November 5, 1973:

Double mapping cylinder:

Given maps of spaces

$$X_0 \xleftarrow{a} X_{01} \xrightarrow{b} X_1$$

one can form the associated double mapping cylinder

$$C = Cyl(X_0 \xleftarrow{a} X_{01} \xrightarrow{b} X_1) = X_0 \amalg X_{01} \times [0,1] \amalg X_1$$

modulo relations:

$$a(x) = (x, 0) \qquad (x, 1) = b(x)$$

There are two topologies one can put on C.

Fine top: C is a quotient space of $X_0 \amalg X_{01} \times I \amalg X_1$. In this case a map $C \to Z$ is the same as a pair of maps $u_i : X_i \to Z$ plus a homotopy $u_0 a \sim u_1 b$.

Coarse top: Here C has the fewest open sets such that the map $C \to I$ and the evident maps

$$C_{[0,1)} \longrightarrow X_0$$

$$C_{(0,1)} \longrightarrow X_{01}$$

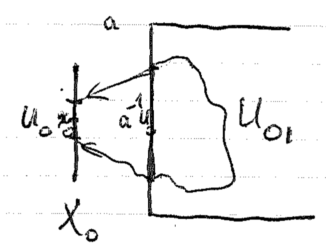
$$C_{(0,1]} \longrightarrow X_1$$

are continuous. With this topology, a map $Z \to C$ is the same as a map $Z \to I$ together with a compatible family of maps

$$\begin{array}{ccccc} Z_{[0,1)} & \supset & Z_{(0,1)} & \subset & Z_{(0,1]} \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \longleftarrow & X_{01} & \longrightarrow & X_1 \end{array}$$

(Stupid remarks to justify the above. i) Given a family $\mathcal{F}$ of a set X there is coarsest top. on X so that all the sets in $\mathcal{F}$ are open - $\bigcap$ of topologies is a topology + or one closest $\mathcal{F}$ under finite $\cap$, then arb. unions. Next ii) given $f: X \to Y$ with X a space, one gets a top on Y by calling a subset of Y open iff its inverse image is.)

The difference of the two topologies is as follows: Take a point $(x,0) \in C$ $x \in X_0$. A fine open nbd. of $(x,0)$ consists of an open nbd. U_0 of x_0 in X_0, plus an open set U_{01} in $X_{01} \times I$ such that $U_{01} \cap X_{01} \times 0 = a^{-1}(U_0)$



Thus shrinking U_{01} we can suppose it is of the form

$$U_0 \cup \{(x,t) \mid x \in a^{-1}(U_0),\ 0 < t < f(x)\}$$

where $f: a^{-1}(U_0) \longrightarrow (0,1]$ is semi-continuous from below.

To get a coarse nbd. of $(x,0)$ one takes a finite intersection of nbds. forced on one, such as

$C_{[0,\varepsilon)}$ and $j^{-1}(U_0)$ $\qquad j: C_{[0,1)} \to X_0$

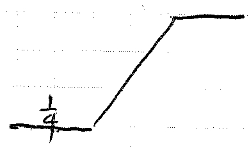
and so one gets a basis for the course nbds. of $(x_0, 0)$ of the form

$$U_0 \cup \{ (x,t) \mid x \in a^{-1}(U_0) \quad 0 < t < \varepsilon \}$$

for some $\varepsilon > 0$ and U_0 is an open nbd. of x_0 in X_0.

<u>Prop.</u> The map $C_{\text{fine}} \longrightarrow C_{\text{coarse}}$ is a homotopy equivalence.

<u>Proof.</u> Let $\varphi : [0,1] \to [0,1]$ have the graph



and define

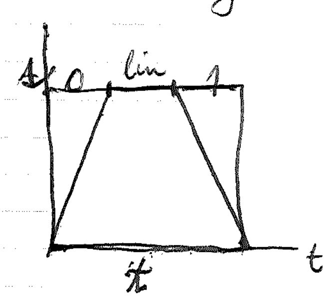
$$C \xrightarrow{\hat{\varphi}} C$$
$$(x,t) \longmapsto (x, \varphi(t)).$$

I claim this is continuous from C_{coarse} to C_{fine}. In effect

$$\hat{\varphi}^{-1} \{ U_0 \cup \{ (x,t) \mid x \in a^{-1}(U_0) \quad 0 < t < f(x) \} \}$$
$$= U_0 \cup \{ (x, \bar{t}) \mid x \in a^{-1}(U_0) \quad 0 < \varphi(\bar{t}) < f(x) \}$$
$$\supset U_0 \cup \{ (x, \bar{t}) \mid x \in a^{-1}(U_0) \quad 0 < \bar{t} < \tfrac{1}{4} \}.$$

Next have to show that $\hat{\varphi}$ from C_{coarse} to itself is homotopy to id, and also for C_{fine}. Pretty clear.

$$C \times I \longrightarrow C$$
$$((x,t), s) \longmapsto x, h_s(t)$$



~~The point of the coarse topology is that it behaves well with respect to pull-backs: Thus suppose that $g: K \longrightarrow I$ is a map and I form $g^*(Cyl(X_0 \leftarrow X_{01} \rightarrow X_1))$. Then a map $Z \longrightarrow g^*(C)$ is the same as a map $Z \xrightarrow{f} K$ together with a compatible family of maps~~

~~$$(gf)^{-1}[0,1) \supset (gf)^{-1}(0,1) \subset (gf)^{-1}(0,1]$$~~

~~Now suppose given a map $g: K \longrightarrow I$, whence an open covering $g^*[0,1)$, $g^*(0,1]$ of K. Form~~

~~$$Cyl(g^*[0,1) \times X_0 \longleftarrow g^*(0,1) \times X_{01} \longrightarrow g^*(0,1] \times X_1)$$~~

Now the point of the coarse topology is that it behaves well wrt pullbacks. For example let $K = U \cup V$ be an open covering and let $g: K' \longrightarrow K$ be a map. Then we have an induced map

$$\begin{array}{c} Cyl(g^*U \times X_0 \longleftarrow g^*(U \cap V) \times X_{01} \longrightarrow g^*V \times X_1) \\ \downarrow \\ K' \times_K Cyl(U \times X_0 \longleftarrow U \cap V \times X_{01} \longrightarrow V \times X_1) \end{array}$$

which is a homeomorphism for the coarse topologies. In effect a map of Z to the former consists of giving $Z \to I$ + maps

$$\begin{tikzcd} Z_{[0,1)} \arrow[d] & Z_{(0,1)} \arrow[l, phantom, "\supset"] \arrow[r, phantom, "\subset"] \arrow[d] & Z_{(0,1]} \arrow[d] \\ g^*U \times X_0 & g^*(U \cap V) \times X_{01} \arrow[l] \arrow[r] & g^*(V) \times X_1 \end{tikzcd}$$

and this is the same as giving a map $Z \to I$, and maps

$$\begin{tikzcd}
Z_{[0,1]} \arrow[r] \arrow[d] & Z_{(0,1)} \arrow[d] & Z_{(0,1]} \arrow[l, hook'] \arrow[d] \\
U \times X_0 & (U \cap V) \times X_{01} \arrow[l] \arrow[r] & V \times X_1
\end{tikzcd} \quad *$$

and a map $Z \to K'$ $\ni$ ~~compatible with the maps~~ $\ni$

$$\begin{tikzcd}
Z \arrow[d] \arrow[dr] & \\
K' \arrow[r] & K
\end{tikzcd}$$

where $\searrow$ is obtained from $*$. This is same as giving a map $Z \to Cyl(U \times X_0 \leftarrow \cdots\cdots)$ and $Z \to K'$ which agree in K.

Now I am in a good position to prove this:

<u>Proposition</u>: Given maps $(X_0 \leftarrow X_{01} \to X_1)$ belonging to a class S satisfying conditions .... listed below. Then for any $g : K \to I$, the map

$$\begin{tikzcd}
Cyl(K_{[0,1)} \times X_0 \leftarrow K_{(0,1)} \times X_{01} \to K_{(0,1]} \times X_1) \arrow[d] \\
K \times Cyl(X_0 \leftarrow X_{01} \to X_1)
\end{tikzcd}$$

is in S. In other words up to a map in S, $Cyl(K_{[0,1]} \times X_0 \leftarrow K_{(0,1)} \times X_{01} \to K_{(0,1]} \times X_1)$ is the h-pull-back of $Cyl(X_0 \leftarrow X_{01} \to X_1)$ by $g : X \to I$.

Assumptions:

i) S closed under composition + contains all heq's.

ii) S closed under homotopy cobase change

Should then be able to prove the "Brown" lemma.

iii) $f: X \to Y \in S$ then $\forall K \quad K \times X \to K \times Y \quad \in S$.

These imply

$$\mathrm{Cyl}\left(K_{[0,1)} \times X_{01} \leftarrow K_{(0,1)} \times X_{01} \rightarrow K_{(0,1]} \times X_{01}\right)$$

$$\Big\downarrow \in S$$

$$\mathrm{Cyl}\left(K_{[0,1)} \times X_0 \leftarrow K_{(0,1)} \times X_{01} \rightarrow K_{(0,1]} \times X_{01}\right)$$

and so via

(iv) $fg \in S, \; f \in S \Longrightarrow g \in S$

$$\begin{tikzcd} \cdot \arrow[r, "\in S"] \arrow[d, "\in S"', "f"] & \cdot \arrow[d, "\in S"] \\ \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$

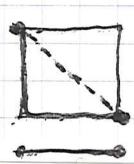
we reduce to case $X_0 = X_{01} = X_1$. In this case

$$\begin{tikzcd} \mathrm{Cyl}\left(K_{[0,1)} \times X_0 \leftarrow K_{(0,1)} \times X_{01} \rightarrow K_{(0,1]} \times X_1\right) \arrow[d, "\text{homeo}"] \arrow[dr] & \\ \mathrm{Cyl}\left(K_{[0,1)} \leftarrow K_{(0,1)} \rightarrow K_{(0,1]}\right) \times X_{01} \arrow[r, "\text{heq}"'] & K \times X_{01} \end{tikzcd}$$

so its clear. One needs

<u>Lemma</u>: Given $g: K \to I$, then
$$\mathrm{Cyl}\left(K_{[0,1)} \leftarrow K_{(0,1)} \rightarrow K_{(0,1]}\right) \longrightarrow K$$
is a heq.

<u>Proof</u>. $\mathrm{Cyl}\left(K_{[0,1)} \leftarrow K_{(0,1)} \rightarrow K_{(0,1]}\right) = K \times_I \mathrm{Cyl}\left([0,1) \leftarrow (0,1) \rightarrow (0,1]\right)$
and the rest is clear
restrict from the closed case.



Onto the classifying space of a top. monoid M.

Consider

$$\Sigma(M) = \mathrm{Cyl}(pt \leftarrow M \rightarrow pt).$$

A map $K \to \Sigma(M)$ is the same thing as a fn. $K \to I$ together with a map $K_{(0,1)} \to M$.

Now suppose I have an open covering $K = U \cup V$ and a map $c: U \cap V \to M$. Then I can form over K

$$\begin{array}{ccccc} \mathrm{Cyl}\,(U \times M & \longleftarrow & U \cap V \times M & \longrightarrow & V \times M) \\ (x, c(x)m) & \longleftarrow\!\shortmid & (x,m) & \longmapsto & (x,m) \end{array}$$

and moreover M acts to the right on this space.

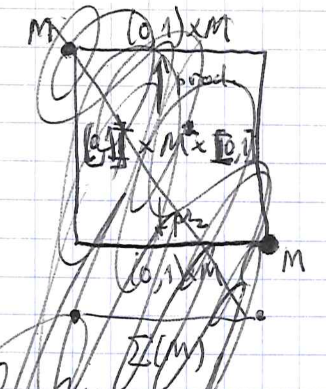
The above space is compatible with pull-backs: Given $g: K' \to K$ we have

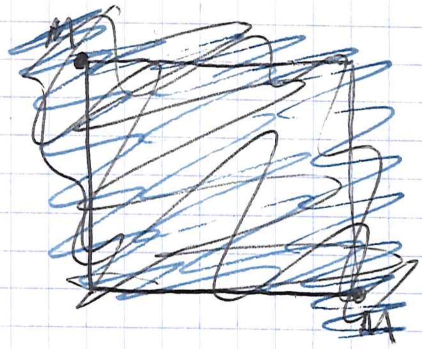
$$\begin{array}{c} \mathrm{Cyl}\left(g^*(U) \times M \leftarrow g^*(U \cap V) \times M \rightarrow g^*(V) \times M\right) \\ \| \\ g^* \,\mathrm{Cyl}\left(U \times M \leftarrow (U \cap V) \times M \rightarrow V \times M\right) \end{array}$$

Now perform this construction over $\Sigma(M)$ and we get

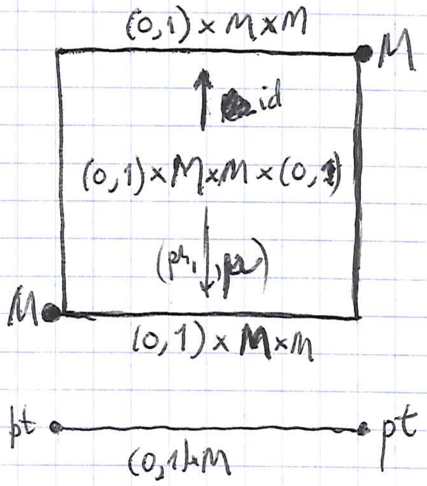
$$\mathrm{Cyl}\left(\Sigma(M)_{[0,1)} \times M \longleftarrow \Sigma(M)_{(0,1)} \times M \longrightarrow \Sigma(M)_{(0,1]} \times M\right)$$

which sits over $\Sigma(M)$. Picture:





~~Next point:~~



Thus over the point (t,m) $0<t<1$ one has

$$Cyl\left(M \xleftarrow{m\cdot} M \xrightarrow{id} M\right)$$

which has the homotopy type of M.

~~Assume now that f is compatible~~

~~Assume~~ Assume $\pi_0(M) = S$ has a filtering ~~[illegible]~~ translation cat + that it acts invertibly on $H_*(M)\cdot S^{-1} = \varinjlim\left(s \mapsto H_*(M)\right)$ right mult. ~~Then if P is a space with right action we will consider equivalences to maps $P \to P'$ $\ni$ $H_*(P)S^{-1} \xrightarrow{\sim} H_*(P')S^{-1}$. The axioms are satisfied. Now we have~~

~~Lemma~~ ~~Proposition.~~

Assume $\pi_0(M) = S$ has a filtering right translation category and that it acts invertibly on $H_*(M)S^{-1}$.

<u>Assertion</u>: Given $g: K' \to K = U \cup V$ and $c: U \cap V \to M$. If g is a homotopy equivalence then

$$\begin{array}{c} Cyl(g^*U \times M \longleftarrow g^*(U \cap V) \times M \hookrightarrow g^*V \times M) \\ \downarrow \\ Cyl\ (U \times M \xleftarrow{c} U \cap V \times M \hookrightarrow V \times M) \end{array}$$

induces ~~[illegible]~~ an isom. for $H_*(M)S^{-1}$.

Proof. ~~[illegible]~~ To simplify notation denote by ~~[illegible]~~ $g(K)$ the set of pairs consisting of a covering $K = U \cup V$ and $c: U \cap V \to K$, and if $\xi \in g(K)$ put P_ξ = the cylinder constructed above. Then we wish to prove that if $g: K' \to K$ is a homotopy equivalence, then the induced map

$$H_*(P_{g^*(\xi)})S^{-1} \longrightarrow H_*(P_\xi)S^{-1}$$

is an isomorphism.

<u>First Reduction</u>: Enough to worry about the case where g is the embedding $K' \xrightarrow{i_0} K' \times I$. In effect by symmetry it will ~~[illegible]~~ also be true for i_1, hence given $K' \times I \xrightarrow{H} K$ $\xi \in g(K)$, one has

$$\begin{tikzcd} h(H_0^*\xi) \arrow[dr, "\sim"] & & \\ & h(H^*\xi) \arrow[r] & h(\xi) \\ h(H_1^*\xi) \arrow[ur, "\sim"] & & \end{tikzcd}$$

so that if we have two homotopic maps $H_0, H_1: K' \to K$, then $h(H_0^*\xi) \xrightarrow{\sim} h(\xi)$ is an isom. $\iff$ $h(H_1^*\xi) \xrightarrow{\sim} h(\xi)$ is an isom. Next for a general heq $g: K' \to K$ one has $f: K \to K'$ $\ni$ $gf \sim id \Rightarrow g_* f_*$ isom $\Rightarrow$ g_* surjective. Replacing f by g one sees that f_* surj. $\Rightarrow$ g_* inj.

<u>2nd Reduction:</u> Call ξ_0 over K_0 "good" if $\forall$ $K' \xrightarrow{g} K \xrightarrow{f} K_0$ with g a heq we have $h((fg)^*\xi) \xrightarrow{\sim} h(f^*\xi)$. If there exists a numerable covering $\{U_i\}$ of K_0 such that $(U_i, \xi/U_i)$ is good for all i, then ξ_0 is good.

In effect given $K \times I \xrightarrow{H} K_0$, $\{H^{-1}(U_i)\}$ is a numerable covering of $K \times I$. One knows then that there exists a covering $\{V_j\}$ of K such that for each j $\exists$ $0 < t_1 < \dots < t_k = 1$ with $V_j \times [t_\nu, t_{\nu+1}]$ contained in some member of $\{H^{-1}(U_i)\}$. Then we know $\forall\, W \subset V_j$

$$h(W \times \{t_\nu\}, H^*\xi) \xrightarrow{\sim} h(W \times [t_\nu, t_{\nu-1}], H^*\xi)$$

so we can prove by induction on ν that

$$h(W \times \{0\}, H^*\xi) \xrightarrow{\sim} h(V_j \times [0, t_\nu], H^*\xi);$$

Thus for any $W \subset$ some V_j we will have $h(W \times \{0\}, H^*\xi) \subset h(W \times I, H^*\xi)$, and so by induction if W is contained in any finite union of V_j, hence passing to the limit for any W.

~~But now suppose we have want to show ξ over K is good.~~

One would want to know that for any numberable covering of $K \times I$ it is refined by a "stacked" covering over a numerable covering of K.

~~The method~~ In the situation to which we apply this we will consider the covering by two open associated to a map $K \times I \longrightarrow I$. Thus we ~~will~~ ~~be reducing to~~ can reduce to the **universal** case: where $K = I^I$ which is a metric space, hence paracompact. Image inverse of a numerable covering is numerable

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<u>Third step:</u> ~~To consider the canonical bundle P_ξ over $\Omega(M)$ we see this is good~~ Show that any ξ over K with $K = U \cup V$ a numerable covering is good. So by second step it is enough to consider separately the cases $K = U$ & $K = V$. In the ~~latter~~ former, one has the inclusion

$$
\begin{array}{c}
P_\xi = Cyl\big(U \times M \xleftarrow{c_\circ} (U \cap V) \times M \xhookrightarrow{id} V \times M \big) \\
\big\updownarrow \\
U \times M = K \times M
\end{array}
$$

homotopy equivalence, so ξ is obviously good.
In the latter $K = V$ one has that $c_\circ$ induces an isomorphism on localized homology:

$$
\begin{array}{ccc}
H_*(U \times M)S^{-1} & \xrightarrow{(u,m) \mapsto (u, c(u)m)} & H_*(U \times M)S^{-1} \\
\Uparrow & & \\
H_*(U, H_*(M))S^{-1} & &
\end{array}
$$

clear from the spectral sequence in homology. Hence one has that the ~~map~~ inclusion

$$\begin{tikzcd} K\times M \arrow[r, equal] & V\times M \arrow[d] \\ & \mathrm{Cyl}\,(U\times M \leftarrow (U\cap V)\times M \rightarrow V\times M) \end{tikzcd}$$

induces an isom. ~~of~~ on localized homology. So again ~~things are good~~ ξ is good.

Serre's course : Lecture 1 Nov. 6, 1973

Les mardis : Cohomologie des groupes discrets

un groupe discret Γ : cela veut dire un groupe abstrait qui est un sousgroupe d'un groupe de Lie.

on obtient information sur la cohomologie de Γ using a space $X \ni X/\Gamma$ est compact.

Probleme : On appelle un group Γ coherent si chaque sousgroupe de type fini ~~est~~ de presentation finie. $GL_n(\mathbb{Q})$ est-il coherent? en particular, est $SL_3(\mathbb{Z})$ coh.? On ne sais aucun contre-exemple.

Ex: $SL_2(\mathbb{Z}[i])$ est coherent. On utilise:

Thm: X variete de dim $\leq 3 \Rightarrow \pi_1 X$ coherent.

Topics:

1) Arbes

2) Groups with duality:

$$H^i(\Gamma, M) \simeq H_{N-i}(\Gamma, I \otimes M)$$

One shows (petit exercise) that this follows from some conditions of finitude +

$$H^i(\Gamma, \mathbb{Z}[\Gamma]) = \begin{cases} 0 & i \neq N \\ \text{sans torsion} & i = N \end{cases}$$

3) Euler characteristics (Brown's results)

$$\chi(E_8(\mathbb{Z})) = \frac{\cdots\cdots}{31} \quad \text{après Harder}$$

+ no element of prime order $l > h+1$ ($h+1 = 30$ for E_8) h = Coxeter no.

+ Brown's results $\Rightarrow E_8(\mathbb{Z})$ has an element of order 31.

4) (time permitting). Proof of Kummer's criterion that p is regular (i.e. $p \nmid \mathrm{card}\, \mathrm{Pic}(\mathbb{Z}[\zeta_p])$) $\Leftrightarrow$ p doesn't divide $b_1, \dots, b_{p-3}$ (Bernoulli numbers). Plus a generalization to other no. fields.

ARBRES.

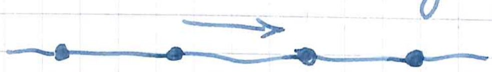
Un arbre X est un complexe simplicial de dim. 1 qui est connexe, non-vide, et sans circuit ($\Leftrightarrow$ contractile).

Si un groupe Γ agit sur X, on pose la condition that it doesn't reflect any arret. This can always be achieved by subdividing once.

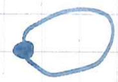
X/Γ est un graphe:

$\Gamma = \mathbb{Z}$

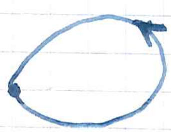
X



X/Γ :



Def. graphe: Ensembles de vertices et d'arrêts. On donne $y \mapsto \bar{y}$, $\bar{\bar{y}} = y$, $y \neq \bar{y}$ sur les arrêts, et on donne pour chaque arret y, its initial vertex.



Double barycentric subdivision of a graphe est toujours un complexe simplicial.

Suppose now Γ agit sur un arbre X.

a) Γ agit librement, $\Rightarrow$ $X \to X/\Gamma$ est the universal covering de X/Γ, alors $\Gamma \simeq \pi_1(X/\Gamma)$ c'est un groupe libre pour chaque graphe. On choisit un arbre maximal de X/Γ (= un arbre dont les vertices sont ceux de X/Γ), est $\pi_1(X/\Gamma)$ est le groupe libre avec generateurs les arrêts qui ne sont pas dans l'arbre maximal.

Inversement chaque groupe libre agit librement sur un arbre. Prends le covering universal de .

One proves in this way.

<u>Schreier thm:</u> Un sousgroupe d'un groupe libre est libre.

as well as getting Schreier's recette for the generateurs.

b) X/Γ est un arbre. Soit $A \subset X$ un arbre $\ni$ $A \to X \to X/\Gamma$ isom. Alors: Γ est <u>la somme amalgamée</u> des stabilisateurs de X: i.e. inductive limit

$$\begin{tikzcd} & \Gamma_P \arrow[dr] & \\ \Gamma_{PQ} \arrow[ur] \arrow[dr] & & \Gamma \\ & \Gamma_Q \arrow[ur] & \end{tikzcd}$$

c) Cas general: Let Γ' be the sous groupe engendre par les stabilsateurs. Then X/Γ' est un arbre on which Γ/Γ' acts freely, so

$$0 \to \Gamma' \to \Gamma \to \underset{\text{free}}{\pi_1(X/\Gamma')} \to 0$$

Proof of this result to be given next time.
On donnera la demontration la prochaine fois.

Example: Let Γ be the subgroup of $GL_2(\mathbb{R})$ gen by

$$x = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

Using fact that π is transcental one sees

$$\Gamma = \underset{y}{\mathbb{Z}[\mathbb{Z}]} \rtimes \underset{x}{\mathbb{Z}}$$

$$x^i y x^{-i} = \begin{pmatrix} 1 & \pi^i \\ & 1 \end{pmatrix}$$

$$\prod_i (x^i y x^{-i})^{n_i} = \begin{pmatrix} 1 & \sum n_i \pi^i \\ 0 & 1 \end{pmatrix}$$

and $\sum n_i \pi^i$ are distinct for different (n_i).

One can show Γ is not finitely presented.

PROOF: (Γ finitely pres. $\Rightarrow \Gamma = F/R$, F free f.t., R f.g. as a normal subgroup of F. Then can finite t. free subgroup F' of F $\ni$ R = normal subgroup gen by F and we have. Then $X = BF/BF'$ a finite 2 complex $\ni$ $\pi_1(X) = \Gamma$. Then

$$X' \longrightarrow X \longrightarrow B\Gamma \qquad \pi_1 X' = 0.$$

$$H_*(\Gamma, H_* X') \Rightarrow H_*(X).$$

$$\begin{array}{|lll} H_0(\Gamma, H_2X') & & \\ \Gamma & 0 & 0 \\ \hline H_0\Gamma & H_1\Gamma & H_2\Gamma \end{array}$$

$$H_3(\Gamma) \longrightarrow H_0(\Gamma, H_2X') \rightarrow H_2(X) \longrightarrow H_2(\Gamma) \longrightarrow 0$$

$\therefore$ Γ finitely presented $\Longrightarrow H_2(\Gamma)$ f.t. over $\mathbb{Z}$.

So in the case of the above semi-direct prod use spec. seq. of $\mathbb{Z}[\mathbb{Z}] \to \Gamma \to \mathbb{Z}$

get
$$E^2_{pq} = H_p(\mathbb{Z}, H_q(\mathbb{Z}[\mathbb{Z}])).$$

so

$$0 \to H_0(\mathbb{Z}, H_2(\mathbb{Z}[\mathbb{Z}])) \longrightarrow H_2(\Gamma) \longrightarrow H_1(\mathbb{Z}, H_{2}(\mathbb{Z}[\mathbb{Z}])) \longrightarrow 0$$

$$\Lambda^2 \mathbb{Z}[\mathbb{Z}]_{\mathbb{Z}} \xrightarrow{\sim} H_2(\Gamma) \longrightarrow \mathbb{Z}[\mathbb{Z}]^{\mathbb{Z}} = 0$$

If $\mathbb{Z}[\mathbb{Z}]$ has basis e_n

then $\Lambda^2 \mathbb{Z}[\mathbb{Z}]$ ——— $e_n \wedge e_{n'}$ $(n < n')$.

and $\mathbb{Z}$ acts by $e_n \wedge e_{n'} \mapsto e_{n+1} \wedge e_{n'+1}$.

$\therefore$ Clearly $\Lambda^2 \mathbb{Z}[\mathbb{Z}]/\mathbb{Z}$ infinite type.) $\therefore$ $H_2(\Gamma)$ inf. type so Γ not f.p.

Serre's course – lecture 2 – Nov. 13, 1973

Th: X un arbre, G opère sur X sans inversion d'arrêt.
H = sous groupe engendré par les stab G_P P sommet.
alors:

X/H est un arbre

$H = \lim\limits_{P \in T} (G_P)$ $\quad$ T sous-arbre de X t.q. $T \simeq X/H$

$$1 \longrightarrow \underset{\text{amalgam}}{H} \longrightarrow G \longrightarrow \underset{\text{libre}}{\pi_1(X/G)} \longrightarrow 1$$

Proof: Useful idea signalled by Deligne: If Y is a space and $X \xrightarrow{f} Y$ is a space over Y with X simply-connected, then one can view X as a point in Y in some sense. Precisely for each $x \in X$ we have the group $\pi_1(Y, f(x))$ and for two different x's we have a canonical transitive isomorphism between these groups, so we can set

$$\pi_1(Y, X \xrightarrow{f} Y) = \varinjlim \pi_1(Y, f(x)).$$

With this idea one has a canon. map

$$G \longrightarrow \pi_1(X/G, X \to X/G)$$

defined as follows: Choose $x \in X$. Given $g \in G$ one takes a path joining x to gx whose image in X/G is a loop at $f(x)$. This is independent of the choice of path because in a tree one has no circuits, hence two paths with the same endpoints can differ only by cancellation T/+

and this doesn't affect the element in $\pi_1(X/\Gamma, f)$. Similarly the map is a homo, provided one composes paths as one has to. (Also ind of x).

It is surjective: Can lift paths. (Serre said something to the effect that this is a general fact about $X \to X/\Gamma$ in a simplicial situation)

Now G/H acts freely on X/H for if $gx = hx$ then $h^{-1}g \in H \Rightarrow g \in H$. Thus X/H is a covering of X/G, so we get a homom.

$$\begin{tikzcd}
\pi_1(X/G) \arrow[rr] & & G/H \\
& G \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and on the other hand H goes to 1 in $\pi_1(X/H)$. (If x is fixed by g, then can take x as basepoint so it's clear.)

Thus one sees that $G/H = \pi_1(X/G)$ and $X/H =$ univ. covering of $X/G \Rightarrow X/H$ is a tree.

To finish the theorem we are thus reduced to the case $H=G$ in which case I guess one has to argue by coverings.

Def: G n'est pas un amalgame si $G = G_1 *_A G_2$ (où $A \subset G_1$, $A \subset G_2$) $\Longrightarrow \begin{cases} G = G_1 \text{ ou } G = G_2 \\ \Updownarrow \\ G_1 = A \text{ ou } G_2 = A. \end{cases}$

Def: G a la propriete (FA) (fixpt sur les arbres): chaque fois G opère sur un arbre, il existe une pointe fixe.

Thm: G denumerable. Alors pour que G ait la prop. (FA), il faut et il suffit que

(i) G est de type fini

(ii) G n'a pas $\mathbb{Z}$ comme quotient

(iii) G n'est pas un amalgame (grace que G est denumerable)

Pf: (FA) $\Longrightarrow$ (i) sinon $\exists$ $G_1 < G_2 < \cdots\cdots$ une suite strictement croissante de sous groupes dont la reunion est G. Form the telescope of

$$n \longmapsto G/G_n$$

and you get a tree (as $\bigcup G_n = G$) on which G acts without fixpts.

(FA) $\Longrightarrow$ (ii) $\mathbb{Z}$ opère sur 

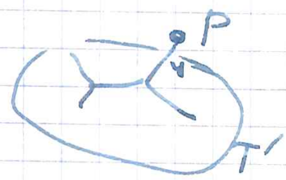
(FA) $\Longrightarrow$ (iii) follows from description of $G_1 *_A G_2$ in terms of groups acting on trees with fundamental domain .

(i), (ii), (iii) $\Longrightarrow$ (FA): Supposons G opère sur l'arbre X. $G \twoheadrightarrow \pi_1(X/G)$ libre + (ii) $\Longrightarrow$ $\pi_1(X/G) = 1$ et $G = \varinjlim_{T} G_p$ où T est un sous arbre de X.

$$G = \bigcup_{T_\alpha} \varinjlim_{T_\alpha} G_p$$

where T_α runs over finite sous-arbres

+ G f.t. $\Longrightarrow$ $G = \varinjlim_{T_\alpha} G_P$ T_α finite. Let $T = T_\alpha$ be minimal such that this is true. Then if P is an extreme point we have $T : T' \cup_y P$



$$G = G_P *_{G_y} \varinjlim_{T'} G_Q$$

so by (iii) T has a single element $\Longrightarrow$ G has a fixpoint.

It seems that preceding argument shows that in general (FA) $\Longleftrightarrow$ (ii), (iii) + (i)': $G = \bigcup_{n \in \mathbb{N}} G_n \Rightarrow G = G_n$. Serre said I think that if one took a product of copies of $\mathfrak{A}_5$ one got a non-countable group having the property (FA). Here this group is torsion, so it satisfies (ii) + (iii), and he claims that it can't be a union of an increasing sequence of proper subgroups. I don't see this last point.

From now on all groups will be assumed to be of finite type.

Exemples de groupes (FA):

1) G fini, (i) (ii) are clear. As for (iii) one knows for $G = G_1 *_A G_2$ that if $s_i \in G_i - A$, that $s_1 s_2$ has infinite order. Thus G torsion $\Longrightarrow$ (iii) always.

Serre has an elem. arg. in the case G finite. One takes an orbit and adds in all the geodesics between different pairs of points thus getting a finite tree invariant under G (any connected (non-void) subset of a tree is a tree — since there still are no circuits). Now remove the extreme points & continue until you reach an invariant point.

2) G torsion (f.t. recall). The preceding elementary argument does not work here, but the theorem does.

3) $1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$ exact, $H, G/H$ have (FA) $\Longrightarrow$ G has (FA).

In effect let G act on a tree X. As H has (FA) X^H is $\neq \phi$, and it is connected because given $P, Q \in X^H$ there is a unique geodesic between them $\Rightarrow$ this geodesic belongs to X^H. Thus X^H is a tree, and as $X^G = (X^H)^{G/H}$ one wins.

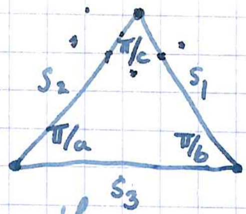
4) H finite index in G, H has (FA) $\Longrightarrow$ G has (FA).

In effect, let H' be the intersection of the conjugates of H, so H' is of finite index. If G acts on the tree X, then H has (FA) $\Rightarrow$ X^H is a tree $\Rightarrow$ $X^{H'}$ is a tree $\Rightarrow$ $X^G = (X^{H'})^{G/H'}$ $\neq \phi$ because G/H' is finite.

Counterexample to H finite index in G, G has (FA) $\Rightarrow$ H has (FA). Let a, b, c be integers ≥ 2. One has three plane geometries

spherical	euclidean	hyperbolic
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > 1$	$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$	$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 1$

and one takes in the corresponding plane a triangle with angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$.



Let G' be the group of rigid motions generated by the reflections in the sides. G' is a Coxeter group with presentation

$$s_1^2 = s_2^2 = s_3^2 = 1$$

$$(s_1 s_2)^c = (s_2 s_3)^a = (s_3 s_1)^b = 1$$

Let G be the subgroup of orientation preserving motions. One know that if $x = s_2 s_3$, $y = s_3 s_1$, $z = s_1 s_2$ then G has the presentation

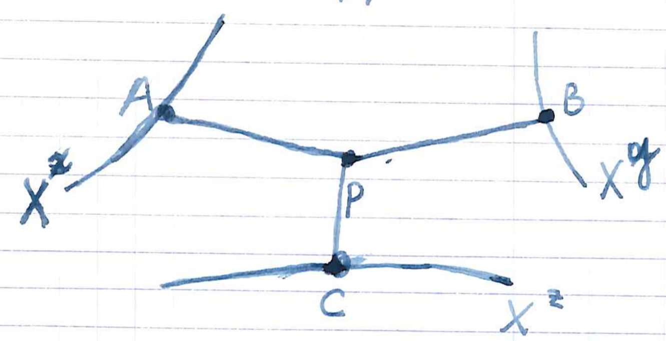
$$x^a = y^b = z^c = 1$$

$$xyz = 1.$$

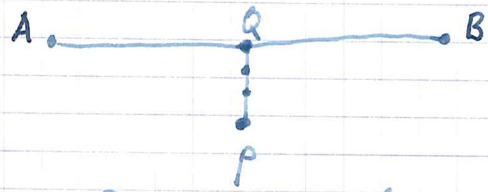
In any case Serre shows as a consequence of these of these relations that G has the property (FA). On the other hand in the euclidean + hyperbolic cases the group G is linear, so one knows that it has torsion free subgroups H of finite index. Since G has compact fundamental domain, it follows that X/H is a compact oriented surface so H has $\mathbb{Z}$ for quotient, so H will not have the prop. (FA).

Let G act on X a tree. Then X^x X^y X^z are subtrees (finite case) which we can assume are mutually disjoint as $xyz = 1$, $(P \in X^x \cap X^y \Rightarrow P \in X^z)$. Observe that if Y is a subtree and P is a vertex, $\exists$ always a ! minimal geodesic from P to Y, hence we can speak of the distance from P to Y.

Choose P so that $d(P, X^x) + d(P, X^y) + d(P, X^z)$ is minimum. Can suppose that $P \notin X^x$, $P \notin X^y$.



Claim A—P—B is the geodesic from A to B. In effect the only way it couldn't be is for one to have



and then moving P toward Q one step decreases the sum of the distances, because it decreases the distance to both A and B. (More precisely one compares the geodesic from A to P followed by the geodesic from P to B with the geodesic from geodesic from X^x to X^y. The latter must be the irredundant version of the former, ~~meaning in the former we have~~ ~~$P_1 \xrightarrow{y} P_2 \xrightarrow{y} P_1$.~~ ~~the Since from A to P + P to B are geodesics so $P_2 = P$ whence~~ so if the former is redundant it contains a $P_1 \xrightarrow{y} P_2 \xrightarrow{y} P_1$, and since A to P, P to B are irredundant, this can happen only if $P_2 = P$, and then we get a contradiction to minimality. Thus A to P + P to B is irredundant, hence it is the geodesic joining X^x to X^y,

~~This holds even if $P \in X^x$ or $P \in X^y$. Similarly, APC, BPC are the geodesics between X^x and X^z, X^y and X^z respectively.~~

~~Now suppose x, y chosen so that $P \notin X^x$, $P \notin X^y$. From $xyz = 1$ one has $xyC = C$ or $x^{-1}C = yC$.~~

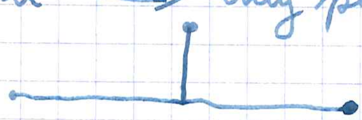
provided one knows that all its points except B are outside of X^y, and all its points except A are outside of X^x. This is clear from minimality if $P \notin X^x, X^y$ as we have assumed.

Thus we have est. that $AP+PB$ is the geodesic from X^x to X^y. A similar argument shows $AP+PC$ is the geodesic from A to C and $CP+PB$ is the geodisic from C to B.

Now start with $xyz=1 \Rightarrow xyC=C \Rightarrow x^{-1}C = yC$. Thus $x^{-1}(AP+PC) = A\cdot x^{-1}P + x^{-1}P\cdot x^{-1}C$ is the geodesic from A to $x^{-1}C = yC$ and $y(BP+PC) = B\cdot yP + yP\cdot yC$ is the geodesic from B to yC, and hence AB is the irredendant version of

$$A \quad x^{-1}P \quad x^{-1}C=yC \quad yP \quad B$$

Since $A\cdot x^{-1}P\cdot x^{-1}C$ and $yC\cdot yP\cdot B$ are irredendent $\Rightarrow$ only possible cancellation occurs in the form



so counting distances this implies $x^{-1}P = yP$ and also $=P$ by uniqueness of the geodesic from A to B. done

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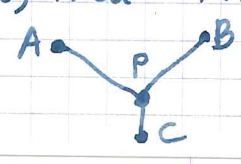
It seems that given three points A, B, C in a tree and P the point such that $d(P,A)+d(P,B)+d(P,C)$ is minimum, then $AP+PB$ is the geodesic joining A to B etc.



Serre's course – November 20, 1973 – Lecture 3.

In the first lecture he stated problem of whether $GL_n(\mathbb{Q})$ is coherent, i.e. whether every subgroup of fin. type is of finite pres. Counterexample: Will show $F \times F$ is not coherent where F = free group on two generators x, y. Then one has

$$F \times F \subset SL_2\mathbb{Z} \times SL_2\mathbb{Z} \subset SL_4\mathbb{Z}$$

so $SL_4\mathbb{Z}$ is not coherent.

Let $H \subset F \times F$ be the subgroup gen. by $(x,x), (y,1), (1,y)$. Will show H not of f.p. by showing $H_2(H)$ n'est pas de type fini. Let Y = normal subgp of F gen. by y:

$$1 \longrightarrow Y \longrightarrow F \longrightarrow \mathbb{Z} \longrightarrow 1$$
$$x \longmapsto 1$$
$$y \longmapsto 0$$

Then

$$\begin{tikzcd} 1 \arrow[r] & Y \times Y \arrow[r] & F \times F \arrow[r] & \mathbb{Z} \times \mathbb{Z} \arrow[r] & 1 \\ 1 \arrow[r] & Y \times Y \arrow[r] \arrow[u, equal] & H \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u, "\Delta"'] & 1 \end{tikzcd}$$

So

$$E^2_{pq} = H_p(\mathbb{Z}, H_q(Y \times Y)) \Longrightarrow H_{p+q}(H)$$

gives

$$0 \longrightarrow H_0(\mathbb{Z}, H_2(Y \times Y)) \longrightarrow H_2(H) \longrightarrow H_1(\mathbb{Z}, H_1(Y \times Y)) \longrightarrow 0$$

Now $H_1(Y)$ free abelian with base the images of $x^i y x^{-i}$ $i \in \mathbb{Z}$, so as a $\mathbb{Z} = \mathbb{Z}[x]$ module it is the group ring $\mathbb{Z}[\mathbb{Z}]$.

$$Y \text{ free} \Rightarrow H_2(Y) = 0 \Longrightarrow H_2(Y \times Y) = H_1(Y) \times H_1(Y)$$
$$= \mathbb{Z}[\mathbb{Z}] \times \mathbb{Z}[\mathbb{Z}]$$
$$= \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}] \quad \text{diag. action}$$

$$\therefore \; H_0(\mathbb{Z}, H_2(Y \times Y)) = \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]/\mathbb{Z} \simeq \mathbb{Z}[\mathbb{Z}].$$

done.

In the last time one should has stated the following

Lemme. G gen by subgps A, B, C acting on a tree $X \ni$ $X^A, X^B, X^C \neq \emptyset$ and

$$A \subset \langle B, C\rangle$$
$$B \subset \langle C, A\rangle$$
$$C \subset \langle A, B\rangle$$

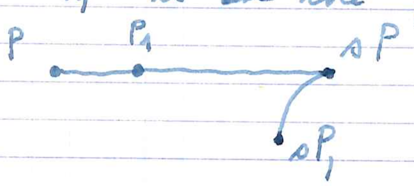
Then $X^G \neq \emptyset$.

Let s be an auto. of a tree X. If s has no fixpoints, claim there exists a unique line L (geodesic infinite in both directions) in X stable under s. Also s acts as translations on L. Serre gives two characterizations of L:

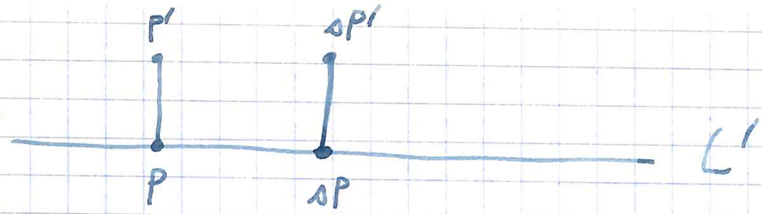
1) L is the set of points P such that $d(P, sP)$ is minimum.

2) The group $\mathbb{Z}$ with gen. acting as s acts freely on X hence $\pi_1(X/\mathbb{Z}) = \mathbb{Z}$ so $X/\mathbb{Z}$ has a unique circuit and L is the inverse image of this circuit.

Try to establish 1). Put m = min. distance and let $L = \{P \mid d(P, sP) = m\}$. Clearly L stable under s. Now suppose P_1 is on the geodesic $P - sP$



By minimality P_1 to sP to sP_1 must be a geodesic without cancellation. It is thus clear that the union of $\overline{s^nP, s^{n+1}P}$ is a line L' on which s acts as translations. Now if $P' \notin L'$, then one has:



and so it is clear that $d(P', sP') > d(P, sP)$. done with 1).
2) is fairly clear.

<u>Prop</u>: G nilpotent of fin. type acting on X a tree. Then either

i) $X^G \neq \phi$.

ii) $\exists$ line L in X invariant under G such that G acts as translations on this line through a non-trivial homo. $G \to \mathbb{Z}$. L is the unique line in X stable under G.

Proof. Use central series

$$0 \subset G_1 \subset \cdots\cdots \subset G_n = G$$

with cyclic quotients. By induction have two cases:

1) $X^{G_{n-1}} \neq \phi$. In this case one takes G_n/G_{n-1} acting on the tree $X^{G_{n-1}}$, and one gets either fixpts. or a line stable under G.

2) $\exists$ unique line L stable under G_{n-1}. Then G preserves L.

The only thing to be sure of is that L is unique. But given $L \neq L'$ stable under G, then the geodesic joining L, L' would be invariant, so G would have fixpoints.

Finally because G is nilpotent, its image in the group of autos of a line, which is the dihedral group, is a subgroup of translations.

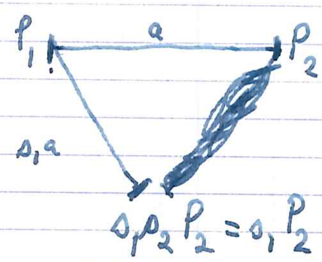
<u>Remark</u>: It seems that if G has a series of normal subgroups $0 \subset G_1 \subset \cdots \subset G_n = G$ such that G_i/G_{i-1} is cyclic or finite (of odd order), then either $X^G \neq \phi$ or $\exists$ unique line L stable under G on which G acts by translations. In effect

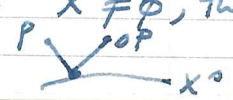
<u>Case 1.</u> $X^{G_{n-1}} \neq \phi$. Now use G_n/G_{n-1} acting in $X^{G_{n-1}}$ to get the desired result.

<u>Case 2.</u> $X^{G_{n-1}} = \phi$ but $\exists$ line L stable under G_{n-1}. Then L is unique so L is also stable under G. G has to act as translations, because $G/G_{n-1} \longrightarrow$ dihedral, where G/G_{n-1} is either $\mathbb{Z}$ or finite of odd order & no fixpts $\Rightarrow$ G acts as translations.

I forgot:

Example: $G = G_1 *_A G_2$ $\quad s_1 \in G_1 - A,\ s_2 \in G_2 - A \Longrightarrow s_1 s_2$ not conjugate to elements of G_1 or G_2. For consider the tree



If $s_1 s_2$ has a fixpoint, then by previous arg. which showed that when one has s auto of $X \ni X^s \neq \phi$, then the midpoint of $P - sP$ is in X^s for all P  $\Longrightarrow s_1 s_2 P_1 = P_1 \Rightarrow$

$s_2 P_1 = P_1 \Rightarrow s_2 \in G_2 \cap G_1 = A$ et ce n'est pas le cas.

2nd Cor. of proposition: If as before G nilp. de t.f. agissant sur X un arbre, and $s^n \in (G,G)$, then $X^s \neq \emptyset$.

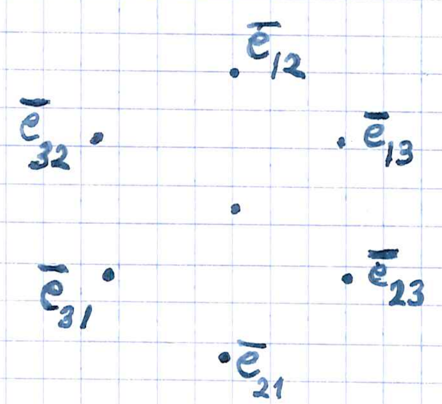
Application: $SL_3(\mathbb{Z})$ a la propriété (FA).

Pf: Suppose $SL_3(\mathbb{Z})$ acts on X. Elementary matrices.

$$(\bar e_{ij}, \bar e_{jk}) = \bar e_{ik} \qquad i,j,k \text{ distinct}$$

$$(\bar e_{ij}, \bar e_{kj}) = \bar e_{ik}^{-1}$$

$$\bar e_{ij} = 1 + e_{ij}$$



each element commutes with its neighbor and is essentially the commutator of its neighbors.

1rst Cor. of Prop. If G is generated by $s_i \ni X^{s_i} \neq \emptyset \Rightarrow X^G \neq \emptyset$.

(Clear because if $X^G \neq \emptyset$ we know $\exists!$ line L stable under G on which G acts by translations. And if s has fixpoints then taking $P \in L$ we know the midpoint of $P - sP$ is fixed $\Rightarrow$ L contains fixpoints of s and since it acts by translations L consists of fixpts.)

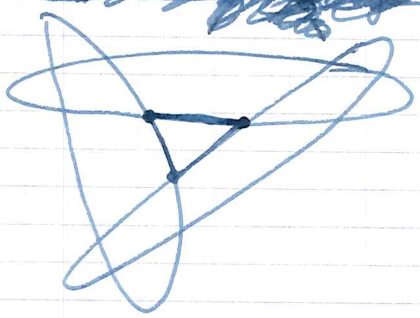
Thus 2nd cor. applied to $\langle \bar e_{12}, \bar e_{13}, \bar e_{23} \rangle \Rightarrow \bar e_{13}$ has fixpts. Thus all $\bar e_{ij}$ have fixpts. So 1rst corollary $\Rightarrow$ each Borel $\langle \bar e_{ij}, \bar e_{ik}, \bar e_{jk} \rangle$ has fixpts. Thus have

$X^{\bar e_{12}}, X^{\bar e_{23}}, X^{\bar e_{31}}$ mutually intersecting trees

But finally have

<u>Lemma</u>: If X_i is a finite family of subtrees of X $\ni$ $X_i \cap X_j \neq \phi$ $\forall\, i,j$, then $\bigcap X_i \neq \phi$.

For $n=3$



one chooses $P_{12} \in X_1 \cap X_2$, $P_{23} \in X_2 \cap X_3$, $P_{13} \cap X_{13}$ such that the sum of the distances $P_{12} - P_{13} - P_{23} - P_{12}$ is min. If $X_1 \cap X_2 \cap X_3 = \phi$, then none of these distances are zeros. And one sees that a redundancy in the above loop is impossible by its minimality. This contradicts fact we have a tree. For $n > 3$ use induction replacing X_{n-1}, X_n by $X_{n-1} \cap X_n$.

Serre's course : Lecture 4, November 27, 1973

Theorem : G a "Chevalley" group (simple & simply conn.) over $\mathbb{Z}$ of rank ≥ 2 $\Rightarrow$ $G(\mathbb{Z})$ has the property (FA).

Lemma: Let R be the root system of G and B a set of simple roots. Then G is generated by the set of $x_\alpha = x_\alpha(1)$ where α runs over the subset T of R given by

$$T : \begin{cases} \alpha \in B & \text{(i.e. } \alpha > 0 \text{ simple)} \\ \alpha < 0,\ -\alpha \notin B & \text{(i.e. } \alpha < 0 \text{ non-simple)}. \end{cases}$$

Granted this the proof goes as for $SL_3(\mathbb{Z})$. $X^\alpha = X^{x_\alpha}$

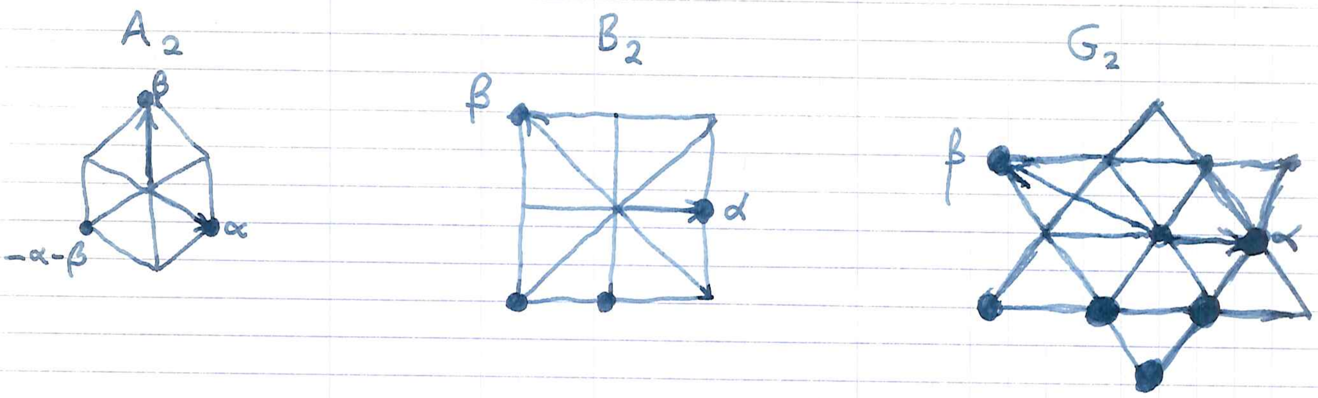
1) $X^\alpha \neq \phi$ $\forall \alpha$. In effect one can always choose an ordering so that $\alpha > 0$ and α is not a simple root. It follows that x_α will be a commutator in the corresponding Borel subgroup's nilpotent part. But Serre showed that when a nilpotent f.t. group acts on a tree, the commutator subgp. has fixpts.

2) $X^\alpha \cap X^\beta \neq \phi$ for $\alpha, \beta \in T$. In this case one knows that by the choice of T, the roots of the form $i\alpha + j\beta$ lie on one side of a hyperplane. This means that the subgroup generated by x_α, x_β is nilpotent. But when a nilpotent f.t. subgp acts on a tree & all the generators have fixpts, the whole group has fixpts.

3) $\bigcap_{\alpha \in T} X^\alpha = X^G$ because the x_α $\alpha \in T$ generate. This intersection is $\neq \phi$ by lemma on finite $\cap$'s of trees.

By Steinberg $G(\mathbb{Z})$ gen. by x_α.

Proof of the lemma: He looks at all groups of rank 2



And works with the identity

$\alpha+\beta\neq 0$
$$(x_\alpha(t), x_\beta(u)) = \prod_{i,j\geq 1} x_{i\alpha+j\beta}(c_{ij}\, t^{e_{ij}} u^{f_{ij}})$$

which gives

$$(x_\alpha, x_\beta) = x_{\alpha+\beta}^{\pm 1} \prod_{\substack{i,j\geq 1\\ |i+j|\geq 3}} x_{i\alpha+j\beta}^{N_{ij}}$$

apply this to $\beta, -\alpha-\beta$ and get $-\alpha\in T$

Yuck! so have symetry s_α.

Same argument works for $G(\mathbb{Z}[\frac{1}{N}])$. The only point to be checked is that the $x_\alpha(\frac{1}{N})$ generate, which means you have to show that $x_\alpha(\frac{1}{N^2})$ lies in grp gen. by $x_\alpha(\frac{1}{N})$, $x_\alpha(-\frac{1}{N})$; this reduces to a calculation in $SL_2(\mathbb{Z}[\frac{1}{N}])$.

Problème: Do the congruence groups in $G(\mathbb{Z})$ have the property (FA), e.g. for those in $SL_3(\mathbb{Z})$.

Positive interpretation of (FA):

Prop: Given $\rho: G \to GL_2(k)$ with G f.t. having (FA), then the propre values of $\rho(g)$ are integral over $\mathbb{Z}$.

Proof: Have to show $\mathrm{Tr}\,\rho(g)$, $\det \rho(g) \in \overline{\mathbb{Z}}$ (or $\overline{\mathbb{F}}_p$ if char $k = p$). To simplify suppose k char. 0. As G fin. type can suppose k finite type extension of $\mathbb{Q}$.

Since G^{ab} is finite, $\det \rho(g) \in \overline{\mathbb{Z}}$ so all we have to do is show that $\mathrm{Tr}\,\rho(g)$ is integral. If not, because K f.t. over $\mathbb{Q}$ $\exists$ valuation discret on K $\ni$ $v(\mathrm{Tr}\,\rho(g)) < 0$. Then $\rho(G) \subset \{\alpha \in GL_2(k_v) \mid v \det(\alpha) = 1\}$, and this acts on the building of k^2. Since G has (FA) $\exists$ fixpt meaning that $\rho(G)$ leaves fixed a lattice $\Rightarrow$ $v(\mathrm{Tr}\,\rho(g)) \geq 0$.

Above may not be too useful. Cong. subgp. problem for $SL_3(\mathbb{Z}) \Rightarrow$ Given $SL_3(\mathbb{Z}) \longrightarrow GL_n(k)$ it agrees on a finite index subgp with an algebraic rep. of SL_3 on k^n. Hence can't have such homoms.

Serre's course: Dec. 4, 1973

Ends:

Let X be locally compact, locally connected, and connected. Let K be a compact subset of X, and Ω_K the set of non-rel.-compact components of $X-K$. Assume Ω_K finite for all K. (perhaps this follows in general). If $K \subset K'$, then $\Omega_{K'}$ maps onto Ω_K so we have a projective system of finite sets. Then the space of ends of X is:

$$X^b = \varprojlim \Omega_K .$$

It is compact and totally-disconnected.

(Reason Ω_K is finite. Choose a compact K_1 such that $K \subset \mathrm{Int}(K_1)$. Let U_i be the non rel. comp. components of $X-K$, $i \in \Omega_K$, and suppose Ω_K is infinite. Claim $\partial K_1 \cap U_i \neq \phi$. Otherwise $\partial K_1 \cap U_i \neq \phi \Rightarrow U_i = (U_i \cap X-K_1) \amalg (U_i \cap \mathrm{Int}\, K_1) \overset{U_i \text{ conn.}}{\Longrightarrow}$ either $U_i \subset X-K_1$ or $U_i \subset \mathrm{Int}\, K_1$; latter impossible as U_i is not rel. compact; $U_i \subset X-K_1 \Rightarrow U_i$ is closed in X which contradicts X connected. ~~Now suppose Ω_K is infinite and let $x \in \partial K_1$ be an accumulation point of the $U_i \cap \partial K_1$~~

But then $X-K = \coprod_{i \in \Omega_K} U_i \amalg$ other $U_i \Rightarrow \partial K_1 = \coprod_{\text{all } i} \partial K_1 \cap U_i$ and as ∂K_1 compact $\Rightarrow$ only finitely many $\partial K_1 \cap U_i \neq \phi$ $\Rightarrow \Omega_K$ finite. This argument shows $\Omega_{K_1} \twoheadrightarrow \Omega_K$).

Now one can compactify X by putting

$$\tilde{X} = X \cup X^b$$

with the evident topology (a nbd. of an end is given by an element of $\mathcal{I}_K$ same K and it consists of U together with all other ends "belonging" to U.) One can characterize this compactification as follows:

Suppose $Y = X \cup F$, X open, $\overline{X} = Y$, Y compact, F tot. disc. such that U open conn. in $Y \to U \cap X$ conn. Then $Y = \tilde{X}$

~~(Why: let U run over the comm. nbds. of $y \in F$, then $U \cap X$ is a conn. open set~~

Examples: 1) Y compact Riemann surface, $X = Y - F$ where F is totally disconnected.

2) (non-trivial) Let S be a compact R. surface of genus g. $\pi_1(S)$ gen. by $a_1, b_1, \ldots, a_g, b_g$ relation $\prod_{i=1}^{g}(a_i, b_i) = 1$. Let X = covering corresp to normal subgp gen by $b_1, \ldots, b_g$, so that $X \to S$ is a principal covering for the group Γ = free gp on g-generators.

Thm: (Koebe): $X \cup X^b = S_2$ and action of Γ on X extends to an analytic action on S_2. (Γ is a Shottky group)

3) X tree. Then ends are the same as equivalence classes of half-lines.

Proposition: Exact sequence

$$0 \longrightarrow A \longrightarrow H^0(X^b, A) \longrightarrow H^1_c(X, A) \longrightarrow H^1(X, A)$$

Proof: Let F be a closed set $\ni$ $X - F$ rel. comp.

$$0 \to \underbrace{H^0_c(X-F; A)}_{=0} \longrightarrow H^0(X, A) \longrightarrow H^0(F, A) \longrightarrow H^1_c(X-F; A) \longrightarrow H^1(X, A) \longrightarrow H^1(F, A)$$

Now pass to the limit over F.

(In general one sees that ends have something to do with the map $H^q_c(X,A) \longrightarrow H^q(X,A)$. Precisely one has $X \subset X \cup \{\infty\}$ so one has

$$\begin{array}{ccccc} H^q_c(X,A) & \longrightarrow & \tilde{H}^q(X\cup\{\infty\},A) & \longrightarrow & \tilde{H}^q(\{\infty\},A) \\ \| & & \downarrow & & \downarrow \\ H^q_c(X,A) & \longrightarrow & \tilde{H}^q(X\cup X^b,A) & \longrightarrow & \tilde{H}^q(X^b,A) \end{array}$$

$$\begin{array}{ccccc} H^q_{\{\infty\}}(X\cup\{\infty\},A) & \longrightarrow & \tilde{H}^q(X\cup\{\infty\},A) & \longrightarrow & \tilde{H}^q(X,A) \\ & & \uparrow\wr & & \\ & & H^q(X_c,A). & & \end{array}$$

Thus perhaps you should think of the map $X \to X\cup\{\infty\}$ and forming the cone $\mathrm{Cone}(X)\cup_X(X\cup\{\infty\})$. Thus you collapse the compact ~~parts of~~ subsets of X to a point in $X\cup\{\infty\}$ and

$$\pi_1\left(\mathrm{Cone}(X\to X\cup\{\infty\})\right) = X^b.\ )$$

<u>Ends for groups.</u> Let G act properly on X (conn., loc. conn., loc. compact) and suppose X/G is compact. <u>Claim</u> <u>G is finitely generated</u>: <u>Proof</u>: Let $K \subset X$ be $\ni \mathrm{Int}\,K \twoheadrightarrow X/G$. and let $S = \{g\in G \mid gK\cap K \neq \phi\}$. S finite since action is proper. To show S gen. G. Let G' = subgp. gen. by S. Then $G'K$ is closed and open hence $G'K = X$. Given $g\in G$ then $gK \cap g'K \neq \phi$ $g'\in G'$ $\Rightarrow (g')^{-1}g \in S \Rightarrow g\in G'$. Done.

Now given G finitely generated, let S be a fin. set of generators such that $S = S^{-1}$ and make an oriented graph whose vertices are G and such that one has an edge from g_1 to g_2 for each $s\in S$ such that $g_1 s = g_2$

More precisely take ~~F(S) acting on universal cover~~

$Y =$ bouquet of circles with one loop for each element of S, ~~let~~ so that $\pi_1(Y) = F(S)$ and let $X =$ covering with $\pi_1(X) = \operatorname{Ker}(F(S) \to G)$.

Define the space of ends of G to be

$$G^b = X^b.$$

To show this is independent of X one ~~proves~~ argues

~~Prop: $H^0(G^b, A) = H^1(G, A[G])$~~

~~The point is that one considers functions $f: G \to A$~~ as follows. ~~Call a function $f: G \to A$~~ One identifies $H^0(X^b, A) =$ cont. functions $f: X^b \to A$ with functions $G \to A$ which are almost invariant (differs from its translate by a function with finite support) ~~mod the~~ modulo ~~constants~~. functions ~~which~~ with finite support.

Serre's course, January 8, 1974.

Groupes with duality – paper of Bieri-Eckmann to appear in Inventiones.

Γ discrete group. One says Γ has the property (FP) if $\exists$ a finite $\mathbb{Z}[\Gamma]$–resolution of $\mathbb{Z}$

$$0 \to P_n \to \cdots\cdots \to P_0 \to \mathbb{Z} \to 0$$

where the P_i are $\mathbb{Z}[\Gamma]$-modules projective of finite type. One says Γ has the property (FL) if in addition the P_i can be chosen free. Obstruction for a Γ with (FP) to be (FL) is an element of $\tilde{K}_0(\mathbb{Z}[\Gamma])$, but there are no examples known where it is $\neq 0$.

The way one obtains such Γ: X finite complex conn. with basepoint $\pi_1(X) = \Gamma$, $\pi_i(X) = 0$, $i > 0$, hence $X = B\Gamma$. Then $\tilde{X}$ = universal cover of X is contractible and the complex of chains

$$0 \to C_n(\tilde{X}, \mathbb{Z}) \to \cdots\cdots \to C_0(\tilde{X}, \mathbb{Z}) \to \mathbb{Z} \to 0$$

is a finite resolution of $\mathbb{Z}$ by free f.t. $\mathbb{Z}[\Gamma]$-modules. Conversely, if Γ is finitely presented and of type FL we can construct such an X. (Thus $B\Gamma$ is ~~a~~ a finite complex $\Longleftrightarrow$ Γ f.p. + of type FL).

Suppose now that Γ has the property FP. If M is a Γ-module put

$$M^* = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(M, \mathbb{Z}[\Gamma]).$$

It is naturally a right Γ-module, hence a left one using the involution $\omega : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ $\quad \gamma \mapsto \gamma^{-1}$.

If M is f.t. proj, so is M^*. One calls $P^{\cdot *}$ the dualizing complex.

$$H^i(P^{\cdot *}) = H^i(\Gamma, \mathbb{Z}[\Gamma]) \qquad \text{(clear)}$$

$$= H^i_c(\tilde{X})$$

To see this one uses the Leray spectral sequence of the map $f: \tilde{X} \longrightarrow X$ with compact supports. Since the map is loc. trivial with discrete fibre, it degenerates yielding

$$\begin{array}{ccc} H^p(X, f_! \mathbb{Z}) & = & H^p_c(\tilde{X}) \\ \| & & \\ H^p(\Gamma, \mathbb{Z}[\Gamma]) & & \end{array}$$

Duality thm: Assume $C^i = H^i(P^{\cdot *})$ or M are without torsion. Then $\exists$ sp. seq.

$$E_2^{pq} = H_{-p}(\Gamma, C^{+q} \otimes_{\mathbb{Z}} M) \Longrightarrow H^{+p+q}(\Gamma, M)$$

Proof.

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(P_\cdot, M) = (P_\cdot)^* \otimes_{\mathbb{Z}[\Gamma]} M$$

so

$$H^i(\Gamma, M) = H_{-i}(\Gamma, P_\cdot^* \otimes_{\mathbb{Z}} M)$$

$$\Uparrow$$

$$E^2_{pq} = H_p(\Gamma, H_q(P_\cdot^* \otimes M))$$

$$\|$$

$$C_q \otimes_{\mathbb{Z}} M.$$

etc.

<u>Remark</u>: Serre wants a proof of

$$E_2^{pq} = H_p(\Gamma, H_c^q(\bar{X}, \bar{F})) \Longrightarrow H^{p+q}(X, F)$$

when $\bar{X} \longrightarrow X$ is a covering with group Γ, X is compact and F is a sheaf on X with inverse image $\bar{F}$. At the moment he proves this for (X, F) triangulable, or when $cd(\Gamma) < \infty$, but the demonstration is unpleasant.

<u>Cor</u>: If $C^i = 0$ $i \neq d$ and $C = C^d$ is torsion-free we have

$$H_{d-q}(\Gamma, C \otimes M) \xleftarrow{\sim} H^q(\Gamma, M)$$

This isomorphism is given by capping with a fundamental class in $H_d(\Gamma, C)$.

<u>Def</u>: Γ is a <u>group with duality</u> if it of type (FP) and $H^i(\Gamma, \mathbb{Z}[\Gamma]) = \begin{cases} 0 & i \neq d \\ C & i = d \end{cases}$ where C is torsion-free.

<u>Example</u>: 1. Poincaré group: Here $C \simeq \mathbb{Z}$.
If X comp. manifold without boundary which is a $K(\Gamma, 1)$, then $\tilde{X}$ is a contractible manifold, so $H_c^i(\tilde{X}) = \begin{cases} 0 & i \neq d \\ \simeq \mathbb{Z} & i = d \end{cases}$
but P.D. for $\tilde{X}$.

<u>Ex. 2</u>: Suppose $X = K(\Gamma, 1)$ is now a comp. manifold with boundary of dimension n.

$$H^i_c(\tilde{X}) \xrightarrow{\sim} H_{n-i}(\tilde{X}, \partial\tilde{X}) \otimes \Omega_{\tilde{X}}$$

where $\Omega_{\tilde{X}}$ is the orientation module of $\tilde{X}$; (it is $\simeq \mathbb{Z}$).
Because $\tilde{X}$ is contractible, this is

$$\xrightarrow{\sim} \tilde{H}_{n-i-1}(\partial\tilde{X}) \otimes \Omega_{\tilde{X}}$$

<u>Example</u>: Let N be a knot in S^3, $\Gamma = \pi_1(S^3 - N)$, $X = S^3 -$ open tubular nbd. of N. One knows X is a $K(\Gamma, 1)$ (asphericity of knots). ∂X is a torus, and one knows

$$\pi_1(\partial X) = \mathbb{Z}\times\mathbb{Z} \longrightarrow \pi_1(X) = \Gamma$$

is <u>injective</u> provided N is knotted. Thus $\partial\tilde{X}$ is a disjoint union of 2-planes. Hence one has

$$H^i_c(\tilde{X}) \xrightarrow{\sim} \tilde{H}_{3-i-1}(\partial\tilde{X}) \otimes \Omega_{\tilde{X}} = \begin{cases} 0 & i \neq 2 \\ \text{torsion free} & i = 2 \end{cases}$$

so Γ is a group with duality of dimension 2.

<u>Note</u>: If Γ is a duality group of dim $d \geq 2$, then $H^1(\Gamma, \mathbb{Z}[\Gamma]) = 0$, so Γ has <u>one</u> end. Eckmann + B.... show using Stallings thm:

<u>Theorem</u>: Γ fin. pres. $cd(\Gamma) = 2 \Longrightarrow \Gamma = \Gamma_1 * \cdots * \Gamma_k$ where Γ_i are groups with duality of dim 2 or $\mathbb{Z}$.

(Thus $cd(\Gamma) = 2 \Longrightarrow$ (Γ has duality $\Longleftrightarrow$ Γ has one end).

Serre's lecture at IHES, January 8, 1974

Serre gives two reasons for being interested in the cohomology of discrete groups.

① Construction of admissible representations of p-adic groups. Not very much known here, but here is an example:

$$\Gamma = Sp_{2n}(\mathbb{Z})$$

$$\Gamma_{p^n} = \text{subgroup} \equiv 1 \mod p^n.$$

$$V^i = \varinjlim_n H^i(\Gamma_{p^n}, \mathbb{Q})$$

Since $\Gamma_{p^{n+1}}$ is normal of finite index in Γ_{p^n} one has

$$H^i(\Gamma_{p^n}, \mathbb{Q}) \xrightarrow{\sim} H^i(\Gamma_{p^{n+1}}, \mathbb{Q})^{\Gamma_{p^n}/\Gamma_{p^{n+1}}}$$

whence the fixpts of Γ_{p^n} on V^i are finite-dimensional. First Serre remarks that $Sp_{2n}(\mathbb{Z}[\frac{1}{p}])$ acts on V^i in a natural way, because it acts continuously with respect to the topology defined by the Γ_{p^n}. Then he remarks that since each element of V^i is fixed by some Γ_{p^n} the action extends to the completion $Sp_{2n}(\mathbb{Q}_p)$. This representation is admissible (stabilizer of each vector open + fixpts of open subgroups are finite dimensional)

The case of SL_2 shows these representations are highly interesting. One doesn't know anything about them for $SL_3(\mathbb{Z})$.

(2) Euler characteristics: Suppose Γ such that $cd(\Gamma) < \infty$ and such that $H_*(\Gamma, \mathbb{Z})$ fin. generated. Then one puts

$$\chi(\Gamma) = \sum (-1)^q \dim_k H_q(\Gamma, k)$$

where k is any field (the universal coeff. formula shows this doesn't depend on k).

More generally if Γ is a group having a subgroup Γ' of finite index such that Γ' satisfies the above two conditions one puts (following C.T.C. Wall)

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']} \in \mathbb{Q}.$$

Brown's work shows this doesn't depend on the choice of Γ'. (In the case where Γ' is of type (FL) this is clear using the fact that if $P_\cdot$ is a f.t. free $\mathbb{Z}[\Gamma']$-resolution of $\mathbb{Z}$, then

$$\chi(\Gamma') = \sum (-1)^q \, rg(P_q).)$$

Theorem 1: Suppose $\chi(\Gamma)$ defined (i.e. $\exists\, \Gamma'$ of fin. index $\ni cd(\Gamma') < \infty$ and $H_*(\Gamma, \mathbb{Z})$ fin. gen.) Let m be the g.c.d. of the orders of the torsion subgroups of Γ (note $T \subset \Gamma$ torsion and if $\Gamma' \triangleleft \Gamma$ is torsion-free, then $T \hookrightarrow \Gamma/\Gamma'$ so m is finite). Then

$$m\,\chi(\Gamma) \in \mathbb{Z}.$$

This reduces to

Theorem 2: Suppose G is a finite group acting on a simplicial complex X simplicially. Assume $\dim(X) < \infty$, ~~and let m be an integer such that~~ and that $H_*(X, \mathbb{Z})$ finitely generated. Let m be an integer which divides the cardinality of every orbit of X. Then m divides $\chi(X)$.

Proof of thm. 2: Can assume $m = p^k$, p prime. Let H be a Sylow p-subgroup of G, $x \in X$, H_x, G_x its stabilizers in H, G respectively. By assumption

$$p^k \mid [G:G_x] \mid [G:H_x] = [G:H]\cdot[H:H_x] \implies p^k \mid [H:H_x]$$

since $[G:H]$ is prime to p. Thus I can ~~assume that~~ replace G by H and so I reduce to the case where G is a p group, $m = p^k$. Here we will need only that $H_*(X, \mathbb{F}_p)$ is fin. gen.

Now I propose to use induction on order of G. Let C be a cyclic group of order p contained in the center of G.

(Floyd)
Lemma: C cyclic group of order p acting simplicially on X, $\dim(X) < \infty$, $H_*(X, \mathbb{F}_p)$ fin type. Then X^C and X/C also satisfy these conditions and

$$\chi(X) = p\,\chi(X/C) - (p-1)\,\chi(X^C)$$

Assuming this, one ~~takes~~ applies induction hyp. to the G/C spaces X/C and X^C. $p^k \mid$ card orbits of X $\implies p^k \mid$ card orbits $X^C \implies p^k \mid \chi(X^C)$; —— —— $= p^{k-1} \mid$ card orbits $X/C \implies p^{k-1} \mid \chi(X/C)$, so done.

Proof of Floyd lemma. Take $p=2$, $\Lambda = \mathbb{F}_2[C] = \mathbb{F}_2[\pi]/\pi^2$ where $\pi = 1-$ generator. Let L denote chains mod 2. ~~Thus we have~~ Subdivide X if necessary so that

$$\cancel{0 \to L(X^C) \oplus \pi L(X) \to L(X) \xrightarrow{\pi} \pi L(X) \to 0}$$

X^C, X/C are simplicial complexes (this is always possible with the 2nd barycentric subdivision.) Then $L_i(X)$ has as base the i-simplices which are permuted by C, and there are two types – the free and fixed orbits.

$$0 \longrightarrow \pi L(X) \longrightarrow L(X) \longrightarrow L(X/C) \longrightarrow 0$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\pi) & \longrightarrow & L(X) & \longrightarrow & \pi L(X) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & L(X^C) \oplus \pi L(X) & & & & & & \end{array}$$

Then one gets

$$\longrightarrow H_{i+1}(X) \longrightarrow H_{i+1}(\pi L(X)) \xrightarrow{\partial} H_i(X^C) \oplus H_i(\pi L(X)) \longrightarrow \cdots$$

So working mod finite groups

$$H_{i+1}(\pi L(X)) \simeq H_i(X^C) \oplus H_i(\pi L(X))$$

Because X finite dimensional $\pi L(X), L(X^C) = 0$ in large degrees $\Rightarrow$ by decreasing induction on i that $H_i(\pi L(X))$, $H_i(X^C)$ are fin. dimensional. $\therefore$ Also $H_i(X/C)$ is fin. diml. So the χ are defined and

$$2\chi(X) = 2\chi(\pi L(X)) + 2\chi(X/C)$$

$$-[\chi(X) = \chi(X^C) + 2\chi(\pi L(X))]$$

$$\therefore \boxed{\chi(X) = 2\chi(X/C) - \chi(X^C)}$$

(Digression – A way to think of the above proof: Recall

$$\longrightarrow H^*_C(X, X^C) \longrightarrow H^*_C(X) \longrightarrow H^*_C(X^C) \longrightarrow \cdots$$

$$\begin{array}{c} \wr\!\wr \\ H^*(X/C, X^C) \end{array} \qquad \text{finite dimensional}$$

which implies the localization thm.

$$H^*_C(X)\,[e^{-1}] \simeq H^*_C(X^C)\,[e^{-1}] = H^*(X^C) \otimes H^*_C[e^{-1}]$$

which shows since ($H^*(X)$ f.d. $\Rightarrow$ $H^*_C(X)$ f.t. over H^*_C) that $H^*(X^C)$ is finite-dimensional.

~~certainly the sequence~~

$$\cancel{\longrightarrow H^*(X, X^C) \longrightarrow H^*(X) \longrightarrow H^*(X^C) \longrightarrow \cdots}$$

~~Suppose~~ If C acts freely on X, then $H^*_C(X) = H^*(X/C)$, and one has a Gysin sequence $(p=2)$

$$\cdots \longrightarrow H^g(X/C) \longrightarrow H^g(X) \longrightarrow H^g(X/C) \xrightarrow{\cdot w} H^{g+1}(X/C) \longrightarrow \cdots$$

which since all the terms are finite-dimensional gives $\chi(X) = 2\chi(X/C)$. So it seems clear now that we can get a topological version of the Floyd proof, i.e. valid for finite dimensional paracompact G-spaces.)

__Cor:__ G p-group, X G-space finite dimensional $H^*(X, \mathbb{F}_p)$ fin. dim. $\Rightarrow$ same true for X^G. Moreover $H^*(X^G, \mathbb{F}_p)$ is a __weak^G homotopy invariant__ of X.

(__weak__ here means that $X \to Y$ a G-map which is an heq forgetting the G-action)

(

Serre's course: January 15, 1974

Complements on groups with duality:

Recall that for Γ $\Rightarrow X = B\Gamma$ is a finite complex, one has a canonical isom

$$H^i(\Gamma, \mathbb{Z}[\Gamma]) \simeq H^i_c(\tilde{X})$$

so one has for Γ' of finite index in Γ a canon. isom

$$H^i(\Gamma', \mathbb{Z}[\Gamma']) = H^i(\Gamma, \mathbb{Z}[\Gamma]).$$

This holds in general without finiteness assumptions by the Shapiro lemma, the point being that $f_!$ and f_* are the same for finite coverings.

Let the $C(\Gamma)$ denote the group of germs of autos. of Γ, a germ being an isom $\Gamma' \xrightarrow{\sim} \Gamma''$ between two subgroups of finite index of Γ. Then it is clear that $C(\Gamma)$ acts on $H^*(\Gamma, \mathbb{Z}[\Gamma])$. Example: If $\Gamma = SL_n(\mathbb{Z})$, $C(\Gamma) = PSL_n(\mathbb{Q})$ for $n \geq 3$.

Examples of groups with duality:

1.) $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$, Γ', Γ'' have duality $\Rightarrow$ Γ also with $cd(\Gamma) = cd(\Gamma') + cd(\Gamma'')$ and $C_\Gamma = C_{\Gamma'} \otimes C_{\Gamma''}$. In particular Γ', Γ'' Poincaré $\Rightarrow$ same for Γ.

2). One-relator groups. Take $F = F\{x_1, \dots, x_n\}$ a free group and r a relation which is not an n-th power for any $n \geq 2$. Put Γ = F/normal subgp gen by r. One knows (Lyndon) that $cd(\Gamma) = 2$, in fact that $B\Gamma$ is a 2-complex with n 1-cells and one 2-cell.

Serre conjectures that Γ is a Poincaré group iff r is equivalent to one of the standard relations giving a Riemann surface.

3). (Wall & Thompson). Take a knotted knot in S^3, and let X be the complement of a tubular nbd. of the knot, so that $\partial X = T^2$. Because the knot is knotted $\pi_1 \partial X \hookrightarrow \pi_1 X$. Now take two such knots with complements X_1, X_2 and form ~~the connected sum.~~ $X = X_1 \cup_{T^2} X_2$. It is a compact 3-manifold which is a $K(\Gamma, 1)$; in general $B\Gamma_1 \cup_{BA} B\Gamma_2 = B(\Gamma_1 *_A \Gamma_2)$ if $A \hookrightarrow \Gamma_1$, $A \hookrightarrow \Gamma$. So one gets a non-trivial Poincaré group of dimension 3.

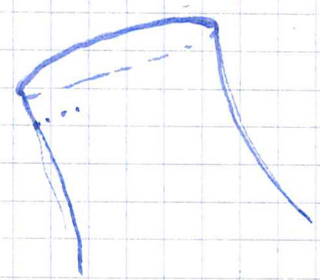
Now and for the next ~~lecture~~ time Serre will try to explain why arithmetic groups without torsion are groups with duality.

G alg. group over $\mathbb{Q}$ (perhaps simple)
Γ arithmetic subgroup
X = symmetric space of $G(\mathbb{R}) \sim \mathbb{R}^n$

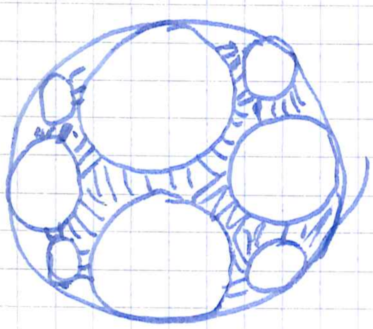
~~One constructs a variety with corners $\bar{X}$ such that if Γ has no torsion, then $\bar{X}/$~~

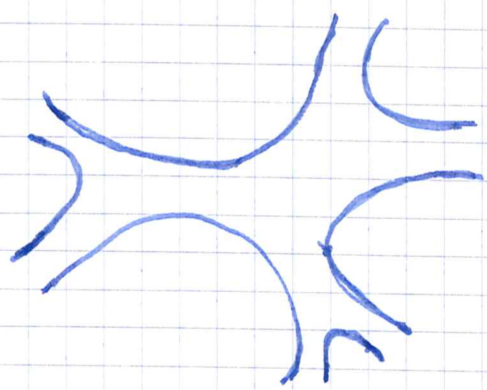
One adds "corners" to X to get a manifold with corners $\bar{X}$. $\partial \bar{X}$ is stratified with one stratum $e(P)$ for each parabolic P in G, $P < G$. ~~$\bar{X}$ is the~~ Each $\overline{e(P)}$ is contractible $\Rightarrow$ $\partial \tilde{X}$ has the homotopy type of the Tits building of G.

Pictures for $SL_2(\mathbb{Z})$. Here X = upper half plane or the unit disc. A parabolic corresponds to a rational point P on the real axis. One will add a line to X for each such point P so as to obtain a manifold with boundary $\bar{X}$. Recall that if one removes an open collar around

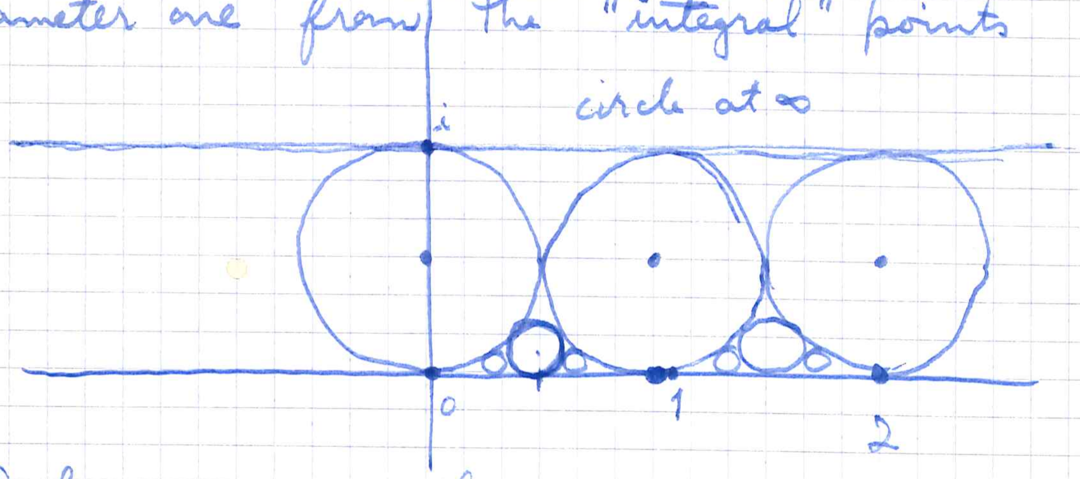


$\partial \bar{X}$ ~~one~~ obtain something diffeomorphic to $\bar{X}$. Thus we can visualize $\bar{X}$ by removing suitable nbds from X of the pts P. Picture





Better to use half plane: First one remove circles of diameter one from the "integral" points



Here I have removed the maximal circles. The diameter around the point with diameter n is essentially given by the nth term in the Fary sequence
See Rademacher where this picture occurs in the theory of the partition function

Serre's seminar, Nov 7, 1973

The subject is to present work of Manin in three papers, the first two of which have been translated + deal with $\Gamma_0(N)$, the last which deals with $SL_2(\mathbb{Z})$. Serre will present the last first, since it serves as an introduction to the subject.

Bibliography, book of Shimura (Princeton), Ogg (Benjamin) and a Springer Lecture Notes, also elementary stuff in Serre's course of arithmetic. For ~~more~~ intense work one has to read Hecke's works.

$G = SL_2(\mathbb{R})$ acts on $H =$ upper half plane by

$$z \longmapsto \frac{az+b}{cz+d}$$

It is useful to ~~identify H with a disk, for~~ recall that H can be conformally mapped onto any disk in $\mathbb{C}$, preserving circles and angles. Thus when we think of H as the upper half plane, we have specified a point on the boundary.

$$\partial H = \mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$$

$$= SL_2(\mathbb{R}) \Big/ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} .$$

Better: one puts $X =$ space of max comp. subgroups of G, so that H is X with an ∞ singled out

~~Let $\Gamma \subseteq G/\{\pm 1\}$ be a discrete subgroup. Then if $P \in \partial H$ has stabilizer Γ_P, then conjugating P to ∞ $\Gamma_P \subseteq \{az+b\}_{a>0}$ and there are the~~

Let $\Gamma \subset G/\{\pm 1\}$ be a discrete subgroup, then X/Γ is a Riemann surface. In effect there is a unique analytic structure on X/Γ such that $X \to X/\Gamma$ is holomorphic. No problem at points where Γ acts freely, and at a fixpoint Γ_P is cyclic of finite order, so no problem here.

However X/Γ is not usually compact, and so it is necessary to add infinite points (il est necessaire d'ajouter)

Let Γ_P be the stabilizer of $P \in \partial X$. Transforming it to ∞, we get a disc. subgroup of the group $\{az+b \mid a > 0\}$, and there are three possibilities:

i) $\Gamma_P = \{e\}$

ii) Γ_P is a group of translations $z \mapsto z+n$ $\quad n \in \mathbb{Z}$.

iii) Γ_P is $z \mapsto c^n z$ $\quad$ some $c \neq 1$, $c > 0$.

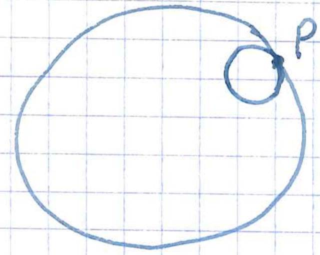
In the case $\Gamma = SL_2(\mathbb{Z})$ which is the only P such that $\Gamma_P \neq 1$ are rational lines:

$$P_1(\mathbb{Q}) \subset P_1(\mathbb{R})$$

which are all conjugate under Γ so case iii) doesn't occur. In the following on takes $\Gamma \subset SL_2(\mathbb{Z})$ and calls "pointe" any P such that $\Gamma_P \neq 1$, in which case it is a group of translations. (Case iii) called hyperbolic fixpoint, Case ii) parabolic).

$\tilde{X} = X \cup$ points $\quad$ topologized so that the nbd of a point P is like $\mathrm{Im}(z) > y_0$ in the case $P = \infty$

Picture of a nbd. of P:



On proves $\tilde{X}/\Gamma$ is separated.

Siegel's thm: $\tilde{X}/\Gamma$ compact $\iff$ $vol(X/\Gamma) < \infty$

I guess this is true for any discrete subgroup Γ of $SL_2(\mathbb{R})/\{\pm 1\}$, which means that above one must define "pointe" for a general discrete Γ. Serre mentions that Siegel's thm. isn't useful because in practice (with $\Gamma \subset SL_2(\mathbb{Z})$) one shows $\tilde{X}/\Gamma$ compact directly, however it is the first result in the general program of finding a natural good compactification of X/Γ for any disc. Γ in a semi-simple real Lie gp. G, such that X/Γ has finite volume, a problem which is on the verge of being solved by Margoulis, etc.

One calls Γ Fuchsian of first kind (première espèce) if $\tilde{X}/\Gamma$ is compact. This is the interesting case.

To show $\tilde{X}/\Gamma$ is a Riemann surface one needs a local parameter at the new pointes. Thus if P is a pointe, conjugate it to ∞, in such a way that Γ_P becomes a group of translations $z \mapsto z+n$. Then a nbd. of P is of the form $y > y_0$ and its orbit by Γ_P becomes isom to the disc $|q| < e^{-2\pi i y_0}$

$$z \longmapsto e^{2\pi i z} = q \qquad |e^{2\pi i z}| = e^{-2\pi \, Im(z)}$$

(If I remember correctly, a Fuchsian group Γ is a discrete subgroup of $SL_2(\mathbb{R})$. ~~[illegible]~~ A "pointe" is a parabolic fixpt. of Γ and $\tilde{X} = X \cup$ pointes. ~~[illegible]~~ It should be so that $\exists$ hyperbolic fix point $\Rightarrow X/\Gamma$ has ∞ volume. So in the good case $\tilde{X}/\Gamma$ compact, ~~[illegible]~~, i.e. Fuchsian of first kind, one gets a compact Riemann surface).

<u>Example:</u> $SL_2(\mathbb{Z})/\{\pm 1\}$. Here one knows $\tilde{X}/\Gamma$ is the Riemann sphere, the identification being given by the j-function. One has one cusp at ∞, and ram. pts i of order 2

$\frac{1}{2}+i\frac{\sqrt{3}}{2}$ —— 3

<u>Modular forms:</u> Let k be an even integer ≥ 0. A modular form of weight k is a holomorphic function f on H s.t.

i) $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \frac{az+b}{cz+d} \in \Gamma$$

ii) f should be holomorphic at the pointes.

Other ways of interpreting i).

$$d\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2}dz = (cz+d)^{-2}dz$$

of the sort that condition i) expresses that ~~[illegible]~~

$$f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right)^{k/2} = f(z)\, dz^{k/2} \qquad \frac{az+b}{cz+d} \in \Gamma$$

i.e. the form $f(z)\,dz^{k/2}$ is Γ-invariant. If Γ acts freely on X, this means f is the same as a section of $\Omega_{X/\Gamma}^{\otimes k/2}$.

Digression: Suppose Γ is the cyclic group of order e acting on $\mathbb{C}$ by $z \longmapsto \zeta z$ $\quad \zeta$ primitive e-th root of 1. Then $\mathbb{C}/\Gamma \simeq \mathbb{C}$ the map being $z \longmapsto z^e = w$, and we have

$$\left(\mathcal{O}_{\mathbb{C},0}\right)^{\Gamma} \xleftarrow{\sim} \mathcal{O}_{\mathbb{C}/\Gamma,0}.$$

i.e. any series $f(z) = \sum a_n z^n \ni f(\zeta z) = f(z)$ is a series in z^e. Now what is the map

$$(*) \qquad \left(\Omega_{\mathbb{C},0}^{\otimes k/2}\right)^{\Gamma} \longleftarrow \Omega_{\mathbb{C}/\Gamma,0}^{\otimes k/2}$$

Suppose $z^n (dz)^{k/2}$ is γ-invariant, i.e.

$$\zeta^{n+k/2} = 1 \qquad \text{i.e.} \qquad n+k/2 \equiv 0 \quad (e)$$

Therefore you get all n of the form

$$n = -\frac{k}{2} + me$$

which are ≥ 0, i.e. such that

$$m \geq \frac{k}{2e}$$

Suppose on the other hand that we consider something coming from $\Omega_{\mathbb{C}/\Gamma,0}^{\otimes k/2}$ i.e.

$$(z^e)^{\nu} (dz^e)^{k/2} = z^{e\nu} (e z^{e-1} dz)^{k/2}$$
$$= z^{e\nu + (e-1)k/2} (dz)^{k/2} \text{ const.}$$

Thus we get all n of the form

$$n = e\nu + (e-1)k/2 = -\frac{k}{2} + \Big(\overbrace{\nu + \frac{k}{2}}^{m}\Big)e$$

where

$$m = \nu + \frac{k}{2} \geq \frac{k}{2}.$$

Thus the cokernel of the map $(*)$ is of dimension $=$ number of integers m with $\frac{k}{2e} \leq m < \frac{k}{2}$ $=$ number of integers m with

$$\frac{k}{2} - \frac{k}{2e} \geqslant \frac{k}{2} - m > 0$$

which is $\left[\frac{k}{2}\left(1-\frac{1}{e}\right)\right]$.

Now condition ii) that f should be holom. at the pointes can be interpreted as follows. If P is a point, transform it to $P=\infty$ so that $\Gamma_P = \{z \mapsto z+n\}$, and put $q = e^{2\pi i z}$ as usual. Then $f(z+1) = f(z)$ implies f has a Laurent expansion

$$f(z) = \sum_n a_n q^n$$

and for f to be holomorphic means that $a_n = 0$ for $n<0$.

How unique are the coeffs. a_n? We choose g carrying P to ∞ ~~and~~ Γ_P to $\{z \mapsto z+n\}$. Two different g's differ by a g of the form $az+b$ $a>0$ which normalize the subgroup $z \mapsto z+n$. But

$$a\left(n + \frac{z-b}{a}\right) + b = az + an$$

so $a=+1$ if it is to preserve the subgroup Γ_P. So we are allowed to change z to $z+b$ b real which changes q to $e^{2\pi i b} q$. Thus a_0 alone has an invariant meaning.

It makes good sense to speak of the value of a modular form at a pointe. One defines a modular form to be a cusp form at P if it vanishes at P.

Suppose now we try to relate a form $f(z)(dz)^{k/2}$ invariant under $z \mapsto z+1$ to a form $g(q)(dq)^{k/2}$.

$$q = e^{2\pi i z}$$

$$dq = e^{2\pi i z}\, 2\pi i\, dz$$

$$g(q)(dq)^{k/2} = (2\pi i)^{k/2} (e^{2\pi i z})^{k/2} g(e^{2\pi i z})(dz)^{k/2}$$

$$\therefore \quad g(q)(dq)^{k/2} = f(z)(dz)^{k/2}$$

$$\Rightarrow \quad f(z) = (2\pi i)^{k/2} (e^{2\pi i z})^{k/2} g(e^{2\pi i z})$$

or

$$g(q) = (2\pi i)^{-k/2} q^{-k/2} \sum_{n \geq 0} a_n q^n$$

If therefore f is modular it has a pole of order $-k/2$ on the Riemann surface, and one sees therefore that

$$\left(\Omega^{\otimes k/2}_{\tilde{X}, P}\right)^{\Gamma} \longleftarrow \left(\Omega^{\otimes k/2}_{\tilde{X}/\Gamma, \bar{P}}\right)$$

has codimension $k/2$.

Now we are in a position to compute the dimension M_k of the space of modular forms of weight k, by using R-R on the Riemann surface $\tilde{X}/\Gamma$ for the line bundle $\Omega^{\otimes k k}_{\tilde{X}/\Gamma}$, or ~~...~~ rather for the line bundle E_k whose global sections are the modular forms. We have

$$0 \to \Omega^{\otimes k/2}_{\tilde{X}/\Gamma} \longrightarrow E_k \longrightarrow Q_k \to 0$$

supported at the ramification + pointes.

and

$$\mathrm{length}(Q_k) = \sum_{P_i} \left[\frac{k}{2}\left(1 - \frac{1}{e_i}\right)\right]$$

where P_i runs over the bad points and e_i = index of ramification, $e_i = \infty$ if P_i is a point.

~~[illegible]~~ Recall $\deg(\Omega) = 2g-2$ $\quad g$ = genus
and that any line bundle of degree $> 2g-2$ has no H^1.
Thus

$$\deg(E_k) = \frac{k}{2}(2g-2) + \sum_i \left[\frac{k}{2}\left(1-\frac{1}{e_i}\right)\right]$$

$$\dim(M_k) = \deg(E_k) + 1 - g$$

$$\boxed{\dim(M_k) = (k-1)(g-1) + \sum_i \left[\frac{k}{2}\left(1-\frac{1}{e_i}\right)\right] \qquad k \geq 1}$$

If P_k is the space of parabolic forms of weight k, one has

$$\dim(P_k) = \dim(M_k) - (\text{no. of points}) \qquad k \geq 2$$

$$\dim(P_2) = g = \dim(M_2) - (\text{no. of points}) + 1$$

because P_2 = space of holom. 1-forms on $\tilde{X}/\Gamma$ and M_2 = space of 1-forms with simple poles at points and one has sum of residues $= 0$.

Serre said something about the genus being related to $\chi(\Gamma)$. For example suppose Γ' is of finite index in Γ and we consider the map

$$f : \tilde{X}/\Gamma' \longrightarrow \tilde{X}/\Gamma$$

of Riemann surfaces. (Note that P a point for Γ $\Rightarrow$ P also a point for Γ' since $\Gamma'_P = \Gamma_P \cap \Gamma'$ finite index in Γ_P.)

Compute: First suppose $f: Y' \to Y$ is a map of Riemann surfaces of degree d. Then if $P' \in Y'$ lies over $P \in Y$ has ram. index e one sees that

$$(f^*\Omega^1_{Y})_{P'} \longrightarrow \Omega^1_{Y', P'}$$

has codim $e-1$: $\quad d(z^e) = e z^{e-1} dz$. Thus from

$$0 \to f^*\Omega^1_Y \to \Omega^1_{Y'} \to Q \to 0$$

one gets the following formula from taking degrees

$$\boxed{2g'-2 = (2g-2)\deg f + \sum_{P'} (e_{P'}-1)}$$

Example: Take the elliptic curve $y^2 = x(x-1)(x-\lambda)$ and the map to $\mathbb{P}^1$ given by x. This ramifies at $x = 0, 1, \lambda$ with ramification index 2 and at $x = \infty$ with ram. index 2 ~~(take coords $\frac{1}{x}, \frac{1}{y}$ at $x=y=\infty$ & equation becomes $\left(\frac{1}{y}\right)^2 = \frac{1}{x(x-1)(x-\lambda)}$ ... eqn. is ... in homogeneous coords,~~

(the equation in homog. coords is

$$x_0 x_2^2 = x_1(x_1 - x_0)(x_1 - \lambda x_0) \qquad x = \frac{x_1}{x_0} \quad y = \frac{x_2}{x_0}$$

So to get equation at $x = \infty$ put $\quad \frac{1}{x} = \frac{x_0}{x_1} \quad z = \frac{x_2}{x_1}$
and it becomes

$$\frac{1}{x} z^2 = \left(1 - \frac{1}{x}\right)\left(1 - \frac{\lambda}{x}\right)$$

or

$$z^2 = \frac{\left(\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)\left(1-\frac{\lambda}{x}\right)}$$

which is quadratic at $\frac{1}{x} = 0$.)

Thus $$2g'-2 = (2\cdot 0-2)\cdot 2 + 4 = 0$$
and so $g'=1$.

Suppose Γ' is of finite index in Γ and let us consider the map $f : \tilde{X}/\Gamma' \longrightarrow \tilde{X}/\Gamma$. Suppose we determine first whether any inf. point of $\tilde{X}/\Gamma'$ is ramified with respect to f. So let $P \in \partial H$ be a pointe of Γ i.e. $\Gamma_P \sim \mathbb{Z}$. Then if $\pi : \tilde{X} \longrightarrow \tilde{X}/\Gamma$ is the projection we have
$$\pi^{-1}(\pi P) = \Gamma \cdot P \leftarrow\!\!\!\sim \Gamma \times_{\Gamma_P} pt$$
and to understand $f^{-1}(\pi P)$ we have to take the Γ' orbits on this:
$$f^{-1}(\pi P) \simeq \Gamma' \backslash \Gamma / \Gamma_P .$$
So one sees therefore that if we decompose $\Gamma' \backslash \Gamma$ into orbits for the right Γ_P action then each orbit corresponds to a point of $f^{-1}(\pi P)$ whose ramification will be the size of the orbit, e.g. $\Gamma'_P \backslash \Gamma_P$ in the case of $\pi' P$.

Better way of writing the genus formula: Let S be an finite set of points containing the ramified points. Then
$$2g'-2 + \operatorname{card} f^{-1}(S) = (\deg f)(2g-2+\operatorname{card} S)$$

~~If Γ is such that there are no fin. ram. points ... the S ... the set of ... points, so~~

~~$2g-2+\operatorname{card} S =$~~

Special case: Assume Γ has no finite ramification points (equivalent to Γ being without torsion). Then we can take S to be the set of infinite points and we find that $2g-2+\mathrm{card}\{\text{inf. points}\}$ is multiplicative.

Next suppose $Q \in \tilde{X}^{\circ}/\Gamma$ is a finite ramification point, i.e. $Q = \pi P$ where Γ_P is cyclic: Then from $f^{-1}(\pi P) \simeq \Gamma'\backslash\Gamma/\Gamma_P$ we want to manufacture something multiplicative as Γ' varies. Let γ_i be double coset representatives:

$$\Gamma = \coprod_{i=1}^{r} \Gamma'\gamma_i\Gamma_P$$

Thus

$$\Gamma'\backslash\Gamma = \coprod_{i=1}^{r} \Gamma'\backslash\Gamma'\gamma_i\Gamma_P = \coprod_{i=1}^{r} \Gamma'\cap\gamma_i^{-1}\Gamma_P\gamma_i\backslash\Gamma_P$$

$$\Gamma'\cap\Gamma_{\gamma_i^{-1}P} = \Gamma'_{\gamma_i^{-1}P}$$

so

$$[\Gamma:\Gamma'] = \sum_{i=1}^{r} \frac{e_Q}{e_{Q_i}}$$

where $\{Q_i\} = f^{-1}(\pi P)$. Thus

$$[\Gamma:\Gamma']\frac{1}{e_Q} = \sum_{i=1}^{r}\frac{1}{e_{Q_i}}$$

has the desired multiplicativity property. Thus if there is one fin. ram. point Q

$$2g'-2+r+\{\text{inf. points for } \Gamma'\} = [\Gamma:\Gamma']\{2g-2+1+\text{inf. pts for } \Gamma\}$$
$$-\sum\frac{1}{e_{Q_i}} = -[\Gamma:\Gamma']\frac{1}{e_Q}$$

And we finally get:

$$\boxed{2g-2+\sum{}'\left\{1-\frac{1}{e_i}\right\}}$$ is multiplicative with respect to subgroups of finite index.

Notice that this number is

$$\lim_{k\to\infty}\frac{\dim(M_k)}{k/2}.$$

Now for $\Gamma = SL_2(\mathbb{Z})/\{\pm 1\}$ one has

$$2\cdot 0-2+\quad 1+\left(1-\tfrac{1}{2}\right)+\left(1-\tfrac{1}{3}\right) = 1-\tfrac{1}{2}-\tfrac{1}{3} = \tfrac{1}{6}$$

$$= -\chi(\Gamma)$$ known because $SL_2(\mathbb{Z})/\{\pm 1\} = \mathbb{Z}/2 * \mathbb{Z}/3$

Thus one has established for any Γ of finite index in $SL_2(\mathbb{Z})/\{\pm 1\}$ the formula

$$\boxed{2g-2+\sum_{Q\in\tilde{X}_\Gamma}\left\{1-\frac{1}{e_Q}\right\} = \frac{1}{6}\left[PSL_2(\mathbb{Z}):\Gamma\right] = -\chi(\Gamma)}$$

Serre says also that there exists a recipe for calculating the ~~number of points~~ e_Q in concrete cases.

November, 1973

Dan,

Bob Williams could use some help on a problem in category theory (there is some sense of urgency because his theorem just came out in the Annals — confidential). He has a specific situation (or category) where he wants to show the following categorical property:

$\mathcal{C}$ is a category and we define two equivalence relations on endomorphisms

1) given $A \underset{s}{\overset{r}{\rightleftarrows}} B$ we say $rs \sim_s sr$

and extend this by transitivity to an equivalence relation on endomorphisms denoted $f \approx_S g$ (strong shift equivalence)

2) given $A \circlearrowleft f$ and $B \circlearrowleft g$ we say $f \sim_s g$ (weak shift equivalence) if there are diagrams and an integer m ~~so that~~

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "r"'] & A \arrow[d, "r"] \\ B \arrow[r, "g"'] & B \end{tikzcd} , \quad \begin{tikzcd} A \arrow[r, "f"] & A \\ B \arrow[u, "s"] \arrow[r, "g"'] & B \arrow[u, "s"'] \end{tikzcd}$$

so that $rs = g^m$ and $sr = f^m$.

Bob wants to show in his situation (nonnegative integer matrices are the morphisms) that $f \approx_S g$ $\Longleftrightarrow$ $f \sim_s g$. $\Longrightarrow$ is clear but $\Longleftarrow$ is false in a general category. The problem would be to isolate perhaps working categorical sufficient conditions for $\sim_s \;\Longrightarrow\; \approx_S$. (This is related to interesting problems in symbolic dynamics, probability, and Markov chains.)

Bob is around today and goes to England tomorrow.

Dennis

Williams problem:

Let C be a category and let C' be the cat of functors $\mathbb{N} \longrightarrow C$, i.e. objects A of C equipped with an endo. f_A. We will be interested in morphisms $r : (A, f_A) \longrightarrow (B, f_B)$ in C' such that $\exists\, r' : (B, f_B) \to (A, f_A)$ with $r'r = f_A^m$, $rr' = f_B^m$ for some integers m. These r are precisely the ones which become isos. in the Artin-Rees category of C, hence we will call such an r ~~AR-invertible of filtration~~ an AR-isom. of filtration $\leq m$.

Check: 1) If r is an AR-isom of filt. $\leq m$, then it is an AR-isom of filt. $\leq m+1$. Indeed $(f_A r')r = f_A^{m+1}$, $r(f_A r') = f_B rr' = f_B^{m+1}$.

~~are AR-isom. of filt. $\leq m_1, m_2$ resp,~~

2) If $A \xrightarrow{r} B \xrightarrow{s} C$ are AR-isoms of filts $\leq m, n$ resp., then sr is an AR-iso of filt. $\leq m+n$. Indeed from $r'r = f_A^m$, $rr' = f_B^m$, $s's = f_B^n$, $ss' = f_C^n$ we get

$$r's'sr = r' f_B^n r = r'r f_A^n = f_A^{m+n} \quad \text{etc.}$$

3) Suppose $r : A \to B$ is an AR isom and $r', r'' : B \to A$ are AR-inverses:

$$r'r = f_A^m \qquad rr' = f_B^m$$
$$r''r = f_A^n \qquad rr'' = f_B^n$$

Then $\quad r'rr'' = f_A^m r''$ and also $= r' f_B^n = f_A^n r'$

This is the usual uniqueness of an inverse

Actually I notice now that the Artin-Rees stuff is without point. If

$$\begin{tikzcd} A \arrow[r] \arrow[d, "r"'] & A \arrow[r] \arrow[d] & A \cdots\cdots \arrow[d] \\ B \arrow[r] & B \arrow[r] & B \cdots \end{tikzcd}$$

is an isomorphism of ind. objects, then ~~we get $r': B \to A$ with~~

$$f_A^{-1}\,\mathrm{Hom}(?, A) \xrightarrow{\sim} f_B^{-1}\,\mathrm{Hom}(?, B)$$

So taking $? = B$ we get $f_A^{-n} r' : B \to A$ such that ~~$rr' = id_B$ is~~ $r(f_A^{-n} r') = id_B$ in $f_B^{-1}\,\mathrm{Hom}(B,B)$, i.e. $\exists$

$$f_B^N r\,(f_A^{-n} r') = f_B^N$$

$$\| $$

$$r\left(f_A^{N-n} r'\right)$$

so replacing n $\therefore$ get r' such that $rr' = f_B^n$. Moreover r' is unique up to mult. by f_A. Thus

$$r(r' f_B) = f_B^{n+1}$$

$$r(f_A r') = f_B r r' = f_B^{n+1}$$

$\Rightarrow$ $f_A^{N+1} r' = f^N r' f_B$, so replacing r' by $f_A^N r'$ can suppose $f_A r' = r' f_B$. Similarly can arrange $r'r = f_A^n$.

~~[illegible]~~

Conclusion: The $r^*: (A, f_A) \to (B, f_B)$ such that $\exists\, r': (B, f_B) \to (A, f_A)$ with $rr' = f_B^n$, $r'r = f_A^n$ are precisely those maps which become isos. in the ind category.

Williams problem is to show under suitable conditions that ind-isomorphic objects of $\mathcal{C}'$ can be connected by a chain of ind isos. of filtration ≤ 1.

<u>EXAMPLES</u>:

1) Impossible in general: One takes the cat with two objects A, B and morphisms

$$\mathrm{Hom}(A,A) = \{f_A^\nu ; \nu \geq 0\} \qquad \mathrm{Hom}(B,B) = \{f_B^\nu ; \nu \geq 0\}$$

$$\mathrm{Hom}(A,B) = \{f_B^\nu r = r f_A^\nu ; \nu \geq 0\}$$

$$\mathrm{Hom}(B,A) = \{f_A^\nu r' = r' f_B^\nu ; \nu \geq 0\}$$

with composition defined so that

$$r r' = f_B^n$$

$$r' r = f_A^n$$

Then one has no way of factoring r.

~~2) OKAY if f_A^n and f_B^n are isos. for some n. For then if we have $rr' = f_B^m$, $r'r = f_A^m$ changing n to $kn \geq m$ and r' to $f_A^{kn-m} r'$, we can suppose $m = n$ whence r, r' are isos.~~

2) OKAY if f_A and f_B are isos. For then $rr' = f_B^n$, $r'r = f_A^n \Rightarrow r, r'$ are isos.

~~3). Assume that $fA \subset A$ exists. Then~~

$$\begin{tikzcd} fA \arrow[r] & fA \\ A \arrow[u, phantom, "\cap"] \arrow[r, "f"] \arrow[ur, dotted] & A \arrow[u, phantom, "\cap"] \end{tikzcd}$$

3) Let B be a sub object of A, and $r: B \to A$

the inclusion. Assume that $f_A : A \to A$ factors thru r and let $r : A \to B$ be the unique map $\ni$ $f_A = r'r$. Define: $f_B = rr'$. Then

$$r f_A = r r' r = f_B r$$

$$f_A r' = r' r r' = r' f_B$$

and so $r' : B \to A$ is an ind. isom. of filtration ≤ 1.

<u>Conclude</u>: If one has a filtration

$$B = A_n \subset \cdots\cdots \subset A_1 \subset A_0 = A$$

such that $f_A(A_{i-1}) \subset A_i$ in the evident sense, then $(B, f_B = f_A|B)$ is connected by a chain of filt. ≤ 1 isos. to A. (equivalent to A).

<u>Examples</u>: 1) $\mathcal{C}$ = finite sets. In this case one knows f induces an isom. on $f^n A$ for n large. Thus any A is equivalent to a A' such that $f_{A'}$ is an isomorphism. In fact $A' = \varinjlim (A \xrightarrow{f} A \to \cdots)$. Thus A ind. isom. to $B \Rightarrow A' = B' \Rightarrow A, B$ equivalent.

2) Same argument works for vector spaces over a field.

Now try to understand finitely generated free abelian groups, where we have images.

Claim any A is equivalent to one such that f_A is injective. We know that A equivalent to $f^n A$ and $\operatorname{Ker} f^n = \operatorname{Ker} f^{n+1}$ for n large $\Rightarrow$ f injective on $f^n A$. ($f f^n x = 0 \Rightarrow f^n x = 0$).

Suppose now that A and B are ind. isom. and we want to show they are equivalent. Can assume f_A, f_B are injective. Observe then that if we have $h : A \to B$, $h' : B \to A$ with

$$hh' = f_B^n \qquad h'h = f_A^n$$

then we have

$$\begin{tikzcd} A \arrow[r, hook, "h"] & B \\ & f_B^n B \arrow[u, phantom, "\cup" description] \arrow[ul, dotted, "h' f_B^{-n}"] \end{tikzcd}$$

in other words A is an invariant subobject sandwiched between $f^n B$ and B. One then has the filtration

$$A = A + f^n B \subset A + f^{n-1} B \subset \cdots \subset A + fB \subset B$$

such that $f B_{i-1} \subset B_i$. So thus we have A equiv. to B.

Next try finitely gen. free abelian monoids — the interesting case of non-negative integer valued matrices.

Here we do not necessarily have images nor can we add free submonoids to get another free submonoid.

Case to look for a counterexample is among the nilpotent things. (A, f_A) $f_A^n = 0$.

Try $n=2$. There is a possibility here of enlarging from A to some bigger monoid where some ~~things~~ might be simpler. Suppose $n=2$.

$$f(A) \subset f^{-1}(0)$$

~~If A is freely generated by $a_1, a_2, \ldots, a_n$, then A is~~

Let's see what we can join with 0. The first step we must have

$$\begin{tikzcd} 0 \arrow[r, shift right, "r'"'] & A \arrow[l, shift right, "r"'] \end{tikzcd} \qquad r'r = 0$$

$$0 = rr' = f_A$$

Thus we can get to any A with $f_A = 0$. Next if $f_B = 0$, then

$$\begin{tikzcd} B \arrow[r, shift right, "r'"'] & A \arrow[l, shift right, "r"'] \end{tikzcd} \qquad \left.\begin{array}{l} r'r = f_A \\ rr' = f_B = 0 \end{array}\right\} \Rightarrow f_A^2 = 0.$$

If r' is injective then we can identify r with f_A and so B is a free monoid sandwiched:

$$f(A) \subset B \subset f^{-1}(0)$$

But suppose r' is not injective. Thus ~~suppose we choose B to be a free monoid mapping onto $f(A)$.~~

$rr' = 0$ $\qquad r'r = f_A$

$r'(B) \supset f_A(A)$

?