarrow k^m \longrightarrow 0$$

and G = those autos. of this sequence inducing identity on the quotient k^m. $\therefore$ $G = k^* \,\tilde{\times}\, \mathrm{Hom}(k^m, k)$ and $T(k^{m+1}, k) = G/k^*$, so

$$0 \longrightarrow J(m+1,1) \longrightarrow \mathbb{Z}[G/k^*] \longrightarrow \mathbb{Z} \longrightarrow 0$$

and so

$$H_*(G, J(m+1,1)) \longrightarrow H_*(k^*, \mathbb{Z}) \longrightarrow H_*(G, \mathbb{Z}),$$

which shows that $H_*(G, J(m+1,1))$ is essentially the same mod p as

$$H_*\left(\begin{array}{c|c} id_m & 0 \\ \hline * & k^* \end{array}\right)$$

the sort of group I get rid of with my splitting theorem when k is infinite.

<u>Idea:</u> Let $n \to \infty$ in the above spectral sequence. One gets ~~ ~~ with field coefficients.

$$E^1_{p*} = H_*\left(\begin{pmatrix} id_m & 0 \\ * & GL_p \end{pmatrix}, J(k^{m+p}, k^p)\right) \otimes H_*(GL_\infty)$$

converging to $H_*(GL_\infty)$. This suggests that there should be a very reasonable way to get a contractible complex ~~ ~~ exhibiting the groups $H_*\left(\begin{pmatrix} id_m & \\ * & GL_p \end{pmatrix}, J(k^{m+p}, k^p)\right)$

<u>Theorem</u>: Let $k = \mathbb{F}_q$, $q = p^d$. Then for each $i > 0$ one has $H_i(GL_n(\mathbb{F}_q), \mathbb{F}_p) = 0$ for n sufficiently large.

We will prove by induction on q that for any m and $i \le q$ one has

$$\tilde{H}_i \left(\begin{array}{c|c} I_m & * \\ \hline 0 & GL_n \end{array} \right) = 0$$

(mod p coefficients) for n sufficiently large. Assume this has been established for $i < q$, and we want to get it for q.

<u>Lemma 1:</u> <u>It is possible to fit the spectral sequences</u>

$$E^1_{st}(n) = H_t\left(\begin{pmatrix} id_m & 0 & 0 \\ * & GL_s & * \\ 0 & 0 & GL_{n-s} \end{pmatrix}, J(k^{m+s}, k^s) \right) \Longrightarrow H_{s+t}\begin{pmatrix} id_m & \\ * & GL_n \end{pmatrix}$$

<u>into an inductive system with the obvious effect on E_1 + abutment</u>

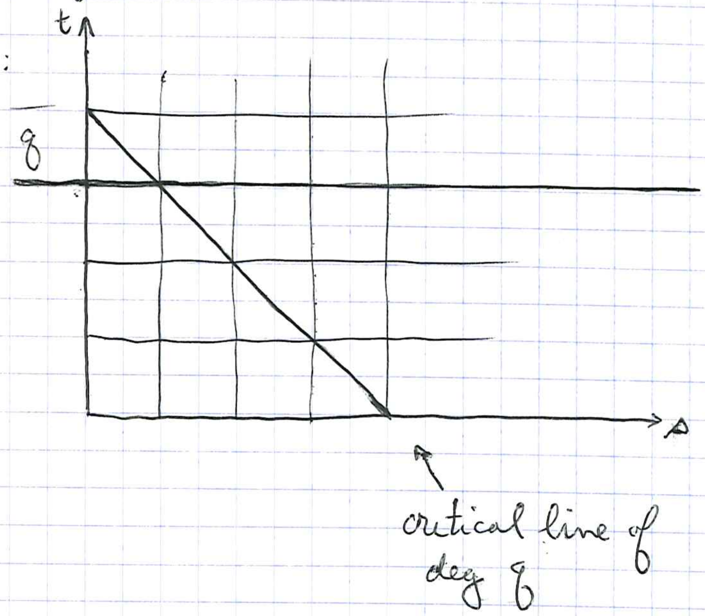
Assume this for the moment. By induction we know that in degrees $i < q$ the group $\begin{pmatrix} I_s & * \\ 0 & GL_{n-s} \end{pmatrix}$ has trivial homology if n is large. Hence

$$E^1_{st}(n) \xrightarrow{\sim} E^1_{st}(\infty) \qquad t < q \qquad s \le r$$

for n large. ~~But I know that the spectral sequence for $n = \infty$ has trivial abutment, hence~~

__Lemma:__ Let $E^r_{st}(f): E^r_{st} \longrightarrow {}'E^r_{st}$ be a map of spectral sequences (first quadrant–homology type) such that $E^1_{st}(f)$ is an isomorphism for $s+t \leq q$, $t<q$ and is surjective for $s+t = q+1$, $t<q$. Then $E^r_{st}(f)$ is an isomorphism for $s+t \leq q$, $t<q$ and all r, and it is surjective for $s+t = q+1$, $t \leq q-r$.

Picture:



Proof by induction on r, the case $r=1$ being hypothesis. Assume true for r suppose $s+t \leq q$, $t \leq q-1$

$$E^r_{s+r,t-r+1}(f) \xrightarrow{d_r} E^r_{st}(f) \xrightarrow{d_r} E^r_{s-r,t+r-1}(f)$$

total degree $\leq q+1$ $\qquad\qquad \cong \qquad\qquad \cong$ because degree $<q$

$t-r+1 \leq q-1-r+1 = q-r$

hence this is __onto__.

Conclude in this case that same cycle & boundary groups $\Rightarrow E^{r+1}_{st}(f)$ isom for $s+t \leq q$, $t \leq q-1$.

On the other hand, suppose $s+t = q+1$, $t \leq q-r-1$

$$\underset{\substack{\text{onto as}\\ t<q}}{E^r_{st}(f)} \xrightarrow{d_r} \underset{\substack{\cong\ (\text{total degree} = q,\ t+r-1 \leq q-2}}{E^r_{s-r,\,t+r-1}(f)}$$

so one gets $\underline{E^{r+1}_{st}(f)}$ is onto.

(Actually the proof seems to give surjectivity in the range $s+t=q+1$, $t \leq q-r+1$ once $r \geq 2$.)

Now returning to the situation on page 8, I know in the case of a finite field of char p and mod p homology that $E^\infty_{st}(\infty) = 0$, $(s,t) \neq (0,0)$. Therefore for n large one has $E^\infty_{st}(n) = 0$ for $s+t=q$ $t<q$. So the only term of degree q is $E^\infty_{0q}(n)$ which is a quotient of $E^1_{0q}(n) = H_q\begin{pmatrix} id_m & 0 \\ 0 & GL_n \end{pmatrix}$.
Thus I get

$$H_q(GL_n) \twoheadrightarrow H_q\begin{pmatrix} id_m & \\ * & GL_n \end{pmatrix}$$

for n large. On the other hand, it is injective obviously.

~~[illegible]~~

By duality one has

$$H_q(GL_n) \xrightarrow{\sim} H_q\begin{pmatrix} id_m & * \\ & GL_n \end{pmatrix}$$

for n large.

Now having obtained this result I can go to the spectral sequence obtained using the Tits building:

$$E^1_{st} = H_t\left(\begin{pmatrix} GL_s & * \\ 0 & GL_{n-s}\end{pmatrix}, I_s\right) \Longrightarrow H_{s+t}(GL_n)$$

For $t \leq q$ now we know that GL_s acts trivially on the homology of $\begin{pmatrix} I_s & * \\ & GL_{n-s}\end{pmatrix}$ for s large.

And since I have seen that $H_*(GL_s, M \otimes I_s) = 0$ for a trivial M of characteristic p, we have $E^1_{st} = 0$ for s bdd, $t \leq q$, $s \geq 2$. This gives then the isom

$$H_q\begin{pmatrix} 1 & * \\ & GL_{n-1}\end{pmatrix} \xrightarrow{\sim} H_q(GL_n)$$

so one wins.

January 7, 1974: Lusztig's D(V).

~~Lusztig~~ Let k be a finite field ~~~~ with q elements, let $W(k)$ be the ring of Witt vectors, ~~Lusztig~~ ~~constructs~~ and let V be a vector space over k of dim n. ~~~~ Lusztig has constructed a basic representation $D(V)$ of $GL(V)$ over $W(k)$ which I really should understand at least for $n=2$.

It has rank $(q-1)\cdots(q^{n-1}-1)$, hence rank ~~~~ $q-1$ when $n=2$. When $n=2$ one has an exact sequence

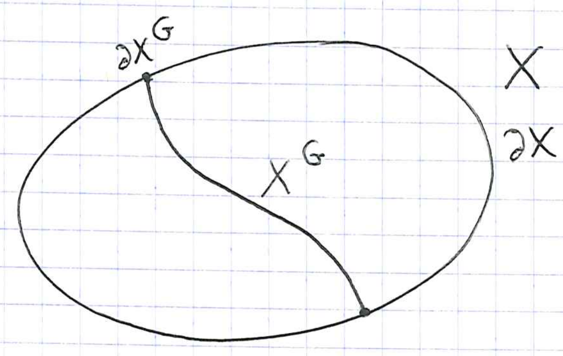
$$0 \longrightarrow D(V)\otimes_W k \longrightarrow \sum_{L\subset V} D(L)\otimes_W k \longrightarrow V \longrightarrow 0$$

Here is Lusztig's definition when $n=2$. He ~~~~ considers the ~~~~ simplicial complex of affine subspaces A of V not containing zero, and looks at the 1-cycles, that is, functions $f(A_0 \subset A_1)$ with values in W_F which are cycles. This gives an H_1 of rank $(q^2-1)(q-1)$ which he cuts down first by put on a homogeneity condition, then he takes an eigenspaces of some operator.

I would like to [redacted] understand Stiefel-Whitney homology classes [redacted] of Euler spaces. Perhaps there is some way to get at them using the diagonal embedding $X \to X \times X$ and the action of the cyclic group of order 2 on $X \times X$.

So [redacted] let $G = \mathbb{Z}/2\mathbb{Z}$ and let X be a G-polyhedron and work out what relations you can.

First suppose X is an n-manifold with boundary ∂X; [redacted] then X^G is a submanifold with boundary; let d be its codimension.



$$\begin{array}{ccccccc} H_G^*(X, X^G) & \longrightarrow & H_G^*(X) & \longrightarrow & H_G^*(X^G) & \xrightarrow{\delta} & \\ \| & & & & \| & & \\ H^*(X/G, X^G) & & & & H_G^*[\text{redacted}] \otimes H^*(X^G) & & \end{array}$$

The localization thm. says $H_G^*(X)[e^{-1}] \xrightarrow{\sim} H_G^*[e^{-1}] \otimes H^*(X^G)$.

One knows also that

$$\chi(X) = [\text{redacted}]\ 2\chi(X/G) - \chi(X^G)$$

In particular, $\chi(X) \equiv \chi(X^G) \mod 2$. So far we've only use homotopy results, and so we have used that X is a manifold. But now consider

$$\begin{tikzcd} {} \arrow[r] & H^*_G(X, X-X^G) \arrow[r] & H^*_G(X) \arrow[r] & H^*_G(X-X^G) \arrow[r, "\delta"] & \cdots \\ & H^{*-d}_G(X^G) \arrow[u, "\int \text{ Gysin}"'] & & & \end{tikzcd}$$

and the composition

$$H^{*-d}_G(X^G) \longrightarrow H^*_G(X) \longrightarrow H^*_G(X^G)$$

is multiplication by the Euler class of the normal bundle $\nu(X^G \subset X)$. As a G-bundle over the trivial G-space X^G, it is the tensor product of the non-trivial rep. of G with the normal bundle, i.e.

$$\underset{\text{G-bundle}}{\nu(X^G \subset X)} = \eta \otimes \nu(X^G \subset X),$$ so its Euler class is

$$w_d(\eta \otimes \nu) = e^d + w_1(\nu) e^{d-1} + \cdots + w_d(\nu)$$

where $w_i(\nu) \in H^i(X^G)$ are the Stiefel-Whitney classes of the normal bundle of X^G in X.

since $$H^*_G(X^G) = H^*_G \otimes H^*(X) \qquad H^*_G = \mathbb{F}_2[e]$$

and the classes $w_i(\nu)$ are nilpotent, one sees that $w_d(\eta \otimes \nu)$ is a non-zero-divisor in $H^*_G(X^G)$. In particular one gets short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^*_G(X, X-X^G) \arrow[r] & H^*_G(X) \arrow[r] & H^*_G(X-X^G) \arrow[r] \arrow[d, equal] & 0 \\ & H^{*-d}_G(X^G) \arrow[u, "\int"'] & & H^*(X/G - X^G) & \end{tikzcd}$$

One should note that ~~the splitting~~ this sequence is not homotopy-invariant because if we multiply X by $\mathbb{R}$ with

-1 action, then d increases by one.

Suppose now that X is a G-polyhedron. Again we have the exact sequence

$$\to H^*_G(X, X-X^G) \longrightarrow H^*_G(X) \longrightarrow H^*_G(X-X^G) \longrightarrow$$

but now we don't have a Gysin homomorphism.

Example: $X = \mathrm{Cone}(Z)$ where G acts freely on Z. Then $(X, X-X^G) \sim (X, Z)$ so we get

$$\begin{tikzcd}
\to H^*_G(X, Z) \arrow[r] & H^*_G(pt) \arrow[r] \arrow[d, equal] & H^*_G(Z) \arrow[r] \arrow[d, equal] & {} \\
 & H^*(BG) \arrow[r] & H^*(Z/G) \arrow[r] & {}
\end{tikzcd}$$

This shows that the long exact sequence needn't split.

<u>Observation:</u> Let p be an odd prime, X a smooth manifold on which $G = \mathbb{Z}/p\mathbb{Z}$ acts, and assume $H^*(X, \mathbb{F}_p)$ finite dimensional. From the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^*_G(X, X-X^G) \arrow[r] \arrow[d, "\cong"] & H^*_G(X) \arrow[r] & H^*(X/G - X^G) \arrow[r] & 0 \\
 & H^{*-d}_G(X^G) & & &
\end{tikzcd}$$

derived above

one sees that if $\beta \in H^1_G$ is the generator, then on taking the homology of $H^*_G(X)$ with respect to β (recall one has $\beta^2 = 0$), one gets that $\chi(X/G - X^G)$ can be computed from $H^*_G(X)$. (This perhaps is not surprising since by duality $\chi(X/G - X^G) = \pm\chi_c(X/G - X^G) = \chi(X/G) - \chi(X^G)$.)

January 10, 1974 — Stability for a field.

Let k be a field. If k is infinite, I think I can prove the stability of $H_i(GL_n(k), \mathbb{Z})$ starting with $n > i$, i.e. $H_i(GL_i(k)) \twoheadrightarrow H_i(GL_{i+1}(k)) \xrightarrow{\sim} \cdots$, except possibly for 2-torsion.

Recall the proof mod torsion. One lets $GL_n(k)$ act on the unimodular complex of k^n which gives a resolution

$$0 \to H_{n-1}(X(k^n)) \to L_n \to \cdots\cdots \to L_0 \to 0$$

where
$$L_s = C_{s-1}(X(k^n)) = \mathbb{Z}[X_s(k^n)] \otimes_{\mathbb{Z}[\Sigma_s]} \mathbb{Z}^{sgn}$$

where $X_s(k^n)$ is the set of s-frames in k^n. Now

$$H_t(GL_n, L_s) = H_t\left(\begin{array}{c|c} \Sigma_s & * \\ \hline & GL_{n-s} \end{array}, \mathbb{Z}^{sgn}\right)$$

Because k is infinite I know that

$$H_*\left(\begin{array}{c|c} 1_s & * \\ \hline & GL_{n-s} \end{array}\right) \xleftarrow{\sim} H_*(GL_{n-s})$$

by using (Casimir operator) style arguments. Hence I know that Σ_s acts trivially on $H_*\begin{pmatrix} 1_s & * \\ & GL_{n-s} \end{pmatrix}$. Now over the rationals Σ_s is coh. trivial so I get

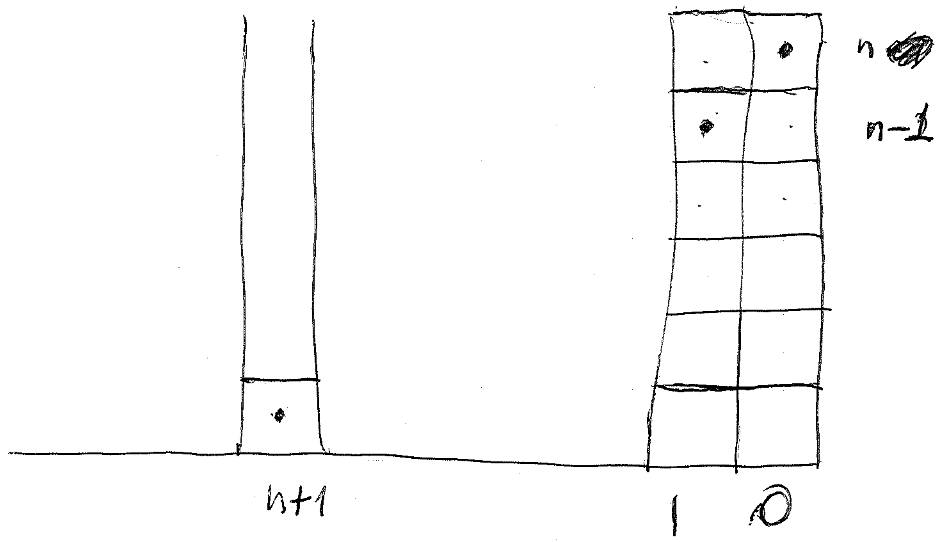
$$H_t(GL_n, L_s) = H_0\left(\Sigma_s, H_t\begin{pmatrix} 1_s & * \\ 0 & GL_{n-s} \end{pmatrix} \otimes \mathbb{Z}^s\right) \qquad \text{mod torsion}$$

$$= \begin{cases} 0 & s \geq 2 \\ H_t\begin{pmatrix} 1_s & * \\ 0 & GL_{n-s} \end{pmatrix} & s = 0, 1 \end{cases}$$

Thus in the spectral sequence assoc. to the above complex one has three non-zero lines.

$$H_t(GL_n, H_{n-1}X(k^n)) \to \cdots\cdots \to H_t(GL_{n-1}) \to H_t(GL_n)$$

$$s = n+1 \qquad\qquad\qquad \uparrow s=1 \qquad \uparrow s=0$$



hence one gets

$$H_i(GL_{n-1}) \xrightarrow{\sim} H_i(GL_n) \qquad i \leq n-2$$

$$\begin{array}{l} H_1(GL_n, H_{n-1}X(k^n)) \longrightarrow H_n(GL_{n-1}) \longrightarrow H_n(GL_n) \\ \to H_0(GL_n, H_{n-1}X(k^n)) \longrightarrow H_{n-1}(GL_{n-1}) \longrightarrow H_{n-1}(GL_n) \to 0 \end{array}$$

Thus one has what I claimed –

$$H_n(GL_n) \twoheadrightarrow H_n(GL_{n+1}) \xrightarrow{\sim} H_n(GL_{n+2}) \xrightarrow{\sim} \cdots$$

at least modulo torsion. As an example one has

$$H_1(GL_1) \twoheadrightarrow H_1(GL_2) \xrightarrow{\sim} H_1(GL_3) \to \cdots$$

modulo torsion.

Now ask whether the torsion really matters. The situation involves calculating d_1 it seems. So let us set up the situation carefully.

We are considering the complex of chains on the unimodular simplicial complex X of k^n

$$0 \to H_{n-1}(X) \to L_n \to \cdots\cdots \to L_0 \to 0$$

where

$$L_s = C_{s-1}(X) \qquad s = 1, \ldots, n$$
$$= \mathbb{Z} \qquad s = 0 .$$

Because GL_n acts transitively on X_s = s-frames in k^n, I know that L_s is an induced module from the stabilizer of $(e_1, \ldots, e_s)$ which is $\begin{pmatrix} \Sigma_s & * \\ 0 & GL_{n-s} \end{pmatrix} \subset GL_n$ and so therefore I get the Shapiro isomorphism

$$(*) \qquad H_*(GL_n, L_s) = H_*\left(\begin{pmatrix} \Sigma_s & * \\ 0 & GL_{n-s} \end{pmatrix}, \mathbb{Z}^{sgn}\right)$$

Now in terms of this isomorphism I wish to compute the map induced by $d: L_s \to L_{s-1}$.

<u>Lemma</u>: With respect to the Shapiro isoms. $(*)$ The homom. $H_*(GL_n, L_s) \to H_*(GL_n, L_{s-1})$ induced by d is the composition of the transfer for

$$\left(\begin{array}{cc|c} \Sigma_{s-1} & & * \\ & 1 & \\ \hline & & GL_{n-s} \end{array}\right) \subset \left(\begin{array}{c|c} \Sigma_s & * \\ \hline & GL_{n-s} \end{array}\right)$$

with the inclusion

$$\left(\begin{array}{cc|c} \Sigma_{s-1} & & * \\ & 1 & \\ \hline & & GL_{n-s} \end{array}\right) \subset \left(\begin{array}{c|c} \Sigma_{s-1} & * \\ \hline & GL_{n-s+1} \end{array}\right)$$

I will assume this for now and see what stability range results. If I work over a field coefficients

$$E^1_{s*} = H_*\left(\left(\frac{\Sigma_s \,\big|\, *}{\big|\, GL_{n-s}}\right), I_s\right) = H_*(\Sigma_s, I_s) \otimes H_*(GL_{n-s})$$

so $d_1 : E^1_{s*} \longrightarrow E^1_{s-1,*}$ is the product of transfer

$$H_*(\Sigma_s, I_s) \longrightarrow H_*(\Sigma_{s-1}, I_s)$$

with ind. map $H_*(GL_{n-s}) \longrightarrow H_*(GL_{n-s+1})$

<u>Lemma 2</u>: Over $\mathbb{F}_p$ one has $H_*(\Sigma_s, I_s) = 0$ except if $s \equiv 0, 1$ (p), and the transfer homo.

$$H_*(\Sigma_s, I_s) \longrightarrow H_*(\Sigma_{s-1}, I_{s-1})$$

for $s \equiv 1$ (p) is an isomorphism.

For $p=2$, one has I_s is trivial

$$\begin{tikzcd} H_*(\Sigma_s) \arrow[r, "tr"] \arrow[rr, bend right, "\text{mult. by index} = s"'] & H_*(\Sigma_{s-1}) \arrow[r, hook, "in", "\text{known by Dold}"'] & H_*(\Sigma_s) \end{tikzcd}$$

Thus transfer $H_*(\Sigma_s) \longrightarrow H_*(\Sigma_{s-1})$ is an isom for s odd $(s \equiv 1\ (2))$ and zero for s even. OKAY.

~~For p odd same argument shows that the transfer~~

<u>p odd:</u>

$$\begin{tikzcd} H_*(\Sigma_s, I_s) \arrow[r, "tr"] \arrow[rr, bend right, "\text{mult by } \binom{s}{2} = \frac{s(s-1)}{2}"'] & H_*(\Sigma_{s-2} \times \overset{0}{\Sigma_2}, I_{s-2} \otimes I_s) \arrow[r] & H_*(\Sigma_s, I_s) \end{tikzcd}$$

Thus $H_*(\Sigma_s, I_s) = 0$ unless $s(s-1) \equiv 0$, i.e. $s \equiv 0, 1$ (p).
If $s \equiv 1$ (p), then transferring down to Σ_{s-1} and restriction

back multiplies by $s \not\equiv 0$, hence $tr: H_*(\Sigma_s, I_s) \hookrightarrow H_*(\Sigma_{s-1}, I_{s-1})$ is injective. Finally the other composition

$$H_*(\Sigma_{s-1}, I_{s-1}) \xrightarrow{res} H_*(\Sigma_s, I_s) \xrightarrow{tr} H_*(\Sigma_{s-1}, I_{s-1})$$

is given by a double coset formula. Now $\Sigma_s / \Sigma_{s-1} = \{1, \dots, s\}$, and Σ_{s-1} has two orbits, one fixpt and another with stabilizer Σ_{s-2}. However we know that $H_*(\Sigma_{s-2}, I_{s-2}) = 0$ if $s \equiv 1 \ (p)$. Thus above composition is the identity. QED.

Suppose now I try to see what follows when p is odd. Fix an integer q and assume that

$$H_i(GL_i) \twoheadrightarrow H_i(GL_{i+1}) \xrightarrow{\sim} \cdots$$

for each $i < q$. Now go to spectral sequence ~~Make assumption~~ To prove that $H_q(GL_q) \twoheadrightarrow H_q(GL_{q+1})$ I take $n = q+1$, and I want to show that

$$E^2_{s, q-s+1} = 0 \quad ? \qquad \text{for } s \geq 2.$$

If we take $s \equiv 0 \ (p)$ so that $tr: H_*(\Sigma_{s+1}, I_{s+1}) \xrightarrow{\sim} H_*(\Sigma_s, I_s)$ then and such that the transfer $H_*(\Sigma_s, I_s) \longrightarrow H_*(\Sigma_{s-1}, I_{s-1})$ is zero, then from

$$E^1_{s,*} = H_*(\Sigma_s, I_s) \otimes H_*(GL_{q+1-s})$$

etc. one sees that

$$E^2_{s*} = H_*(\Sigma_s, I_s) \otimes \operatorname{Coker}\{H_*(GL_{q+1-s-1}) \to H_*(GL_{q+1-s})\}$$

this is only valid if $s \leq q$. But if $s = q+1$, $E^1_{q+1,0} = 0$

Because p odd $\Rightarrow H_0(\Sigma_s, I_s) = 0$ this begins in degree $1 + q + 1 - s = q - s + 2$

and so $E^2_{s,q-s+1} = 0$. Now for $s+1 \equiv 1 \ (p)$

$$E^2_{s+1,*} = H_*(\Sigma_s, I_s) \otimes \mathrm{Ker}\{H_*(GL_{q+1-s-1}) \to H_*(GL_{q+1-s})\}$$

which begins in degree $1 + q+1-s-1 = q-s+1$, in particular $E^2_{s+1,q-s} = 0$.

Thus we see all of the E^2 terms of degree $q+1$ are zero, except for $E^2_{1,q}$ and so we deduce that $H_q(GL_q) \twoheadrightarrow H_q(GL_{q+1})$. Analogously we will get that

$$H_q(GL_{q+1}) \xrightarrow{\sim} H_q(GL_{q+2}) \xrightarrow{\sim} \cdots\cdots$$

This argument doesn't work for the prime 2, because we now have $H_0(\Sigma_s, I_s) \neq 0$. What is the range you get? Again to get $H_q(GL_n) \twoheadrightarrow H_q(GL_{n+1})$ I want $E^2_{s,q-s+1} = 0$ for $s \geq 2$. Now one has

s even $\quad E^2_{s,*} = H_*(\Sigma_s, I_s) \otimes \mathrm{Coker}\{H_*(GL_{n+1-s-1}) \to H_*(GL_{n+1-s})\}$

$s+1$ odd $\quad E^2_{s+1,*} = H_*(\Sigma_s, I_s) \otimes \mathrm{Ker}\{H_*(GL_{n+1-s-1}) \to H_*(GL_{n+1-s})\}$

$q=1$ get $\quad \cancel{H_1(GL_1)} \twoheadrightarrow H_1(GL_2) \xrightarrow{\sim} H_1(GL_3) \xrightarrow{\sim} \cdots$

$q=2$ get $\quad E^2_{2,1}$ begins in degree 1 if $n=2$

$E^2_{3,0} = 0$

So for surj. $\quad H_2(GL_3) \twoheadrightarrow H_2(GL_4) \twoheadrightarrow$

To get injectivity I need $\quad E^2_{s,q-s+2} = 0 \quad s \geq 3$.

$\cancel{E^2_{2,*} = H_*(\Sigma_2, I_2) \otimes \mathrm{Coker}\{H_*(GL_{n-2}) \to H_*(GL_{n-1})\}}$

$\cancel{E^2_{2,2} = 0 \text{ if } n \geq 5}$

$$E^2_{3,*} = H_*(\Sigma_2, I_2) \otimes \operatorname{Ker}\{H_*(GL_{n-2}) \to H_*(GL_{n-1})\}$$

$$E^2_{3,1} = 0 \quad \text{if} \quad n \geq 4$$

Thus we get

$$H_2(GL_3) \twoheadrightarrow H_2(GL_4) \xrightarrow{\sim} H_2(GL_5) \xrightarrow{\sim} H_2(GL_6) \xrightarrow{\sim} \cdots\cdots$$

$q=3$: For surjectivity I need $E^2_{s,q-s} = 0 \quad s = 2,3$

$$E^2_{2,2} = H_*(\Sigma_2, I_2) \otimes \operatorname{Coker}\{H_*(GL_{n-2}) \to H_*(GL_{n-1})\}$$
$$= 0 \quad \text{if} \quad n \geq 5$$

$$E^2_{3,1} = \qquad \operatorname{Ker} \qquad H_1(GL_{n-3}$$
$$= 0 \quad \text{if} \quad n \geq 5$$

So $\quad H_3(GL_5) \twoheadrightarrow H_3(GL_6) \twoheadrightarrow \cdots$

For injectivity need $E^2_{s,5-s} = 0 \quad s = 3,4$

$$E^2_{3,2} = 0 \quad \text{if} \quad n \geq 6$$

$$E^2_{4,1} = 0 \quad \text{if} \quad n \geq 6$$

Thus I get $\quad H_3(GL_5) \twoheadrightarrow H_3(GL_6) \xrightarrow{\sim} H_3(GL_7) \xrightarrow{\sim} \cdots\cdots$

So it appears that this method proves

$$H_q(GL_{2q-1}) \twoheadrightarrow H_q(GL_{2q}) \xrightarrow{\sim} \cdots\cdots$$

One should notice the for the symmetric groups this result is best possible. In effect one knows $\bigoplus_n H_*(B\Sigma_n)$ is a polynomial ring, and $\exists$ generator in $w \in H_1(B\Sigma_2)$ whose powers $w^n \in H_n(B\Sigma_{2n})$ is not in the image of $H_n(B\Sigma_{2n-1})$. ERROR.

The error was caused by fact that ~~for n small~~ to ~~...~~ get $H_n(GL_n) \twoheadrightarrow H_n(GL_{n+1})$ we forgot to consider $E^2_{n,0}$

correctly. $E^1_{n,0} = H_*(\Sigma_n, I_n) \otimes H_*(GL_0)$ and if n is even ($p=2$ here), there is no $E^1_{n+1,0}$ term to cancell it. Thus one has to start with $n \geq 2$ to get $H_1(GL_2) \twoheadrightarrow H_1(GL_3)$ in which case we get

$$H_1(GL_2) \xrightarrow{\sim} H_1(GL_3) \xrightarrow{\sim} H_1(GL_4) \xrightarrow{\sim} \cdots$$

Then we get $H_2(GL_4) \xrightarrow{\sim} H_2(GL_5) \xrightarrow{\sim} \cdots\cdots$ and similarly

$$H_q(GL_{2q}) \xrightarrow{\sim} H_q(GL_{2q+1}) \xrightarrow{\sim} \cdots\cdots\cdots$$

in general. And this is best possible for the symmetric groups.

One should recall that for p odd $H_*(\Sigma_p)$ begins in dimension $2(p-1)-1 = 2p-3$. Thus for the symmetric groups one has a basic element in $H_3(\Sigma_3)$.

<u>January 11, 1974.</u>

I want to know if the 2-torsion peculiarities, which I have obtained via the symmetric groups, actually is a K-phenomenon. Thus consider the ring $\mathbb{Z}$, let $GL_n(\mathbb{Z})$ act on the Tits building of $\mathbb{Q}^n$, and consider the groups $H_*(GL_n(\mathbb{Z}), I_n)$ where $I_n = \tilde{H}_{n-2}(T(\mathbb{Q}^n))$. These are analogous to the groups $H_*(\Sigma_n, \mathbb{Z}^{sgn})$, and yesterday's work showed that the 2-torsion peculiarities in stability are a consequence of the fact that

$$H_0(\Sigma_n, \mathbb{Z}^{sgn}) = \mathbb{Z}/2 \qquad n \geq 2$$

So I now will compute $H_0(GL_2\mathbb{Z}, I_2)$. Now

$$0 \to I_2 \to \mathbb{Z}[P_1\mathbb{Q}] \to \mathbb{Z} \to 0$$

$$\mathbb{Z}[P_1\mathbb{Q}] = \mathbb{Z}[P_1\mathbb{Z}] = \mathbb{Z}\left[GL_2\mathbb{Z}\Big/\begin{pmatrix}\pm 1 & * \\ & \pm 1\end{pmatrix}\right]$$

since any f.t. torsion free $\mathbb{Z}$-module is free. (all this would also go through for a PID). So

$$H_1\begin{pmatrix}\pm 1 & * \\ & \pm 1\end{pmatrix} \to H_1(GL_2\mathbb{Z}) \to H_0(GL_2\mathbb{Z}, I_2) \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$$

One has

$$0 \to \{\pm 1\} \to SL_2\mathbb{Z} \to PSL_2\mathbb{Z} \to 1, \qquad PSL_2\mathbb{Z} = \mathbb{Z}/2 * \mathbb{Z}/3$$

central ext. so

$$H_2(PSL_2\mathbb{Z}) \to \mathbb{Z}/2 \to H_1(SL_2\mathbb{Z}) \to H_1(PSL_2\mathbb{Z}) \to 0.$$

$$H_2(PSL_2\mathbb{Z}) = 0, \qquad H_1(PSL_2\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/3$$

But in fact $SL_2\mathbb{Z} = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ so $H_1(SL_2\mathbb{Z}) = \mathbb{Z}/12$ generated by the matrices corresponding to mult. by i in $\mathbb{Z}[i]$ and $\omega = \sqrt[3]{1} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ in $\mathbb{Z}[\sqrt[3]{1}]$

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

To see that ω generates a cyclic subgp of order 3 in $H_1(SL_2\mathbb{Z})$, one uses the map

$$\begin{array}{c} SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/3\mathbb{Z}) \\ \qquad\qquad\qquad\qquad \wr\!\! \mid \\ \qquad\qquad\qquad\qquad \mathbb{Z}/3\mathbb{Z} \tilde{\times} \text{ quat. gp order } 8 \end{array}$$

And if one uses

$$\begin{tikzcd} SL_2\mathbb{Z} \arrow[r, two heads] \arrow[d] & GL_2(\mathbb{Z}/2\mathbb{Z}) = \Sigma_3 \\ SL_2\mathbb{Z}/\{\pm 1\} \arrow[ur] & \end{tikzcd}$$

which shows ω is non-trivial in $H_1(PSL_2\mathbb{Z})$ hence ω is non-trivial of order 4 in $H_1(SL_2\mathbb{Z})$. Now from the extension

$$1 \longrightarrow SL_2\mathbb{Z} \longrightarrow GL_2\mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

One has

$$H_1(GL_2\mathbb{Z}) = H_0(\mathbb{Z}/2, H_1(SL_2\mathbb{Z})) \oplus \mathbb{Z}/2$$

So take

$$\omega^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \omega^2$$

Thus on the $\mathbb{Z}/3$ part the quotient $\mathbb{Z}/2$ acts as the inverse, and so there is no 3-torsion in $H_1(GL_2\mathbb{Z})$. But

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underset{i}{\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underset{i^3}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

so one sees that i gives rise to a element of order 2 in $H_0(\mathbb{Z}/2, H_1(SL_2\mathbb{Z}))$. Thus

$$H_1(GL_2\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \qquad \text{generators } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note in fact the one of these is detected by the determinant and the other by

$$H_1(GL_2\mathbb{Z}) \longrightarrow H_1(GL_2(\mathbb{Z}/2)) = \mathbb{Z}/2$$
$$\qquad\qquad\qquad\qquad \overset{\parallel}{\Sigma_3}$$

Now one can compute the map

$$H_1\begin{pmatrix} \pm 1 & * \\ & \pm 1 \end{pmatrix} \longrightarrow H_1(GL_2\mathbb{Z})$$

and one finds that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ goes to the generator $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ while $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ goes to an element of order 2 of $GL_2(\mathbb{Z}/2)$.

Conclude $H_1\begin{pmatrix} \pm 1 & * \\ & \pm 1 \end{pmatrix} \twoheadrightarrow H_1(GL_2\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

<u>Summary:</u> $H_1(GL_2\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$$H_0(GL_2\mathbb{Z}, I(\mathbb{Q}^2)) = 0$$

I see now that the preceding calculation has been stupid. In effect by the Euclidean algorithm any matrix in $GL_2(\mathbb{Z})$ is the product of elementary matrices and $\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus the cokernel of $H_1\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \to H_1(GL_2\mathbb{Z})$ is clearly zero.

Prop: k field: $H_0(GL_n(k), I_*(k^n)) = 0$ for $n \geq 2$.

Proof: If
$$0 \to V' \to V \to V'' \to 0$$
is an exact sequence of vector spaces, and if we choose a splitting, so that we get a homo.
$$GL(V') \times GL(V'') \subset GL(V, V')$$
then I know that as $GL(V,V')$-modules
$$I(V) = \mathbb{Z}[GL(V,V')] \otimes_{\mathbb{Z}[GL(V') \times GL(V'')]} I(V') \otimes I(V'').$$
Thus
$$\begin{aligned} H_0(GL(V,V'), I(V)) &= H_0(GL(V') \times GL(V''), I(V') \otimes I(V'')) \\ &= H_0(GL(V'), I(V')) \otimes H_0(GL(V''), I(V'')) \end{aligned}$$
so this will be zero if say $\dim(V') \geq 2$. But $H_0(GL(V), I(V))$ is a quotient of $H_0(GL(V,V'), I(V))$, so one wins.

Question: Is $H_0(GL_n(\mathbb{Z}), I(\mathbb{Q}^n)) = 0$ for $n \geq 3$?

Assume now that we are over a field k, and let us consider the spectral sequence associated to GL_n acting on the Tits building $T(k^n)$.

$$E^1_{s*}(n) = H_*\left(\left(\begin{array}{c|c} GL_s & * \\ \hline 0 & GL_{n-s}\end{array}\right), I_s\right) \Longrightarrow 0$$

I recall this is obtained by taking the Lusztig sequence

$$L_\cdot(V): 0 \to I(V) \longrightarrow \bigoplus_{\mathrm{cod}\, V'=1} I(V') \longrightarrow \cdots\cdots \longrightarrow \bigoplus_{d(L)=1} I(L) \longrightarrow \mathbb{Z} \longrightarrow 0$$

If now V is a subspace of W, then $L_\cdot(V)$ is a subcomplex of $L_\cdot(W)$ and it is stable under the action of $GL(W,V)$. Thus one has

$$\begin{array}{ccc} L(V) & \longrightarrow & L(W) \\ \uparrow{\scriptstyle GL(V)} & & \\ GL(W,V) & \subset & GL(W) \end{array}$$

so that if we choose a complement for V in W, so that we get a hom. $GL(V) \to GL(W)$, then $L(V) \to L(W)$ is equivariant for $GL(V) \to GL(W)$, and so we obtain an induced map of spectral sequences

So now I have then a map of spectral sequences

$$\begin{array}{c} E^1_{s*}(n) = H_*\left(\left[\begin{array}{c|c} GL_s & * \\ \hline 0 & GL_{n-s}\end{array}\right], I_s\right) \Longrightarrow 0 \\ \downarrow \\ E^1_{s*}(\infty) = H_*\left(\left[\begin{array}{c|c} GL_s & * \\ \hline 0 & GL_\infty\end{array}\right], I_s\right) \Longrightarrow 0 \end{array}$$

[redacted] Now then suppose we try to use this to prove a stability theorem. First try for H_1.

line $q=0$ is $\qquad 0 \qquad 0 \qquad \mathbb{Z} \qquad \mathbb{Z}$

line $q=1$ is $\qquad H_1(GL_{n-1}) \qquad H_1(GL_n)$

$$H_1(GL_3, I_3) \to H_1(GL_2, I_2) \to \oplus\, H_1(GL_1, I_1) \to$$

The point is similar – one has in the limit. ?

It would seem that to use the comparison theorem for spectral sequences is the wrong approach. I would prefer to understand what is happening.

Question 1: I seem to be finding that there is a spectral sequence

$$E^1_{st} = H_t(GL_s, I_s) \Longrightarrow 0$$

Recall that by filtering the Q-category I got a spectral sequence. If $F_n(Q) =$ modules of [redacted] rank $\leq n$, then

$$\to H_i(F_{n-1}Q) \to H_i(F_nQ) \to H_{\underset{t}{i-n}}(GL_{\underset{s}{n}}, I_{\underset{s}{n}}) \xrightarrow{\partial} H_{\underset{s+t-1}{i-1}}(F_{\underset{s-1}{n-1}}Q) \to$$

$$E^1_{pq} = H_q(GL_p, I_p)$$

$$E^1_{st} = H_t(GL_s, I_s) \Longrightarrow H_{s+t}(Q)$$

Can check this as follows. $\quad F_1 = \Sigma(Bk^{\cdot})$, $\quad F_0 = pt$

January 12, 1974. Computation of Steinberg homology.

Consider the Q-category of a field k. Forgetting unipotent subgroups, it is the simplicial space which is the bar construction of the monoid

$$M = \coprod_{n\geq 0} BGL_n$$

and it looks like this:

$$BM: \quad \cdots \; M\times M \overset{pr_2}{\underset{pr_1}{\xrightarrow{\;\mu\;}}} M \rightrightarrows pt$$

$$M^p = \coprod_{a_1,\ldots,a_p\geq 0} B(GL_{a_1}\times\cdots\times GL_{a_p})$$

with faces determined by the rules

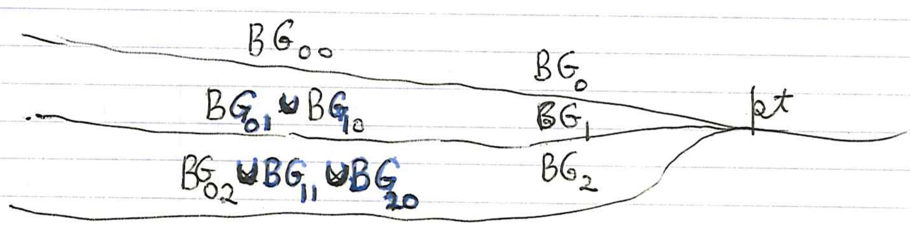
$$d_0(m_1,\ldots,m_p) = (m_2,\ldots,m_p)$$
$$d_i(\qquad\qquad) = (\ldots,m_im_{i+1},\ldots) \qquad 1\leq i\leq p-1$$
$$d_p(\qquad\qquad) = (m_1,\ldots,m_{p-1})$$

One filters BM by

$$F_n(BM) = \left\{ F_n(M^p) = \coprod_{a_1+\cdots+a_p\leq n} BGL_{a_1}\times\cdots\times BGL_{a_p} \right\}$$

whence

$$F_n(BM)/F_{n-1}(BM) = \left\{ p \mapsto pt \amalg \coprod_{a_1+\cdots+a_p=n} BGL_{a_1}\times\cdots\times BGL_{a_p} \right\}$$



Algebraically what is happening: Put

$$R = \bigoplus_{n \geq 0} H_*(BG_n)$$

coefficients in a field k. Then the simplicial space BM gives a spectral sequence

$$E^1_{s*} = \underset{\underbrace{R \otimes \cdots \otimes R}_{s}}{\overset{}{H_*(M^s)}} \Longrightarrow H_*(BM)$$

where d^1 is given by the formula

$$d^1(r_1 \otimes \cdots \otimes r_s) = \varepsilon(r_1) r_2 \otimes \cdots \otimes r_s - \sum_{i=1}^{s-1} (-1)^i r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_s$$
$$+ (-1)^s r_1 \otimes \cdots \otimes r_{s-1} \, \varepsilon(r_s).$$

I should recall that $\varepsilon : R \to k$ is the map sending $H_*(BG_n) \longrightarrow H_*(pt)$, hence it is the augmentation

$$R \longrightarrow \bigoplus_n H_0(BG_n) = k[T] \longrightarrow k$$
$$T \longmapsto 1$$

Now I know that (E^1, d^1) is just the bar construction of the augmented algebra R, hence

$$E^2_{s*} = \mathrm{Tor}^R_s(k, k)$$

k being regarded as an algebra via the augmentation ε. (The proof is to exhibit the semi-simp resolution

$$\cdots \equiv R \otimes R \rightrightarrows R \longrightarrow k$$

with

$$d_i(r_0 \otimes \cdots \otimes r_s) = \cdots \otimes r_i r_{i+1} \otimes \cdots$$
$$d_s(\qquad) = r_0 \otimes \cdots \otimes r_s$$

which is contractible because $s_{-1} = (r_0 \otimes \cdots \otimes r_s) = 1 \otimes r_0 \cdots \otimes r_s$

Because k is a field, this is a free resolution of k regarded as an R-module via ε.

Now we have a natural increasing filtration of the ring R:

$$F_n R = \bigoplus_{a \leq n} H_*(BGL_a)$$

and what I am doing is to consider the induced filtration of the bar construction. In passing from R to $gr(R)$ the ring doesn't change, but rather the augmentation. ~~so that in fact I~~
Thus as a space

$$F_n(BM)/F_{n-1}BM = \{\not{p} \mapsto pt \amalg \coprod_{a_1+\cdots+a_p=n} BGL_{a_1}\times\cdots\times BGL_{a_p}\}$$

whence we have a spectral sequence

$$E^1_{s*}(n) = \bigoplus_{a_1+\cdots+a_p=n} H_*(GL_{a_1})\otimes\cdots\otimes H_*(GL_{a_p}) \Longrightarrow \tilde{H}_*(F_n(BM)/F_{n-1}(BM))$$

where here $d_0(r_1\otimes\cdots\otimes r_p) = 0$ if $a_1 = \deg(r_1) > 0$
$d_p(r_1\otimes\cdots\otimes r_p) = 0$ if $a_p = \deg(r_p) > 0$.

Thus here

$$E^2_{s*}(n) = \operatorname{Tor}^R_s(k^{(0)}, k^{(0)})_{*,n}$$

where here $k^{(0)}$ denotes k regarded as an R module via the augmentation sending $H_*(GL_n) \mapsto 0$ for $n > 0$.

Now suppose we compute this for the algebraic closure of $\overline{\mathbb{F}}_p$ with coeffs in $\mathbb{F}_\ell$. Then one has

$$R = \bigoplus H_*(GL_n) = k[\xi_0, \xi_1, \dots]$$

where $\xi_i \in H_{2i}(GL_1)$ is dual to $u^i \in H^{2i}(GL_1)$, u being a generator of $H^2(GL_1)$. Thus in the bigrading of R, ξ_i has degree $(2i, 1)$. Now because R is a polynomial ring

$$\mathrm{Tor}^R_s(k,k) = \Lambda^s \, \mathrm{Tor}^R_1(k,k)$$

where

$$\mathrm{Tor}^R_1(k,k) = I/I^2$$

I denoting the augmentation ideal. Thus in the case at hand $\mathrm{Tor}^R_1(k,k)$ is concentrated in the dimension degree 1, which means that $E^2_{st}(n) = 0$ for $s \neq n$. So the spectral sequence degenerates & we find that

$$\bigoplus_n \tilde{H}_*(F_n BM / F_{n-1} BM)$$

is an exterior algebra on $\tilde{H}_*(F_1 BM / F_0 BM) = \tilde{H}_*(\Sigma BGL_1)$ $= \bigoplus \mathbb{F}_\ell \cdot \eta_i$ where $\eta_i \in H_{2i+1}$.

Basic fact is that for $\bar{\mathbb{F}}_p$

$$\bigoplus_n H_*(GL_n, I_n) = \Lambda[\eta_1, \eta_2, \dots\dots]$$

coeffs. over $\mathbb{F}_\ell$

where η_i is a basis for $H_{2i}(GL_1, I_1) = H_{2i}(GL_1) \simeq \mathbb{F}_\ell$.

January 14, 1974

Stability proof in the "direct sum" situation.

Consider the following example: Let $M = \coprod_{n\geq 0} B\Sigma_n$ act on $M' = \coprod_{n\geq 0} BGL_n$ in the obvious way and form the simplicial category which plays the role of M'/M. It corresponds to the simplicial space

$$M^2 \times M' \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; M \times M' \rightrightarrows M'$$

which looks like

$$\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \coprod_{a_0,a_1,a_2} B\Sigma_{a_0}\times B\Sigma_{a_1}\times BG_{a_2} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \coprod_{a_0,a_1} B\Sigma_{a_0}\times BG_{a_1} \rightrightarrows \coprod_{a_0} BG_{a_0}.$$

Filter this in the obvious way and then

$$F_n/F_{n-1}: \quad \Big(\overset{*}{\underset{\smile}{}}\!\!\coprod_{\substack{a_0+a_1+a_2\\ =n}} B\Sigma_{a_0}\times B\Sigma_{a_1}\times BG_{a_2}\Big) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \Big(\overset{*}{\underset{\smile}{}}\!\!\coprod_{a_0+a_1=n} B\Sigma_{a_0}\times BG_{a_1}\Big) \rightrightarrows \Big(\overset{*}{\underset{\smile}{}} BG_n\Big)$$

I want to identify this gadget.

Maybe it would be clearer if I introduced the Q-category. It has as objects those of M, in this case, vector spaces over k, and a morphism from V to V' is an isomorphism $V \oplus kS \xrightarrow{\sim} V'$ where S is a set. (i.e. an injection $V \hookrightarrow V'$ plus a basis for a complement for V in V'). Now filter this category according to its dimension. F_n = full subcat. consisting of V of rank $\leq n$. Then have inclusion

$$F_{n-1} \xhookrightarrow{i} F_n$$

and the spectral sequence

$$E^2_{pq} = \varinjlim_{F_n}{}^{(p)} \left(V \mapsto H_q(i/V) \right) \Longrightarrow H_{p+q}(F_{n-1})$$

Now i/V has a final object if $\mathrm{rg}(V) < n$, and if $V = k^n$, then i/V is equivalent to the ordered set $Z(V)$ consisting of splittings

$$V = W \oplus kS \qquad W < V$$

in which

$$(W,S) \leq (W',S') \Leftrightarrow W \subset W', \ S \supset S'$$

Now one has the functor

$$GL_n \xrightarrow{j} F_n$$

such that if M is a GL_n module, then one has

$$(j_! M)(V) = \varinjlim_{j/V} M = \begin{cases} 0 & \text{if } \mathrm{rg}(V) < n \\ \cong M & \text{if } \mathrm{rg}(V) = n. \end{cases}$$

Thus the Leray spectral sequence for j degenerates yielding

$$H_q(F_n, j_! M) = H_q(GL_n, M).$$

So we have

$$E^2_{pq} = H_p\left(F_n, V \mapsto H_q(i/V)\right) \Longrightarrow H_{p+q}(F_{n-1})$$

$$H_q(i/V) = \begin{cases} H_q(pt) & \mathrm{rg}(V) < n \\ H_q(J(V)) & \mathrm{rg}(V) = n \end{cases}$$

Thus for $q > 0$ one has

$$H_q(i/V) = j_! H_q(J(V))$$

and we have:

Thus one has an exact sequence of complexes

$$0 \longrightarrow \mathring{j}_!\{V \mapsto \tilde{C}_\cdot(J(V))\} \longrightarrow Li_!(\mathbb{Z}_{F_{n-1}}) \longrightarrow \mathbb{Z}_{F_n} \longrightarrow 0$$

and we get a cofibration situation:

$$\begin{array}{ccc} (J(V), GL(V)) & \subset & (\mathrm{Cone}\,J(V), GL(V)) \\ \downarrow & & \downarrow \\ F_{n-1} & \subset & F_n \end{array}$$

Now semi-simplicial, one constructs the cone by shifting one dimension. Precisely the non-degenerate simplices σ of the cone $C(X)$ are of the form σ or $\sigma \cup v$ where σ is a non-degenerate simplex of X. Thus if we realize $(J(V), GL(V))$ as the simplicial space ~~without deg~~

$$\Rrightarrow \coprod_{\substack{a_0+a_1+a_2=n \\ a_0>0}} B\Sigma_{a_0} \times B\Sigma_{a_1} \times BG_{a_2} \rightrightarrows \coprod_{\substack{a_0+a_1=n \\ a_0>0}} B\Sigma_{a_0} \times BG_{a_1}$$

then $\mathrm{Cone}\,J(V), GL(V)$ is the simplicial space ~~without deg~~.

$$\Rrightarrow \coprod_{a_0+a_1+a_2=n} B\Sigma_{a_0} \times B\Sigma_{a_1} \times BG_{a_2} \rightrightarrows \coprod_{a_0+a_1=n} B\Sigma_{a_0} \times BG_{a_1}$$

and the cofibre is the simplicial space

$$\cdots\cdots\cdots \Rrightarrow \left(\begin{matrix} * \\ \cup \\ \coprod_{a_1+a_2=n} B\Sigma_{a_1}\times BG_{a_2}\end{matrix}\right) \rightrightarrows \left(\begin{matrix} * \\ \cup \\ BG_n\end{matrix}\right).$$

which we obtained before.

Suppose now we put in the fact that $J(V)$, which is an ordered set of dimension $n-1$, is spherical of this dimension, so that we get then an exact sequence

$$\cdots \longrightarrow H_i(F_{n-1}) \longrightarrow H_i(F_n) \longrightarrow H_{i-n}(GL_n, \tilde{H}_{n-1}J(k^n)) \longrightarrow \cdots$$

How can we deduce from this information the stability of $H_*(GL_n)$?

Idea: We consider the result of right multiplying by the basepoint of BG_1, call this ξ. It gives us a map of simplicial spaces

$$\begin{array}{ccccc} M^2\times M' & \Rrightarrow & M\times M' & \rightrightarrows & M' \\ \downarrow{\cdot\xi} & & \downarrow & & \downarrow{\cdot\xi} \\ M^2\times M' & \Rrightarrow & M\times M' & \rightrightarrows & M' \end{array}$$

which is a cofibration in each dimension. Now my idea was to work with the simplicial pair of spaces $p\mapsto M^p\times(M', M'\xi)$, and to filter it as before:

$$p \longmapsto \coprod_{a_0+\cdots+a_p\le n} B\Sigma_{a_0}\times\cdots\times B\Sigma_{a_{p-1}} \times (BG_{a_p}, BG_{a_{p-1}}\cdot\xi)$$

The point now is that we know by the connectivity of F_n, that $F_n/F_{n-1} \xrightarrow{\cdot\xi} F_{n+1}/F_n$ is a h.eq. in a range, and this gives us a spectral sequence

$$E^2_{pq}(n) = \mathrm{Tor}_p^{\oplus H_*(\Sigma_n)}\Big(k, \bigoplus_n H_*(G_n, G_{n-1})\Big)_{q,n} \Longrightarrow \Big\{ 0 \quad \text{in a range increasing with } n.$$

Now we compute the Tor using as pleasant a resolution as we can. Suppose therefore I have a minimal resolution of k over $R = \oplus H_*(\Sigma_n)$

$$\cdots \longrightarrow R\otimes V_{2*} \longrightarrow R\otimes V_{1*} \longrightarrow R \longrightarrow k$$

where $V_{p*} = \mathrm{Tor}_p^R(k,k)$. This will give an E^1 term for computing the Tor:

$$\cdots \longrightarrow V_{1*} \otimes \oplus H_*(G_n, G_{n-1}) \longrightarrow \bigoplus_n H_*(G_n, G_{n-1})$$

which hopefully yields the desired stability result. For example if $k = \mathbb{Q}$, than $\oplus H_*(\Sigma_n) = k[\xi]$, so one has a $\mathrm{Tor}_0 = \oplus H_*(G_n, G_{n-1})$ and a Tor_1, hence all the differentials are zero.

January 18, 1974 (still groggy)

I now seem to understand better what is going on in the stability proof for $H_*(GL_n)$ using the unimodular complex.

First of all I have succeeded in relating the two different stability proofs that I had before, one based on the unimodular complex and which involved $H_*(\Sigma_s, \text{sign})$, the other using the Q-category which involved $H_*(GL_s, \text{steinberg})$. And also I have made the proof less dependent on special features, i.e. the complex of simplicial chains.

Consider the unimodular complex of F^n, denote it $X(F^n)$. It is a simplicial complex, but for purposes of generalization, it is perhaps desirable to view it as a poset, whose elements are finite non-empty unimodular subsets of F^n under inclusion. Now then since $X(F^n)$ is spherical of dim $n-1$, one gets a complex of GL_n-modules

$$0 \longrightarrow 0 \longrightarrow L_n \longrightarrow \cdots\cdots\cdots \longrightarrow L_0 \longrightarrow 0$$

, which is acyclic except in degree n, where

$$L_s = \coprod_{\sigma_1 < \cdots < \sigma_s} \mathbb{Z}$$

Using transitivity of GL_n on unimodular subsets & splitting of exact sequences one has

$$H_*(GL_n, L_s) = \bigoplus_{\substack{a_0+\cdots+a_s=n \\ a_0,\dots,a_s>0}} H_*\left(\Sigma_{a_0}\times\cdots\times\Sigma_{a_{s-1}}\times GL_{a_s}\right)$$

which I can recognize in the following way. Assume over a field k of coefficients, so that we have Künneth

$$H_*\left(\Sigma_{a_0}\times\cdots\times\Sigma_{a_{s-1}}\times GL_{a_s}\right) = H_*(\Sigma_{a_0})\otimes\cdots\otimes H_*(GL_{a_s}).$$

Then if I put

$$R = \bigoplus_a H_*(\Sigma_a) \qquad R' = \bigoplus_a H_*(GL_a)$$

and grade these by the degree a to be called from now on the "size", one finds that the complex

$$s \longmapsto H_*(GL_n, L_s)$$

is the size n part of the standard bar complex

$$\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\rightrightarrows}} \bar R\otimes\bar R\otimes R' \;\underset{id\otimes\mu}{\overset{\varepsilon\otimes id=0}{\underset{\mu\otimes id}{\rightrightarrows}}}\; \bar R\otimes R' \;\overset{\varepsilon\otimes id=0}{\underset{\mu}{\rightrightarrows}}\; R'.$$

which one knows computes

$$\mathrm{Tor}_s^R(k, R'). \qquad k = R/\bar R \qquad \bar R = \bigoplus_{a>0} H_*(\Sigma_a)$$

Question: Is the expression just derived:

$$H_s\left(\nu\mapsto H_*(GL_n, L_\nu)\right) = \mathrm{Tor}_s^R(k, R')_*(n)$$

independent of the choice of chain complex used for computing the homology of $X(F^n)$? In other words if I had used another complex $L_\bullet$, would I have gotten the same result?

Answer – yes in some sense, because any two such complexes, if obtained geometrically, will be equivariantly homotopy equivalent so the two complexes of the form $s \mapsto H_*(GL_n, L_s)$ will be also h.eq.

This is the first part of the argument, namely the use of the unimodular complexes to tell us something about $\mathrm{Tor}^R_s(k, R')$. This part of the argument is quite general and I can easily replace symmetric groups by the general linear groups of a subfield.

Why this is related to the Q-construction: suppose I let the symmetric groups act by direct sum on the general linear groups and form the quotient, i.e. the simplicial space

$$\coprod \Sigma_{a_0} \times \Sigma_{a_1} \times GL_{a_2} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \coprod \Sigma_{a_0} \times GL_{a_1} \;\substack{\longrightarrow\\\longrightarrow}\; \coprod GL_{a_0}$$

which is related to Q- of the symmetric groups. Now filtering this according to total rank leads to quotients

$$F_n/F_{n-1} \qquad \begin{array}{cc} * & * \\ \coprod_{a_0+a_1=n} \Sigma_{a_0} \times GL_{a_1} & GL_n \end{array}$$

which is essentially the unimodular complex of F^n divided out by the action of GL_n.

January 18, 1974.

Idea: Make a detailed study of categories obtained by the following glueing procedure

$$\begin{tikzcd} (X, G) \arrow[r, hook] \arrow[d] & (CX, G) \arrow[d] \\ \mathcal{C}' \arrow[r, hook] & \mathcal{C} \end{tikzcd}$$

where X is a "G-space", in practice, a poset.

Example: Let F be a field and let k be a subfield. Let $\mathcal{C}$ be the category whose objects are F-vector spaces in which a map from V to V' is an isomorphism $V \oplus F \otimes_k W \xrightarrow{\sim} V'$, W an independent k-subspace of V'. Then if $\mathcal{C}_n$ is the full subcategory of ~~[illegible]~~ V of rank $\leq n$, then one should have a cocartesian square

$$\begin{tikzcd} (f/F^n, GL_n F) \arrow[r, hook] \arrow[d] & (\overset{\text{cone}}{C} f/F^n, GL_n F) \arrow[d] \\ \mathcal{C}_{n-1} \arrow[r, hook, "f"'] & \mathcal{C}_n \end{tikzcd}$$

So here $X = f/F^n$ is the ordered set consisting of decompositions

$$F^n = V \oplus F \otimes_k W$$

with $V < F^n$.

Question: Given a vector space V over a field F and a subfield k, consider the poset consisting of non-zero k-subspaces $W \ni F \otimes_k W \hookrightarrow V$. Is this spherical?