December 19, 1973. More Grassmannians.

$\dim(V) = N$. Let $W' \subset W'' \subset V$ and consider in $G_p(V)$ the cycle $Z = \{A_p \mid \dim(A \cap W') \geq 1, \dim(A \cap W'') \geq 2\}$. I want to compute the cohomology class of this cycle in $H^*(G_p(V))$. Introduce the desingularization $\tilde{Z} = \{(\ell_1 \subset m_2 \subset A_p) \mid \ell \subset W', m \subset W''\}$. Then we have

$$\begin{tikzcd}
\tilde{Z} \arrow[r, hook, "j'"] \arrow[d] \arrow[dr, phantom, "\text{trans.}"] & \{(\ell \subset m \subset A)\} \arrow[r, "f"] \arrow[d, hook, "i"] & \{A\} \\
\{\ell \subset m \mid {\ell \subset W' \atop m \subset W''}\} \arrow[r, hook, "j"] & \{\ell \subset m \subset V\} \times \{A\} \arrow[ur, "pr_2"'] &
\end{tikzcd}$$

$$\{\ell \subset m \subset V\} = G_{11}(V)$$

so the class we are after $f_*(j'_*1)$ is $pr_{2*}(j_*1 \cdot i_*1)$. Now the image of i is where the map

$$m \subset V \longrightarrow V/A$$

vanishes, so if we denote by L_1, L_2 the line bundles on $G_{11}(V)$ with fibres $\ell, m/\ell$ respectively, and by T the two plane bundle with fibre m, then

$$\begin{aligned}
i_*1 &= e(\mathrm{Hom}(pr_1^* T, pr_2^* Q)) \\
&= e(pr_1^* L_1^\vee \otimes pr_2^* Q)\, e(pr_2^*(L_2^\vee) \otimes pr_2^* Q) \\
&= (x_1^q + \cdots + c_q(Q))(x_2^q + \cdots + c_q(Q))
\end{aligned}$$

where Q is the quotient bundle on the Grassmannian and $x_i = c_1(L_i^\vee)$.

On the other hand the image of j is first where $\ell \subset W'$, i.e. where a section of $L_1 \otimes V/W'$ vanishes, and then on this set where $m/\ell \to V/W' \to V/W''$

is zero, i.e where a section of $L_2 \otimes V/W''$ vanishes. ~~Hence~~ Hence

$$j_* 1 = x_1^{e'} x_2^{e''} \qquad \begin{array}{l} e' = \dim(V/W') \\ e'' = \dim(V/W'') \end{array}$$

(We take $W' < W'' \Rightarrow e' > e''$). So now we want

$$(pr_2)_* \left[x_1^{e'} x_2^{e''} (x_1^q + \cdots + c_q Q)(x_2^q + \cdots + c_q Q) \right]$$

which is an integration over $G_{11}(V)$ which we solved yesterday.

$$= \text{coeff of } (x_1 x_2)^{N-1} \text{ in } \quad (x_2 - x_1)\, x_1^{e'} x_2^{e''} (x_1^q + \cdots + c_q Q)(x_2^q + \cdots + c_q Q)$$

$$x_1^{e'} \cancel{x}_2^{e''+1}\, x_1^{\overbrace{N-e'-1}^{d'}} c_{q-d'+1} Q \;\; x_2^{\overbrace{N-e''-2}^{d''}} c_{q-d''+2}(Q)$$

$$- x_1^{e'+1} x_2^{e''}\, x_1^{\overbrace{N-e'-2}^{d'}} c_{q-d'+2} Q \;\; x_2^{N-e''-1} c_{q-d''+1}(Q)$$

$$= \begin{vmatrix} c_{q-d'+1}(Q) & c_{q-d''+1}(Q) \\ c_{q-d'+2}(Q) & c_{q-d''+2}(Q) \end{vmatrix} .$$

So it is now more or less clear that the same method will yield the following result in general.

<u>Proposition:</u> Let Q be the canonical quotient bundle over $G_p(V)$, $\dim(V) = N = p+q$, $q = \dim(Q)$. Let

$$0 < W_{d_1} < \cdots < W_{d_r} \leq V$$

be a chain of subspaces, and let

$$Z = \left\{ A_p \in G_p(V) \;\middle|\; \dim(A \cap W_{d_i}) \geq i \;, \; 1 \leq i \leq r \right\}$$

Then the cohomology class in $H^*(G_p(V))$ corresponding to this cycle is

$$\begin{vmatrix} c_{q-d_1+1}(Q) & \cdots & c_{q-d_s+1}(Q) \\ \vdots & & \vdots \\ c_{q-d_1+s}(Q) & \cdots & c_{q-d_s+s}(Q) \end{vmatrix}$$

Examples: If one takes a sub-space W of dimension d and considers the cycle

$$\{A \mid \dim(A \cap W) \geq s\}$$

then this is an example of the preceding with $W_s = W$, so $d_s = d$ and $(d_1, \ldots, d_s) = (d-s+1, \ldots, d)$. Thus the cohomology class where a generic subspace of d sections has rank $\leq d-s$ is the determinant

$$\begin{vmatrix} c_{q-d+s} & \cdots & c_{q-d+1} \\ \vdots & & \vdots \\ c_{q-d+2s-1} & \cdots & c_{q-d+s} \end{vmatrix}$$

e.g. if $d=q$, then this has degree $= s^2$ as it should.

December 20, 1973. More Grassmannians (groggy encore)

Let E be a vector bundle of rank q over ~~a~~ a manifold X and V a vector space mapping to $\Gamma(E)$ such that $V_X \longrightarrow E$ is surjective, whence we have a map $f: X \longrightarrow G_p(V)$ $\quad p+q = N = \dim(V)$
~~[illegible]~~ $f: x \longmapsto A(x) = \mathrm{Ker}\{ev_x : V \longrightarrow E(x)\}$ inducing E from the quotient bundle on the Grassmannian.

Given a subspace W of V (or more generally a flag $W_{d_1} < \cdots < W_{d_r}$) in V there should be a notion of when this subspace is "generic" which roughly should mean that the map f is transversal to the strata + their resolutions defined by W.

Suppose $\dim(W) = d$. Then for each integer ~~s~~ s one gets a cycle

$$Z_s = \{A \mid \dim(A \cap W) \geq s\} \qquad s = 1, \ldots, p$$

in $G_p(V)$ of dimension $= \dim\{B^s \subset W\} + \dim\{A^p \supset B^s\}$

$$= s(d-s) + (p-s)(N-p)$$

hence of codim $= s(q-d+s)$ whose cohom. class we computed yesterday.

Suppose A now ~~[illegible]~~ such that $\dim(A \cap W) = s$, i.e. A is a good point of the cycle Z_s. ~~[illegible]~~ The tangent space to $G(V)$ at A is $\mathrm{Hom}(A, V/A)$. To compute the tangent space to Z_s at A, one first chooses where $A \cap W$ goes, which gives $\mathrm{Hom}(A \cap W, W/A \cap W)$, and then if this ~~[illegible]~~ is zero one sees where A goes, which gives $\mathrm{Hom}(A/A \cap W, V/A)$. $\therefore$ one has

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A/A\cap W, V/A) \arrow[r] & T_{Z_s}(A) \arrow[r] \arrow[d, hook] & \mathrm{Hom}(A\cap W, W/A\cap W) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{Hom}(A/A\cap W, V/A) \arrow[r] & \mathrm{Hom}(A, V/A) \arrow[r] & \mathrm{Hom}(A\cap W, V/A) \arrow[r] \arrow[d] & 0 \\
& & & \mathrm{Hom}(A\cap W, V/A+W) &
\end{tikzcd}$$

Better: A tangent vector $\theta \in \mathrm{Hom}(A, V/A)$ will be tangent to Z_s provided it keeps $A\cap W$ inside of W, which means $\theta(A\cap W) \subset A+W/A$.

So we see the normal space to Z_s at A is $\mathrm{Hom}(A\cap W, V/A+W)$ which has dim $s(q-d+s)$ as it should. Thus we have:

<u>Proposition</u>: The subspace W of V is "generic" in the sense that the map f is transversal to the stratification of $G(V)$ defined by W if and only if for each point $x \in X$, the map composee

$$T_X(x) \xrightarrow{df} \mathrm{Hom}(A(x), V/A(x)) \longrightarrow \mathrm{Hom}(A(x)\cap W, V/A(x)+W)$$

is surjective.

<u>Variant</u>: A collection of d sections $\{s_1, \dots, s_d\}$ of a projective module of rank q is "generic" if the set where they have rank $\leq d-s$ has codimension $\geq s(q-d+s)$.

Suppose now we give two subspaces $W' < W'' < V$ of dimensions d' and d'' and put

$$Z = \{A \mid \dim(A \cap W') \geq s', \ \dim(A \cap W'') \geq s''\}.$$

(require $s' \leq d'$, $s'' \leq d''$, $s' \leq s'' \leq p$ for this to be ~~interesting~~ non-empty. $0 \leq d'-s' \leq d''-s'' \leq q$.

Before we analyzed in the case of $W_{d_1} < \cdots < W_{d_p}$ the cycle

$$\{A \mid \dim(A \cap W_{d_i}) \geq i\}$$

~~[illegible]~~ which is the closure of the cell with canonical forms

$$\begin{array}{c} \begin{array}{ccccc} d_1 & & d_2 & & d_p \end{array} \\ \begin{pmatrix} & 1 & 0 & \text{———} & & & & \\ & & & 1 & 0 & \text{—} & & \\ & & & & & 1 & 0 & \text{———} \end{pmatrix} \end{array}$$

hence it has codimension $(N-d_1-p+1)+\cdots+(N-d_p)$ $= (q-d_1+1)+\cdots+(q-d_p+p)$. To get the cell I am interested in, I put

$$\begin{array}{ccccccc} d_1 & \text{need } s' \leq d' \text{ here} & d_{s'} & \text{need } d' \leq d''-s''+s' & d_{s''} & \text{need } d'' \leq q+s'' & d_p \\ & & \| & & \| & & \| \\ d'-s'+1, & \cdots\cdots, & d', & d''-s''+s'+1, \cdots\cdots, & d'', & N-p+s''+1, \cdots, & N \end{array}$$

so I find the cycle Z I am interested in has codimension

$$s'(q-d'+s') + (s''-s')(q-d''+s'')$$

and its cohomology class is the determinant

$$\left|\begin{array}{ccc|cc}
c_{q-d'+s'}(Q) & & c_{q-d'+1}(Q) & & c_{q-d''+1}(Q) \\
 & & & & \vdots \\
 & & c_{q-d'+s'}(Q) & & \vdots \\
\hline
 & & & c_{q-d''+s''}(Q) & \vdots \\
 & & & & \vdots \\
 & & & & c_{q-d''+s''}(Q) \\
c_{q-d'+s''+2s'-1}(Q) & & c_{q-d'+s''}(Q) & &
\end{array}\right|$$

(Yuck! The point is the diagonal entries are the Chern classes of degrees $q-d_1+1, \ldots, q-d_p+p$ which here is $q-d'+s'$ s'-times and $q-d''+s''$ $(s''-s')$ times and 0 $(p-s'')$ times. In each column the degree of the Chern class increases by ~~one~~ each step going down the column.)

So now given ~~[illegible]~~ a subspace A put $s' = \dim(A \cap W')$, $s'' = \dim(A \cap W'')$ and ~~[illegible]~~ ask when a tangent vector $\Theta \in \mathrm{Hom}(A, V/A)$ will be tangent to the strata $Z_{s's''} = \{A' \mid \dim(A' \cap W') = s', \dim(A' \cap W'') = s''\}$ through A. This will be the case iff Θ keeps the intersection $A \cap W'$ in W' and $A \cap W''$ in W''. Thus

$$0 \longrightarrow T_{Z_{s's''}}(A) \longrightarrow \mathrm{Hom}(A, V/A) \longrightarrow \begin{array}{c} \mathrm{Hom}(A\cap W', V/A+W') \\ \times \\ \mathrm{Hom}(A \cap W'', V/A+W'') \end{array} ~~[illegible]~~$$

~~[illegible]~~

~~Now one has~~

$$\begin{tikzcd}
 & & \arrow[d] & & \\
 & \arrow[r] & \mathrm{Hom}(A, V/A) \arrow[r] \arrow[d] & \mathrm{Hom}(A \cap W', V/A+W') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\arrow[r] & \mathrm{Hom}(A \cap W'', V/A+W'') \arrow[r] \arrow[d] & \mathrm{Hom}(A \cap W', V/A+W'') \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

So therefore what seems to be happening is this:
One has two filtrations

$$0 \subset W' \cap A \subset W'' \cap A \subset A$$

$$0 \subset W'+A/A \subset W''+A/A \subset V/A$$

and the tangent space to the (W', W'')-stratum containing A is the subspaces of $\mathrm{Hom}(A, V/A)$ consisting of endos compatible with the filtration. Therefore it is clear that this generalizes to:

<u>Proposition</u>. Given $0 < W_{d_1} < \cdots < W_{d_r} \leq V$, and a subspace A, the tangent space to the stratum Z of $G(V)$ associated to this flag containing A

$$Z = \{A' \mid \dim(A' \cap W_{d_i}) = \dim(A \cap W_{d_i})\}$$

is the subspace of θ in $\mathrm{Hom}(A, V/A)$ which are compatible with the filtrations $\{W_{d_i} \cap A\}$ and $\{W_{d_i} + A/V\}$.

Now I want to understand what it means for $W' = ke_1$, $W'' = ke_1 + ke_2$ to be generic. Given $A = A(x)$ I investigate the possibilities

$$\underset{s'}{0 \subset W' \cap A} \subset \underset{s''}{W'' \cap A} \qquad s' \le s'' \le p$$

$$0 \le 1 - s' \le 2 - s'' \le q$$

There are four possibilities $(s', s'') = (0,0), (0,1), (1,1), (1,2)$.

Case: $(s', s'') = (0,0)$; here e_1, e_2 are independent at x and there is no condition.

Case: $(s', s'') = (0,1)$; here e_1 vanishes at x but e_2 doesn't.

$$0 \subset ke_1 = ke_1 \subset A$$

$$0 = 0 \subset ke_2(x) \subset E(x)$$

$\theta \in \mathrm{Hom}(A, E(x))$ preserves the filtration iff $\theta(e_1) = 0$, hence transversality means that the map

$$T_X(x) \xrightarrow{e_1} E(x)$$

be onto. The condition for transversality is thus that e_1 be transversal to 0 at x.

Case: $(s', s'') = (1,1)$; here $e_1(x) \neq 0$ and e_1, e_2 are dependent at x.

$$0 = 0 \subset k(\lambda e_1 + e_2) \subset A$$

$$0 \subset ke_1(x) = ke_1(x) \subset E(x)$$

Here θ preserves filtration iff $\theta(\lambda e_1 + e_2) \subset k(e_1(x))$. Thus trans. means

$$T_X(x) \longmapsto \mathrm{Hom}(k(\lambda e_1 + e_2), E(x)/ke_1(x))$$

is onto. So transversality means that e_2 as a section of E/ke_1 is transversal to zero at x.

<u>Case.</u> $(s',s'') = (1,2)$; here e_1, e_2 both vanish at x and the filtration is

$$0 < ke_1 < ke_1 + ke_2 \subset A$$

$$0 = 0 = 0 \subset E(x)$$

so transversality means that

$$\underline{T(x) \twoheadrightarrow Hom(ke_1 + ke_2, E(x))}$$

__December 20, 1973:__ Lusztig - Kervaire approach to buildings.

Let $V = ke_1 + \cdots + ke_n$ and let $X(V)$ be the simplicial complex of ~~[illegible]~~ independent subsets in V. One puts

$$F_\ell = \{(v_0,\dots,v_q) \in X(V) \mid e_\ell \notin ke_1 + \dots + ke_{\ell-1} + kv_0 + \dots + kv_q\}$$

This is a subcomplex contained in the link of e_ℓ, hence it contracts to a point in $X(V)$. But notice that if we contract F_ℓ to a point, then the contraction moves $F_\ell \cap F_{\ell+h}$ through $F_{\ell+h}$. In effect if $(v_0,\dots,v_q) \in F_\ell \cap F_{\ell+h}$, then $(e_\ell, v_0, \dots, v_q) \in F_{\ell+h}$.

Put another way

$$\begin{tikzcd} F_\ell \cap \bigcup_{h>0} F_{\ell+h} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{Cone}(F_\ell \cap \bigcup_{h>0} F_{\ell+h})"] & F_\ell \arrow[d, hook] \arrow[r, phantom, "\mathrm{Cone}(F_\ell)"] & {} \\ \bigcup_{h>0} F_{\ell+h} \arrow[r, hook] & \bigcup_{h>0} F_{\ell+h} & \end{tikzcd}$$

__Lemma:__ Suppose $X = A \cup B$, and that there exists a contraction of A in X to a point of B such that the contraction moves $A \cap B$ through B. Then $B \hookrightarrow X$ is a homotopy equivalence.

__Proof:__ Let $h: A \times I \longrightarrow X$ be the homotopy. $h_0 = \text{incl.}$ The restriction $h' : (A \cap B) \times I \longrightarrow B$ can be extended to a homotopy $\tilde{h}' : B \times I \longrightarrow B$ such that $\tilde{h}'_0 = id$. Putting these together one gets an extension $\tilde{h} : X \times I \longrightarrow X$

starting with id_X and ending with a map $\tilde{h}_1$ of X into B. Then $X \xrightarrow{\tilde{h}_1} B \longrightarrow X$ is homotopic to id_X via $\tilde{h}$ and $B \longrightarrow X \xrightarrow{\tilde{h}_1} B$ is homotopic to the identity of B via $\tilde{h}' = \tilde{h}/B$.

From the ~~[illegible]~~ viewpoint of homology one argues that the pair $(A, A\cap B)$ contracts to a point in (X, B), hence the induced map $H_*(A, A\cap B) \to H_*(X,B)$ is zero, but on the other hand it is ~~[illegible]~~ an isomorphism by excision, so $H_*(X,B) = 0$

So to apply the lemma in the situation at hand one takes

$$X = F_\ell \cup F_{\ell+1} \cup \cdots \cup F_n$$
$$B = F_{\ell+1} \cup \cdots \cup F_n$$
$$A = F_\ell$$

and the contraction which assigns to a simplex $(v_1, \ldots, v_q)$ of F_ℓ the simplex $(e_\ell, v_1, \ldots, v_q)$. ~~[illegible]~~ Check this belongs to some $F_{\ell+h}$. Now I know $e_\ell \notin ke_1 + \cdots + ke_{\ell-1} + kv_1 + \cdots + kv_q$

Doesn't work. In fact it might very well happen that we get $ke_1 + \cdots + ke_{\ell-1} + ke_\ell + kv_1 + \cdots + kv_q = V$ hence $(e_\ell, v_1, \ldots, v_q)$ not in any $F_{\ell+h}$, $h \geq 0$.

Thus what we want is the

~~Lemma: Given $f: A \cup B \longrightarrow X$. Assume f/A and f/B null-homotopic and that the contract~~

Lemma: Given $A \cup B \subset X$, assume A can be contracted in X to a point of B, and that the contraction moves $A \cap B$ thru B. If B contracts to a point in X, then $A \cup B$ contracts to a point in X.

Proof: Take the homotopy $h: A\times I \to X$ which contracts X to a point of B, and whose restriction to $A\cap B$ is: $h': (A\cap B)\times I \longrightarrow B$ and extend h' to a homotopy $\tilde{h}': B\times I \longrightarrow B$ starting with id_B. By putting h and $\tilde{h}'$ together we get an extension $h: (A\cup B)\times I \longrightarrow X$ which pulls $A\cup B$ into B. Then followed by a contraction of B to a point we finish.

Now apply this when

$$B = F_{\ell+1} \cup \cdots \cup F_n$$

$$A = F_\ell$$

and the contraction of A to e_ℓ given by embedding a simplex $(v_0, \dots, v_i) \ni e_\ell \notin ke_1 + \cdots + ke_{\ell-1} + kv_0 + \cdots + kv_i$ into $(e_\ell, v_0, \dots, v_i)$. Note that if $(v_0, \dots, v_i) \in A\cap B = \bigcup_{h>0} F_\ell \cap F_{\ell+h}$ then also $e_{\ell+h} \notin ke_1 + \cdots + ke_{\ell+h-1} + kv_0 + \cdots + kv_i$, so we have that $(e_\ell, v_0, \dots, v_i) \in F_{\ell+h}$. This shows the contraction of A to e_ℓ moves $A\cap B$ through B. So by induction one obtains that

Proposition: In $X(V)$, the subcomplex $F_\ell \cup \cdots \cup F_n$ where $F_\ell = \{(v_1, \dots, v_i) \mid ke_1 + \cdots + ke_{\ell-1} + kv_1 + \cdots + kv_i \not\ni e_\ell\}$ contracts to a point.

But note that if $kv_1 + \cdots + kv_i < V \iff (v_1, \dots, v_i) \in F_\ell$ for some ℓ. Thus $F_1 \cup \cdots \cup F_n$ is exactly the $(n-2)$-skeleton of $X(V)$ and so $X(V)$ is a bouquet of $(n-1)$-spheres.

December 21, 1973. Stability

In the following Λ is a prime field and homology is understood as having coefficients in Λ. Recall

Prop 1: $$H_*(Aut(P) \tilde{\times} Hom(V,P)) \simeq H_*(Aut(P))$$

if either i) $\Lambda = \mathbb{Q}$ and A is an alg. over $\mathbb{Z}[p^{-1}]$ for some prime p.

ii) $\Lambda = \mathbb{F}_p$ and p is a unit in A.

Proof. Use spectral sequence
$$E^2_{pq} = H_p(Aut(P), H_q(Hom(V,P)) \Longrightarrow H_{p+q}(Aut(P) \tilde{\times} Hom(V,P))$$

In case (ii), $Hom(V,P)$ is a $\mathbb{Z}[p^{-1}]$-module because p is a unit in A, ~~[illegible]~~ hence $H_*(Hom(V,P))$ is trivial.

In case (i) consider the auto of the spectral sequence produced by conjugation with the element $p \cdot id_P$. One has
$$H_q(Hom(V,P)) = \Lambda^q(Hom(V,P) \otimes_{\mathbb{Z}} \mathbb{Q})$$
this auto. acts as multiplication by p^q. ~~[illegible]~~ ~~[illegible]~~ Since $p^q \neq p^r$ for $q \neq r$ and the differentials commute with the action it follows that all differentials in the spectral sequence are zero. As the action is trivial on the abutment, one must have $E^2_{pq} = 0$ for $q > 0$, and the proposition follows.

~~In the following I want~~

Suppose now that A is a Dedekind domain with fraction field F, and let E be a finite type projective A-module. I let $GL(E)$ act on the building $X(E)$ = ordered set of submodules P of E such that P is a direct factor of E and $0<P<E$. Now because A is a Dedekind domain, I know that $X(E)$ = Tits building of $F \otimes_A E$.

Now because $X(E)$ is a bouquet of spheres, etc. one gets a complex

$$0 \to I(E) \to \cdots\cdots \to \bigoplus_{rg(P)=2} I(P) \longrightarrow \bigoplus_{rg(P)=1} I(P) \to \mathbb{Z} \to 0$$

where I denotes the Steinberg module:

$$I(P) = \begin{cases} \tilde{H}_{n-2}(X(P), \mathbb{Z}) & rg(P) = n \geq 2 \\ \mathbb{Z} & rg(P) = 0, 1 \end{cases}$$

This gives me then a spectral sequence

$$E^1_{pq} = H_q\Big(GL(E), \bigoplus_{rg(P)=p} I(P)\Big) \Rightarrow 0$$

$$\|$$

$$\bigoplus_{\{P_\alpha\}} H_q(GL(E, P_\alpha), I(P_\alpha))$$

where $\{P_\alpha\}$ run over representatives for the $GL(E)$ orbits on $Grass_p(E) = \{P \subset E \mid P \text{ direct summand } rg(P) = p\}$, and where $GL(E, P_\alpha)$ is the stabilizer of P_α so that on choosing a splitting $E = P_\alpha \oplus Q_\alpha$ one has

$$GL(E,P) = \begin{pmatrix} GL(P) & Hom(Q,P) \\ 0 & GL(Q) \end{pmatrix}$$

Now recall in the case of a Dedekind domain that a finite type projective is determined up to isomorphism by its rank and determinant line bundle. Thus it is clear now that for $0 < p < rg(E)$, the sum is taken over the ideal classes of A.

Now assuming we are in the situation of Proposition 1, we have

$$H_*\big(GL(Q) \tilde{\times} Hom(Q,P)\big) = H_*(GL(Q))$$

and hence

$$H_*(GL(E,P), I_\Lambda(P)) \simeq H_*(GL(P), I_\Lambda(P)) \otimes H_*(GL(E/P))$$

where $I_\Lambda(P) = I(P) \otimes \Lambda$.

~~At this point it should be clear that we get some kind of information on stability. Thus first of all I get~~

Now I want to deduce stability for $H_*(GL(E))$. Let H be a hyperplane in E, i.e. E/H invertible. Then the building for H is contained in that of E, hence the exact sequences of Lusztig for H should map to that for E, so we get a map of complexes

$$\begin{array}{ccccc} \longrightarrow & \displaystyle\bigoplus_{\substack{P \subset H \\ rg(P) = p}} I(P) & \longrightarrow & \cdots \\ & \downarrow & & \\ \longrightarrow & \displaystyle\bigoplus_{\substack{P \in E \\ rg(P) = p}} I(P) & \longrightarrow & \cdots \end{array}$$

In fact the former is a subcomplex of the latter. This map is equivariant for the action of $GL''(E,H) = \{\alpha \in GL(E,H) \mid \alpha = \mathrm{id}$ on $E/H\}$. Call $G = GL(E)$, $G' = GL''(E,H)$. One has a map of complexes of G-modules

$$(*) \qquad \mathbb{Z}[G] \otimes_{\mathbb{Z}[G']} \Big(\bigoplus_{\substack{P \subset H \\ rg(P) = p}} I(P) \Big) \longrightarrow \bigoplus_{\substack{P \subset E \\ rg(P) = p}} I(P)$$

Now given $P \subset E$ of rank $p \leq n-2$, $n = \text{rang}\, E$, ~~let Q be a complement for P in E. Then as $rg(Q) \geq 2$ one~~ ~~since $rg(Q) \geq 2$~~ ~~since $rg(E/P) \geq 2$, one has an isomorphism $E/P \simeq E/H \oplus Q$, hence an isomorphism~~

$$\cancel{H \oplus E/H \simeq E \simeq E/P \oplus P \simeq P \oplus (Q \oplus E/H)}$$

~~and so cancelling an isomorphism $H \simeq P \oplus Q$~~

since $rg(P) < rg(H)$, P is isomorphic to a direct factor of H, say $P \simeq H_1$, where $H = H_1 \oplus Q$. It follows ~~that a complement P' for P in E is isomorphic to a complement of H_1 in E, whence we obtain an isomorph~~ from cancellation that there is an element of G carrying P to H_1. This shows the map $(*)$ is surjective in degrees $p \leq n-2$. Let J be the kernel of $(*)$ in degree $\leq n-2$. It is acyclic in degrees $\leq n-4$, so it will give us a spectral sequence in this range with

$$E^1_{pq} = H_q(GL(E), J_p) \Longrightarrow 0.$$

Now by the exact sequence

$$0 \longrightarrow J_p \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G']} \bigoplus_{\substack{P \subset H \\ rg(P)=p}} I(P) \longrightarrow \bigoplus_{\substack{P \subset E \\ rg(P)=p}} I(P) \longrightarrow 0$$

one gets a long exact sequence

$$H_q(GL(E), J_p) \longrightarrow \bigoplus_\alpha H_q(G'_\alpha, I(P_\alpha)) \longrightarrow \bigoplus_\alpha H_q(G_\alpha, I(P_\alpha)) \rightarrow$$

where $G_\alpha = GL(E, P_\alpha)$ and $G'_\alpha = G_\alpha \cap G'$. But since $P_\alpha \subset H$, if we write $H = P_\alpha \oplus Q_\alpha$, $E = H \oplus L$, we have

$$G'_\alpha = \left(\begin{array}{cc|c} GL(P_\alpha) & * & \\ 0 & GL(Q_\alpha) & * \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right)$$

whence by prop. 1 we get

$$H_q(G'_\alpha, I(P_\alpha)) = H_q(GL(H, P_\alpha), I(P_\alpha))$$

$$\simeq H_*(GL(P_\alpha), I(P_\alpha)) \otimes H_*(GL(H/P_\alpha)).$$

So now assume that provided n is sufficiently large then $H_q(GL(H)) \longrightarrow H_q(GL(E))$ is an iso for $q < r$ and onto for $q = r$. Then this implies that $H_q(GL(E), J_p) = 0$ for $q < r$ by the above sequence

?

December 22, 1973. Stability

In the following Λ is a prime field and homology is understood as having coefficients in Λ. Recall first the basic method for killing unipotent radicals.

One has an extension of a group G by an abelian subgroup

$$1 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

and one wants to prove $H_*(E) \xrightarrow{\sim} H_*(G)$. ~~In the center of G one finds a subgroup C such that~~ It suffices to find a normal subgroup G' of G such that $H_*(E') \xrightarrow{\sim} H_*(G')$ where E' = inverse image of G'. In effect one has a map of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & G/G' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G/G' & \longrightarrow & 1 \end{array}$$

hence a map of spectral sequences

$$\begin{array}{ccc} E^2_{pq} = H_p(G/G', H_q(E')) & \Longrightarrow & H_{p+q}(E) \\ \downarrow & & \downarrow \\ E^2_{pq} = H_p(G/G', H_q(G')) & \Longrightarrow & H_{p+q}(G). \end{array}$$

so that $H_*(E') \xrightarrow{\sim} H_*(G') \implies H_*(E) \xrightarrow{\sim} H_*(G)$.

(If G' is not normal then one uses the covering

$$\Rrightarrow G/G' \times G/G' \rightrightarrows G/G' \longrightarrow pt$$

which gives a spectral sequence

$$E^1_{pq} = H$$

which should show that if G'_α is any intersection of conjugates of G', and E'_α is the inverse image of G'_α, then $H_*(E'_\alpha) \xrightarrow{\sim} H_*(G'_\alpha)$ for all $\alpha \Longrightarrow H_*(E) \xrightarrow{\sim} H_*(G)$.)

In the examples, we will take G' ~~to~~ be a central subgroup of G, usual the multiplicative group B^* of the center B of the ring A under consideration. Also M will be an A-module such that in the extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow B^* \longrightarrow 0$$

the B^*-action on M comes from the underlying B-module structure. <u>Cases</u>:

i) $B \otimes_{\mathbb{Z}} \Lambda = 0$ (i.e. $n.1_B = 0$ with $n \neq 0$ and $\Lambda = \mathbb{Q}$, or $n.1_B = 0$ where $n \not\equiv 0 \pmod p$ $\Lambda = \mathbb{F}_p$.) In this case $H_*(M, \Lambda) = \Lambda$.

ii) $\Lambda = \mathbb{Q}$ and B is a algebra over $\mathbb{Z}[p^{-1}]$. Here

$$H_q(M, \Lambda) = \Lambda^q (M \otimes_{\mathbb{Z}} \mathbb{Q})$$

and $p \in B^*$ acts on $H_q(M, \Lambda)$ as multiplication by p^q. In the spectral sequence

$$E^2_{pq} = H_p(B^*, H_q(M)) \Longrightarrow H_{p+q}(B^* \tilde{\times} M)$$

one considers the effect of conjugating by an element of B

<u>December 24, 1973</u>: <u>Stability revisited</u> (Still groggy).

Suppose A is a ring and I wish to prove that $H_*(GL_n(A), \Lambda)$ stabilizes, where say Λ is a prime field.

Let $X(A^n)$ be the simplicial complex of unimodular vectors in A^n.

<u>Hypothesis 1:</u> The connectivity of the complex $X(A^n)$ $\uparrow \infty$ as $n \to \infty$.

Granted this one gets a complex

$$(*) \qquad \cdots\cdots \longrightarrow C_1(X(A^n)) \longrightarrow C_0(X(A^n)) \longrightarrow \mathbb{Z} \longrightarrow 0$$

which is more and more acyclic as n increases. The group $GL_n(A)$ acts on this complex and so one gets relations between the homology of $GL_n(A)$ acting on the ~~groups~~ different modules of the complex and the homology of the complex. In general, if $K.$ is a chain complex of G-modules, then one has two spectral sequences with common abutment.

$$E^1_{pq} = H_q(G, K_p) \Longrightarrow H_{p+q}(G, K)$$

$$E^2_{pq} = H_p(G, H_q(K)) \Longrightarrow H_{p+q}(G, K)$$

Hypothesis one implies (taking K to be $(*)$, $G = GL_n(A)$), that

$$H_q(K) = 0 \qquad \text{in a range} \qquad q \leq \varphi(n)$$

~~increasing with~~ tending to infinity with n. Thus the second spectral sequence $\Rightarrow$ $H_q(G, K) = 0$ $\quad q \leq \varphi(n)$

in a range tending to infinity with n.

Let $U_p(A^n)$ denote the set whose elements are sequences $(v_1, \ldots, v_p)$ of independent unimodular vectors in A^n (this means that the homo. $A^p \longrightarrow A^n$ defined by the v_i is an injection onto a direct summand of A^n). The symmetric group Σ_p acts freely on $U_p(A^n)$, and clearly one has

$$C_{p-1}(X(A^n)) = \mathbb{Z}[U_p(A^n)] \otimes_{\mathbb{Z}[\Sigma_p]} I_p$$

where I_p is the sign representation of Σ_p. Put

$$X_{p-1}((A^n)) = U_p(A^n)/\Sigma_p$$

for the set of p-simplices of the unimodular complex.

<u>Hypothesis 2</u>: $GL_n(A)$ acts transitively on $U_p(A^n)$ for $p \leq \psi(n)$, where $\psi(n) \nearrow \infty$ as $n \nearrow \infty$.

Given $A^p \xrightarrow{v} A^n$ in $U_p(A^n)$, put $C = \mathrm{Coker}(v)$ so that $A^p \oplus C \xrightarrow{\sim} A^n$. In order that there exist a $\theta \in GL_n(A)$ transforming v into $(e_1, \ldots, e_p)$ it is clearly necessary and sufficient that C be isomorphic to A^{n-p}. Thus hypothesis 2 is equivalent to cancellation:

If $A^p \oplus C \simeq A^n$, then $C \simeq A^{n-p}$ for all $p \leq \psi(n)$ where $\psi(n) \nearrow \infty$ as $n \longrightarrow \infty$.

Let us note this: Assume $GL_n(A)$ trans. on $U_p(A^n)$ for $p \leq r$, i.e. $A^p \oplus C \simeq A^n$, $p \leq r \Longrightarrow C \simeq A^{n-p}$. Then given $A \oplus M \simeq A^{m+1} \Rightarrow A^{n-m} \oplus M \simeq A^n \Rightarrow M \simeq A^m$ provided $n-m \leq r$ or $m \geq n-r$. Thus it would seem that the natural version of Hyp. 2 might be

Cancellation Property: $\exists$ integer λ such that $(A \oplus M \simeq A^{m+1}$ with $m \geq \lambda) \Longrightarrow M \simeq A^m$.

This implies: $(A^p \oplus M \simeq A^n,\ n-p \geq \lambda) \Rightarrow M \simeq A^{n-p}$ which as we have seen is equivalent to having $GL_n(A)$ acts transitively on $U_p(A^n)$ for $p \leq n-\lambda$.

Bass' condition: For all $n \geq \lambda$ If $u = (a_1, \dots, a_n)$ is a unimodular vector in A^n, then there exists $b_1, \dots, b_{n-1} \in A$ such that $(a_1 + b_1 a_n, \dots\dots, a_{n-1} + b_{n-1} a_n)$ is unimodular in A^{n-1}.

Bass' condition says that any unimodular vector in A^n can be moved by a transvection fixing A^{n-1} until it projects to a unimodular vector in A^{n-1}, i.e. such that it is independent of e_n. It implies the cancellation result because if u, v are two independendent unimodular vectors in E, then we have

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A \arrow[r, equal] \arrow[d, "v"] & A \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, equal] & E \arrow[r] \arrow[d] & E/Au \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & E/Av \arrow[r] \arrow[d] & E/Au+Av \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and so

$$E/Av \simeq A \oplus E/Au+Av \simeq E/Au$$

It appears that the Bass condition is not so easy to deduce from Serre's theorem, which can be formulated as follows:

Given M finitely generated over A noetherian, with $X = \mathrm{Max}(A)$ of dimension d, suppose $\dim(M(x)) > d$ for every $x \in X$. Then $\exists\, m \in M$ such that $m(x) \neq 0$ for every $x \in X$.

(Recall the proof for $d = 1$. One chooses $s_1 \in M$ to be $\neq 0$ at one maximal ideal in each irreducible component of X. Then s_1 vanishes at a finite subset of X. One chooses s_2 to be independent of s_1 at one pt in each irred. component of X, and to be $\neq 0$ at the finite set of points where s_1 vanishes. Then one considers $s_1 + f s_2$. Since s_1, s_2 are generically independent, $s_1 + f s_2$ can vanish at the finite set where s_1, s_2 become dependent. At these points either s_1 or s_2 is $\neq 0$, so by choosing f suitably at these points, one wins. (Observe that there is a single bad value for f at each point.))

Error: I assumed above that given $s \in M$ the set of $x \in X \ni s(x) \neq 0$ is open. However if A is local, $M = k$, $s : A \longrightarrow k$ the augm., then $\{x \mid s(x) \neq 0\} = \{\mathfrak{m}\}$ is closed.

Motivation: If E is a vector bundle of rank $\geq d+2$, $d = \dim(X)$, and if u, v are unimodular vectors in E, then to show that u, v are conjugate under $GL(E)$, I must show $E/Au \simeq E/Av$. My idea for doing this is to find a unimodular vector w which is independent of u and of v, and then use that for independent vectors u, w we have isos.

$$E/Au \simeq A \oplus E/Au+Aw$$
$$\simeq E/Aw$$

So I was going to construct w by applying Serre's theorem to $M = E/Au+Av$, whose fibre at each point has rank $\geq d$. It seems now that my scheme doesn't work because d is wrong. Probably trying to find a w independent of u, v is more than one needs.

For example: Suppose A is a dimension 1 and rank $(E) = 3$, and u, v are two unimodular vectors in E. Then I would proceed to construct a section s independent of both u and v. So make the choice s_1 at the generic points whence one knows that there are a finite set of points where (s_1, u) or (s_1, v) is dependent. Next make a choice s_2 ind. of (s_1, u), (s_1, v) at the generic pts, and at the finite set where either (s_1, u) or (s_1, v) become dependent arrange the independence of s_2. Now consider $s_1 + f s_2$. This will be independent of u, v at all points where (s_1, s_2, u), (s_1, s_2, v) are ind. So consider one of the finitely many bad points, and suppose for all values of f that $s_1 + f s_2$ is dep. on either u or v. In part. s_1 is dep. on u or v, so we know s_2 is ind of (s_1, u) and (s_1, v), so for $s_1 + s_2 \in ku$ or kv is impossible. So we wins.

It seems to be desirable to get around the error on page 4. Bass does this by considering instead of dim $M(x)$, he considers the number of free factors of M_x; he calls this the free rank at x.

It appears desirable to review his theory.

First the theory of the Jacobson radical. Put

$$\mathrm{rad}(A) = \{a \mid \forall \text{ simple } A\text{-module } M,\ aM = 0\}$$

~~Since every simple $M \simeq A/\mathfrak{m}$ where $\mathfrak{m}$ is a max. left ideal, and thus~~

~~$\mathrm{Rad}(A) = \{a \mid \forall \mathfrak{m}\ aA \subset \mathfrak{m}\}$~~

~~$= \{a \mid 1 + aA$~~

Since for every $0 \neq m \in M$ simple, one has $A/\mathrm{Ann}(m) \simeq M$ where $\mathrm{Ann}(m)$ is a maximal left ideal, it is clear that

$$\mathrm{rad}(A) = \bigcap \mathfrak{m} \qquad \mathfrak{m} \text{ max. left ideal}$$

But

$a \in \bigcap \mathfrak{m}$, $x \in A \Rightarrow xa \in \bigcap \mathfrak{m} \Rightarrow A(1+xa)$ is a left ideal not contained in any $\mathfrak{m} \overset{\text{(Zorn)}}{\Longrightarrow} A(1+xa) = A \Rightarrow \exists \lambda \ni (1+\lambda)(1+xa) = 1 \Rightarrow \lambda + xa + \lambda xa = 0 \Rightarrow \lambda \in Aa \Rightarrow \exists \mu \ni (1+\mu)(1+\lambda) = 1$. Then by equality of left unit and right unit of $(1+\lambda)$, one sees $(1+\mu) = 1+xa$, so $1+xa \in A^*$.

$\therefore$ We have

$$a \in \bigcap \mathfrak{m} \Longrightarrow (1 + Aa) \subset A^*.$$

But also if $a \notin \mathfrak{m} \Rightarrow Aa + \mathfrak{m} = A \Rightarrow -xa + m = 1 \Rightarrow 1 + xa \in \mathfrak{m} \Rightarrow (1+xa) \notin A^*$. Thus

$$\mathrm{rad}(A) = \bigcap \mathfrak{m} = \{ a \mid (1 + Aa) \subset A^* \}$$

But $\mathrm{rad}(A)$ is a two-sided ideal, so $a \in \mathrm{rad}(A) \Rightarrow$ $aA \subset \mathrm{rad}(A) \Rightarrow (1 + aA) \subset A^* \Rightarrow a \in \bigcap$ max. right ideals.
And by symmetry, one thus has

$$\mathrm{rad}(A) = \bigcap_{\substack{\text{max} \\ \text{left}}} \mathfrak{m} = \bigcap_{\substack{\text{max} \\ \text{right}}} \mathfrak{m}$$

$=$ smallest ideal modulo which any element $\equiv 1$ is a unit.

Now consider a semi-local ring A, i.e. such that $A/\mathrm{rad}(A)$ is semi-simple. Suppose we have

$$A \oplus M \simeq A \oplus M'.$$

~~Consider~~ ~~$A \oplus M \simeq A \oplus M' \longrightarrow A$~~

Consider the map

$$f: \quad A \oplus M \simeq A \oplus M' \xrightarrow{pr_1} A$$
$$(x, m) \longmapsto xa + \lambda(m)$$

To prove that $M \simeq M'$ it will suffice to split the sequence

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & A \oplus M \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[l, dotted, bend right, "s"'] \arrow[dl, dotted, "fs"] & 0 \\
& & A & &
\end{tikzcd}$$

such that fs is an isomorphism. Thus I have to find $s(1) = 1 + m$ such that $f(s(1)) = a + \lambda(m)$ is a unit in A. Thus I want to prove

<u>Bass Lemma</u>: If $Aa + I = A$, where I is a left ideal, then $a + I$ contains a unit.

Proof: First note that if $u \in A$ is a unit modulo rad(A), then u is a unit. In effect $\exists v \ni uv, vu \in 1 + \mathrm{rad}(A) \subset A^* \Longrightarrow u \in A^*$. Thus to prove the lemma we can replace A by $A/\mathrm{rad}(A)$, and hence we can suppose A semi-simple. Since A, I, etc. decompose according to the simple components of A, one can suppose A is simple, hence a matrix algebra over a skew-field.

Since A is semi-simple, one can find a left ideal $I' \subset I$ such that $Aa \oplus I' = A$. (In effect write I as a sum of minimal left ideals and take I' to be a minimal sum $\ni Aa + I' = A$.) Then since A is simple, we have

$$A = \underbrace{l_1 \oplus \cdots \oplus l_p}_{Aa} \oplus \underbrace{l_{p+1} \oplus \cdots \oplus l_n}_{I'}$$

and so we can identify A with ~~$(n \times n)$ matrices with coefficients in a skew field D, such that~~ the ring of endos of a vector space V over a skew-field D, Aa, I' with the left ideals of endos. vanishing on W_1, W_2 resp. where $V = W_1 \oplus W_2$. In particular $\mathrm{Ker}(a) = W_1$, and so if we choose b so that $\mathrm{Im}(b)$ is complementary to $\mathrm{Im}(a)$, $\mathrm{Ker}(b) = W_2$, then $b \in I'$ and $a+b$ is an isomorphism. QED.

December 27, 1973 Stability

Let V be a vector space over a field and W a subspace. Let $Y_2(V,W)$ be the simplicial complex whose vertices are 2-dimensional subspaces L of V such that $L \cap W = 0$, and whose simplices are subsets $L_1, \dots, L_q$ of 2-dimensional subspaces such that the sum $L_1 + \dots + L_q + W$ is direct, i.e. of dim $2q + \dim W$. Then $Y_2(V,W)$ is a simplicial complex of dimension $\left[\frac{1}{2}\operatorname{cod}(W)\right] - 1$ which I would like to show is spherical, using induction on the codimension of W in V.

Thus let $e \notin W$ and divide the simplices σ of $Y_2(V,W)$ into two groups: ~~[illegible]~~ according to whether $e \in W + k\sigma$ or not. As $W + k\sigma = L_1 \oplus \dots \oplus L_q \oplus W$, those σ such that $e \notin W + k\sigma$ form the subcomplex

$$Y_2(V, ke+W) \subset Y_2(V,W)$$

If $\overset{(L_1, \dots, L_q)}{\overset{\shortparallel}{\sigma}}$ is not in the subcomplex, then

$$e \in L_1 \oplus L_2 \oplus \dots \oplus L_q \oplus W$$

and so one sees that σ has a minimum face with this property. Let Γ be the set of $\sigma = (L_1, \dots, L_q)$ in $Y_2(V,W)$ such that $e \in L_1 \oplus \dots \oplus L_q \oplus W$ and such no proper subset of σ has this property. Thus

$$Y_2(V,W) - Y_2(V, ke+W) = \coprod_{\sigma \in \Gamma} \text{Open star}(\sigma)$$

Now if $\sigma \in \Gamma$, then the link of σ is simply $Y_2(V, k\sigma + W)$, which is a subcomplex of $Y(V, ke+W)$. Thus using our earlier analysis of the situation, ~~we have a cocartesian square~~

we know that $Y_2(V, W)$ is obtained from $Y_2(V, ke+W)$ by attaching a cone on $S^{\dim \sigma} Y(V, k\sigma + W)$ for every $\sigma \in \Gamma$. In particular we have a cofibration

$$Y_2(V, ke+W) \longrightarrow Y(V, W) \longrightarrow \bigvee_{\sigma \in \Gamma} S^{\mathrm{card}(\sigma)} Y(V, k\sigma + W)$$

Now I want to see if this implies that $Y(V, W)$ is spherical by induction on codimension.

Put $\mathrm{cod}(W) = m = 2j + \begin{cases} 1 \\ 0 \end{cases}$ so that $Y(V, W)$ has dimension $j-1$. Then for $\sigma \in \Gamma$, $k\sigma + W$ has codimension $m - 2\,\mathrm{card}(\sigma)$, so $Y(V, k\sigma + W)$ by induction will be a bouquet of $(j - \mathrm{card}(\sigma) - 1)$-spheres and so $S^{\mathrm{card}(\sigma)} Y(V, k\sigma + W)$ will indeed be a bouquet of $j-1$ spheres. Unfortunately $Y_2(V, ke+W)$ will give us trouble, for when m is even, we will have to show the inclusion $Y_2(V, ke+W) \longrightarrow Y_2(V, W)$ is null-homotopic.

For example consider low codimensions

$$\left.\begin{array}{l} m=0 \\ m=1 \end{array}\right\} \quad Y(V,W)=\phi$$

$$\left.\begin{array}{l} m=2 \\ \;\;=3 \end{array}\right\} \quad Y(V,W) \text{ is a non-empty set}$$

So the next case is $m=4$. So consider $V = ke_1 + \cdots + ke_4$, $W=0$. Then $Y(V,0)$ is a graph whose vertices are the 2 planes in V and whose edges are pairs of complementary 2-planes. I want to show this graph is connected.

~~Now the preceding argument shows that $Y(V,0)$ is obtained from $Y(V, ke_1+0)$, which is a set of points, by attaching a cone with base $Y(V,L)$ for each 2 plane L containing e_1.~~
~~(Review: I consider each σ such that $\sigma \in Y(V,0)$~~
~~i.e. such that $e_1 \in k\sigma$~~

Now as before one considers $Y(V,0) - Y(V,ke_1)$, i.e. one considers those σ such that $e_1 \in k\sigma$, and one lets Γ be the minimal such σ. There are 2-types:
<u>vertices</u>: these are 2-planes L containing ke_1
<u>1-simplices</u>: these are complementary 2-planes L_1, L_2 such that $e_1 \notin L_1$, $e_1 \notin L_2$.

To get $Y(V,0)$ one attaches the cone on the set of L' complementary to L for each $L \supset ke_1$ and one joins a 1-simplex between each comp. pair L_1, L_2 such that $e_1 \notin L_1, e_1 \notin L_2$.

December 29, 1973.

Poincaré duality
Zeeman spectral sequence

~~Lemma: Let X be a simp complex which is spherical and whose links are spherical. Let S be a set of~~

It is clearly necessary at this point to have the sorites on polyhedra with spherical links.

General case: Suppose X is a nice space such as a finite polyhedron. What does duality mean for X? How does one prove Poincaré duality for a manifold.

If X is a C^∞-manifold, one computes its real cohomology using C^∞-forms, that is, by means of the soft resolution

$$0 \longrightarrow \mathbb{R} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots\cdots$$

where $I^q = \Omega^q$. One also uses this resolution to compute the cohomology with compact supports. Thus

$$H^q\{\Gamma_c(U, I^\cdot)\} = H^q_c(U, \mathbb{R}).$$

Now this is covariant in the open set U. So if V is an $\mathbb{R}$-module, one gets a presheaf

$$\mathrm{Hom}(\Gamma_c(U, I^\cdot), V)$$

in fact a complex of presheaves.

Lemma: $U \longmapsto \mathrm{Hom}(\Gamma_c(U, I^q), V)$ is a sheaf.

The proof roughly amounts to the following. It suffices to verify the sheaf property for a locally finite family $\{U_i \mid i \in I\}$.

Denote by F_U the sheaf obtained by restricting F to U and then extending by zero, so that $\Gamma(X, F_U) = \Gamma_c(U, F)$ when X is compact. Then we have an exact sequence of sheaves

$$\longrightarrow \bigoplus_{i,j} F_{U_{ij}} \longrightarrow \bigoplus_i F_{U_i} \longrightarrow F_U \longrightarrow 0$$

if $U = \bigcup U_i$. If F is soft so that $H_c^+(U, F) = 0$, then the same is true of the sheaves F_{U_i}. If then X is of finite coh. dimension, the above complex is then a resolution of arb. length length of the kernel sheaf Z_n, etc. So if X is of fin. coh. dimension, we conclude that on applying Γ, it is exact: i.e.

$$\longrightarrow \bigoplus_{i,j} \Gamma_c(U_{ij}, F) \longrightarrow \bigoplus \Gamma_c(U_i, F) \longrightarrow \Gamma_c(U, F) \longrightarrow 0$$

which shows that $U \longmapsto \mathrm{Hom}(\Gamma_c(U, F), V)$ is a sheaf.

Now if $U' \subset U$, then $\Gamma_c(U', F) \subset \Gamma_c(U, F)$, so as V is injective,

$$\mathrm{Hom}(\Gamma_c(U, F), V) \longrightarrow \mathrm{Hom}(\Gamma_c(U', F), V)$$

is surjective. Thus $U \mapsto \mathrm{Hom}(\Gamma_c(U, I), V)$ is a flask sheaf. So therefore I find that

$$U \longmapsto \mathrm{Hom}(\Gamma_c(U, I^\cdot), V)$$

is a complex of flask sheaves, which if I denote this complex by $\omega(V)$, then I have

$$\Gamma(U, \omega(V)) = \mathrm{Hom}(\Gamma_c(U, I^\cdot), V)$$

so taking homology groups of these two complexes

$$H^{-i}(U, \omega(V)) = \mathrm{Hom}(H^i_c(U, \mathbb{R}), V)$$

But now one can use this formula to compute the stalks of $\mathcal{H}^{i}(\omega(V))$. Thus

$$\begin{aligned} \mathcal{H}^{-i}(\omega(V))_x &= \varinjlim_{U \ni x} H^{-i}(U, \omega(V)) \\ &= \varinjlim_{U \ni x} \mathrm{Hom}(H^i_c(U, \mathbb{R}), V) \end{aligned}$$

X manifold of dim n

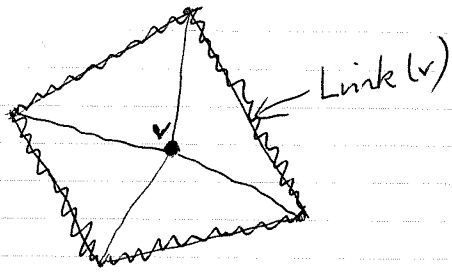
$$= \begin{cases} 0 & i \neq n \\ \omega \otimes V & i = n \end{cases}$$

where ω is the orientation sheaf. Thus we get

$$H^{n-i}(U, \omega \otimes V) = \mathrm{Hom}(H^i_c(U, \mathbb{R}), V)$$

which is the usual Poincaré duality formula.

Now I am above all interested in the case of a simplicial complex of dimension n, whose links are spherical. Now in a simplicial complex the local geometry around a point is given by the link of the open simplex containing the point. Picture for a vertex



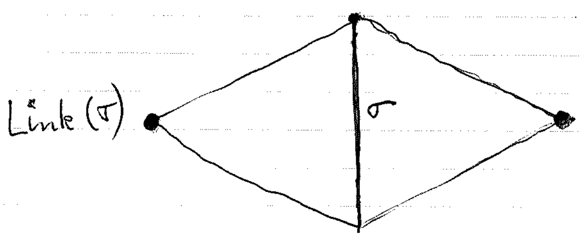
$$\text{Closed Star}(v) = \text{Link}(v) * v$$

This is a cone on Link(v).

Then

$$H^*_{\{v\}}(X, A) = H^*(\text{Cone}(\text{Link}(v)), \text{Link}(v); A) = \tilde{H}^*(\text{Susp}\{\text{Link}\, v\}; A)$$

In the general case, suppose x belongs to an open simplex σ. Picture:



so
$$\text{Closed star}(\sigma) = \text{Link}(\sigma) * \sigma$$

and
$$\begin{aligned} \partial\,\text{Closed star}(\sigma) &= \text{Cl st}(\sigma) - \text{Op st}(\sigma) \\ &= \text{Link}(\sigma) * \partial\sigma \\ &= \underbrace{S^0 * \cdots * S^0}_{\dim(\sigma)\text{ - times}} * \text{Link}(\sigma) \end{aligned}$$

It is clear from the picture that if we have an interior point x of σ, then the closed star of σ is the cone on its boundary with base $\mathrm{Susp}^{\dim(\sigma)}\{\mathrm{Link}(\sigma)\}$, so

$$H^*_{\{x\}}(X, A) = \tilde{H}^*(\mathrm{Susp}^{\dim(\sigma)+1}\{\mathrm{Link}(\sigma)\}; A)$$

As a check if X is a manifold of dim n, and if σ has dim q, then $\mathrm{Link}(\sigma)$ has dimension $n-q-1$, and so $\mathrm{Susp}^{\dim(\sigma)+1}\{\mathrm{Link}(\sigma)\} \sim S^{q+1}(S^{n-q-1}) = S^n$ as it should be.

Suppose now I try to understand duality for a polyhedron, and then afterward, for a nice partially ordered set. Start with a polyhedron X, and work with the category of sheaves on X which are constant over each open simplex; these are the same as covariant functors to abelian groups on the category of simplices ordered by inclusion. Thus

$$\sigma \longmapsto \underset{\substack{\text{open star} \\ \text{of } \sigma}}{U_\sigma} \longmapsto \Gamma(U_\sigma, F)$$

is a composition of two contravariant functors.

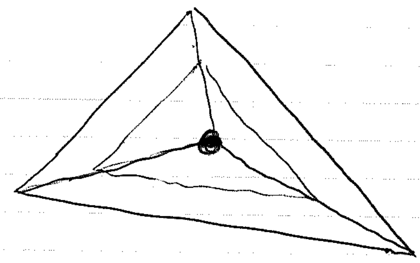
To compute $H^*(X, F)$, we can use the Čech covering given by the open stars of the vertices. Not immediately promising, because as yet we don't know that over U_σ any such F has trivial cohomology.

~~So the other method is to filter by the skeleta~~:

$$\cancel{\phi = X_{(-1)} \subset X_{(0)} \subset X_{(1)} \subset \cdots \subset \cdots}$$

~~which are closed sets.~~

But we can argue as follows: Let us use the local conical structure of X



to get a cofinal system of nbds of the point x.

$$\mathcal{U}_t = \{ z \in X \mid \lambda_x(z) > t \}$$

where λ_x is the coordinate barycentric of z at the vertex x. Then as $t \uparrow 1$, the $\mathcal{U}_t$ shrink down to x. And we know that

$$\varinjlim_t H^q(\mathcal{U}_t; F) = \begin{cases} F_x & q = 0 \\ 0 & q > 0 \end{cases}$$

so the only thing to prove is that because F is constant on the open simplices, one has $H^q(\mathcal{U}_t, F) \xrightarrow{\sim} H^q(\mathcal{U}_{t'}, F)$ for $t < t'$. But this is clear by homotopy. Consider the radial deformation $\mathcal{U}_t \times I \xrightarrow{h} \mathcal{U}_t$ which shrinks $\mathcal{U}_t$ down to $\mathcal{U}_{t'}$. But the assumption on F, if we pull F back to $\mathcal{U}_t \times I$ by h we get the same as pulling it back via $pr_1 : \mathcal{U}_t \times I \to \mathcal{U}_t$. Rest is clear from the spectral sequence for h.

So it follows that for any σ

$$H^q(\mathcal{U}_\sigma, F) = \begin{cases} F(\mathcal{U}_\sigma) = \text{stalk of } F \text{ at any int. point of } \sigma & q = 0 \\ 0 & q > 0 \end{cases}$$

and so we can use

the ~~open~~ open covering given by stars of vertices. Thus we get a complex

$$C^q(X,F) = \prod_{\dim(\sigma)=q} F(U_\sigma)$$

and

$$H^*(X,F) = H^*(C^\bullet(X,F)).$$

We have proved:

Prop: Let X be a simplicial complex, and let F be a sheaf which is constant on each open simplex. Then if U_σ is the open star of the simplex σ, we have

$$H^q(U_\sigma, F) = \begin{cases} F(U_\sigma) & q=0 \\ 0 & q>0 \end{cases}$$

This implies that if we let Y be the space which is the quotient of X obtained by collapsing each open simplex to a point (say X is finite to avoid problems), then the above says

$$R^q f_*(f^*G) = \begin{cases} G & q=0 \\ 0 & q>0. \end{cases}$$

whence it follows that cohomology of F coincides with derived functors of $\varprojlim$ on the cat of functors on the ordered set of simplices.

Now take up duality. First with field coefficients Λ assuming that the links of all points are spherical of dimension $n-1$. Then one considers

$$U \longmapsto \mathrm{Hom}(H^i_c(U,\Lambda),\Lambda).$$

Precisely, I should know that

$$U \longmapsto \mathrm{Hom}(H^n_c(U,\Lambda),\Lambda)$$

is a sheaf, which is constant on each open simplex. If I call this ω, then one has an isom.

$$H^0(U,\omega) = \mathrm{Hom}(H^n_c(U,\Lambda),\Lambda)$$

which one wants to extend to higher dimensions.

To do this one needs a resolution to compute the cohomology with compact supports.

<u>Example</u>: suppose to get an understanding, that I assume for each point $x \in X$, that

$$H_q(X, X-x;\mathbb{Z}) = \begin{cases} 0 & q \neq n \end{cases}$$

where $n = \dim X$. From our local discussion above this means that the links are spherical.

Then put

$$\omega_x = H_n(X, X-x;\mathbb{Z})$$

I claim this is a sheaf constant on each open simplex. The way to see this is to notice, that by homotopy $\omega_x = H_n(X, X - U_x;\mathbb{Z})$

where U_x is the open star of x. Thus if x specializes to y in the sense that y belongs to the smallest closed simplex containing x, then $U_y \supset U_x$, hence $X - U_y \subset X - U_x$ and so we have a map $\omega_y \longrightarrow \omega_x$. In fact I guess the way to say this is to say that

$$U \longmapsto H_n(X, X-U; \mathbb{Z})$$

is a sheaf on X constant on each open simplex.

But now consider the complex used to compute $H_*(X, X-U; \mathbb{Z})$, namely chains on X modulo those vanishing on $X-U$.

(Note: with field coefficients instead of $\mathbb{Z}$

$$H_q(X, X-U) \text{ dual to } H^q(X, X-U) = H^q_c(U))$$

$$C_q(X, X-U; \mathbb{Z}) = \bigoplus_{\substack{\sigma \subset U \\ \dim(\sigma)=q}} H_q(\bar{\sigma}, \partial\sigma; \mathbb{Z})$$

Thus this is a flask sheaf. Thus one finds that

$$U \longmapsto C_{-q}(X, X-U; \mathbb{Z})$$

is a flask complex of sheaves on X, so we get a spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{H}_{-q}) \Longrightarrow H_{-p-q}(X, X-U)$$

where $\mathcal{H}_{-q}$ is the sheaf associated to the presheaf

$$U \longmapsto H_{-q}(X, X-U; \mathbb{Z})$$

hence its stalk at x is $H_{-q}(X, X-x; \mathbb{Z})$.

<u>Summary</u>: Let X be a finite simplicial complex. If U is a simplicial open subset of X, that is, the complement of a subcomplex, or equivalently a union of open stars of simplices, put

$$\omega^q(U) = C_{-q}(X, X-U; \mathbb{Z})$$

Then $U \mapsto \omega^{\cdot}(U)$ is a complex of flask simplicial sheaves on X such that for any U:

$$\boxed{H^q(U, \omega^{\cdot}) = H_{-q}(X, X-U; \mathbb{Z})}$$

So one gets the spectral sequence of Zeeman

$$E_2^{pq} = H^p(U, \mathcal{H}_{-q}) \Longrightarrow H_{-p-q}(X, X-U; \mathbb{Z})$$

$$\mathcal{H}_i = \text{sheaf assoc. to presheaf } U \mapsto H_i(X, X-U; \mathbb{Z})$$

$\omega^{\cdot}$ is the dualizing complex. One has more generally a duality formula (style Groth-Verdier)

$$R\,Hom_{/pt}\big(R\Gamma(X,F), G\big) = R\,Hom_{/X}(F, \omega \otimes G)$$

So now I suppose X spherical of dim n so that $\omega^{\cdot}$ is concentrated in degree $-n$. Put ω_X for this sheaf, so that the spectral sequence degenerates yielding:

$$H^i(U, \omega_X) = H_{n-i}(X, X-U; \mathbb{Z})$$

and in the case of field coefficients this gives:

$$H^i_\bullet(X, F) \text{ dual to } \mathrm{Ext}^{n-i}(F, \omega_X \otimes \Lambda)$$

or

$$H^i_c(U, \Lambda) \text{ dual to } H^{n-i}_\bullet(U, \omega \otimes \Lambda)$$

The point is that the duality is given by a trace map $H^n_c(X, \omega_X) \longrightarrow \mathbb{Z}$ in general.

December 30, 1973. Grassmannians again.

Suppose E and Q are two vector bundles over X and that $\theta : E \longrightarrow Q$ is a generic homomorphism. ('generic' means not special in any way. In the present case it means that θ as a section of $\mathrm{Hom}(E,Q)$ is transversal to the stratification by rank).

I think for the following it is enough to assume that for each x the derivative map

$$d\theta(x) : T_X(x) \longrightarrow \mathrm{Hom}\left(\mathrm{Ker}\,\theta(x), \mathrm{Coker}\,\theta(x)\right)$$

is onto. This should amount to transversality to the strata which $\theta(x)$ lies on.

For each r we therefore have the cycle where the rank drops by r or more

$$Z_r = \{ x \mid \dim \mathrm{Ker}\,\theta(x) \geq r \}$$

and its desingularization

$$\tilde{Z}_r = \{ (x, A_r) \mid A \text{ is a } r\text{-plane} \subset \mathrm{Ker}\,\theta(x) \}$$

(The reason this should be non-singular is that is the pull back of a desingularization of the same stratum in $\mathrm{Hom}(E,Q)$, by a map which is transversal to this stratification.)

I want to compute the coh. class in $H^*(X)$ determined by the cycle Z_r. First of all

$$\mathrm{codim}(Z_r) = \dim \mathrm{Hom}(\mathrm{Ker}, \mathrm{Coker}) = r(r + \mathrm{rg}\,Q - \mathrm{rg}\,E)$$

Now introduce the bundle of flags in E

$$\underbrace{G_{1 1 \cdots 1}}_{r}(E) = \{(x, l_1 \subset \cdots \subset l_r \subset E(x))\}.$$

has $\dim = \dim(X) + \cdots + (e-1) + \cdots + (e-r)$

$e = \dim(E)$

and the sequence of submanifolds

$$W_1 = \{(x, l_1 \subset \cdots \subset l_r \subset E(x)) \mid \theta(x)(l_1) = 0\}$$

$$\cup$$

given by vanishing of $\theta \in Hom(F_1, Q)$

$$\{ \quad '' \quad \mid \theta(x)(l_2) = 0\}$$

given by vanishing of section of $Hom(F_2/F_1, Q)$ ind. by θ.

$$\cup$$

$$\cup$$

$$W_r = \{ \quad '' \quad \mid \theta(x)(l_r) = 0\}$$

Thus the coh. class in $G_{11\cdots1}(E)$ belonging to W_r should be

$$e(Hom(F_1, Q)) \cdot \cdots \cdots e(Hom(F_r/F_{r-1}, Q))$$

$$= \prod_{i=1}^{r} (T_i^{\,q} + \cdots + c_q(Q))$$

has dim qr.

Now W_r is the full flag bundle of the vector bundle $(x, A_r) \longmapsto A_r$ over $\tilde{Z}_r$. So to get the coh. class in X corresponding to $\tilde{Z}_r$, call it $[Z_r]$, I can take $[Z_r] \cdot 1 = f_*(\alpha f^*[Z_r])$ where $f_*(\alpha) = 1$. $\alpha = T_1^{r-1} \cdots T_{r-1}$

$$\begin{tikzcd}
\{(x, l_1 \subset \cdots \subset l_r \subset Ker\, \theta(x))\} = W_r \arrow[r] \arrow[d] & G_{1\cdots1}(E) = \{x, l_1 \subset \cdots \subset l_r \subset E(x)\} \arrow[d, "f"] \\
\{x, l_r \subset Ker\, \theta(x)\} = \tilde{Z}_r \arrow[r] & X
\end{tikzcd}$$

So one should have

$$[Z_r] = \operatorname{res} \frac{T_1^{r-1}\left(T_1^{q}+\dots+c_q Q\right)}{T_1^{e}+\dots+c_e(E)}\, dT_1 \cdots \operatorname{res} \frac{\left(T_r^{q}+\dots+c_q Q\right)\, dT_r}{T_r^{e-r+1}+\dots+c(E/F_{r-1})}$$

and so since

$$T_r^{r-1}+\dots+c_{r-1}(F_{r-1}) = \prod_{i=1}^{r-1}(T_r - T_i)$$

one gets

$$[Z_r] = \operatorname{res} \frac{\prod_{i=1}^{r} T_i^{q}+\dots c_q(Q)}{\prod_{i=1}^{e} T_i^{e}+\dots+c_e(E)}\ \overbrace{\left(T_1^{r-1}\cdots T_{r-1}^{1}\cdot 1\right)\prod_{i<j}(T_j - T_i)}^{\begin{vmatrix} T_1^{r-1} & \cdots & 1 \\ \vdots & & \vdots \\ T_1^{2r-1} & \cdots & T_r^{r-1} \end{vmatrix}} dT_1 \cdots$$

But if $E \oplus E' = N$ is trivial, then

$$\frac{T_i^{q}+\dots+c_q Q}{T_i^{e}+\dots+c_e E} = T_i^{N+q-e}+\dots+c_{N+q-e}(Q\oplus E')$$

so now it should be clear that we get

<u>Thm</u>: Let Z_r be the cycle where the rank of a generic homomorphism $\theta: E \to Q$ drops by r. Then the coh. class corresponding to Z_r is

$$\begin{vmatrix} c_{q-e+r}(Q-E) & \cdots & c_{q-e+1}(Q-E) \\ \vdots & \ddots & \vdots \\ c_{q-e+2r-1}(Q-E) & \cdots & c_{q-e+r}(Q-E) \end{vmatrix}$$

In particular one gets the formula derived before when E is trivial. Special cases: $\dim(E) = \dim(Q)$. Then the classes are the determinants:

$$r=1 \qquad c_1(Q-E)$$

$$r=2 \qquad \begin{vmatrix} c_2(Q-E) & c_1(Q-E) \\ c_3(Q-E) & c_2(Q-E) \end{vmatrix}$$

etc.

Now to see if these formulas are of any use for the MacPherson problem. Recall that if $f: X \longrightarrow Y$ is proper & smooth with Y connected of rel. dim d, and if χ is the Euler characteristic of any fibre of f, then

$$f_* c_i(X) = \chi \cdot c_{i-d}(Y).$$

In effect

$$f_* c_t(X) = f_* c_t(\tau_f) \cdot f^* c_t(\tau_Y)$$
$$= f_* c_t(\tau_f) \cdot c_t(\tau_Y)$$

and

$$f_* c_t(\tau_f) = t^d f_*(c_d(\tau_f)) = t^d \cdot \chi$$

Now MacPherson has proved this formula to be true for any proper map provided the Euler characteristic of any fibre of f is constant with value χ. If we put $\tau_f = \tau_X - f^*\tau_Y$ as a virtual bundle, one has

$$c_t^{-1}(Y) \cdot f_* c_t(X) = f_*\left(\frac{c_t(X)}{f^* c_t(Y)}\right) = f_*\left(c_t(\tau_f)\right)$$

so what he has proved is that $f_*(c_t(\tau_f)) = 1$
if $\chi(f^{-1}(y)) = 1$ for all y.

Now we have some information on the Chern classes of τ_f using the map $df : \tau_X \to f^*\tau_Y$. In particular if I assume that df is generic, then I know the cohomology classes of the singularity sets.

~~Translate the problem by factoring~~.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Y \times P \arrow[d, "p"] \\ & Y \end{tikzcd}$$

?

~~Why not as a start assume X is a complete intersection.~~

December 30, 1973

Removing simplices in a quasi-spherical complex

Prop: Let X be an n-diml simplicial complex whose links are spherical. Let S be a set of simplices of X such that if σ, σ' are different members of S, then $\sigma \cup \sigma'$ is not a simplex of X. Let X' be the complement of the open stars of the simplices in X. (Thus our assumptions imply

$$X - X' = \coprod_{\sigma \in S} \operatorname{Op\,st}(\sigma)$$

and hence X is obtained from X' by attaching a cone with vertex σ on $\partial\sigma * \operatorname{Link}(\sigma) \subset X'$ for each $\sigma \in S$, and so we have a cofibration

$$(*) \qquad X' \longrightarrow X \longrightarrow \bigvee_{\sigma \in S} \operatorname{Susp}^{\dim(\sigma)+1} \operatorname{Link}(\sigma))$$

Assume now that X' has dimension $n-1$. Then one has the implications:

$$X \text{ spherical} \iff X' \text{ spherical} \ \& \ X' \to X \text{ null-homotopic}$$

Proof: The homology picture is clear from the exact sequence

$$0 \to \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/X') \longrightarrow \tilde{H}_{n-1}(X') \longrightarrow \tilde{H}_{n-1}(X) \to 0$$

Now to prove $\Rightarrow$: The fact $X' \to X$ is null-homotopic because X' is contained in the n-skeleton. To show X' spherical, suppose given $f: K \to X'$ with $\dim(K) \le n-2$. Then in X this map extends to $C(K)$ which has dim. $\le n-1$.

Because links are spherical, one can push this map off points, hence off the barycenters of the σ in S, and so one can push it down into X'.

Next ($\Leftarrow$). Given $K \longrightarrow X$ with $\dim K \leq n-1$, we see as before that it can be pushed into X'. But then it contracts to a point in X.

~~Summary: It seems that under the assumption that all the links of X are spherical, and that X is a retract~~

~~NEW TERMINOLOGY: QUASI-SPHERICAL FOR A BOUQUET OF SPHERES.~~

~~Summary. Geometrically X is supposed to have (quasi-) spherical links of dim (n−1), and on removing a set of points it deforms down to X' which is of dimension (n−1). Then one has the implications~~

~~X quasi-spher. $\Longrightarrow$ X' quasi-sphr. & $X' \to X$ null-hom.~~

~~$X' \to X$ null-homot $\Longrightarrow$~~

Suppose $\dim X = n$ and X is obtained from X' by attaching cones on bouquets of $(n-1)$-spheres. If $X' \to X$ is null-homotopic, then X is a bouquet of n-spheres.

<u>Proof</u>: Let $f: K \longrightarrow X$ be a map where $\dim(K) < n$. Then because the link of the vertex of each attached cone is $(n-2)$-connected, one can deform f off of these points, hence down into X'. But then it contracts to a point.

Corollary: If X is a simplicial complex of dimension n with quasi-spherical links of dimension $(n-1)$, and if on removing a set of points from X the rest contracts to a point in X, then X is a bouquet of n-spheres.

This may be useful, because in the past you have always tried to show X' is contractible.

Suppose now I let $0<p_0<\cdots<p_r<n$ and denote by $T_{p_0,\dots,p_r}(V)$ the simplicial complex associated to the ordered set of subspaces of W of dimensions $\neq p_0,\dots,p_r$. Thus where one has ~~$0<1<\cdots<n-1<n$~~ $r=0$ one gets the Tits building $T(V)$.

I want to show that $T_{p_0,\dots,p_r}(V)$ is quasi-spherical of dim ~~r~~ $n-r-2$ by decreasing induction on r, starting from the fact that it is true for ~~$r=n-1$~~, $r=0$.

Since $T_{p_0,\dots,p_r}(V)$ is obtained from $T_{p_0,\dots,p_{r-1}}(V)$ by removing the vertices corresp. to the subspaces of dimension p_r, and since the dimension drops by one, all I have to do is check that the links are q-spherical of the vertices I remove. Actually it would be nice to check that every link of $T_{p_0,\dots,p_r}(V)$ is q-spherical. Now suppose $\sigma: 0<W_{q_0}<\cdots<W_{q_s}<W$ is a simplex of $T_{p_0,\dots,p_r}(V)$, i.e. $\{p_0,\dots,p_r\}\cap\{q_0,\dots,q_r\}=\emptyset$. Then its link is the simp. complex assoc. to the ordered set of subspaces W which refine σ and which are not of dim $= p_0,\dots,$ or p_r. Then if we divide

$$\underbrace{p_0\cdots p_a}_{q_0}\quad \underbrace{p_{a+1}\cdots p_b}_{q_1}\quad \underbrace{\qquad\qquad}$$

this link is the join of $T_{p_0,\dots,p_a}(W_{q_0})$ with $T_{p_{a+1}-q_0,\dots,p_b-q_0}(W_{q_1}/W_{q_0})$ with ...

and so one wins by induction.

December 31, 1973 Euler characteristics, and Stiefel-Whitney homology classes.

~~Consider the following question.~~

Let Z be a cycle embedded in a sphere S. Consider now a map $f: M \longrightarrow S$ where M is a closed manifold. Then I can pull Z back by f, at least after moving f transversal to Z in some sense, and take the Euler characteristic of the inverse image $f^{-1}(Z)$. I would like to have conditions guaranteeing that the result depends only on the bordism class of f.

First case to understand is where $Z \subset S$ is a submanifold. Then $f^{-1}(Z)$ is a manifold which is closed, and so I can compute its Euler class by integrating the highest Stiefel-Whitney class of its tangent bundle, in fact the whole S-W class

$$\begin{tikzcd}
f^{-1}(Z) \arrow[r, "f'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\
M \arrow[r, "f"] \arrow[d, "\pi"'] & S \arrow[dl] \\
pt &
\end{tikzcd}$$

$$f'^{*}\nu_i = \nu_{i'}$$

$$\tau_{f^{-1}(Z)} + \nu_{i'} = i'^{*}\tau_M$$

$$\chi(f^{-1}Z) = \int_{f^{-1}(Z)} w(\tau_{f^{-1}(Z)}) = \pi_* i'_* \, w(\tau_M - \nu_{i'})$$

$$= \pi_* i'_* \left(i'^{*} w(\tau_M) / f'^{*} w(\nu_i) \right)$$

$$= \pi_* \left(w(\tau_M) \cdot f^* i_* w^{-1}(\nu_i) \right)$$

$$= \int_S \left(f_* w^{-1}(\nu_M) \cdot i_* w^{-1}(\nu_i) \right).$$

It is now clear that the numbers we get depends only on $f_*(\omega^{-1}(\nu_M)) = f_*\,\omega(\tau_M) \in H^*(S)$. Now I want to interpret this properly, so it will be convenient to write down where the things are to end up.

Ultimately for a space X satisfying some local conditions roughly to the effect that the Euler characteristics of the links are 0 mod 2, one will have global Stiefel-Whitney classes which will be homology classes with infinite support. For example if X is a manifold, its fundamental class mod 2 is such a class. Put another way we expect to have classes

$$w_i \in H_i(X, X-U) \qquad \text{mod } 2$$

if U is a relative compact open set such that the local conditions are true inside of U. In cohomological terms, then, one expects classes

$$w^{-i} \in H^{-i}(X, \omega_X^{\bullet})$$

where $\omega_X^{\bullet}$ is the dualizing complex. Thus if X is a manifold of dimension n, we have

$$\omega_X^{\bullet} = \mathbb{Z}/2\,[n]$$

so

$$H^{-i}(X, \omega_X^{\bullet}) = H^{n-i}(X)$$

and $w^{-i} =$ $(n-i)$-th Stiefel-Whitney class of τ_X.

(Now the amazing thing is that these classes will come out of the Euler characteristic.)

Suppose now we return to the first calculation. To define the S-W classes for X, I embed X in a manifold S. Then the images of the S-W classes of X will be homology (cocompact) classes which correspond to cohomology classes of S which I can get at by intersection.

To fix the ideas suppose $X \overset{i}{\hookrightarrow} S$ is a closed submanifold. Then for any closed man. M and map $f: M \longrightarrow S$, one forms $f^{-1}(X)$ after moving transversally. Then we have

$$\chi(f^{-1}X) = \int_{f^{-1}X} w(\tau_{f^{-1}X})$$

$$= \int_S f_* w^{-1}(\nu_M) \cdot i_* w^{-1}(\nu_i)$$

~~In other words, the operation $[M,f] \longmapsto \chi(f^{-1}X)$ factors through the map~~

$$M \longmapsto f_* w^{-1}(\nu_M)$$

$$MO_c^*(S) \longrightarrow H_c^*(S). \qquad \text{(not degree-preserving)}.$$

~~Now one has that~~

$$\mathbb{Z}_2 \otimes MO_c^*(S) \xrightarrow{\sim} H_c^*(S)$$

$x \nwarrow MO(pt)$

~~Finally I want to be able to interpret $i_*(w^{-1}(\nu_i))$ as the image of the Stiefel-Whitney classes of X. This is clear from $\tau_X + \nu_i = \tau_S$~~

Do this more carefully:

$$\begin{tikzcd} f^{-1}X \arrow[r, "f'"] \arrow[d, "i'"'] & X \arrow[d, "i"] \\ M \arrow[r, "f"'] & S \end{tikzcd}$$

$$\chi(f^{-1}X) = \int_{f^{-1}X} w(\tau_{f^{-1}X})$$

[redacted]

$$\tau_{f^{-1}X} + \underset{\displaystyle i'^*\nu_f}{\overset{\shortparallel}{\nu_{f'}}} = \tau_X$$

[redacted]

$$= \int_S f_* i'_* \, w(\tau_{f^{-1}X}) = \int_S f_* i'_* \left[i'^* w^{-1}(\nu_f) \cdot f'^* w(\tau_X) \right]$$

$$= \int_S f_* \left(w^{-1}(\nu_f) \cdot i'_* f'^* w(\tau_X) \right)$$

$$= \int_S f_*\left(w^{-1}(\nu_f)\right) \cdot i_* w(\tau_X)$$

Thus: If you use the isomorphism

$$\mathbb{Z}_2 \otimes_{MO^\bullet(pt)} MO_\bullet(S) \xrightarrow{\sim} H_\bullet(S)$$

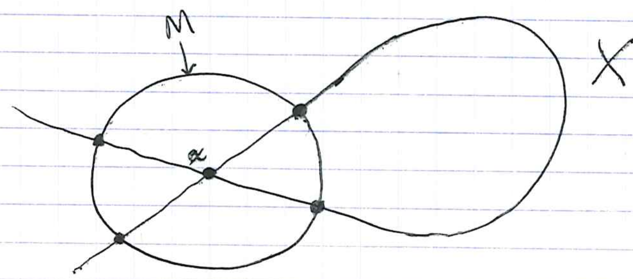
$$1 \otimes (f_! 1) \longmapsto f_* w^{-1}(\nu_f)$$

Then we can recover the class $i_* w(\tau_X)$ $\in H_c^\bullet(S)$ by the [redacted] above intersection method.

So now I suppose X is a cycle in S. Then I want to know when $\chi(f^{-1}X)$, $f: M \longrightarrow S$ transversal to X, depends only on the cobordism class of f. (Note that there is never any problem is making f transversal to X; this results from the theory of stratified sets — one inducts on the strata - transversal to a stratum $\Rightarrow$ transversal to higher strata near the point).

So what I have to prove now is that given a compact manifold with boundary $(M, \partial M)$ and a map $f: M \longrightarrow S$ such that f and $f/\partial M$ are transversal to X, then $\chi(\partial M \cap f^{-1}(X)) \equiv 0 \mod 2$.

Obvious necessary condition: Let $x \in X$ and take the plane orthogonal to the stratum of X passing through x. Take a small disk in this plane to be M.



Then $\partial M \cap f^{-1}(X)$ is just the link of x in X. Perhaps it is useful to recall that in a stratified set the local structure is that of a product: the product of the stratum going through the point with a cone.

~~Therefore it is obviously necessary that the Euler characteristic of the link of each point~~

~~Now in general if the stratum through x has dimension d, then~~

Taking M to be a small disk in S around x so that ∂M is transversal to all the strata, we see that an obvious necessary condition for what we want is that the link of each point x have even Euler characteristic.

Now suppose we have $f: M \longrightarrow S$ transversal to X, as well as $f|\partial M$ transversal. Given $m \ni f(m) = x$, then the inverse images of the stratum through x is a submanifold of M, and the normal geometry is the same as that for X. More precisely we know X is locally at x isomorphic to the product of the stratum thru x, call it W_x, with a cone $C(L)$, L being the link of the stratum W_x at x. Thus locally we have ~~a cartesian square~~

a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & C(L) \\ \downarrow & & \cap \\ S & \longrightarrow & S/W_x = W_x^{\perp} \end{array}$$

Now we have $f: M \to S$ transversal to X, which gives a transversal cartesian square

$$\begin{tikzcd} f^{-1}(X) \arrow[r] \arrow[d] & C(L) \arrow[d, hook] \\ M \arrow[r] & W_x^{\perp} \end{tikzcd}$$

locally around m. Note that because $0 \in C(L)$ is actually a stratum, this means that $M \to W_x^{\perp}$ is submersive, so therefore $f^{-1}(X) = f^{-1}(0) \times C(L)$ locally at x.

The conclusion of this discussion is that the normal geometry of $f^{-1}(X)$ in M is the same as that of X in S. Thus because the links L have even Euler characteristic, the same will be true for the links of $f^{-1}(X)$.

So now what we want to know is this: Given a compact manifold M with boundary ∂M and a stratified set X in M transversal to ∂M, show that $\chi(X \cap \partial M) = 0 \mod 2$ assuming that the link of X at each point is even.

~~Translate to a triangulated situation. Suppose M is triangulated~~

Triangulated situation: Suppose M is a simplicial complex and that F is a simplicial sheaf of k-modules on M. For any open set U we put

$$\chi(U,F) = \sum (-1)^q \dim H^q(U,F)$$
$$\chi_c(U,F) = \sum (-1)^q \dim H^q_c(U,F).$$

Both of these satisfy

$$f(U\cup V) + f(U\cap V) = f(U) + f(V)$$

as well as additivity in F. Note that

$$H^*_c(U,F) = H^*(\bar{U}, \partial\bar{U}; F)$$

Thus if $\partial\bar{U}$ is "transversal" to F so that

$$H^*(\bar{U},F) \xrightarrow{\sim} H^*(U,F)$$

one has from the long exact sequence

$$\longrightarrow H^*_c(U,F) \longrightarrow H^*(\bar{U},F) \longrightarrow H^*(\partial\bar{U},F) \longrightarrow \cdots$$

that
$$\chi(U,F) = \chi_c(U,F) + \chi(\partial\bar{U},F)$$

But now one knows from the additivity that one has $\chi(U,F) = \chi_c(U,F)$ if this is true locally i.e. for the open star of a simplex σ. Here however $\partial\bar{U}$ is the link of the center of σ and one is assuming this is 0.

Thus one can prove by building M up as a finite union of open sets that

$$\chi(U\cap X) = \chi_c(U\cap X)$$

and hence taking $U = \mathrm{Int}(M)$, we get $\chi(\partial M \cap X) = 0$ as desired.

So at this point we know that we have a well-defined operation

$$MO_{\cdot}(S) \longrightarrow \mathbb{Z}/2$$

obtained by taking $f: M \to S$, and forming $\chi(f^{-1}X) \bmod 2$. Multiplying M by a closed manifold N multiplies the result by $\chi(N)$, so this is a homom.

$$\begin{tikzcd} \mathbb{Z}_2 \otimes_{MO(pt)} MO_{\cdot}(S) \arrow[r] \arrow[d, "s"] & \mathbb{Z}/2 \\ H_{\cdot}(S) & \end{tikzcd}$$

and so we get an element of $H_{\cdot}(S) \overset{\text{//}}{} H_{\cdot}(X)$ as desired. This class $w \in H_{\cdot}(S)$ is characterized by the formula

$$\langle w, f_*(w^{-1}(\nu_f)) \rangle = \chi(f^{-1}(X))$$

for any $f: M \to S$ transversal to X, M compact.

Suppose X is a compact manifold. Then Wu has shown that the Stiefel-Whitney classes of M are determined by the action of the Steenrod operations in $H^*(X)$ and Poincare duality.

To derive his formula recall one has

$$\boxed{Sq_t(f_* x) = f_*(w_t(\nu_f)\, Sq_t(x))}$$

In effect one has to check this for a line bundle L. Then

$$Sq_t\, w_1(L) = w_1(L) + t\, w_1(L)^2 = w_1(L) \cdot w_t(L)$$

etc.

Thus if $i : X \longrightarrow S$ is an embedding into a sphere we have

$$\int_X Sq(x)\, w(\nu_X) = \int_S i_*(Sq(x)\, w(\nu_X)) = \int_S Sq(i_*(x))$$

$$= \int_S i_*(x) \qquad \text{because } S \text{ is a sphere}$$

$$= \int_X x$$

Hence if $x \in H^i(X)$ and $\dim(X) = n$, we find

$$\int_X Sq^{n-i}(x) \cdot 1 + Sq^{n-i-1}(x) \cdot w_1(\nu_X) + \cdots + x \cdot w_{n-i}(\nu_X) = 0$$

if $i < n$

which by decreasing induction on i and Poincaré duality determines $w_{n-i}(\nu_X)$.

Another form which is perhaps more sympathetic is to substitute $\chi(Sq)x$ for x in

$$\int_X Sq(x)\, w(\nu_X) = \int_X x$$

whence we get

$$\boxed{\int_X x\, w(\nu_X) = \int_X \chi(Sq)x.}$$

Here $\chi(Sq)$ is the antipode of Sq; it is the multiplicative coh. operation such that

$$\chi(Sq)(t) = t + t^2 + t^4 + \cdots$$

if $\deg(t) = 1$, hence $Sq(\chi(Sq)z) = \chi(Sq)(Sq\, z) = z$.

Thus $w_{n-i}(\nu_X) \in H^{n-i}(X)$ is the class $\ni$ $\forall x \in H^i(X)$

$$\int_X x \cdot w_{n-i}(\nu_X) = \int_X \chi(Sq)^{n-i}(x)$$

In other words, it appears that one can get $w_i(\nu_X)$ at least by applying $\chi(Sq)^i$ to the fundamental cycle.

<u>Problem</u>: Let X be a locally compact locally contractible space such that the Euler characteristic of the link at each point is even, or equivalently such that $\chi(H_*(X, X-\{x\})) \equiv 1$ mod 2. The problem is to construct the Sullivan classes $w \in H^{-i}(X, \omega_X^{\cdot})$. One wants a construction entirely in the framework of sheaf theory, but independent of transversality.